# THE RIEMANN SOLUTION FOR THERMAL FLOWS WITH MASS TRANSFER BETWEEN PHASES

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ABSTRACT. We are interested in solving systems of balance equations under the approximation of local thermodynamical equilibrium except at very localized locations. This equilibrium occurs for states on a stratified variety called the "thermodynamical equilibrium variety", which is obtained as the domain of the zero-order approximation of an asymptotic expansion for these balance laws. Waves far from thermodynamical equilibrium occur in thin regions of physical space and they form shocks connecting sheets of the stratified variety. In this scenario, we develop the general theory for fundamental solutions of a large class of systems of balance equations. We study all bifurcation loci, such as coincidence and inflection locus and develop a systematic approach to solve problems described by similar equations.

For concreteness, we exhibit the bifurcation theory for a representative system with four equations. This class of equations models thermal flow with mass interchange between phases in porous media appearing in oil recovery. We find the complete solution of the Riemann problem for two-phase thermal flow in porous media with two chemical species; to simplify the physics, the liquid phase consists of a single chemical species. We give an example of steam and nitrogen injection into a porous medium initially saturated with water, with applications to geothermal energy recovery.

## 1. INTRODUCTION

We are interested in systems of balance equations modeling flows in porous media with total mass conservation. For such flows the source terms represent mass transfer of chemical species between phases with no net gain or loss of chemical species mass; these models are called compositional models in petroleum science, see [10, 19, 24]; there is also an equation representing conservation of energy. In these systems the total volume is not conserved, so there exists an independent unknown $u$ representing volumetric flow rate, which appears in a particular way within the flux terms and is not constant, generically. The systems can be written in the following form for $j = 1, \cdots, l$ and $i = l+1, \cdots, m+1$ :

$$\frac{\partial}{\partial t} \mathcal{G}_j(\mathcal{V}) + \frac{\partial}{\partial x} u \mathcal{F}_j(\mathcal{V}) = \hat{q}_j(\mathcal{V})/\epsilon, \qquad (1.1)$$

$$\frac{\partial}{\partial t} \mathcal{G}_i(\mathcal{V}) + \frac{\partial}{\partial x} u \mathcal{F}_i(\mathcal{V}) = 0. \qquad (1.2)$$

The dependent variables in (1.1)-(1.2) are $\mathcal{V} \in \Omega \subset \mathbb{R}^m$ and $u \in \mathbb{R}$. This is a multi-scale problem, because $\epsilon$ is very small; this quantity is the time scale of physical phase changes

that are active in non-equilibrium regions. The source terms $\hat{q}_j(\mathcal{V})/\epsilon$ represent rates of mass transfer of chemical species between different phases.

The implicit equations $(\hat{q}_1, \hat{q}_2, \cdots, \hat{q}_l, 0, \cdots, 0) = 0$ in the system (1.1)-(1.2) represent equilibrium states, which form a stratified variety. On each smooth sheet of this variety, the system reduces to a simpler system of conservation laws with fewer variables, which are called the "primary variables" of this sheet. In the language of statistical physics each sheet is called a phase; we cannot adopt this nomenclature because this word has a different meaning in the context of flows in porous media. We call each sheet (together with its simpler evolution system) a "physical configuration". This equilibrium variety is continuous and piecewise smooth; it is very important because it represents the locus of states under local thermodynamical equilibrium. In this work we will solve (1.1)-(1.2) on this locus.

In (1.1)-(1.2), generically, we have saturation variables, thermodynamical variables and the speed $u$. In our class of models, we assume that pressure variations are so small that they do not affect gas volume, which varies due to temperature or composition changes; this is a realistic assumption for many flows in porous media, see [19]. So the pressure is fixed and the fluid phases main thermodynamical variables are the temperature $T$ and the compositions of each phase. We assume that the pores in the rock are fully filled with fluids (one of the fluids is gaseous). Different fluid phases do not mix microscopically. Each saturation variable in the system is the fraction of the total volume of a fluid phase relatively to the total volume of the fluid phases. If we denote the saturations by $s_1, s_2, \cdots, s_f$, where $f$ is the number of immiscible phases, e.g., water, oil or gas, we know that, see [19]:

$$s_1 + s_2 + \cdots + s_f = 1. \tag{1.3}$$

Depending on its nature, a chemical species may exist in a single phase or it can coexist in several phases. The quantity of each chemical species that exists in each phase is described by concentration variables, which are denoted by $c_{ij}$, i.e., chemical species $i$ in the phase $j$. In (1.1)-(1.2) there is an equation for each $c_{ij}$. The total concentration $c_i$ of a chemical species is related to the concentrations per volume of this species in the phases in which it may exist by the following expression [19]:

$$c_i = c_{i1}s_1 + c_{i2} + \cdots c_{if}s_f. \tag{1.4}$$

For example the chemical species $H_2O$ can exist in two physical phases: gas (as steam in the gaseous water) and liquid water. Thus there is a equation for the steam mass balance and another equation for the liquid water balance, both in the form (1.1); the concentration of total water is given by:

$$c_w = c_{wl}s_w + c_{wg}s_g.$$

In our notation, the number of balance equations that represents mass transfer is $l$. Equations in (1.1) represent the balance of each chemical species in different phases. Equations in (1.2) represent the conservation of chemical species existing in a single phase; there is also an equation representing the conservation of total energy. The remaining $m - l + 1$ equations represent the conservation of chemical species existing in a single phase, where there is no mass transfer, as well as one equation of the total conservation of energy. The remaining variables in the system are temperature $T$, which is the main unknown in the energy conservation equation, and the Darcy speed $u$.

We are interested in the Riemann-Goursat problem with initial and boundary data of the form:

$$\begin{cases} (\mathcal{V}, u)_L, & \text{if } x = 0, \ t \geq 0, \\ (\mathcal{V}, \cdot)_R, & \text{if } x > 0, \ t = 0. \end{cases} \tag{1.5}$$

The injection point is $x = 0$. If the characteristic speeds are positive, we may expect that this problem can be regarded as a Riemann problem with initial data $(\mathcal{V}, u)_L$ for $x < 0$ defined by the injection boundary data at $x = 0$, because the shock trajectories and rarefactions fans in the solution have positive slopes in the $(x, t)$ plane. However, for certain balance laws, such as systems modeling combustion problems, there are backward moving reaction waves, but this type of models are not considered here, see [26]. We prove that it suffices to provide only one Darcy speed as boundary data. Here we choose to specify $u_L$, however, it is also possible to give $u_R$ instead of $u_L$.

It is well known that conservation laws with Riemann data exhibit solutions that are constant along lines with constant slope $x/t$, i.e., the solutions are self-similar. In the plane $(x, t)$, the rarefactions are continuous solutions, while the shocks are the discontinuous ones. Physical considerations indicate that in our model rarefaction waves occur typically within each physical configuration, where Eqs. (1.1)-(1.2) reduce to a corresponding system of conservation laws. We assume that there is very fast mass transfer in the infinitesimally thin space between regions in distinct physical configurations, so we propose shocks linking such configurations. Therefore, we still seek self-similar solutions for (1.1)-(1.2), (1.5); such self-similarity is established in Section 4.2.

An innovative feature of our model deals with phase transitions. In [7], Colombo et. al. studied a problem with phase transitions in $2 \times 2$ systems of conservation laws. Their physical domain was formed by two disjoint sub-domains, which Colombo called phases. The sub-domains are physical configurations in the present work. The phase transition is the jump in the solution with left and right states belonging to different sub-domains. Our model is physically more adequate because it includes also infinitesimally small phase transitions, because the sub-domains may be adjoining, as they often are in realistic models.

Isaacson, Marchesin, Plohr and Temple developed a very general bifurcation theory to obtain the Riemann solution for conservation laws in [11, 12, 13]. In our work, we generalize that theory for the zero-order approximation of balance equations under local thermodynamical equilibrium, with variable velocity $u$.

In Proposition 4.2 we summarize the theory on the solution structure and show that the Riemann solution can be completely obtained utilizing the spaces of primary variables, i.e., in the sheets of the equilibrium variety. Once the rarefaction and shock waves are found in the spaces of primary variables, he secondary variable $u$ can be recovered in terms of the primary variables. In [22], Mota et. al. examined the internal structure of a single combustion wave in a model where the Darcy speed $u$ changes; in the theory presented here $u$ varies in all waves in the flow. Moreover, we prove that if a sequence of waves and states solves the Riemann problem in the primary variables for a given $u_L > 0$, then this sequence when properly scaled provides the solution in the $(x, t)$ plane for any other $u_L > 0$.

In Section 2, by using singular perturbation techniques on the model equation we obtain the formal zero-order approximation solution; this is the asymptotic expansion that motivates our approach. In Section 3, we present a concrete example for the flow of nitrogen, steam and liquid water in a porous medium. This is a representative and interesting example of our theory because it exhibits a non-trivial stratified variety, as well as a new type of wave. In

Section 4, we prove generalizations of the Triple Shock Rule [11] and of the Bethe-Wendroff theorem. These results are important to compare the speed of shocks of different families and obtain Riemann solutions for general data on the local thermodynamical equilibrium variety. As an example, we introduce the theory for the Riemann problem for a 4x4 system with two chemical species and one mass transfer term, related to either evaporation or condensation. However, this theory is clearly valid for more general systems of equations with more chemical species or more physical configurations. This theory generalizes the classical work of Buckley-Leverett designed for incompressible isothermal two-phase flow in porous media.

In Section 4.2 we define the shocks connecting different configurations. In Section 5, we present the fundamental theorems for the bifurcation theory. In Section 6, we obtain the elementary waves in each physical configuration: shocks, rarefactions or contact discontinuities. A new fact is the presence of a rarefaction wave associated to evaporation in the two-phase configuration. We believe that the solution of the system of balance laws (1.1)-(1.2) tends to the solution on the stratified variety for large times.

In Section 7, we present the Riemann solution for the problem of geothermal energy recovery at moderate temperature. Injecting water and nitrogen at moderate temperature into a hot porous rock, the water in the mixture evaporates and rock thermal energy can be recovered. A new feature of this recovery method is that it can be used also for rock below the water boiling temperature (i.e., even if the rock is not very hot). In Section 8, we draw the conclusions. In Appendix A, we present the appropriate thermodynamic laws used in the model used as example, which provide constraints between thermodynamic variables.

## 2. The models

Systems of type (1.1)-(1.2) model thermal compositional flows in porous media. In such systems the pair $\mathcal{W} = (\mathcal{V}, u)$ in $\Omega \subset \mathbb{R}^{m+1}$ is called *state variable*. $\mathcal{G}$ and $\mathcal{F}$ are the vector-valued functions $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_{m+1})^T$: $\Omega \longrightarrow \mathbb{R}^{m+1}$ and $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_{m+1})^T$: $\Omega \longrightarrow \mathbb{R}^{m+1}$, where $u\mathcal{F}_i$ is the flux for the conserved quantity $\mathcal{G}_i$ and $\partial \mathcal{G}_i/\partial t$ is the corresponding accumulation term, for $i = 1, 2, \cdots m + 1$. On the right hand side the first $l$ components of the source term vector $(\hat{q}_1, \hat{q}_2, \cdots, \hat{q}_l, 0, \cdots, 0)/\epsilon : \Omega \longrightarrow \mathbb{R}^{m+1}$ represent mass transfer between phases of each chemical species, while the $l+1$ to $m$ vanishing entries of source vector indicate the conservation of chemical species that remain in a single phase. Finally, the last component $(m+1)$ indicates conservation of total energy. The conserved quantities, fluxes and source terms are $\mathcal{C}^2$ in the whole domain $\Omega$. Physics dictates that the source terms are defined to vanish precisely for states $\mathcal{W}$ in local thermodynamic equilibrium described by equations of states and thermodynamical constraints.

Generically, in the system (1.1)-(1.2), each equation in (1.1) represents the mass balance of a chemical species in each phase while each but the last equation in (1.2) represents the mass conservation of a chemical species that appears in a single phase. It is useful to rewrite the system (1.1)-(1.2) into equations for total conservation of each chemical species in all phases. For $j = 1, 2, \cdots, m-n$ and $i = m-n+1, m-n+2, \cdots, m+1$ this new system can be written from (1.1)-(1.2) in a minimal form. This procedure is used in Petroleum Science, see [19], in order to minimize the number of balance equations in (1.1) and maximize the

number of conservation laws in (1.2); see Appendix $B$ for more details. Thus we obtain:

$$\frac{\partial}{\partial t}G_j(\mathcal{V}) + \frac{\partial}{\partial x}uF_j(\mathcal{V}) = q_j/\epsilon, \tag{2.1}$$

$$\frac{\partial}{\partial t}G_k(\mathcal{V}) + \frac{\partial}{\partial x}uF_k(\mathcal{V}) = 0, \tag{2.2}$$

with

$$G_j = (S\mathcal{G})_j\,, \quad F_j = (S\mathcal{F})_j\,, \quad G_k = (E\mathcal{G})_k \ \text{and} \quad F_k = (E\mathcal{F})_k\,, \tag{2.3}$$

where $\mathcal{G} = (\mathcal{G}_1, \cdots, \mathcal{G}_l, \mathcal{G}_{l+1}, \cdots, \mathcal{G}_{m+1})^T$ and $\mathcal{F} = (\mathcal{F}_1, \cdots, \mathcal{F}_l, \mathcal{F}_{l+1}, \cdots, \mathcal{F}_{m+1})^T$. The complementary matrices $S$ and $E$ are described in Appendix $B$.

For problems with important applications, local thermodynamical equilibrium is obtained quickly, so the $\epsilon$ scale is very small ($10^{-5}$ or less) and the source terms are large compared to the prevailing scale; thus it is useful to apply singular perturbation theory to the system (2.1)-(2.2). We assume that $\mathcal{V}$ and $u$ can be expanded as:

$$\mathcal{V} = \mathcal{V}^0 + \epsilon\mathcal{V}^1 + \epsilon^2\mathcal{V}^2 + \cdots. \tag{2.4}$$

$$u = u^0 + \epsilon u^1 + \epsilon^2 u^2 + \cdots. \tag{2.5}$$

Substituting (2.4) in (2.1)-(2.2) we have:

$$\frac{\partial}{\partial t}G_j(\mathcal{V}^0 + \epsilon\mathcal{V}^1 + \cdots) + \frac{\partial}{\partial x}\left(u^0 + \epsilon u^1 + \cdots\right)F_j(\mathcal{V}^0 + \epsilon\mathcal{V}^1 + \cdots) = q_j(\mathcal{V}^0 + \epsilon\mathcal{V}^1 + \cdots)/\epsilon, \tag{2.6}$$

$$\frac{\partial}{\partial t}G_k(\mathcal{V}^0 + \epsilon\mathcal{V}^1 + \epsilon^2\mathcal{V}^2 + \cdots) + \frac{\partial}{\partial x}\left(u^0 + \epsilon u^1 + \cdots\right)F_k(\mathcal{V}^0 + \epsilon\mathcal{V}^1 + \epsilon^2\mathcal{V}^2 + \cdots) = 0. \tag{2.7}$$

Multiplying (2.6) by $\epsilon$ and setting $\epsilon = 0$ we obtain the lowest order approximation for $j = 1, 2, \cdots, m-n$ and $i = m-n+1, m-n+2, \cdots, m+1$ :

$$q_j(\mathcal{V}^0) = 0, \tag{2.8}$$

$$\frac{\partial}{\partial t}G_i(\mathcal{V}^0) + \frac{\partial}{\partial t}uF_i(\mathcal{V}^0) = 0. \tag{2.9}$$

Typically, the thermodynamical relationships and equations of state play a central role in multiphase models (2.1)-(2.2), which are represented here implicitly by $q_j(\mathcal{V}^0) = 0$. Each physical configuration, or actually a *physical configuration under local thermodynamical equilibrium or quasi-equilibrium,* is a sheet of the *equilibrium variety.* This equilibrium variety is $q(\mathcal{V}^0) = \left(q_1, q_2, \cdots, q_{n-m}\right) = 0$. It is continuous and smooth by parts, i.e., it is a stratified variety, where the local thermodynamical equilibrium is enforced by relationships among the quantities $\mathcal{V}^0$. We denote these variables as $V$ and the respective state $W = (V, u)$. Note that $V$ has a different meaning in each sheet.

There are three groups of variables in each sheet of the variety, the basic variables $V$, or "primary variables" that are unknowns of (2.10). Notice that the system is "elliptic" in the variable $u$ in the sense that perturbations in $u$ propagate instantaneously; it turns out that $u$ can be found from the primary variables, so we call $u$ a "secondary variable"; and the "trivial variables" are constant or they can be recovered from other variables in a simple way by relationships expressing local thermodynamical equilibrium in each of the $\mathcal{C}^2$ parts of the variety; they complement the variables in $V$. Since in this paper we are interested only in the problems with states at local thermodynamical equilibrium, which is formally the zero-order approximation, we drop the superscript zero from now on.

Now the variables $\mathcal{V}$ lie in the equilibrium variety, so, by abuse of notation, we indicate the dependence of the accumulation and the flux functions only on the primary variables $V$, i.e., we use $G(V)$ and $F(V)$ representing $G(\mathcal{V})$ and $F(\mathcal{V})$ for a sheet, so the system (2.9) can be rewritten in each sheet of this variety in the unknowns $W = (V, u)$ as:

$$\frac{\partial}{\partial t}G_i(V) + \frac{\partial}{\partial t}uF_i(V) = 0. \tag{2.10}$$

Notice that $G$ and $F$ are $\mathcal{C}^2$ in each sheet, but they are only continuous on the variety.

Splitting of thermodynamical from hydrodynamical variables in systems of conservation laws modeling certain isothermal flows in a porous medium appeared in [25].

## 3. CONFIGURATIONS IN A SPECIFIC MODEL

Multiphase models (2.1)-(2.2) can be used to model steam injection in a reservoir, which is widely studied in Petroleum Engineering, see [3]. In [15], we study the steam and water injection in several proportions into a porous medium containing steam (gaseous $H_2O$), water (liquid $H_2O$) or a mixture. In [17], we solve a model for groundwater remediation. As an example of the theory developed in this paper, we use it in another application, the recovery of geothermal energy. We consider the injection of steam and nitrogen in a one-dimensional horizontal porous rock, where we disregard gravity effects and heat conductivity. The rock has constant porosity $\varphi$ and absolute permeability $k$ (see Appendix A). We assume that the fluids are incompressible. This is a good approximation for liquid water; for the gaseous mixture of steam and nitrogen we assume that the gas density does not change due to pressure, but it is expansible and the density is a function of the temperature only; in other words, we assume that the pressure variations along the core are so small compared to the total pressure that they do not affect the physical properties of the gas phase.

3.1. **The model for our example.** Darcy's law for multiphase flows relates the pressure gradient in each fluid phase with its seepage speed:

$$u_w = -\frac{kk_{rw}}{\mu_w}\frac{\partial p}{\partial x}, \qquad u_g = -\frac{kk_{rg}}{\mu_g}\frac{\partial p}{\partial x}. \tag{3.1}$$

The water and gas relative permeability functions $k_{rw}(s_w)$ and $k_{rg}(s_g)$ are considered to be functions of their respective saturations (see Appendix A); $\mu_w$ and $\mu_g$ are the viscosities of liquid and gaseous phases. Since we are interested in large scale problems, with flow rate far from zero we have disregarded capillarity and diffusive effects. The "fractional flow" for water and steam are saturation dependent functions defined by:

$$f_w = \frac{k_{rw}/\mu_w}{k_{rw}/\mu_w + k_{rg}/\mu_g}, \qquad f_g = \frac{k_{rg}/\mu_g}{k_{rw}/\mu_w + k_{rg}/\mu_g}. \tag{3.2}$$

The saturations $s_w$ and $s_g$ add to 1. By (3.2) the same is true for $f_w$ and $f_g$. Using Darcy's law (3.1) and (3.2) yield:

$$u_w = uf_w, \qquad u_g = uf_g, \quad \text{where} \quad u = u_w + u_g \quad \text{is the } total \text{ or } Darcy \text{ velocity.} \tag{3.3}$$

We write the equations of mass balance for water (liquid $H_2O$), steam (gaseous $H_2O$) and nitrogen (gaseous $N_2$) as:

$$\frac{\partial}{\partial t}\varphi\rho_W s_w + \frac{\partial}{\partial x}u f_w \rho_W = +q_{g \longrightarrow a,w}, \tag{3.4}$$

$$\frac{\partial}{\partial t}\varphi\rho_{gw} s_g + \frac{\partial}{\partial x}u f_g \rho_{gw} = -q_{g \longrightarrow a,w}, \tag{3.5}$$

$$\frac{\partial}{\partial t}\varphi\rho_{gn} s_g + \frac{\partial}{\partial x}u f_g \rho_{gn} = 0, \tag{3.6}$$

where $q_{g \longrightarrow a,w}$ is the liquid water mass source term (i.e., the condensation rate of steam from the gaseous to the water phase); here $\rho_W$ is the water density, which is assumed to be constant, $\rho_{gw}$ ($\rho_{gn}$) denote the concentration of steam (nitrogen) in the gaseous phase (mass per unit gas volume).

Globally there are 4 unknowns ($T$, $s_g$, $\psi_{gw}$ and $u$) and 3 equations; it is necessary to specify another equation, representing energy conservation; we formulate it in basis of enthapies, see [1, 2], as we ignore adiabatic compression and decompression effects. We neglect heat conduction and heat losses to the surrounding rock. Thus the energy conservation is given by:

$$\frac{\partial}{\partial t}\varphi\left(\hat{H}_r + H_w s_w + H_g s_g\right) + \frac{\partial}{\partial x}u\left(H_w f_w + H_g f_g\right) = 0, \tag{3.7}$$

here $\hat{H}_r = H_r/\varphi$ and $H_r$, $H_W$ and $H_g$ are the rock, the liquid water and the gas enthalpies per unit volume; their expressions can be found in Appendix $A$.

Notice that the system (3.4)-(3.5) is of form (2.1)-(2.2). We can rewrite equations (3.4)-(3.5) using (3.4) and the conservation of total mass of water by adding (3.4) and (3.5):

$$\frac{\partial}{\partial t}\varphi\left(\rho_W s_w + \rho_{gw} s_g\right) + \frac{\partial}{\partial x}u\left(f_w \rho_W + \rho_{gw} f_g\right) = 0. \tag{3.8}$$

So the system of equations reduces to (3.4), (3.6)-(3.8); this is an example of the reduction from the form (1.1)-(1.2) to (2.1)-(2.2).

3.2. **Physical configurations in the example.** There are three main different physical configurations: a single-phase liquid configuration, $spl$, with water; a single-phase gaseous configuration, $spg$, with steam and nitrogen; and a two-phase configuration, $tp$, with a mixture of liquid water, gaseous nitrogen and steam at boiling temperature. The latter is specified by the concentration of nitrogen in the gas. We assume that each configuration is in local thermodynamical equilibrium, so we can use Gibbs' phase rule, $f_G = c - p + 2$ where $f_G$ represents the Gibb's number of thermodynamical degrees of freedom, $c$ and $p$ are the number of chemical species and phases. Since in our thermodynamical model the pressure is fixed, the remaining number of degrees of freedom is $f = f_G - 1$.

3.2.1. *Single-phase gaseous configuration - spg.* There are two chemical species ($N_2$ and $H_2O$) and one gaseous phase, i.e., $c = 2$ and $p = 1$, so the number of degree of freedom is $f = 2$: temperature and gas composition. This fact allows to define the steam and nitrogen gas compositions $\psi_{gw}$ and $\psi_{gn}$ as follows, see [3, 17]:

$$\psi_{gw} = \rho_{gw}/\rho_{gW}(T) \qquad \text{and} \qquad \psi_{gn} = \rho_{gn}/\rho_{gN}(T), \quad \text{so} \quad \psi_{gw} + \psi_{gn} = 1. \tag{3.9}$$

These gas fractions are related to compositions $c_{ij}$ (Eq. (1.4)); we assume that in the nitrogen and steam there are no effects due to mixing so that the volumes of components are additive.

We rewrite equations (3.4), (3.6)-(3.8) using Eqs. (3.9.$a$) and (3.9.$b$). Since $s_w = 0$, using Eq. (3.2.$a$) and ($A$.9.$a$), we have $f_w = 0$ and $f_g = 1$, so the left hand side of Eq. (3.4) disappears and $q_{g \longrightarrow a,w}$ vanishes. The system (3.5)-(3.7) becomes:

$$\frac{\partial}{\partial t} \varphi \rho_{gW} \psi_{gw} + \frac{\partial}{\partial x} u \rho_{gW} \psi_{gw} T = 0, \tag{3.10}$$

$$\frac{\partial}{\partial t} \varphi \rho_{gN} \psi_{gn} + \frac{\partial}{\partial x} u \rho_{gN} \psi_{gn} = 0, \tag{3.11}$$

$$\frac{\partial}{\partial t} \varphi \left( \hat{H}_r + \psi_{gw} H_{gW} + \psi_{gn} H_{gN} \right) + \frac{\partial}{\partial x} u \left( \psi_{gw} H_{gW} + \psi_{gn} H_{gN} \right) = 0; \tag{3.12}$$

where we have defined $H_{gW}(T)$ and $H_{gN}(T)$:

$$H_{gW}(T) = \frac{M_W p_{at} h_{gW}}{RT} \quad \text{and} \quad H_{gN}(T) = \frac{M_N p_{at} h_{gN}}{RT}, \tag{3.13}$$

where $M_N$, $M_W$ are the constant molar masses of nitrogen and water; $R$ is the universal gas constant; $p_{at}$ is the atmospheric pressure; $h_{gW}$ and $h_{gN}$ are functions of $T$ given in ($A$.1)-($A$.2).

Since $s_w = 0$, Eq. (3.4) is trivially satisfied, so the system (3.10)-(3.12) is exact rather than only the zero-order approximation of the system (3.4), (3.6)-(3.8).

**Remark 3.1.** *The spg configuration consists of a 2-Dimensional (2-D) set of triplets ($s_w = 0, T, \psi_{gw}$) on a sheet of a stratified variety. The states should satisfies $\psi_{gw} \leq \Gamma(T) \equiv \rho_{gw}(T)/\rho_{gW}(T)$, with $\rho_{gw}$ and $\rho_{gW}$ given by ($A$.7.$a$) and ($A$.8.$a$), see Figure 3.1 and [3, 17]. Here, $V = (T, \psi_{gw})$ are the primary variables, $u$ is the secondary variable of the system (3.10)-(3.12) and $s_w = 1$ and $\rho_{gn} = 1 - \rho_{gw}$ are the trivial variables.*
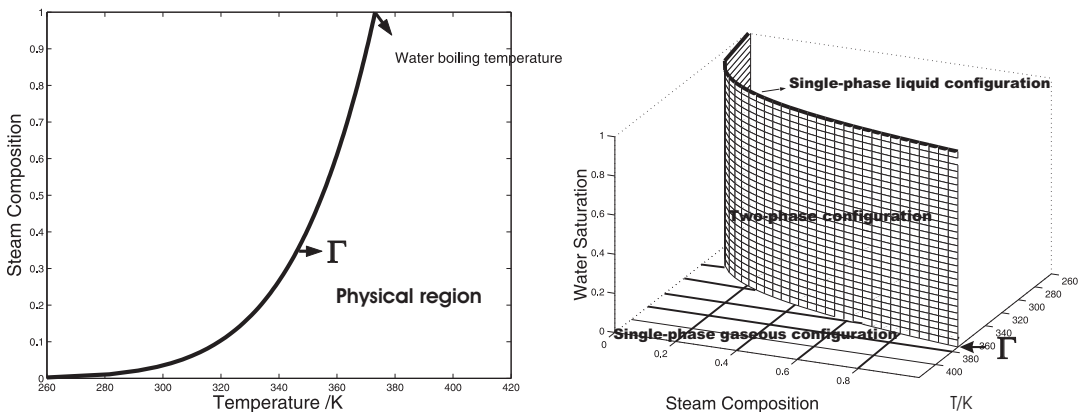


FIGURE 3.1. *a) left:* The physical region for the *spg* configuration of steam and nitrogen is formed by the pairs $(T, \psi_{gw})$ satisfying $\psi_{gw} \leq \Gamma(T)$. The continuous graph represents the composition of a mixture of nitrogen and steam in equilibrium with liquid water. *b) right:* Phase space for $\mathcal{V} = (s_W, \psi_{gw}, T)$ and physical configurations sheets: single phase liquid (*spl*), single phase gaseous (*spg*), and the two-phase (*tp*) configurations. The union of the sheets for *spl*, *spg* and *tp* sheets is the stratified variety.

3.2.2. *Single-phase liquid configuration - spl.* There is one chemical species ($H_2O$) and one phase, so there is only $f = 1$ thermodynamical degree of freedom, which is the temperature. Since $s_w = 1$ and $s_g = 0$, using (3.2) and (A.9) we have $f_w = 1$ and $f_g = 0$. From (3.5), $q_{g\longrightarrow a,w}$ vanishes. The liquid is incompressible and composition changes have no volumetric effects, so that the total Darcy velocity $u$ is independent of position. As we assume that rock and water enthalpies depend linearly on temperature, Eqs. (3.4), (3.6) and (3.8) are satisfied trivially and (3.7) reduces to:

$$\frac{\partial}{\partial t}T + \lambda_T^W \frac{\partial T}{\partial x} = 0, \quad \text{where} \quad \lambda_T^W = \frac{u_W}{\varphi}\frac{C_W}{C_W + \hat{C}_r}, \tag{3.14}$$

where we use $u_W$ to indicate that the velocity $u$ is spatially constant in the *spl* water configuration; here $C_W$ is the water heat capacity and $\hat{C}_r$ is the rock heat capacity $C_r$ divided by $\varphi$. All quantities are given in Appendix A.

Notice here that Eq. (3.14) is exact rather than only the zero-order approximation of the system (3.4), (3.6)-(3.8).

**Remark 3.2.** *The spl configuration consists of a 1-D set of triplets $(s_w = 1, T, \psi_{gw}(T))$, Figure 3.1. Here, $V = T$ is the primary variable, $u$ is the secondary variable, $s_w = 0$ and $\psi_{gw} = \Gamma(T)$ are the trivial variables. Actually $\psi_{gw}$ is unnecessary, however we specify it as in tp configuration to ensure continuity of the equilibrium variety.*

3.2.3. *Two-phase configuration - tp.* There are two chemical species ($N_2$ and $H_2O$), $c = 2$, and two-phases (liquid water and gas), $p = 2$; so $f = 1$ and the thermodynamical free variable is the temperature. The composition in Eqs. (3.9.$a$) and (3.9.$b$) are prescribed by the temperature: $\psi_{gw}(T) = \Gamma(T)$.

Here we disregard equation (3.4) and the expression $q_{g\longrightarrow a,w} = 0$ enforces that $\psi_{gw} = \Gamma(T)$. The resulting system (3.6)-(3.8) is the zero-order approximation of the system (3.4), (3.6)-(3.8). The solution of this system satisfies (3.4) only asymptotically.

**Remark 3.3.** *The tp configuration consists of a 2-D set of triplets $(s_w, T, \psi_{gw}(T))$, on a sheet of stratified variety, because of thermodynamic constraints, see Figure 3.1. Here, $V = (s_w, T)$ are the primary variables, $u$ is the secondary variable of the system (3.6)-(3.8) and $\psi_{gw} = \Gamma(T)$ is the trivial variable.*

Fig. 3.1.$b$ shows the three physical configurations in the variables $V = (T, \psi_{gw}, s_w)$, i.e., in the equilibrium variety. As basic variable, we arbitrarily choose $s_w$ instead of $s_g$. In Table 1, we summarize the primary, secondary and trivial variables.

| Physical configuration \ Variable types | Primary $V$ | Secondary | Trivial |
|---|---|---|---|
| Single-phase gaseous configuration | $T, \psi_{gw}$ | $u$ | $s_g = 1$ |
| Two-phase configuration | $s_g, T$ | $u$ | $\psi_{gw}(T)$ |
| Single-phase liquid configuration | $T$ | $u$ | $s_g = 0, \psi_{gw}$ |

**Table 1:** Classification according to configuration.

## 4. General theory of Riemann Solutions

We are interested in the Riemann problem associated to (2.1)-(2.2), with initial data (1.5). The speed $u_L > 0$ is specified at the injection point. In the next sections we show that $u_R$ can be obtained in terms of $u_L$ and the primary variables.

The general solution of the Riemann problem associated to Eq. (2.10) consists of a sequence of elementary waves, rarefactions and shocks [27]; they are studied now. The rarefaction waves are calculated on the each sheet of the equilibrium variety.

4.1. **Characteristic speeds.** In each physical configuration, systems of conservation laws in appropriate form must be used to find the characteristic speeds. If we assume that the solution is sufficiently smooth, we differentiate all equations in (2.10) with respect to their variables, obtaining a system of the form:

$$B\frac{\partial}{\partial t}\begin{pmatrix} V \\ u \end{pmatrix} + A\frac{\partial}{\partial x}\begin{pmatrix} V \\ u \end{pmatrix} = 0, \tag{4.1}$$

where the matrices $B(V)$ and $A(V)$ are the derivatives of $G(V)$ and $uF(V)$ with respect to the variables $V$ and $u$. Since $G(V)$ does not depend on $u$, the last column in the matrix $B$ is zero. For the pair of primary and secondary variables $W = (V, u)$, the characteristic values $\lambda^i := \lambda^i(W)$ and vectors $\vec{r}^i := \vec{r}^i(W)$, (where $i$ is the label of each eigenvector or family) for the following system are the characteristic speeds and vectors:

$$A\vec{r}^i = \lambda^i B\vec{r}^i \quad \text{where } \lambda^i \text{ is obtained by solving} \quad det(A - \lambda^i B) = 0. \tag{4.2}$$

Similarly the left eigenvectors $\vec{\ell}^i = (\ell_1^i, \ell_2^i, \ell_3^i)$ satisfy:

$$\vec{\ell}^i A = \lambda^i \vec{\ell}^i B, \quad \text{for the same} \quad \lambda^i. \tag{4.3}$$

**Remark 4.1.** *Notice that in the left and right eigenvectors we represent only the components for the primary and secondary variables, because we assume that the trivial variables are obtained from these variables.*

**Remark 4.2.** *Here we derive the formulae for a general $3 \times 3$ system because it is the smallest non-trivial system of type (2.10), it will be used in the tp configuration. For such a system, generically the right and left eigenvectors have three components, corresponding to the two primary variables and to the secondary variable. (However, in the spl configuration we do not need this formalism because the system (2.10) reduces to a single equation).*

Hereafter, the word "eigenvectors" means "right eigenvectors". For brevity, we will write the right and left eigenvectors without the arrow, i.e., we replace $\vec{r}$ and $\vec{\ell}$ by $r$ and $\ell$.

The following lemma is important to characterize the eigenpairs, see [16]; for another derivation see [6].

**Lemma 4.1.** *Assume that $u \neq 0$ and that the flux vector $F(V)$ never vanishes for all $V$. The eigenvalue, right and left eigenvectors for the system (4.1) have the form:*

$$\lambda = u\vartheta(V), \quad r = (g_1(V), g_2(V), ug_3(V))^T \quad and \quad \ell = (\ell_1(V), \ell_2(V), \ell_3(V)), \tag{4.4}$$

*where $\vartheta$, $g_i$ and $\ell_i$ (for all $i \in \mathcal{C}$) are functions of $V$ only. Moreover, there are at most 2 eigenvalues and eigenvectors associated to this system of 3 equations.*

For any $\vartheta \in \mathbb{R}$, let us define the following $3 \times 2$ matrix:

$$C(V; \vartheta) = \frac{\partial F}{\partial V}(V) - \vartheta\frac{\partial G}{\partial V}(V); \tag{4.5}$$

We also define the following $3 \times 3$ matrix, where $F(V)$ is the flux column vector:

$$A - \lambda B = \left( uC(V; \lambda/u) \,\Big|\, F(V) \right). \tag{4.6}$$

**Proof:** The eigenvalues $\lambda$ of (4.1) are the roots of $det(A - \lambda B) = 0$. Since $u \neq 0$, we divide the first 2 columns of (4.1) by $u$ and set $\vartheta = \lambda/u$, obtaining the characteristic equation:

$$det\left( C(V;\vartheta) \,\big|\, F(V) \right) = 0. \tag{4.7}$$

Since Eq (4.7) depends on $V$ only, the eigenvalues $\vartheta$ are functions of $V$, i.e., $\vartheta = \vartheta(V)$, they do not depend on $u$. Using $\lambda = u\vartheta$, the eigenvalues have the form (4.4.$a$).

The eigenvectors are solutions of $(A - \lambda B)r = 0$, or $\left( uC(V;\vartheta) \,\big|\, F(V) \right) r = 0$, for each $V$ there exists an index $k \in \{1, 2, 3\}$ and an open neighborhood $\mathcal{B}_V$ of $V$ in which $F_k \neq 0$, so we can write $r_3$ as:

$$r_3 = u\frac{1}{F_k(V)}\sum_{l=1}^{2} C(V;\vartheta)r_l, \tag{4.8}$$

where $C_k(V;\vartheta)$ is the $k$-$th$ row of $C(V;\vartheta)$ and $r = (r_1, r_2, r_3)^T$. By suitable compactness arguments we can extend the result (4.8) for all $V \in \Omega$.

Substituting $r_3$ given by Eq. (4.8) into $(A - \lambda B)r = 0$, we obtain a linear system in the unknowns $r_j$ for $j = 1, 2$. We can cancel $u$ in this system, showing that the $r_l$ for $l = 1, 2$ depend on $V$ only. So (4.4.$b$) follows from (4.8).

For $\ell = (\ell_1, \ell_2, \ell_3)$, we solve $\ell \cdot \left( uC(V;\vartheta) \,\big|\, F(V) \right) = 0$, so that $\ell$ is solution of the following system:

$$uC_1(V;\vartheta)\ell_1 + uC_2(V;\vartheta)\ell_2 + uC_3(V;\vartheta)\ell_3 = 0 \quad \text{for } l = 1, 2, \tag{4.9}$$
$$F_1\ell_1 + F_2\ell_2 + F_3\ell_3 = 0. \tag{4.10}$$

We divide Eqs. (4.9) by $u$ and we obtain a system with coefficients that depend only on the variables $V$, leading to (4.4.$c$). $\square$

**Definition 4.1.** *For each $i$, the **integral curves** in the $W = (V, u)$ plane are solutions of*

$$\left(\frac{dV}{d\xi}, \frac{du}{d\xi}\right) = r, \quad i.e, \quad \frac{dV_1}{d\xi} = r_1(V), \quad \frac{dV_2}{d\xi} = r_2(V) \quad and \quad \frac{du}{d\xi} = r_3(V). \tag{4.11}$$

*When $\xi$ satisfies:*

$$\xi = \lambda(W(\xi)), \tag{4.12}$$

*and the the variable $\xi$ increases the integral curve defines a rarefaction curve.*

*A rarefaction wave in the $(x, t)$ plane is defined by $W(\xi)$, with $\xi = x/t$, given a rarefaction curve $W(\xi)$.*

**Proposition 4.1.** *Assume that locally the eigenvector $r$ associated to a certain family forms a vector field. Then we calculate the primary variables $V$ on the rarefaction curve in the $(x, t)$ plane independently of $u$, i.e., first we obtain the primary variables in the classical way by solving (4.11.b)-(4.11.c), and then we calculate the secondary variable $u$ in terms of the primary variables from:*

$$u = u^-exp(\gamma(\xi)), \quad with \quad \gamma(\xi) = \int_{\xi^-}^{\xi} g_3(V(\eta))d\eta, \tag{4.13}$$

$\xi^- = \lambda(W^-)$ *and $u^-$ is the "leftmost" value of $u$ on the rarefaction wave, i.e., $u = u^-$ for $\xi = \xi^-$.*

**Proof:** Using (4.11) and (4.12) with $g_i(V) = r_i$ for $i = 1, 2$ we obtain $V$ independently of $u$ by solving the system of differential equations:

$$\frac{dV}{d\xi} = \left(\frac{dV_1}{d\xi}, \frac{dV_2}{d\xi}\right) = (g_1, g_2), \quad \text{with} \ \ \xi^- = \lambda(W^-).$$

After obtaining $V(\xi)$, we use the expression for the last component of $r$ in (4.4.$b$) to solve $du/d\xi = ug_3$, yielding (4.13). Lemma 4.1 asserts that $\lambda$ has the form $u\vartheta(V)$, so using Eqs. (4.12) and (4.13), we obtain $\xi$ implicitly as:

$$\xi = u^-\vartheta(V(\xi))exp(\gamma(\xi)). \tag{4.14}$$

Since $\xi$ depends only on $u^-$ and $V$ on the integral curves, the result follows.    $\square$

**Definition 4.2.** *Since only the first* 2 *coordinates of the eigenvectors are pertinent to define the integral curves in the space of primary variables* $V$, *it is useful for proving the generalized Bethe-Wendroff theorem in Section* 5 *to define for any i-family the following quantities that do not depend on u:*

$$\tilde{r}^i = (r_1^i, r_2^i), \quad \tilde{\ell}^i = (\ell_1^i, \ell_2^i), \quad \tilde{\lambda}^{i,-}(V) := \lambda_i(W)/u^- \quad and \quad \tilde{\lambda}^{i,+}(V) := \lambda_i(W)/u^+. \tag{4.15}$$

**Corollary 4.1.** *Assume that* $u^- \neq 0$, *so we perform the change of variables:*

$$\xi = x/t \longrightarrow \tilde{\xi} = x/\left(u^-t\right). \tag{4.16}$$

*The system* (2.10) *can be written in the space of variables* $(x, \tau)$, *where* $\tau = u^-t$, *i.e., in these space-time coordinates the rarefaction wave projected on the space of primary variable is independent of u.*

**Proof:** In $V$ space, we have shown that the eigenvalues have the form $\lambda^-(V)$, with $r$ given in Eq. (4.4.$a$). From Eqs. (4.12) and (4.14), it follows that $\tilde{\xi}^-$ and $\tilde{\xi}$ satisfy:

$$\tilde{\xi}^- = \tilde{\lambda}^- = \vartheta(V^-) \quad \text{and} \quad \tilde{\xi} = \tilde{\lambda}(V(\tilde{\xi})) = \vartheta(V(\tilde{\xi}))exp(\gamma(\tilde{\xi})). \tag{4.17}$$

Thus in the $(x, \tau)$ plane, the variables $V$ and $\tilde{\xi}$ do not depend on $u$, so the rarefaction curve does not depend on $u$. The speed keeps the form (4.13).    $\square$

**Remark 4.3.** *A rarefaction wave connecting adjoining sheets of the stratified variety occurs occasionally; such a connection can happen only when there is equality between characteristic speeds of the rarefaction curves at the left and right sheets. The rarefaction curve should be continuous, however generically its derivative is discontinuous at the boundary between the sheets.*

4.2. **Shock waves.** When there is a source term, it acts in the infinitesimal space between regions in different physical configurations, so we are led to propose the existence of shocks linking these configurations. We assume that there are no shocks linking non-adjacent sheets in the variety.

Shocks are special type of discontinuities in the solution of the PDE's. To obtain the discontinuities we define the function $\mathcal{H} = \mathcal{H}(W^-; W^+)$ as:

$$\mathcal{H} := v\left(G^+ - G^-\right) - u^+F^+ + u^-F^-, \tag{4.18}$$

where $W^- = (V^-, u^-)$ and $W^+ = (V^+, u^+)$ are the states on the left and the right side of the discontinuity; $v = v(W^-, W^+)$ is the discontinuity speed; $G^- = G^-(V^-)$, $(G^+ = G^+(V^+))$ and $F^- = F^-(V^-)$, $(F^+ = F^+(V^+))$ are the accumulation and fluxes at the left (right) of the discontinuity. When the states $(-)$ and $(+)$ lie in the same configuration,

the conserved quantities, accumulations and and fluxes arise from a system of conservation laws in a single sheet; while if these states $(-)$ and $(+)$ lie in different configurations, the conserved quantities, accumulations and fluxes arise from different systems of conservation laws, defined in two sheets. For shocks contained in a single sheet, the velocity $u$ usually varies and the formulae in the theory has the same form as if they were contained in different sheets.

For a fixed $W^- = (V^-, u^-)$ the **Rankine-Hugoniot Locus** ($RH$ locus) parametrizes the discontinuous solutions of Eq. (3.4)-(3.7), ie., it consists of the $W^+ = (V^+, u^+)$ that satisfy $\mathcal{H}(W^-; W^+) := 0$. We specify the state $(V^-, u^-)$ on the left hand side, but at the right $u^+$ cannot be specified, and we will see that it is obtained from the condition $\mathcal{H}(W^-; W^+) = 0$.

In this work we assume an extension of hyperbolicity, i.e., except on the coincidence curve between the eigenvalues, the system is hyperbolic in the primary variables, i.e., there exists a basis of characteristic vectors for each state $V$, see [16]. Thus only connected branches of the $RH$ locus are considered (i.e., that contain the $(-)$ state), see [16], thus we use the following criterion to define admissible discontinuities, or shocks, the Liu's criterion, (see Liu [20, 21]).

**Definition 4.3.** *For a fixed $W_L$, we call a **shock curve** each connected part $\{W\}$ formed by the $W_R$ in the $\mathcal{RH}(W_L)$, such that $v(W_L, W_R) \leq v(W_L, W)$, where $W \in \mathcal{RH}(W_L)$ between $W_L$ and $W_R$. In $(x, t)$ space, each point of the shock curve represents a **shock wave**. The shock curve parametrizes the $(+)$ states of admissible shocks between the $(-)$ and the $(+)$ states.*

4.2.1. *The Rankine-Hugoniot locus.* For a fixed $W^- = (V^-, u^-)$ in a configuration and $i = 1, 2, 3$, the $RH$ locus, or $\mathcal{RH}(W^-)$, is obtained setting (4.18) to zero, i.e.:

$$v[G_i] = u^+ F_i^+ - u^- F_i^-, \tag{4.19}$$

where $[G_i] = G_i^+ - G_i^-$, $G_i^\pm = G_i^\pm(V^\pm)$ and $F_i^\pm = F_i^\pm(V^\pm)$ for $i = 1, 2, 3$. The states $W^+$ can be in the same or in a different configuration. We rewrite Eq. (4.19) as a linear system:

$$\begin{pmatrix} [G_1] & -F_1^+ & F_1^- \\ [G_2] & -F_2^+ & F_2^- \\ [G_3] & -F_3^+ & F_3^- \end{pmatrix} \begin{pmatrix} v \\ u^+ \\ u^- \end{pmatrix} = 0. \tag{4.20}$$

We define the set of unordered index pairs $\mathbb{P} = \{\{1, 2\}, \{3, 1\}, \{2, 3\}\}$. We utilize also the notation:

$$\mathcal{Y}_{ij} = F_i^+ F_j^- - F_j^+ F_i^-, \quad \mathcal{X}_{ij}^+ = F_i^+[G_j] - F_j^+[G_i] \quad \text{and} \quad \mathcal{X}_{ij}^- = F_i^-[G_j] - F_j^-[G_i]. \tag{4.21}$$

For a non-trivial solution of the system (4.20) to exist, the determinant of matrix in Eq. (4.20) must vanish; this yields another form of the Rankine-Hugoniot locus denoted by $\mathcal{RH}(V^-)$, namely for each $V^-$ it the set of $V^+$ satisfying $\mathcal{H}_V(V^-, V^+) = 0$ with:

$$\mathcal{H}_V := [G_1]\mathcal{Y}_{32} + [G_2]\mathcal{Y}_{13} + [G_3]\mathcal{Y}_{21}. \tag{4.22}$$

Generically, $\mathcal{RH}(V^-)$ is 1-$D$ structure, see Appendix $C$.

There are two primary variables in $V^+$ for the *spg* and the *tp*, so $\mathcal{RH}(V^-)$ consists of the union of two curves. In the *spl* there is only one scalar equation with the temperature as the primary variable, so $\mathcal{RH}(T^-)$ is the whole physical range of the temperature axis.

Solving the system (4.20), we obtain $u^+$ and $v$ as functions of $V^-$, $V^+$ and $u^-$:

$$u^+ = u^- \frac{\mathcal{X}_{ij}^-}{\mathcal{X}_{ij}^+}, \quad u^+ = u^- \frac{\sum_{\{l,k\}\in\mathbb{P}} \mathcal{X}_{lk}^- \mathcal{X}_{lk}^+}{\sum_{\{l,k\}\in\mathbb{P}} \left(\mathcal{X}_{lk}^+\right)^2}, \quad v = u^- \frac{\mathcal{Y}_{ji}}{\mathcal{X}_{ij}^+} \quad \text{and} \quad v = u^- \frac{\sum_{\{l,k\}\in\mathbb{P}} \mathcal{Y}_{kl} \mathcal{X}_{lk}^+}{\sum_{\{l,k\}\in\mathbb{P}} \left(\mathcal{X}_{lk}^+\right)^2},$$
(4.23)

for any $\{i,j\} \in \mathbb{P}$. Eqs. (4.23.$b$) and (4.23.$d$) are useful for numerical calculations, and are obtained from (4.23.$a$) and (4.23.$c$). Of course, Eqs. (4.23.$a$) and (4.23.$c$) are valid if the corresponding denominator is non-zero, while (4.23.$b$) and (4.23.$d$) require that just one denominator is non-zero. In [16] we give the definition of *regular RH locus*: for such a locus (4.23.$b$) and (4.23.$d$) are well defined if $V^+ \neq V^-$, because we assume that for each $V$ in the primary variables there is $\{p,q\} \in \mathbb{P}$ such that the inequality $\mathcal{X}_{pq}^- \neq 0$ is satisfied.

**Remark 4.4.** *In the definitions that follow, all wave structures can be obtained in the space of primary variables $V$. Using $u^+ = u^+(V^-, u^-; V^+)$ and $v := v(V^-, u^-; V^+)$, we define:*

$$\mathcal{Z}(V^-; V^+) = \frac{u^+}{u^-}, \quad \tilde{v}^-(V^-; V^+) := \frac{v}{u^-} \quad and \quad \tilde{v}^+(V^-; V^+) := \frac{v}{u^- \mathcal{Z}(V^-; V+)}. \quad (4.24)$$

**Lemma 4.2.** *Assume that $u^-$ is positive. If the Darcy speed $u^-$ is modified while $V^-$ and $V^+$ are kept fixed, the Darcy speed $u^+$ as well as the shock waves are rescaled in the $(x,t)$ plane, while the values of $V$ are preserved in the rescaled shock.*

**Proof:** Performing the change of variable $(x,t) \longrightarrow (x, u^- t)$ in $\mathcal{H} = 0$, for $\mathcal{H}$ given by (4.18), we obtain:

$$\frac{v}{u^-}(G^+ - G^-) = \frac{u^+}{u^-} F^+ - F^-. \quad (4.25)$$

The result follows from the relationships in (4.24), because $\mathcal{Z}$ and $\tilde{v}^-$ depend only on the primary variables $V$.   $\square$

4.3. **Wave Sequences and Riemann Solutions.** In the Riemann solution, we denote the state $\mathcal{W}$, $\mathcal{V}$, $W$ and $V$ by a subindex between parenthesis to avoid confusion with vector components. The left and right states are indicated only by the subindex $L$ and $R$ and the $(-)$ and $(+)$ states by the superscript $-$ and $+$.

A Riemann solution is a sequence of elementary waves $w_k$ (shocks and rarefactions) for $k = 1, \cdots, m$ and constant states $\mathcal{W}_{(k)} = (\mathcal{V}, u)_{(k)}$ for $k = 1, \cdots, m$, which are states on the equilibrium variety, plus the secondary variable $u$:

$$\mathcal{W}_L \equiv \mathcal{W}_{(0)} \xrightarrow{w_1} \mathcal{W}_{(1)} \xrightarrow{w_2} \cdots \xrightarrow{w_m} \mathcal{W}_{(m)} \equiv \mathcal{W}_R. \quad (4.26)$$

The subscripts $(k)$ in $\mathcal{W}$ do not indicate the component of a vector, but the position of a state or a wave in the sequence. The wave $w_k$ has left and right states $\mathcal{W}_{(k-1)}$ and $\mathcal{W}_{(k)}$ and speeds $\xi_k^- < \xi_k^+$ in case of rarefaction waves and $v = \xi_k^- = \xi_k^+$ in the case of shock waves, where $v$ is the shock speed. The left state of the first wave $w_1$ is $(\mathcal{V}, u)_L$ and the right state of $w_m$ is $(\mathcal{V}, u)_R$, where $u_R$ needs to be found. In the Riemann solution it is necessary that $\xi_k^+ \leq \xi_{k+1}^-$; this inequality is called *geometrical compatibility*. When $\xi_k^+ < \xi_{k+1}^-$, there is a constant state $\mathcal{W}_{k+1}$ separating $w_k$ and $w_{k+1}$; in this sequence the wave $w_k$ is indicated by $\longmapsto$. If $\xi_k^+ = \xi_{k+1}^-$ there is no actual constant state in physical space, so the wave $w_k$ is a composite with $w_{k+1}$; it is indicated by $\twoheadrightarrow$.

When it is useful to emphasize the waves in the sequence (4.26) rather than the states, we use the notation:

$$w_1 \rightarrowtail w_2 \rightarrowtail \cdots \rightarrowtail w_m, \quad (4.27)$$

where for each $w_k$, $\rightarrowtail$ represents $\mapsto$ for the case $\xi_k^+ < \xi_{k+1}^-$ and $\twoheadrightarrow$ for the case $\xi_k^+ = \xi_{k+1}^-$.

We will see, however, that we can determine the Riemann solution in the space of primary variables $V$, without taking into account the secondary and trivial variables. The values of these variables along the Riemann solutions are fully determined by the values of the primary variables supplemented by a boundary condition on $u$. Indeed, we have proved:

**Proposition 4.2.** *Assume that $u_L$ is positive. The values of primary variables $V$ in the shock and rarefaction curves do not depend on the left Darcy speed $u_L > 0$. If a sequence of waves and constant states solve the Riemann problem for (2.10) with left state $W_L = (V_L, u_L)$ and right state $W_R = (V_R, \cdot)$ in the primary variables for a given $u_L > 0$, then this sequence defines the solution for any other $u_L > 0$ in the plane $(x, u_L t)$ for the problem with data:*

$$\begin{cases} (V_L, \cdot) & \text{if } x < 0 \\ (V_R, \cdot) & \text{if } x > 0. \end{cases} \tag{4.28}$$

*Then $u_R$ is given by:*

$$u_R = u_L \prod_{l=1}^{\varrho_1} exp(\gamma_l) \prod_{m=1}^{\varrho_2} \mathcal{Z}_m, \tag{4.29}$$

We could specify $u_R$ instead of $u_L$; a formula similar to (4.29) is valid. In the Proposition above, $\mathcal{Z}_m$ is given by (4.24.$a$) for the $m$-th shock wave. Similarly $\gamma_l$ is given by (4.13.$b$) for $\xi = \xi^+$ computed along the $l$-th rarefaction curve.

From Prop. 4.1 and Lemma 4.2, a minor modification of the proof of Lemma 4.2 yields, see [16]:

**Theorem 4.1.** *Assume the same hypotheses of Prop. 4.2. If the Darcy speed $u_L$ in the initial data (1.5) is modified while $V_L$ and $V_R$ are kept fixed in the Riemann problem for (2.10), the Darcy speed $u_R$ as well as the Riemann solutions are rescaled in the $(x, t)$ plane, while the values of $V$ in the wave sequence are preserved.*

**Remark 4.5.** *Theorem 4.1 says that the Riemann solution can be obtained in each physical configuration first in the primary variables $V$. Then the Darcy speed can be obtained at any point of the space $(V, u)$ in terms of $V$ and $u_L$ by an equation analogous to Eq. (4.29).*

Remark 4.5 leads us to omit the speed $u$ in the figures.

## 5. BIFURCATION THEORY FOR RIEMANN SOLUTIONS

Riemann solutions bifurcate non-trivially for general systems of conservation law that violate the conditions of the Lax Theorem, see [11, 12, 13].

5.1. **Fundamental theorems for bifurcation theory.** We recall an important theorem for bifurcation of systems of conservation laws of standard form, the *Triple Shock Rule* [11]:

**Proposition 5.1.** *For the system of conservation laws $\partial G(V)/\partial t + \partial F(V)/\partial x = 0$, consider three states $V^M$, $V^+$ and $V^-$. Assume that $V^- \in \mathcal{RH}(V^+)$, $V^M \in \mathcal{RH}(V^-)$ and $V^+ \in \mathcal{RH}(V^M)$, with speeds $v^{+,-}$, $v^{-,M}$ and $v^{M,+}$. Then, either:*
  (1)  $v^{+,-} = v^{M,-} = v^{+,M}$; *or*
  (2)  $G(V^+) - G(V^-)$ *and* $G(V^+) - G(V^M)$ *are linearly dependent.*

Instead of the Triple Shock Rule, we have the following *Quadruple Shock Rule*:

**Proposition 5.2.** *Assume that there are two physical configurations labelled by I and II, with a common boundary. In these configurations, consider four states: $(V^-, u^-)$, with $V^-$ in I, $(V^+, u^+)$ with $V^+$ in II; $(V^M, u^M)$, $(V^*, u^*)$ with $V^M$, $V^*$ free to be in I and II. Assume that the RH condition is satisfied by the following pairs of states:*

   *(i) $(V^-, u^-)$ and $(V^+, u^+)$ with speed $v^{-,+}$,*
   *(ii) $(V^-, u^-)$ and $(V^M, u^M)$ with speed $v^{-,M}$,*
   *(iii) $(V^M, u^M)$ and $(V^*, u^*)$ with speed $v^{M,*}$,*
*such that two speeds coincide, i.e., at least one of the following equalities is satisfied:*

$$\text{either}\quad v^{-,+} = v^{-,M} \quad \text{or}\quad v^{-,+} = v^{M,*} \quad \text{or}\quad v^{-,M} = v^{-,+}. \tag{5.1}$$

*If the following conditions $(a)$ to $(c)$ are satisfied:*
   *(a) $G(V^+) - G(V^-)$ and $G(V^*) - G(V^M)$ are linearly independent (LI);*
   *(b) $V^+$ and $V^*$ have one component $V_k$ with coinciding values;*
   *(c) $\partial \mathcal{H}_V / \partial V_j \neq 0$ for $j \neq k$ for all $V \in \mathcal{RH}(V^-)$, see Eq. (4.22);*
*then:*

   *(1) $V^* = V^+$;*

   *(2) $u^* = u^+$;*

   *(3) all three speeds are equal:*     $v^{-,+} = v^{-,M} = v^{M,*}.$ *   (5.2)*

**Proof:** The $RH$ conditions (4.18) for $(V^-, u^-)$-$(V^+, u^+)$, $(V^-, u^-)$-$(V^M, u^M)$ and $(V^M, u^M)$-$(V^*, u^*)$ are respectively:

$$v^{-,+}(G^+ - G^-) = u^+ F^+ - u^- F^-, \tag{5.3}$$

$$v^{-,M}(G^M - G^-) = u^M F^M - u^- F^-, \tag{5.4}$$

$$v^{M,*}(G^* - G^M) = u^* F^* - u^M F^M. \tag{5.5}$$

Assume that now Eq. (5.1.a) is satisfied. Substituting $v^{-,+} = v^{-,M} = v$ in Eqs. (5.3) and (5.4) and subtracting Eq. (5.3) from (5.4), we obtain:

$$v(G^+ - G^M) = u^+ F^+ - u^M F^M. \tag{5.6}$$

Notice that Eqs. (5.6) and (5.5) define implicitly the $RH$ locus by $\mathcal{H}_V(V^M; V^+) = 0$ in the variables $V^M$ and $V^+$. Since the $RH$ locus depends solely on $V^M$ and the accumulation and flux functions, we obtain that both $RH$ loci defined by Eqs. (5.6) and (5.5) coincide. From the condition $(b)$ the states $V^+$ and $V^*$ have a coinciding coordinate $V_k$. From the conditions $(a)$ and $(c)$, the implicit function theorem ensures that we can write the components $V_j$ in terms of $V_k$. Thus there exists a single $V$ with component $V_k$ satisfying (5.5) and (5.6), so $V^*$ and $V^+$ are equal.

Now from Eq. (4.23.b), we notice for a fixed $u^-$ that the Darcy and shock speeds depend solely on $V^-$ and $V^+$. From Eqs. (5.5) and (5.6), we can see that the $(-)$ and $(+)$ states are the same for each expression and that they define the same $RH$ locus, so $u^* = u^+$ and Eq. (5.2) is satisfied.

The other cases are proved similarly.     $\square$

Another important bifurcation theorem for Riemann solutions is the *Bethe-Wendroff Theorem*, see [28]. We extend this result for the velocity-dependent system (2.10), including shocks connecting different sheets of the stratified variety, obtaining the *generalized Bethe-Wendroff Theorem*; its proof is given in Appendix C.1:

**Proposition 5.3.** *Assume that $F$ and $G$ are $\mathcal{C}^2$. Let $(W^+; W^-; v)$ be a shock between different physical configurations. Assume that $\ell^i(V^+) \cdot (G^+(V^+) - G^-(V^-)) \neq 0$. Then $v$ has a critical point at $W^+$ (and $\tilde{v}^+(V^-; V^+)$ has a critical point at $V^+$), if and only if:*

$$\tilde{v}^+(V^-; V^+) = \tilde{\lambda}^{i,+}(V^+) \qquad for \quad i = 1 \text{ or } 2, \tag{5.7}$$

*where $\lambda^+(V)$ is given by Eq. (4.15) and $v^+(V^-; V^+)$ is given by (4.24). In this case the tangent to $RHL$ carro is parallel to the characteristic vector in the space $W$ to the characteristic vector $r^i$ at $W^+$. Notice also that the $RHL$ in the primary variables is parallel to $\tilde{r}^i$ at $V^+$.*

**5.2. Bifurcation loci.** Assume that there are no high-order degeneracies in any $RH$ locus described in Appendix $C$. For conservation laws of standard form, there are loci which induce topological change in the solution, such as: secondary bifurcation, coincidence, double contact, inflection, hysteresis and interior boundary contact, see [12, 13]. For our class of problems, such loci are equally important.

*5.2.1. Secondary bifurcation locus.* This locus is defined in the space $W = (V, u)$ for conservation laws of standard form, however we will see that it suffices to study it in the space of primary variables $V$. The $RH$ locus for a fixed $V^-$ is obtained implicitly by $\mathcal{H}_V(V^-; V^+) = 0$, where $\mathcal{H}_V : \mathbb{R}^4 \longrightarrow \mathbb{R}$, and $V^+$ are primary variables. At some pairs $(V^-, V^+)$ for fixed $V^-$, this implicit expression fails to define a curve for $V^+$. Following [12], we call the set of the points where there is potential for failure the **secondary bifurcation locus** in $V$. From the implicit function theorem, they are the " $+$ " states for which there exists a " $-$ " state such that the following equalities are satisfied:

$$\mathcal{H}_V(V^-; V^+) = 0 \qquad \text{and} \qquad \frac{\partial \mathcal{H}_V}{\partial V_j^+} = 0, \quad \text{for} \quad j = 1 \text{ and } 2. \tag{5.8}$$

The following theorem yields an equivalent expression for the secondary bifurcation, which is remarkably similar to the expression for standard conservation laws. This Proposition is proved in Appendix $C.2$.

**Proposition 5.4.** *A state $V^+$ belongs to the secondary bifurcation locus for the family $i$ when there exists a state $V^-$ such that:*

$$V^+ \in \mathcal{RH}(V^-) \quad \text{with} \quad \tilde{v}^+(V^-; V^+) = \tilde{\lambda}^{i,+}(V^+) \quad \text{and} \quad \ell^i(V^+) \cdot [G] = 0. \tag{5.9}$$

**Remark 5.1.** *The secondary bifurcation locus in the space $W$ is a ruled surface in $u$. In other words the $RH$ locus for a fixed $W^-$ is obtained implicitly by $\mathcal{H}_V(V^-; V^+) = 0$ and in addition to conditions (5.8) we have $\partial \mathcal{H}_V/\partial u^+ = 0$, which is trivially satisfied because the $RHL$ given by Eq. (4.22) depends only on the variable $V$.*

*5.2.2. Coincidence loci.* There are two important types of speed coincidence: coincidence between eigenvalues and coincidence between eigenvalues and shock speeds. The structures where such coincidences occurs are important because the Riemann solution changes when the $L$ (or $R$) data is prescribed in different regions relatively to these curves.

*Coincidence between eigenvalues in each physical configuration.* This coincidence occurs when eigenvalues of different families have coincident speed in a state $W = (V, u)$:

$$\lambda^1(W) = \lambda^2(W)$$

Notice from Lemma 4.1 that $\lambda^i = ug(V)$, thus it is possible to analyze the system in the variables $V$ without taking into account the Darcy speed $u$. At the coincidence locus between

eigenvalues the Darcy speeds also coincide, so the coincidence locus in the $V$ space consists of the states in a certain physical configuration such that

$$\tilde{\lambda}^1(V) = \tilde{\lambda}^2(V).$$

5.2.3. *Double contact curves.* It consists of the states $W^-$ for which there is a state $W^+$ such that the shock joining $W^-$ and $W^+$ has speed coinciding with the characteristic speed of family $i$ at $(-)$ and of family $j$ at $(+)$:

$$W^+ \in \mathcal{RH}(W^-) \quad \text{with} \quad \lambda^i(W^-) = v^-(W^-, W^+) \quad \text{and} \quad v^+(W^-, W^+) = \tilde{\lambda}^j(W^+).$$

Similarly, we can define this structure only in the space of primary variables:

$$V^+ \in \mathcal{RH}(V^-) \quad \text{with} \quad \tilde{\lambda}^{i,-}(V^-) = \tilde{v}^-(V^-, V^+) \quad \text{and} \quad \tilde{v}^+(V^-, V^+) = \tilde{\lambda}^{j,+}(V^+).$$

Notice that $V^-$ and $V^+$ can be in the same or in different physical configurations.

5.2.4. *Inflection curves.* The rarefaction curves are useful to construct rarefaction waves where the characteristic speed varies monotonically; the inflection curve is the curve where the monotonicity fails, thus rarefaction curves stop at this curve. Any state $W = (V, u)$ on the inflection curve satisfies:

$$\nabla \lambda^i(W) \cdot r^i(W) = 0. \tag{5.10}$$

However, the Darcy speed can be isolated in Eq. (5.10); indeed, in the space of primary variables, the inflection curve of family $i$, for $i = 1, 2$, consists of the states $V$ satisfying the equation:

$$\nabla_V \tilde{\lambda}^i \cdot \tilde{r} = -\tilde{\lambda}^i g_3, \tag{5.11}$$

where $g_i(V)$ for $i = 1, 2, 3$ are the coordinates for the eigenvector $r$ given in Eq. (4.4.$b$).

**Remark 5.2.** *The hysteresis curves are important in general, see* [11, 12, 13]; *we do not discuss then here because they does not arise in our model.*

5.2.5. *Interior boundary contact (extension of the boundary).* Because of the presence of physical region boundaries it is important to obtain the states $W$ joined to points $W'$ on the physical boundary by shock waves that are characteristic at $W$ for the family $i$. They satisfy:

$$W \in \mathcal{RH}(W') \quad \text{with } W' \text{ on the boundary and } \lambda^i(W) = v^-(W, W').$$

In the space of primary variables the interior boundary satisfies:

$$V \in \mathcal{RH}(V') \quad \text{with } V' \text{ on the boundary and } \tilde{\lambda}^i(V) = \tilde{v}^-(V, V').$$

Frequently, for a fixed state $V^-$ there are points in the $RH$ curve of $V^-$ that separate potentially (admissible) shocks from (non-admissible) discontinuities. At these points the shock speed coincides with the characteristic speed and the shock speed has an extremum (see Proposition 5.3) so the Riemann solution usually changes. There are two types of such shocks: left-characteristic and right-characteristic shocks, i.e., the shocks have speed equal to the characteristic speed at the left or at the right state.

5.2.6. *Left and right characteristic shock.* For a fixed state $W^-$, the left characteristic shock is formed by states $W^+$ such that:

$$W^+ \in \mathcal{RH}(W^-) \quad \text{with} \quad \tilde{v}(W^-; W^+) = \lambda^j(W^+) \quad \text{for family } j = 1 \text{ or } 2.$$

In the space of primary variables for a fixed $V^-$, the left shocks are formed by the states $V^+$ such that:

$$V^+ \in \mathcal{RH}(V^-) \quad \text{with} \quad \tilde{v}^+(V^-; V^+) = \tilde{\lambda}^{j,+}(V^+) \quad \text{for family } j = 1 \text{ or } 2.$$

Notice that $V^-$ and $V^+$ can be in the same or different physical configurations.

The right characteristic shock is obtained in similar way.

## 6. Elementary waves for the nitrogen-steam model

The elementary waves are the basic elements in the Riemann solution. In the previous section we obtained some important results to construct the solution and proved that we can obtain this solution in the space of primary variables. Thus in the following sections we describes our structures in the Riemann solution in the space of primary variables only.

6.1. **Elementary waves in single-phase gaseous configuration.** The *spg* configuration in Section 3.2.1 is described by (3.10)-(3.12).

6.1.1. *Rarefaction waves.* We have two eigenvalues and eigenvectors. The first eigenvalue is labeled as $\lambda_c$, because the composition $\psi_{gw}$ changes but the speed and the temperature are constant; the eigenpair is given by:

$$\lambda_c = u/\phi, \qquad r_c = (1, 0, 0)^T, \tag{6.1}$$

which correspond to fluid transport; this wave is actually a contact discontinuity.

The other eigenpair is labelled as $(\lambda_T, r_T)$, because the temperature changes on the rarefaction waves; it is given by:

$$\lambda_T = \left(1 - \hat{C}_r T/\mathbb{F}\right) u/\varphi \quad \text{and} \quad r_T = (0, \mathbb{F}, u\hat{C}_r). \tag{6.2}$$

where using (3.13), $\mathbb{F} := \mathbb{F}(T) = T\psi_{gw}\rho_{gW}h'_{gW} + T\psi_{gn}\rho_{gN}h'_{gN} + T\hat{C}_r$. As the temperature dependet functions $\rho_{gW}, \rho_{gN}, h'_{gW}, h'_{gN}$ and the constant $\hat{C}_r$ are positive and $\psi_{gw}$ and $\psi_{gn}$ are non-negative, $\mathbb{F}(T)$ is positive. The rarefaction wave has constant composition $\psi_{gw}$. Notice that $\lambda_T < \lambda_c$ in the physical range.

6.1.2. *Inflection of thermal wave.* An important structure is the gaseous thermal inflection locus, which consist of the states $(\psi_{gw}, T)$ in *spg* that satisfy $\nabla\lambda_T \cdot r_T = 0$ (or equivalently, Eq. (5.11)). We denote the inflection curve as $\mathcal{I}_T$.

We plot the physical region and $\mathcal{I}_T$ in Figure (6.1.*a*), showing the signs of $\Xi$. We plot the horizontal rarefaction lines associated to $\lambda_c$ and the vertical rarefaction lines associated to $\lambda_T$, (6.2), In Figure 6.1.*b*, w.

6.1.3. *Shock waves.* The *RH* locus is obtained from Eq. $\mathcal{H}_V(V^-, V^+) = 0$ given by (4.22).

The waves associated to $\lambda_c$ are contact discontinuities. A shock occurs with $\psi_{gw}$ constant; it is a thermal shock with speed $v$ and Darcy speed $u^+$ on the right side using Eq. (4.23.*a*), see [16] for a particular expression. The thermal *RH* locus and rarefaction curves are contained in the horizontal lines in Figure (6.1.*b*).
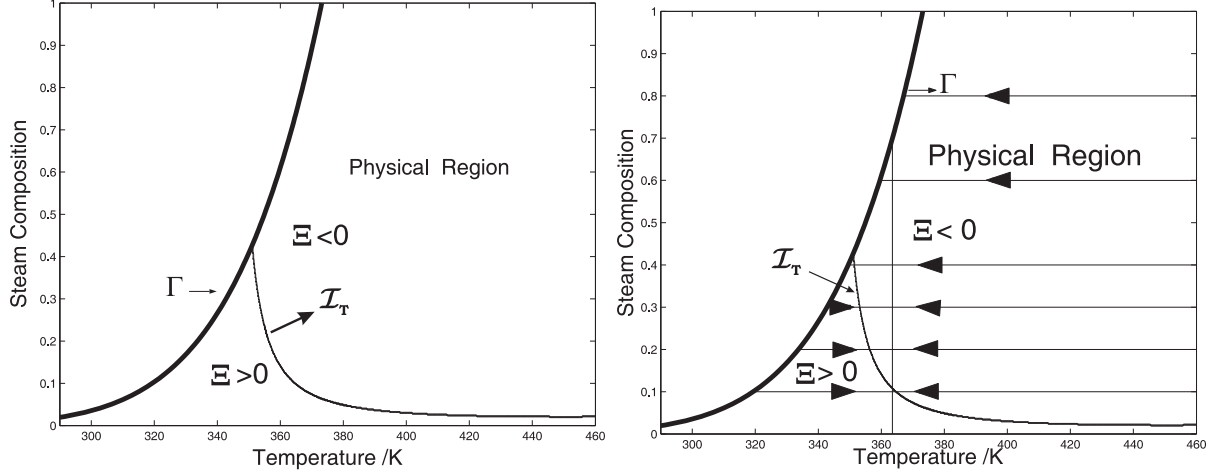
FIGURE 6.1. *a)-left:* The single-phase gaseous configuration, the $\Gamma$ curve and the inflection locus $\mathcal{I}_T$. *b)-right:* Rarefaction curves. The horizontal rarefaction curves are associated to $\lambda_T$; we indicate with an arrow the direction of increasing speed. The vertical lines are contact discontinuity lines associated to $\lambda_c$, in which $\psi_{gw}$ changes, $T$ and $u$ are constant; $\lambda_c$ is constant in each curve.

6.2. **Elementary waves in single-phase liquid configuration.** For the (*spl*) given in Section (3.2.2, Eq. (3.14) is linear, so a single wave is associated to $\lambda_T^W$, given by (3.14.*b*). This wave is a contact discontinuity and there is no genuine rarefaction or shock wave.

**Remark 6.1.** *In the spl there exists a cooling discontinuity between a state with temperature $T^-$ and another state with temperature $T^+$. For the Riemann data: $T^-$ if $x$ is negative and $T^+$ if $x$ is positive, the solution is $T^-$ if $x/t < \lambda_T^W$ and $T^+$ if $x/t > \lambda_T^W$.*

6.3. **Elementary waves in the two-phase configuration.** The *tp* configuration in Section 3.2.3 is described by (3.5)-(3.8).

6.3.1. *The rarefaction waves.* We have two waves: an isothermal wave, defined by the Buckley-Leverett characteristic speed and characteristic vector:

$$\lambda_s = \frac{u}{\varphi}\frac{\partial f_g}{\partial s_g}, \qquad r_s = (1,0,0)^T, \tag{6.3}$$

On this rarefaction curve $T$ and $u$ are constant, only the saturation $s_g$ changes, hence the subscript $s$.

The other eigenpair corresponds to an evaporation wave, where the temperature, saturation and speed change:

$$\lambda_e = \frac{u}{\varphi}\frac{f_g\left[(\gamma_3' - (\rho_{gn}'/\rho_{gn})\gamma_3)\rho_W - H_W(\rho_{gw}' - (\rho_{gn}'/\rho_{gn})\gamma_2)\right] + C_W\rho_W}{\rho_W(C_W + \hat{C}_r) + s_g\left[(\gamma_3' - (\rho_{gn}'/\rho_{gn})\gamma_3)\rho_W - H_W(\rho_{gw}' - (\rho_{gn}'/\rho_{gn})\gamma_2)\right]}, \tag{6.4}$$

where $\gamma_2 = \rho_{gw} - \rho_w$ and $\gamma_3 = H_g - H_W$, with eigenvector

$$r_e = (-\varpi_1, \bar{\gamma}_1, -u\varpi_3)^T, \qquad \text{and we have:} \tag{6.5}$$

$$\bar{\gamma}_1 = \frac{\partial f_g}{\partial s_g} - \frac{f_g\left[(\gamma_3'\rho_{gn} - \rho_{gn}'\gamma_3)\rho_W - H_W(\rho_{gw}'\rho_{gn} - \rho_{gn}'\gamma_2)\right] + C_W\rho_W\rho_{gn}}{\rho_W\rho_{gn}(C_W + \hat{C}_r) + s_g\left[(\gamma_3'\rho_{gn} - \rho_{gn}'\gamma_3)\rho_W - H_W(\rho_{gw}'\rho_{gn} - \rho_{gn}'\gamma_2)\right]},$$

$$\varpi_1 = \bar{\gamma}_4 \left( \frac{(\rho'_{gn}) \rho_W - f_g(\rho'_{gw}\rho_{gn} - \rho'_{gn}\gamma_2)}{\rho_{gn}\rho_W} \right) + \frac{\partial f_g}{\partial T}, \quad \varpi_3 = \frac{\bar{\gamma}_1 \bar{\gamma}_4 \left( \rho'_{gw}\rho_{gn} - \rho'_{gn}\gamma_2 \right)}{\rho_W \rho_{gn}},$$

where

$$\bar{\gamma}_4 = \frac{C_W (f_g - s_g)\rho_W \rho_{gn} + \hat{C}_r f_g \rho_W \rho_{gn}}{\rho_W \rho_{gn}(C_W + \hat{C}_r) + s_g \left[ (\gamma'_3 \rho_{gn} - \rho'_{gn}\gamma_3)\rho_W - H_W(\rho'_{gw}\rho_{gn} - \rho'_{gn}\gamma_2) \right]},$$

In Figs. (6.3.$a$) and (6.3.$b$) we can see that in the region where $\lambda_s > \lambda_e$, the temperature, gaseous water saturation and $u$ increase along the rarefaction wave, while in the region where $\lambda_s < \lambda_e$ the temperature and $u$ increase and the gaseous water saturation decreases. This is an evaporation wave hence the subscript $e$ in $\lambda_e$, $r_e$.

6.3.2. *Shock structures.* The $RH$ locus is obtained from Eq. $\mathcal{H}_V(V^-, V^+) =$ with $\mathcal{H}_V$ given by (4.22). For this isothermal branch of the $RH$ curve the shock speed is

$$v = \frac{u^-}{\varphi} \frac{f_w(s_w^+, T) - f_w(s_w^-, T)}{s_w^+ - s_w^-}.$$

On this wave, the Darcy speed is constant. This is the Buckley-Leverett shock; the $RH$ and rarefaction curves associated to saturation waves lie on the same vertical lines in Fig. 6.3. The other branch of $RH$, associated to $\lambda_e$, is a condensation shock drawn, in Fig. 6.4.

6.3.3. *Coincidence curve.* The coincidence curve between $\lambda_e$ and $\lambda_s$ in the space of primary variables $V$ is:

$$\mathbb{C}_{s,e} = \{(T, s_g) \in tp \text{ such that } \lambda_s(T, s_g) = \lambda_e(T, s_g)\}. \tag{6.6}$$

Computing $\partial\lambda_e/\partial s_g$ from (6.4), equating it to zero and isolating $\partial f_g/\partial s_g$, we obtain:

**Lemma 6.1.** *On the coincidence curve $\mathbb{C}_{s,e}$, the derivative $\partial\lambda_e/\partial s_g$ vanishes.*

We can prove that the points where $\nabla\lambda_e \cdot r_e$ vanishes satisfy:

$$\lambda_e = \lambda_s \qquad \text{or} \qquad \frac{f_g}{s_g} = \frac{C_W}{C_W + \hat{C}_r}. \tag{6.7}$$

We define the curve $\mathbb{I}_e := \{(T, s_g) \in tp \text{ satisfying Eq. } (6.7.b)\}$. We define the inflection curve $\mathcal{I}_e = \{(T, s_g) \text{ satisfying } (6.7.a) \text{ and } (6.7.b)\}$ associated to the evaporation rarefaction in the $tp$.

The following Lemma follows from Definition (6.6) and Eq. (6.7).

**Lemma 6.2.** *The coincidence curve $\mathbb{C}_{s,e}$ is contained in $\mathcal{I}_e$. Moreover $\mathcal{I}_e = \mathbb{C}_{s,e} \bigcup \mathbb{I}_e$.*

## 7. THE RIEMANN SOLUTION FOR GEOTHERMAL ENERGY RECOVERY

We present an example of Riemann solution for the specific model in Section 3. Other examples are found in [16]. We consider the injection of a two-phase mixture of water, steam and nitrogen into a rock containing superheated steam ($\psi_R = 1$) at a temperature $T_R > T^b$. The injection boundary condition and initial data are:

$$\begin{cases} (s_g, \psi(T), T, u)_L & \text{if } x = 0 \text{ (the injection point)}, \qquad \text{with} \quad u_L > 0, \\ (s_g = 1, 1, T, \cdot)_R & \text{if } x > 0 \end{cases} \tag{7.1}$$

From Prop. 4.2, the projection of the solution onto the primary variable space does not depend on $u_L$.
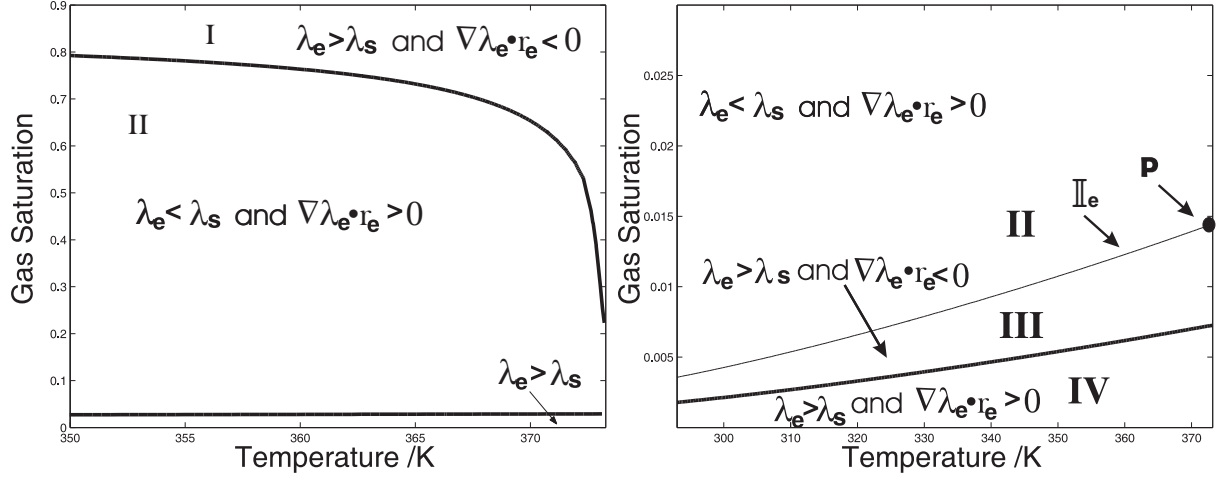
FIGURE 6.2. *a)-left:* Coincidence curve $\mathbb{C}_{s,e}$. Relative sizes of $\lambda_s$ and $\lambda_e$ in a sheet of the *tp* state space. The almost horizontal coincidence curve $\lambda_e = \lambda_s$ is not drawn in scale, because it is very close to the axis $S_g = 0$. *b)-right:* A zoom of the region below the lower coincidence curve. In both figures, all curves form $\mathcal{I}_e$, subdividing the *tp* configuration in four parts; the curve $\mathbb{I}_e$ is the defined satisfying Eq. (6.7.*b*), the point $P$ is the intersection of $\mathbb{I}_e$ with the boundary of the $TS$ plane at the water boiling temperature.
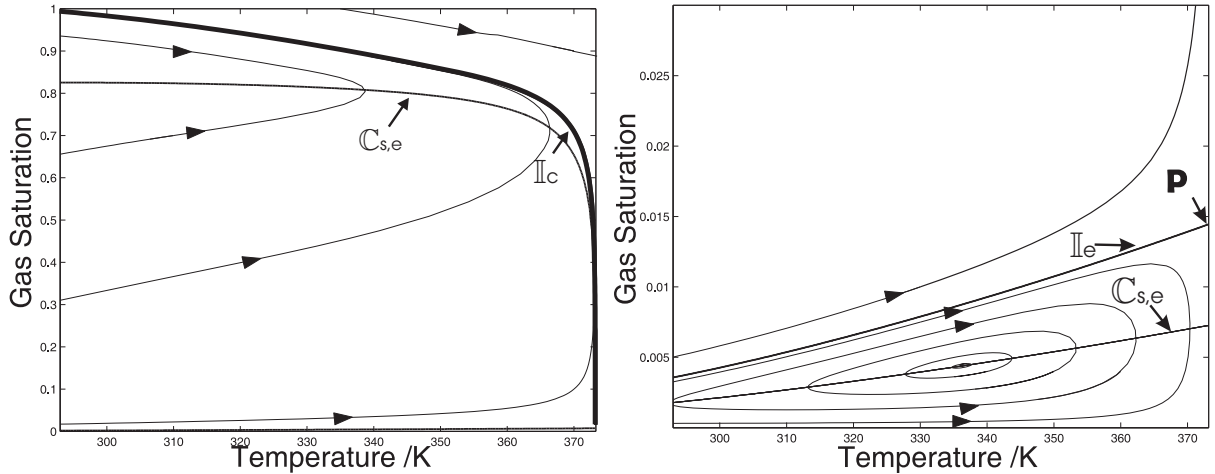


FIGURE 6.3. *a)-left:* The rarefaction curves projected in the $(T, s_g)$ plane. The thin curve without arrows is $\mathbb{C}_{s,e}$. The bold curve $\mathbb{I}_c$ is an invariant curve for the rarefaction field, namely the integral curve through $P$ of Fig. (6.2.*b*). *b)-right:* The rarefaction curves in the regions $III$ and $IV$ shown in the Figure (6.2.*b*). The arrows indicate the direction of increasing speed.

The inflection $\mathcal{I}_e$ defines in *tp* configuration 4. These regions are labeled as: $I$, $II$, $III$ and $IV$, see Figure 6.2.

For a fixed $R$ in the *spg* configuration, we can subdivide each of regions $I - IV$ in subregions with the following property: for any $L$ in a given subregion the Riemann problems with data $R$ of form (7.1) have the same sequence of waves, see Figure 7.1. The corresponding Riemann
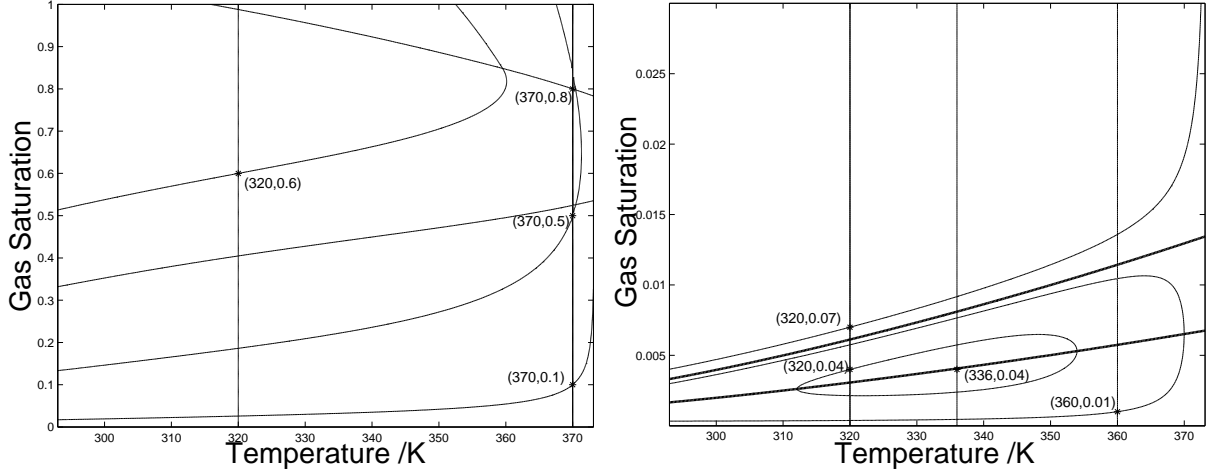
FIGURE 6.4. *a)-left:* The $RH$ curves projected in the $(T, s_g)$ plane for the marked $(-)$ states: $(320, 0.6)$, $(370, 0.1)$, $(370, 0.5)$ and $(370, 0.8)$. The vertical line is the isothermal wave: Buckley-Leverett shock and rarefaction. The other branch is the condensation wave. *b)- right:* The region shown in Figure $(6.2.b)$ is a zoom of the bottom of $(6.2.a)$. Each $RH$ locus is formed by a non-isothermal curve and a vertical line which is the isothermal Buckley-Leverett $RH$ locus. For the state $(336, 0.04)$ the Rankine-Hugoniot locus reduces to the isothermal line only.

solutions are described in Section 7.3, we utilize some results of [15] to obtain the Riemann solutions.

In the following section we indicate the Riemann solution in the space of variables $\mathcal{V}$, but the bifurcation curves are drawn in the space of primary variables $V$.
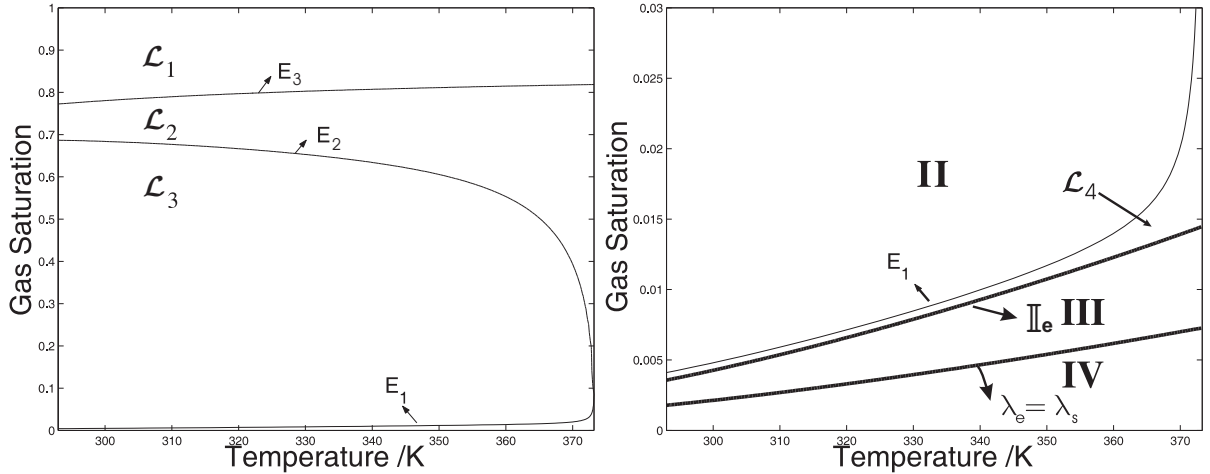


FIGURE 7.1. *a)-left:* The 3 subregions in $I$ and $II$ for a Riemann data of form (7.1). The curves are defined in Secs. 7.1.2, 7.1.1 and 7.1.3. The region $\mathcal{L}_4$ lies below $E_1$. *b)-right:* Zoom of regions $II$ to $IV$. Each curve is explained in Section 7.1. The curve $\mathbb{I}_e$ satisfies $(6.7.b)$; it is an invariant curve for shocks and rarefactions.

7.1. **Subdivision of** $tp$. For the data (7.1) with $T_R$ fixed, the Riemann solutions have the same wave sequence and structure for $L$ in each subregion labelled $\mathcal{L}_1$ to $\mathcal{L}_4$ of Figure 7.1. We obtain the curves that bound subregions: the curve $E_2$, see Fig. (7.1.$a$) and Sec. 7.1.1; the curve $E_1$, see Figs. (7.1.$a$), (7.1.$b$) and Sec. 7.1.2; the curve $E_3$, see Fig. (7.1.$a$) and Sec. 7.1.3.

**Remark 7.1.** *The evaporation wave speed $\lambda_e$ for states in $\mathcal{L}_2$ and $\mathcal{L}_3$ in the tp is larger than the thermal wave speed in the spg. From geometrical compatibility, in the wave solution, there is no thermal wave (rarefaction or shock) after the evaporation shock in the spg, i.e., the shock between regions should reach a state of the form $\mathcal{V} = (1, \psi_{gw}, T_R)$.*

7.1.1. *Curve $E_2$.* This curve consists of the states $V^-$ in the $tp$ such that the evaporation rarefaction speed $\lambda_e(V^-)$ equals the speed $v^{BG}(V^-; V^+)$ of the shock to the state $V^+ = (T_R, \psi_{gw})$, which is the primary variable of $\mathcal{V}^+ = (1, T_R, \psi_{gw})$ in the $spg$. This curve is called extension of the physical boundary, see Section 5.2.5.

We obtain $\psi_{gw}$ as follows. For each primary variable state $V^-$ of the form $\mathcal{V} = (S, \psi_{gw}(T), T)$ in the $tp$, we search the state in the $RH$ locus starting at $V^-$ with right temperature $T_R$. These restrictions yield a state in the $spg$ with steam composition $\psi_{gw}$ found numerically.

7.1.2. *Curve $E_1$.* We find the Riemann solution for left states in $I$ or $II$. From geometrical compatibility, to solve the Riemann problem we choose first the slowest wave whenever possible. In $II$, the slowest wave is the evaporation wave. Since we are interested in connecting a left state at lower temperature to a state at higher temperature, we use an evaporation rarefaction curve instead of a shock, see Section 6.3. We can do this until the speed of the state at the right of the evaporation rarefaction equals the speed of the shock between regions; this occurs when the rarefaction curve crosses $E_2$.

The rarefaction curves starting at certain left states do not cross $E_2$, rather they reach directly the steam region at boiling temperature. We define the curve $E_1$ as the evaporation rarefaction curve that crosses $E_2$ at the boiling temperature of water. Notice that for left states in $II$ above $E_1$, the evaporation rarefaction curve always crosses $E_2$.

7.1.3. *Curve $E_3$.* For states $V^-$ above the curve $E_2$, the evaporation rarefaction speed is larger than $v^{BG}(V^-; V^+)$, for the primary variable $V^+ = (T_R, \psi_{gw})$ of the state $\mathcal{V}^+ = (1, T_R, \psi_{gw})$; in Figures (6.2.$a$) and (7.2.$a$) we show the relatives sizes of $v^{BG}$, $\lambda_e$ and $\lambda_s$. When $s_g$ tends to 1 the Buckley-Leverett shock speed is zero, so there are transition curves, $v^{BL} = v^{BG}$, $v^{BL} = v^T$ and $v^{BG} = v^T$. From the Prop. 5.2, we obtain a bifurcation curve $E_3$ where $v^T = v^{BL} = v^{BG}$. For states above this curve, the Buckley-Leverett shock is slower than $v^{BG}$ and $v^T$, so there is no direct shock between the $tp$ and the $spg$ regions.

7.2. **The Riemann solution.** We use the notation $\mathcal{V}_L = (s_g, \psi_{gw}(T), T)_L$, $\mathcal{V}_R = (s_g = 1, 1, T)_R$ and $V_L$, $V_R$ for the respective primary variables. All the variables for the intermediate states are written out. The states are labelled by 1, 2, etc. We use the following nomenclature: the evaporation rarefaction is $R_E$, the Buckley-Leverett shock and rarefaction are $S_{BL}$ and $R_{BL}$, the shock between the $tp$ and the $spg$ regions is $S_{BG}$ and the compositional contact discontinuity is $S_C$. We use the notation for wave sequences established in Section 4.3.

7.2.1. $V_L$ *in* $\mathcal{L}_4$. There is an evaporation rarefaction curve from $\mathcal{V}_L$ up to $\mathcal{V}_{(1)} = (s_L, 1, T^b)$, where $T^b$ is the boiling temperature. For this state there is no nitrogen and so $\psi_{gw} = 1$.
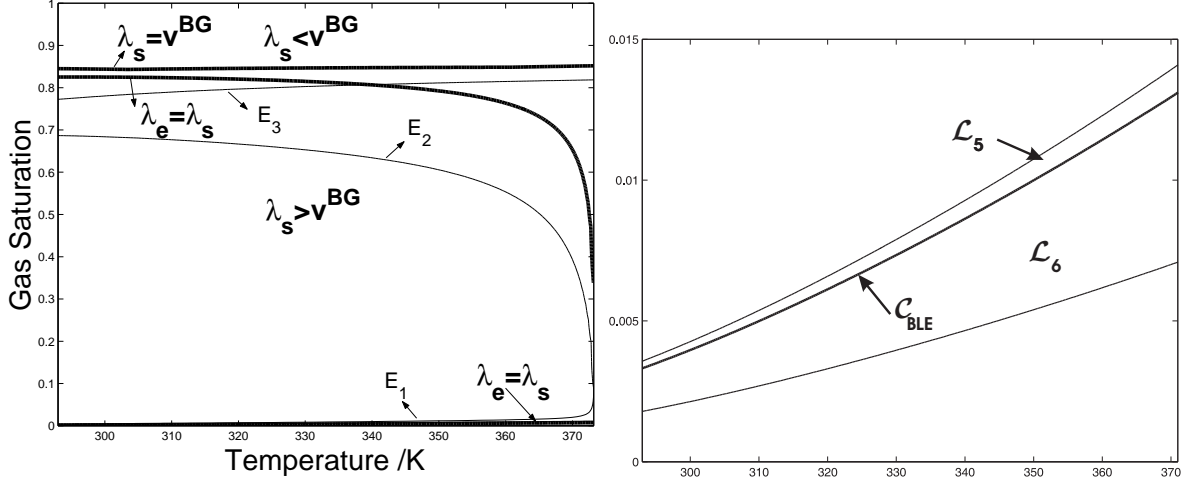
FIGURE 7.2. *a)-left:* The relevant curves for the Riemann solution for the left states in the *tp* configuration. *b)-right:* Subregions $\mathcal{L}_5$ and $\mathcal{L}_6$. $\mathcal{C}_{BLE}$ is the coincidence curve $v_{g,w}^b = \lambda_e$. For states in $\mathcal{L}_5$, we have that $v_{g,w}^b < \lambda_e$; for states in $\mathcal{L}_6$, we have that $v_{g,w}^b > \lambda_e$.

The solution after the intermediate state $(s_M, 1, T^b)$ was found in [15]: an evaporation shock between the boiling configuration to the steam configuration appears, denoted by $S_{VS}$.

The solution consists of the waves $R_e \mapsto R_{BL} \twoheadrightarrow S_{VS}$ with sequence:

$$\mathcal{V}_L \xrightarrow{R_e} \mathcal{V}_{(1)} \xrightarrow{R_{BL}} \mathcal{V}_{(2)} \xrightarrow{S_{VS}} \mathcal{V}_R. \tag{7.2}$$

7.2.2. $\mathcal{V}_L$ *in* $\mathcal{L}_3$. There is a rarefaction from $\mathcal{V}_L$ up to $\mathcal{V}_{(1)} = (s^*, \psi_{gw}(T^*), T^*)$ in the *tp* region; $\mathcal{V}_{(1)}$ is a state where $\lambda_e(V_{(1)}) = v^{BL}(V_{(1)}; V_{(2)})$, with $\mathcal{V}_{(2)} = (1, \psi_{gw}, T_R)$ in the *spg* region. We have described how to obtain $\psi_{gw}$ in Section 7.1.1. Finally, there is a compositional contact discontinuity at temperature $T_R$ with speed $v^C$ from $\mathcal{V}_{(2)}$ to $\mathcal{V}_R$.

The solution consists of the waves $R_E \twoheadrightarrow S_{BG} \mapsto S_C$ with sequence:

$$\mathcal{V}_L \xrightarrow{R_e} \mathcal{V}_{(1)} \xrightarrow{S_{BG}} \mathcal{V}_{(2)} \xrightarrow{S_C} \mathcal{V}_R. \tag{7.3}$$

7.2.3. $\mathcal{V}_L$ *in* $\mathcal{L}_2$. Let $\mathcal{V}_{(1)} = (1, \psi_{gw}, T_R)$ be a state in the *spg* region. Since $\lambda_e(V_L) > v^{BL}(V_L; V_{(1)})$, there is a shock between $\mathcal{V}_L$ and $\mathcal{V}_{(1)}$ with speed $v^{BG}$. From the state $\mathcal{V}_{(1)}$ there is a compositional contact discontinuity with speed $v^C$ to $\mathcal{V}_R$.

The solution consists of the waves $S_{BG} \mapsto S_C$ with sequence:

$$\mathcal{V}_L \xrightarrow{S_{BG}} \mathcal{V}_{(1)} \xrightarrow{S_C} \mathcal{V}_R. \tag{7.4}$$

7.2.4. $\mathcal{V}_L$ *in* $\mathcal{L}_3$. Since $v^{BG}$ and $v^{BL}$ are smaller than $v^T$, there is a Buckley-Leverett saturation shock between $\mathcal{V}_L$ and $\mathcal{V}_{(1)} = (1, \psi_{gw}(T_L), T_L)$. From $\mathcal{V}_{(1)}$ there is a thermal shock with speed $v^T$ to $\mathcal{V}_{(2)} = (1, \psi_{gw}(T_L), T_R)$ and finally there is a compositional contact discontinuity with speed $v^C$ to $\mathcal{V}_R$.

The solution consists of the waves $S_{BL} \twoheadrightarrow S_T \mapsto S_C$ with sequence:

$$\mathcal{V}_L \xrightarrow{S_{BL}} \mathcal{V}_{(1)} \xrightarrow{S_T} \mathcal{V}_{(2)} \xrightarrow{S_C} \mathcal{V}_R. \tag{7.5}$$
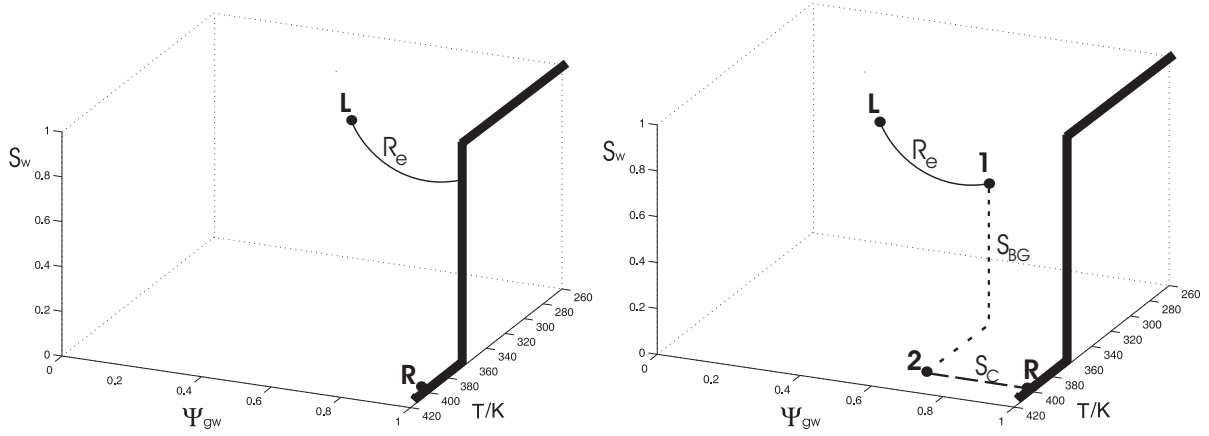
FIGURE 7.3. Riemann solutions in phase space, omitting the surface shown in Fig. 3.1. *a) left:* Solution (7.2) for $V_L \in \mathcal{L}_4$, Sec. 7.2.1.   *b) right:* Solution (7.3) for $V_L \in \mathcal{L}_3$, Sec. 7.2.2. The numbers 1 and 2 indicate the intermediate states $V$ in the wave sequence.
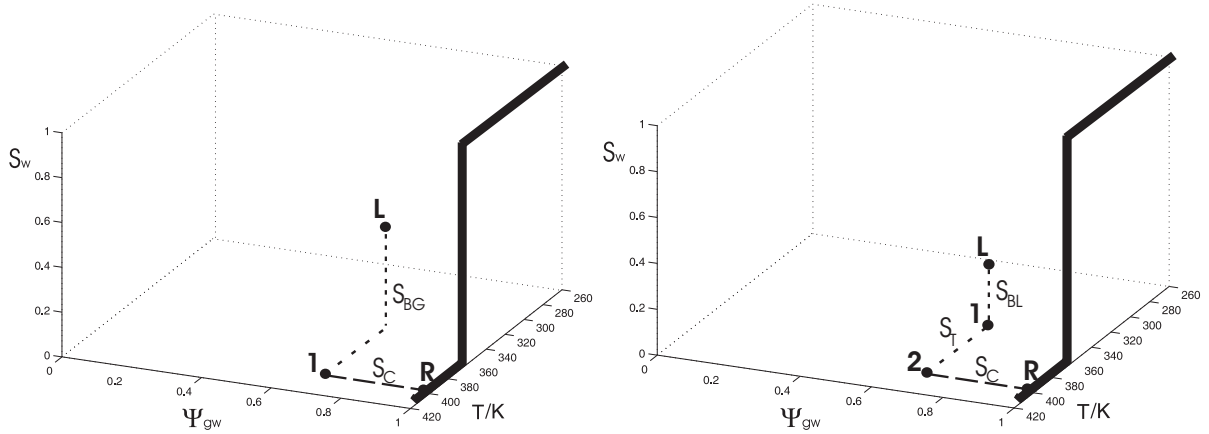
FIGURE 7.4. Riemann solutions in phase space, removing the surface shown in Fig. 3.1. *a) left:* Solution (7.4) for $V_L \in \mathcal{L}_2$, Sec. 7.2.3.   *b) right:* Solution (7.5) for $V_L \in \mathcal{L}_1$, Sec. 7.2.4. The numbers 1 and 2 indicate the intermediate states $V$ in wave sequence.

**7.3. $V_L$ in $III$ and $IV$.** The integral curves or $RH$ locus starting at any point in $III$ or $IV$ do not reach the water boiling temperature. Moreover, the $RH$ locus between the $tp$ and the $spg$ configurations for points in $III$ or $IV$ do not reach temperatures above the temperature $T_L$. So the only possibility to reach the right state is by crossing the curve $B$. However, the evaporation wave does not cross the curve $B$; to cross it we need a Buckley-Leverett shock. The Buckley-Leverett shock for an initial state below $B$ and a final state above $B$ is faster than the evaporation wave $\lambda_e$. Moreover, it is faster than the shock between the $spg$ and the $tp$ configurations. So with a Buckley-Leverett shock of this type, it is only possible to reach states of form $(1, \psi_{gw}(T), T)$ with $T < T^b$. Since the thermal wave is slower than the Buckley-Leverett shock, it is impossible to construct the Riemann solution satisfying the geometrical compatibility principle for waves only in the $tp$ configuration.

To obtain the Riemann solution, we utilize the results of [15]. First we can identify subregions $\mathcal{L}_5$ and $\mathcal{L}_6$ in $III \bigcup IV$. The only way to reach the *spl* region from $III$ and $IV$ is by a Buckley-Leverret shock, see Section 7.3. Notice that the boundary between $III$ and $IV$ is the coincidence curve $\lambda_s = \lambda_e$, in which $\lambda_s$ and $\lambda_e$ are given by Eqs. (6.3) and (6.4). Moreover, there exists a coincidence curve between the evaporation wave and the hot isothermal steam water, HISW, which is the shock between the $(-)$ state $(s_g^-, T^b)$ in the *spg* and the $(+)$ state $(0, T^b)$ in the *spl* with speed $v_{g,w}^b$ defined in [15] as:

$$v_{g,w}^b = \frac{u^b}{\varphi} \frac{f_g^b(s_g^-)}{s_g^-}. \tag{7.6}$$

Here the $T^b$ indicates the water boiling temperature. We denote this coincidence curve by $\mathcal{C}_{BLE}$, see Fig. (7.2.$b$).

For $\mathcal{V}_L$ in $\mathcal{L}_6$ the HISW is slower than the evaporation shock, so we can reach the *spl* directly by a HISW. The proofs of next two lemmas are found in [15]. Notice that in [15] the water saturation is the unknown, but in the current paper we use gas saturation as unknown, to avoid misundersting we indicate all saturation by the sunbindex $g$.

**Lemma 7.1.** *The following inequalities are valid for $(s_g, T)$ in $\mathcal{L}_6$:*

$$v_{g,w}^b < v^{WES}(s_g = 0, T; s_{g\dagger\dagger}, T^b) < v_T^w, \tag{7.7}$$

*where $v^{WES}$ is the shock speed between a $(-)$ state $(s_g = 0, T^-)$ in the spl region and a $(+)$ state $(s_g^+, T^b)$ in the tp region with $T^b$ the boiling temperature of water given by:*

$$v^{WES}(s_g = 0, T; s_g, T^b) = \frac{u^+}{\varphi} \frac{f_g^+ \rho_{gW}^b \left(h_{gW}^b - h_w^-\right) + f_w^+ \rho_w \left(h_w^b - h_w^-\right)}{\hat{H}_r^b - \hat{H}_r^- + s_g^+ \rho_{gW}^b \left(h_g^b - h_w^-\right) + s_w^+ \rho_w \left(h_w^b - h_w^-\right)}, \tag{7.8}$$

*The gas saturation $s_{g\dagger\dagger}$ is obtained from the following equality:*

$$\lambda_s^b(s_{g\dagger\dagger}) = v^{WES}(T^-, s_g = 0; s_{g\dagger\dagger}). \tag{7.9}$$

**Lemma 7.2.** *There exists a state $(\hat{s}_g, T^b)$ in the tp satisfying the following equality:*

$$v^{VS} = \frac{u^b}{\varphi} \frac{\partial f_w}{\partial s_w}(\hat{s}_g, T^b), \tag{7.10}$$

*where $v^{VS}$ is the shock speed between a $(-)$ state $(s_g^-, T^b)$ in the tp and a $(+)$ state $(s_g = 1, T^+ > T^b)$ in the spg.*

$$v^{VS} = \frac{u^-}{\varphi} \frac{f_g^- \rho_{gW}^b \left(h_{gW}^+ - h_{gW}^b\right) + f_w^- \rho_w \left(h_{gW}^+ - h_w^b\right)}{\hat{H}_r^+ - \hat{H}_r^b + s_g^- \rho_{gW}^b \left(h_{gW}^+ - h_{gW}^b\right) + s_w^- \rho_w \left(h_{gW}^+ - h_w^b\right)}. \tag{7.11}$$

In the previous lemmas the superscript $b$ in the functions indicates that they are evaluated at the boiling temperature of water; similarly the superscript $-$ indicates that the functions are evaluated at temperature $T^-$. The enthalpies per unit mass of water and steam are $h_w$, $h_{gW}$, described in appendix A.

In the Riemann solution there a is a HISW shock between the state $\mathcal{V}_L = (s_L, \psi_{gw}(T_L), T_L)$ and a state $\mathcal{V}_{(1)} = (0, \psi_{gw}(T_L), T_L)$. From this state, there exists a shock of speed $v^{WES}$ between the *spl* configuration and a state $\mathcal{V}_{(2)} = (s_{g\dagger\dagger}, 0, T^b)$ in the *tp* configuration without nitrogen and with temperature $T^b$. From $\mathcal{V}_{(2)}$ there is a Buckley-Leverett rarefaction up to the state $\mathcal{V}_{(3)} = (\hat{s}_g, 0, T^b)$ and so there exists a shock between $\mathcal{V}_{(3)}$ up to the right state $\mathcal{V}_R$.

The Riemann solution consists of the waves $HISW \mapsto WES \twoheadrightarrow R_s \twoheadrightarrow VS$ with sequence:

$$\mathcal{V}_L \xrightarrow{HISW} \mathcal{V}_{(1)} \xrightarrow{WES} \mathcal{V}_{(2)} \xrightarrow{R_s} \mathcal{V}_{(3)} \xrightarrow{VS} \mathcal{V}_R, \qquad (7.12)$$

For the left primary variable $V_L$ in $\mathcal{L}_5$, the HISW is faster than the evaporation wave, so from $V_L$ we reach a state $\mathcal{V}_{(0)}$ belongs to $\mathcal{C}_{BLE}$ using an evaporation rarefaction $R_e$. From this state in the coincidence curve, we can reach the $spl$ directly by a HISW a state $\mathcal{V}_{(1)} = (0, \psi_{gw}(T_L), T_L)$; from this state the sequence of wave is the same of (7.12)

The Riemann solution consists of the waves $R^e \twoheadrightarrow HISW \mapsto WES \twoheadrightarrow R_s \twoheadrightarrow VS$ with sequence:

$$\mathcal{V}_L \xrightarrow{R_e} \mathcal{V}_{(0)} \xrightarrow{HISW} \mathcal{V}_{(1)} \xrightarrow{WES} \mathcal{V}_{(2)} \xrightarrow{R_s} \mathcal{V}_{(3)} \xrightarrow{VS} \mathcal{V}_R, \qquad (7.13)$$
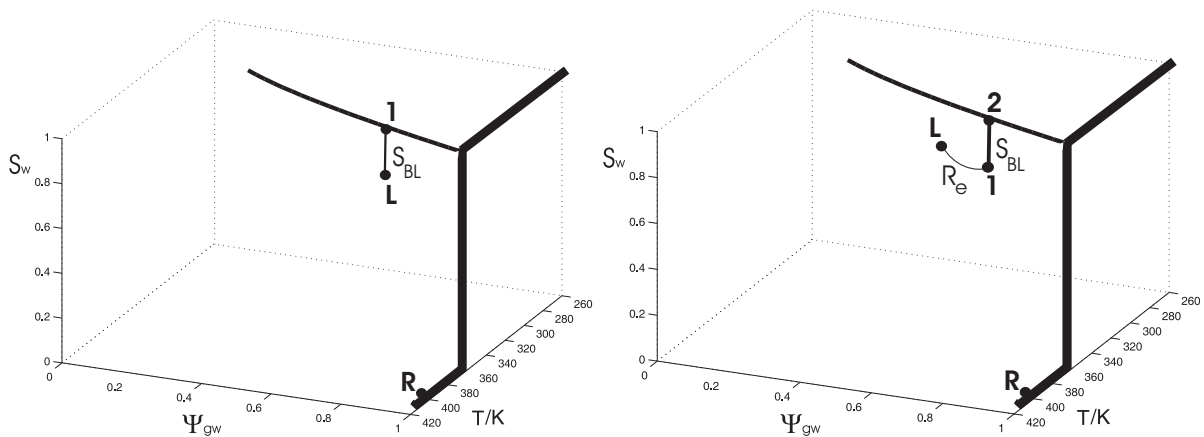


FIGURE 7.5. Riemann solutions in phase space, omitting the surface shown in Fig. 3.1. *a) left:* Solution (7.12) for $V_L \in \mathcal{L}_6$.    *b) right:* Solution (7.13) for $V_L \in \mathcal{L}_5$. The numbers 1 and 2 indicate intermediate states $V$ in the wave sequence. The waves after $spl$ configuration are not represented in this figure.

## 8. SUMMARY AND CONCLUSIONS

We have described a solution of the Riemann problem for the injection of a mixture nitrogen/steam/water into a porous rock filled with steam above boiling temperature. We have developed a systematic theory for the Riemann solution for a $4 \times 4$ balance system, where we assume thermodynamical equilibrium laws. This theory can be applied for a system $n \times n$. The set of solutions depends $L^1$-continuously on the Riemann data. It is a step towards obtaining a general method for solving Riemann problems for a wide class of balance equations with phase changes, [16]. In [18], we intend to understand how small deviations from thermodynamical equilibrium determines the discrepancy between the solution under assumptions of thermodynamical equilibrium and the solution in which we relax this assumption.

Our formalism has applications in numerical simulations for fluid flow such as the "mortar upscaling method". This method consists in dividing the spatial domain in several sub-regions; different models are used in each sub-region, see [23]. The formalism developed in this paper gives physically meaningful conditions to match adjoining sub-regions. Since we need to track the moving discontinuities between sub-domains, we can also use this formalism by a natural way in the front-tracking algorithm, see [8, 9].

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## APPENDIX A. PHYSICAL QUANTITIES; SYMBOLS AND VALUES

| Table 2, Summary of physical input parameters and variables | | | |
|---|---|---|---|
| *Physical quantity* | *Symbol* | *Value* | *Unit* |
| Water, steam fractional functions | $f_w, f_g$ | Eq. (3.2) . | [m$^3$/m$^3$] |
| Porous rock permeability | $k$ | $1.0 \times 10^{-12}$. | [m$^3$] |
| Water, steam relative permeabilities | $k_{rw}, k_{rg}$ | Eq. (A.9) . | [m$^3$/m$^3$] |
| Pressure | $p_{at}$ | $1.0135 \times 10^5$. | [Pa] |
| Water Saturation Pressure | $p^{sat}$ | Eq. $(A.6)$. | [Pa] |
| Water, gaseous phase velocity | $u_w, u_g$ | Eq. (3.1) . | [m$^3$/(m$^2$s)] |
| Total Darcy velocity | $u$ | $u_w + u_g$, Eq. (3.3) . | [m$^3$/(m$^2$s)] |
| Effective rock heat capacity | $C_r$ | $2.029 \times 10^6$. | [J/(m$^3$K)] |
| Steam and nitrogen enthalpies | $h_{gW}, h_{gN}$ | Eqs. $(A.1)$, $(A.2)$. | [J/m$^3$] |
| Water enthalpy | $h_W$ | $h_W = C_W T/\rho_w$. | [J/m$^3$] |
| Rock enthalpy | $H_r$ | Eq. $(A.3)$. | [J/m$^3$] |
| Water, steam saturations | $s_w, s_g$ | Dependent variables. | [m$^3$/m$^3$] |
| Connate water saturation | $s_{wc}$ | 0.15. | [m$^3$/m$^3$] |
| Temperature | $T$ | Dependent variable. | [K] |
| Water, gaseous phase viscosity | $\mu_w, \mu_g$ | Eq. (A.4) , Eq. (A.5) . | [Pa s] |
| Steam and nitrogen densities | $\rho_{gw}, \rho_{gn}$ | Eq. $(A.7.a)$, $(A.7.b)$. | [kg/m$^3$] |
| Constant water density | $\rho_W$ | 998.2. | [kg/m$^3$] |
| Steam and nitrogen gas composition | $\psi_{gw}, \psi_{gn}$ | Dependent variables. | [$-$] |
| Universal gas constant | $R$ | 8.31 | [J/mol/K] |
| Nitrogen and water molar masses | $M_N, M_W$ | 0.28, 0.18 | [kg/mol] |
| Rock porosity | $\varphi$ | 0.38. | [m$^3$/m$^3$] |

A.1. **Temperature dependent properties of steam and water.** Following [3], the water enthalpy per mass unit $h_W = C_W T/\rho_w$. The steam enthalpy $h_{gW}[J/kg]$ as a function of temperature is approximated by

$$h_{gW}(T) = -2.20269 \times 10^7 + 3.65317 \times 10^5 T - 2.25837 \times 10^3 T^2 + 7.3742 T^3$$
$$- 1.33437 \times 10^{-2} T^4 + 1.26913 \times 10^{-5} T^5 - 4.9688 \times 10^{-9} T^6 - \overline{h}_w. \tag{A.1}$$

The nitrogen enthalpy $h_{gN}[J/kg]$ as a function of temperature is approximated by

$$h_{gN}(T) = 975.0T + 0.0935 T^2 - 0.476 \times 10^{-7} T^3 - \overline{h}_{gN}. \tag{A.2}$$

These constants enthalpies $\overline{h}_w$ and $\overline{h}_{gN}$ are chosen such that $h_w(T)$, $h_{gN}(T)$ vanish at a reference temperature $\overline{T} = 293K$. The rock enthalpy $H_r$, $\hat{H}_r$, water and gaseous enthalpies per mass unit $H_w$ and $H_g$ are are given by:

$$H_r = (1 - \varphi)C_r(T - \overline{T}), \quad \hat{H}_r = H_r/\varphi, \quad H_w = \rho_w h_w \text{ and } H_g = \rho_{gw} h_{gW} + \rho_{gn} h_{gN}. \tag{A.3}$$

The temperature dependent liquid water viscosity $\mu_w$ $[Pas]$ is approximated by

$$\mu_w = -0.0123274 + \frac{27.1038}{T} - \frac{23527.5}{T^2} + \frac{1.01425 \times 10^7}{T^3} - \frac{2.17342 \times 10^9}{T^4} + \frac{1.86935 \times 10^{11}}{T^5}.$$
(A.4)

We assume that that the viscosity of the gas is independent of the composition.

$$\mu_g = 1.826\,4 \times 10^{-5} \left(T/T^b\right)^{0.6},$$
(A.5)

where $T^b = 373.14K$ is the boiling temperature of the water.

The water saturation pressure as a function of temperature is given as

$$p^{sat} = 10^3(-175.776 + 2.29272T - 0.0113953T^2 + 0.000026278T^3$$
$$-0.000000273726T^4 + 1.13816 \times 10^{-11}T^5)^2$$
(A.6)

The corresponding concentrations $\rho_{gw}$, $\rho_{gn}$ are calculated with the ideal gas law:

$$\rho_{gw} = M_W \, p^{sat}/\left(RT\right), \quad \rho_{gn} = M_N \, \left(p_{at} - p^{sat}\right)/\left(RT\right),$$
(A.7)

The pure phase densities are:

$$\rho_{gW}(T) = M_W p_{at}/\left(RT\right), \qquad \rho_{gN}(T) = M_N p_{at}/\left(RT\right),$$
(A.8)

where the gas constant $R = 8.31[J/mol/K]$, $M_W$ and $M_N$ are the the nitrogen and water molar masses.

For simplicity the liquid water density is assumed to be constant at $998.2.kg/m^3$.

## A.2. Constitutive relations.
The relative permeability functions $k_{rw}$ and $k_{rg}$ are considered to be power functions of their respective saturations, i.e.

$$k_{rw} = \begin{cases} k'_{rw} \left(\frac{s_w - s_{wc}}{1 - s_{wc} - s_{gr}}\right)^{n_w} \\ 0 \end{cases}, k_{rg} \begin{cases} k'_{rg} = \left(\frac{s_g - s_{gr}}{1 - s_{wc} - s_{gr}}\right)^{n_g} & \text{for} \quad s_{wc} \leq s_w \leq 1, \\ 1 & \text{for} \quad s_w < s_{wc}. \end{cases}$$
(A.9)

For the computations we take $n_w = 2$ and $n_g = 2$. The end point permeabilities $k'_{rw}, k'_{rg}$ are 0.5 and 0.95 respectively. The connate water saturation $s_{wc}$ is given in the table.

## Appendix B. Spliting in the system

For these flows, the principle of conservation of mass for each chemical species implies that there exist two linear maps $E : \mathbb{R}^{m+1} \longrightarrow \mathbb{R}^n$ and $S : \mathbb{R}^{m+1} \longrightarrow \mathbb{R}^{m+1-n}$ that satisfy:

$$E(\hat{q}(\mathcal{V})) = 0 \qquad \text{and} \qquad S(Q(\mathcal{V})) = q(\mathcal{V}) \qquad \forall \, \mathcal{V}.$$
(B.1)

The function $q(\mathcal{V})$ is a $m + 1 - n$ vector. Each component of this vector represents a single type of mass transfer. For example, water can exist in liquid and gaseous form, when the water evaporates, the total amount of mass of water that disappears in the equation for balance of liquid water is the same quantity that appears in the equation for balance of steam (gaseous water), so these mass transfers are dependent: because in the equation for liquid water the transfer term is the negative of the term appearing in the equation for gaseous water (steam). The vector $q(\mathcal{V})$ is obtained from the independent components of the set $(\hat{q}_1, \hat{q}_2, \cdots, \hat{q}_l, 0, \cdots, 0)$. In Eq. $(B.1.a)$, 0 is a $n \times 1$ vector. The linear map $S$ isolates the independent entries of $Q$. One can prove that the matrix $M$ defined as:

$$M = \begin{pmatrix} S \\ E \end{pmatrix}$$
(B.2)

is non-singular and transforms the system (1.1)-(1.2) into the equivalent system (2.1)-(2.2).

## APPENDIX C. DEGENERACIES OF THE $RH$ LOCUS AND BIFURCATIONS

Proposition 4.2 is valid if the denominator of Eq. (4.29) is non zero for some $\{i,j\} \in \mathbb{P}$. So it is necessary to study the behavior of the solution when $\mathcal{X}_{ij}^+ = 0$ for all $\{i,j\} \in \mathbb{P}$. For a fixed pair $\{i,j\} \in \mathbb{P}$, it easy to prove that:

**Lemma C.1.** *Let $\{V^-, V^+\}$ satisfy $\mathcal{H}_V(V^-; V^+) = 0$, where $\mathcal{H}_V$ is given by Eq. (4.22). If $\mathcal{X}_{ij}^+ = 0$, then one of following conditions is satisfied:*

$$(i) \quad \mathcal{Y}_{ij} = 0 \qquad or \qquad (ii) \quad \mathcal{X}_{12}^+ = \mathcal{X}_{31}^+ = \mathcal{X}_{23}^+ = 0. \tag{C.1}$$

From this Lemma it follows immediately that:

**Corollary C.1.** *Let $\{V^-, V^+\}$ satisfy $\mathcal{H}_V(V^-; V^+) = 0$. If $\mathcal{X}_{ij}^+$ vanishes for two index pairs $\{i,j\} \in \mathbb{P}$, then it vanishes for all pairs.*

**Proposition C.1.** *If $\mathcal{X}_{ij}^+ = 0$ for all $\{i,j\} \in \mathbb{P}$ and $(F_1^+, F_2^+, F_3^+) \neq 0$, we obtain:*

$$[G_k] = \varrho_1 F_k^+ \qquad and \qquad F_k^- = \varrho_2 F_k^+, \quad for \ k = 1,2,3. \tag{C.2}$$

*where $\varrho_1$ and $\varrho_2$ are constants depending on $[G]$, $F^-$ and $F^+$, which are obtained in the proof of Proposition. Moreover, for $\mathcal{Z}$ defined in $(4.24.a)$, the shock speed $v$ satisfies:*

$$v = u^- \frac{\mathcal{Z}(V^-; V^+) - \varrho_2}{\varrho_1}. \tag{C.3}$$

**Proof:** Let $u^- > 0$. Since $\mathcal{X}_{ij}^+ = 0 \ \forall \ \{i,j\} \in \mathbb{P}$, it follows that:

$$\mathcal{X}_{23}^+ e_1 + \mathcal{X}_{31}^+ e_2 + \mathcal{X}_{12}^+ e_3 = 0, \tag{C.4}$$

where $e_i$ for $i = 1,2,3$ is the canonical basis for $\mathbb{R}^3$. Eq. $(C.4)$ can be written as:

$$(F_1^+, F_2^+, F_3^+) \times ([G_1], [G_2], [G_3]) = 0, \tag{C.5}$$

where $\times$ represents the outer product. Since Eq. $(C.5)$ is satisfied, it follows that $[G]$ is parallel to $F^+$, so there is a constant $\varrho_1$ so Eq. $(C.2.a)$ is satisfied. Substituting $[G] = \varrho_1 F^+$ into the $RH$ condition (4.18), we obtain, for $i = 1, 2, 3$:

$$v\varrho_1 F_i^+ = u^+ F_i^+ - u^- F_i^-, \tag{C.6}$$

If $F_i^- = 0$ for some $i = 1,2,3$, so $v = u^+/\varrho_1$. If $F_i^- \neq 0$ for all $i = 1,2,3$, multiplying Eq. $(C.6)$ for $i = 1$ by $F_2^-$ and $(C.6)$ for $i = 2$ by $-F_1^-$ and adding, it follows that $v\varphi\varrho_1\mathcal{Y}_{12} = u^+\mathcal{Y}_{12}$. Let us assume temporarily that $\mathcal{Y}_{12} \neq 0$, so Eq. $(C.6)$ yields $v = u^+/\varrho_1$. Substituting $v = u^+/\varrho_1$ into Eq. $(C.6)$ we obtain $u^- F_1^- = 0$. Since $F_1^- \neq 0$, generically, it follows that $u^- = 0$, which is false. So $\mathcal{Y}_{12} = 0$.

Similar calculations show that $\mathcal{Y}_{12} = \mathcal{Y}_{23} = \mathcal{Y}_{31} = 0$, so there exists a constant $\varrho_2$ such that $F^- = \varrho_2 F^+$. Eq. $(C.3)$ can be obtained by substituting $[G] = \varrho_1 F^+$ and $F^- = \varrho_2 F^+$ in the $RH$ condition (4.18). $\square$

**Remark C.1.** *If $(F_1^+, F_2^+, F_3^+) = 0$ and $([G_1], [G_2], [G_3]) \neq 0$, it is easy to prove that:*

$$F_k^- = \rho_3[G_k] \quad and \quad v = u^- \rho_3, \tag{C.7}$$

*where $\rho_3$ is a constant that depends on $[G]$ and $F^-$.*

**Corollary C.2.** *The states $\{V^-, V^+\}$ satisfying the RH condition (4.18) for which $\mathcal{X}^+_{ij} = 0$ for all $\{i, j\} \in \mathbb{P}$, satisfy also:*

$$\mathcal{Y}_{ij} = 0, \quad \forall \{i, j\} \in \mathbb{P}. \tag{C.8}$$

We notice that the system (C.8) has always the trivial solution $V^+ = V^-$.

C.1. **Proof of Proposition** 5.3. **Proof:** Assuming that the $RH$ curve can be parametrized by $\mathcal{Z}$ in a neighborhood of $(V^+, u^+)$, we can write the $RH$ condition as:

$$v(G(V(\zeta)) - G^-) = u(\zeta)F(V(\zeta)) - u^- F^-, \tag{C.9}$$

where $v := v(\zeta)$. Differentiating (C.9) with respect to $\zeta$ we obtain:

$$\frac{dv}{d\zeta}(G(V(\zeta)) - G^-) + v\frac{\partial G(V(\zeta))}{\partial W}\frac{dW}{d\zeta} = \frac{\partial\left(u(\zeta)F(V(\zeta))\right)}{\partial W}\frac{dW}{d\zeta}, \tag{C.10}$$

Setting $\zeta^+$ such that $(V(\zeta^+), u(\zeta^+)) = W(\zeta^+) = W^+ = (V^+, u^+)$, Eq. (C.10) yields:

$$[G]\frac{dv}{d\zeta} + v\frac{\partial G}{\partial W}\frac{dW}{d\zeta} = \frac{\partial\left(uF\right)}{\partial W}\frac{dW}{d\zeta}, \tag{C.11}$$

where $W = (V, u)$. Assume first that (5.7) is satisfied. Notice that if $\tilde{v}^+(V^-, V^+) = \tilde{\lambda}^+$, then for $(V^+, u = u^+)$, we have $\lambda = u^+\tilde{\lambda}^+$ and $v(V^-, u^-; V^+) = \lambda(V^+, u^+)$ (we dropped the family index $p$). Substituting them in (C.11) we obtain at $(+) = (V^+, u^+)$ the expression (C.11) with $v := u\lambda/u$. Let $\ell$ the left eigenvector associated to $\tilde{\lambda}^+$; taking the inner product of (C.11) at $(V^+, u^+)$ by $\ell$, we obtain:

$$\ell \cdot [G]\frac{dv}{d\zeta} + \ell \cdot \left(u\frac{\lambda}{u}\frac{\partial G}{\partial W} - \frac{\partial\left(uF\right)}{\partial W}\right)\frac{dW}{d\zeta} = 0. \tag{C.12}$$

Since $\ell$ is an eigenvector associated to $\lambda$, the second term of (C.12) is zero and it follows that:

$$\ell \cdot [G]\frac{dv}{d\zeta} = 0.$$

Since by hypothesis $\ell \cdot [G] \neq 0$, we obtain that $dv/d\zeta = 0$ and the shock speed is critical.

On other hand, assume that $v$ has a critical point, so $dv/d\zeta = 0$ and Eq. (C.11) reduces to:

$$v\frac{\partial G}{\partial W}\frac{dW}{d\zeta} = \frac{\partial\left(uF\right)}{\partial W}\frac{dW}{d\zeta}, \quad \text{or} \quad \left(\frac{\partial\left(uF\right)}{\partial W} - v\frac{\partial G}{\partial W}\right)\frac{dW}{d\zeta} = 0. \tag{C.13}$$

Notice that $dW/d\zeta$ is parallel to the eigenvector $r$ at $W^+$. Similarly the projection of $dW/d\zeta$ is parallel to the projection of eigenvector $\tilde{r}^i$ at $V^+$.

Eq. (C.13) has a solution if, only if,

$$v(V^-, u^-; V^+) = \lambda(V^+, u^+) \qquad \text{so} \qquad \tilde{v}^+(V^-; V^+) = \tilde{\lambda}^+(V^+).$$

$\square$

C.2. **Proof of Proposition** 5.4. **Proof:** We drop the family index $i$. Assume that (5.8) is satisfied, where the $\partial \mathcal{H}_V / \partial V_j^+$ for $j = 1, 2$ are:

$$\left( F^- \times \frac{\partial F^+}{\partial V_j^+} \right) [G]^T + (\mathcal{Y}_{32}, \mathcal{Y}_{13}, \mathcal{Y}_{21}) \left( \frac{\partial G^+}{\partial V_j^+} \right)^T,$$

where $\times$ is the outer product. Rearranging the terms, we can rewrite $\partial \mathcal{H}_V / \partial V_j^+$ as:

$$\frac{\partial \mathcal{H}_V}{\partial V_j^+} = (\mathcal{X}_{32}^-, \mathcal{X}_{13}^-, \mathcal{X}_{21}^-) \left( \frac{\partial F^+}{\partial V_j^+} \right)^T + (\mathcal{Y}_{32}, \mathcal{Y}_{13}, \mathcal{Y}_{21}) \left( \frac{\partial G^+}{\partial V_j^+} \right)^T. \qquad (C.14)$$

Recall that $\lambda^+ = u^+ \tilde{\lambda}^+$, so $\partial_W(uF) - \lambda \partial_W G$ at $(V^+, u^+)$ is:

$$\begin{pmatrix} u^+ C_{1,1}(V^+; \tilde{\lambda}^+) & u^+ C_{1,2}(V^+; \tilde{\lambda}^+) & F_1^+ \\ u^+ C_{2,1}(V^+; \tilde{\lambda}^+) & u^+ C_{2,2}(V^+; \tilde{\lambda}^+) & F_2^+ \\ u^+ C_{3,1}(V^+; \tilde{\lambda}^+) & u^+ C_{3,2}(V^+; \tilde{\lambda}^+) & F_3^+ \end{pmatrix}. \qquad (C.15)$$

Setting $\tilde{v}^+(V^-; V^+) = \tilde{\lambda}^+(V^+)$ in $(C.15)$ and assuming that $\mathcal{X}_{ij}^- \neq 0$ for all $\{i, j\} \in \mathbb{P}$, we use Eqs. $(4.23.a)$ and $(4.24)$ to write an equivalent but convenient expression for $\tilde{v}^+$ in each matrix element, obtaining for $\mathbb{Y}_{ij} = \mathcal{Y}_{ij}/\mathcal{X}_{ij}$:

$$\begin{pmatrix} u^+ C_{1,1}(V^+; \mathbb{Y}_{32}) & u^+ C_{1,2}(V^+; \mathbb{Y}_{32}) & F_1^+ \\ u^+ C_{2,1}(V^+; \mathbb{Y}_{13}) & u^+ C_{2,2}(V^+; \mathbb{Y}_{13}) & F_2^+ \\ u^+ C_{3,1}(V^+; \mathbb{Y}_{21}) & u^+ C_{3,2}(V^+; \mathbb{Y}_{21}) & F_3^+ \end{pmatrix}. \qquad (C.16)$$

Since $\partial \mathcal{H}_V / \partial V_j^+ = 0$ for $j = 1, 2$, from Eq $(C.14)$, it follows that for $\ell$ given by

$$\ell = \left( \mathcal{X}_{32}^-, \ \ \mathcal{X}_{13}^-, \ \ \mathcal{X}_{21}^- \right), \qquad (C.17)$$

the inner products of columns 1 and 2 of $(C.16)$ by $\ell$ are zero. Since at $(V^-, V^+)$ the expression $\mathcal{H}_V(V^-)$ vanishes, after some calculations we obtain that $\ell \cdot (F_1^+, F_2^+, F_3^+) = 0$, so it is clear that $\ell$ is a left eigenvector of the system. Since the system is hyperbolic, there is only a left eigenvalue associated to the eigenvalue, thus all left eigenvectors of $(C.16)$ with $\lambda(V^+) = v(V^-, V^+)$ are parallel to $\ell$. Notice that $\ell \cdot ([G_1], [G_2], [G_3]) = 0$, because this equality satisfies the $RH$ condition $\mathcal{H}_V = 0$, with $\mathcal{H}_V$ given by (4.22).

   If $\mathcal{X}_{ij}^- = 0$ for a pair $\{i, j\} \in \mathbb{P}$, using Lemma $C.1$, the relationships $\mathcal{Y}_{ij}^- = 0$ or $\mathcal{X}_{ij}^- = 0$ for all $\{i, j\} \in \mathbb{P}$ are satisfied, so $\ell^i(V^+) \cdot [G] = 0$.

   Assume that $\mathcal{X}_{ij}^- \neq 0$ for some $\{i, j\} \in \mathbb{P}$. For concreteness we set $i = 1$ and $j = 3$; the other cases can be proved similarly. Since $\mathcal{X}_{13}^- \neq 0$ and the matrix $(C.16)$ has the form $(a_{ij})$ for $i, j = 1, 2, 3$, and we can write $a_{11}$, $a_{12}$ and $a_{13}$, respectively as:

$$u^+ C_{1,1}(V^+; \mathbb{Y}_{13}), \quad u^+ C_{1,2}(V^+; \mathbb{Y}_{13}) \quad \text{and} \quad F_1^+.$$

Substituting $\ell$ in $(C.17)$ by the the following vector:

$$\ell = \left( 0, \ \ F_1^-[G_3] - F_3^-[G_1], \ \ F_2^-[G_1] - F_1^-[G_2] \right), \qquad (C.18)$$

it is easy to prove that this vector $\ell$ is in the kernel of the transpose of the matrix $(C.16)$.

   Finally assume that $\mathcal{X}_{ij}^- = 0$ for all $\{i, j\} \in \mathbb{P}$. Since $\ell$ is a left eigenvector of the matrix $(C.16)$ it follows that $\ell \cdot (F_1^+, F_2^+, F_3^+) = 0$. From Eq. $(C.2.a)$ in the Proposition $C.1$, we see that $([G_1], [G_2], [G_3]) = \rho_1(F_1^+, F_2^+, F_3^+)$ for any constant $\rho_1 \in \mathbb{R}$, so:

$$\ell \cdot [G] = \ell \cdot (\rho_1 F^+) = \rho_1 \ell \cdot F^+ = 0.$$

The converse can be proved similarly by reversing the order of the calculations. $\square$

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