CRITICAL POINTS FOR SURFACE MAPS AND THE BENEDICKS-CARLESON THEOREM

HIROKI TAKAHASI

Abstract. We give an alternative proof of the Benedicks-Carleson theorem on the existence of strange attractors in Hénon-like families on surfaces. To bypass a huge inductive argument, we introduce an induction-free explicit definition of dynamically critical points. The argument is sufficiently general and in particular applies to the case of non-invertible maps as well. It naturally raises the question of an intrinsic characterization of dynamically critical points for dissipative surface maps.

Contents

Date: March 16, 2007.

1. Introduction

Strange attractors are of fundamental importance in the study of dynamical systems. While they are quite often observed numerically, a theoretical study of them still remains a challenge. The first existence theorem was obtained by Benedicks and Carleson [BC91], on the Hénon family $(x,y)\to(1-ax^2+y,bx)$ for a positive measure set of parameters close to $(2,0)$. Mora and Viana [MV93], Dìaz, Rocha, and Viana [DRV96] pushed their argument further and proved the existence of strange attractors in very general bifurcation mechanisms, such as homoclinic tangencies or critical saddle-node cycles. See also Wang and Young [WY01] for a more geometric treatment which yields advanced properties of the attractor.

A breakthrough in this direction had taken place before in the context of the quadratic family $f_a\colon x\to 1-ax^2$. With a careful control of the recurrence of the critical point $x=0$, Jakobson [J81] constructed a positive measure set of parameters such that the corresponding maps admit absolutely continuous invariant probability measures. See [CE83] [BC85] taking similar approaches.

[BC91] is a very creative extension of their previous argument in one dimension [BC85]. Since the Hénon map is a diffeomorphism, there is no critical point in the usual sense. However, they remarkably invented *dynamically critical points* for certain Hénon maps, which allowed them to develop a parameter selection argument with some partial resemblance to the one dimensional case.

In [BC91] [MV93] [WY01], the construction of critical points involves a huge inductive scheme. To recover the assumption of the induction, parameter selections are made with a careful control of the recurrence of critical points constructed at early stages. As such, the assumption of the induction has to incorporate both phase space dynamics and structures in parameter space relative to the old critical points, and necessarily becomes complicated.

The aim of this paper is to improve this point by providing a conceptually simpler proof of the Benedicks-Carleson theorem. A key ingredient is an induction-free explicit definition of critical points. A strong dissipation and an exponential growth of derivatives along the orbits of critical points together imply the existence of strange attractors with positive Lyapunov exponents (Theorem A). The set of parameters satisfying this growth condition is shown to have positive Lebesgue measure (Theorem B). The definition of critical points is a purely analytic one and makes sense for any smooth dissipative surface maps. It is interesting to ask whether it has any intrinsic meaning. A similar question is addressed and some results are given in [PH].

Our argument is sufficiently general and in particular applies to the case of non-invertible maps such that the unstable manifold intersects itself. While no explicit result has been known in this case (see the next paragraph), non-invertible Hénon-like families often appear in many places: for example, in homoclinic bifurcations of surface maps in which transverse homoclinic orbits intersects fold singularities, studied in [San00]; in certain reaction-diffusion equations, studied in [OP00].

Let us make a technical and historical remark. A crucial fact used in [BC91] [MV93] [WY01] is that tangent directions of two nearby horizontal pieces of the unstable manifold are nearby as well, for them to avoid intersecting each other. A new difficulty in the non-invertible case is the obvious failure of this property.

Meanwhile, the same difficulty appears in dimension higher than two, and Viana [Via95] dealt with this by taking the closeness of tangent directions as an independent assumption. This implies that one can deal with the non-invertible case in two dimension by adapting his argument. See also Remark 2.10.3.

1.1. **Statement of the result.** An *Hénon-like family* is a continuous two parameter family of not necessarily invertible maps $H_{a,b}\colon [-2,2]^2 \to \mathbb{R}^2$, of the form

$$H_{a,b}\colon \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1-ax^2+bu(a,b,x,y) \\ bv(a,b,x,y) \end{pmatrix}, \tag{1}$$

where (a,b) is close to $(2,0)$, and u, v are C^4 with respect to a, x, y. We assume

$$\partial_x v(2,0,0,0) \neq 0. \tag{2}$$

Let P denote the hyperbolic fixed point whose x-coordinate is positive. Regardless of whether H is invertible or not, the unstable manifold $W^u(P)$ is obtained as an immersed real line. To bypass its possible self-intersections, define

$$T_zW^u(P) = \{v \in T_z\mathbb{R}^2\colon \text{there exists a segment in } W^u(P) \text{ which is tangent to } v\}.$$

The result splits into two theorems. The first one gives a sufficient condition for the existence of strange attractors, in the form of *exponential growth condition* $(EG)_n$. It is a condition on the growth of orbits of critical points of order n. We need to wait until Section 4 to correctly define this.

Theorem A. *For an Hénon-like family $(H_{a,b})$, there exists $N > 0$ such that if (a,b), $b > 0$ is sufficiently close to $(2,0)$ and $H = H_{a,b}$ satisfies $(EG)_n$ for all $n \geq N$, then:*
(i) $\mathrm{cl}(W^u(P))$ *is a compact, positively invariant set under H;*
(ii) *there exists a countable set $\mathcal{C} \subset W^u(P)$ near $(0,0)$ such that:*
(iia) *for every $z \in \mathcal{C}$ and $n \geq 1$,*

$$\|DH^n(H(z))\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\| \geq e^{\frac{99}{100}\log 2\cdot n};$$

(iib) *for every $z \in \mathcal{C}$ there exists a unique (up to sign) unit vector $e \in T_{H(z)}W^u(P)$ such that for every $n \geq 1$,*

$$\|DH^n(H(z))e\| \leq (Kb)^n,$$

where $K > 0$ is a uniform constant;
(iic) *for all $z \in W^u(P) \setminus \bigcup_{n=-\infty}^{\infty} H^n(\mathcal{C})$ and $u \in T_zW^u(P)$,*

$$\limsup_{n\to+\infty} \frac{1}{n}\log\|DH^n(z)u\| \geq \frac{\log 2}{3};$$

(iid) *there exists $z \in \mathcal{C}$ whose forward orbit is dense in* $\mathrm{cl}(W^u(P))$*;*
(iii) *For any periodic point $p \in [-2,2]^2$,*

$$\limsup_{n\to+\infty} \frac{1}{n}\log\|DH^n(p)\| \geq \frac{\log 2}{3}.$$

In particular, all periodic points of H are hyperbolic of saddle type.

The following theorem states that the condition in Theorem A is not empty from a measure theoretical point of view. These two theorems together imply the Benedicks-Carleson theorem.

Theorem B. *For an Hénon-like family* $(H_{a,b})$ *and* $b > 0$ *small, there exists a positive measure set* Ω_b *of a-values near* $(2, 0)$ *such that* $H = H_{a,b}$ *satisfies* $(EG)_n$ *for all* $n \geq N$ *whenever* $a \in \Omega_b$*.*

Several remarks are in order on the scope of the theorems. The present setting may be considerably extended along the line that is now well-understood. In the definition of the Hénon-like family, one may replace the quadratic family by the so-called transversal family of uni/multimodal maps and keep the conclusion the same, except transitivity. Moreover, while only the two dimensional case is treated here, the argument may be extended to higher dimensions with additional geometric considerations, as in [Via95] [WY]. We have suppressed these possible extensions for simplicity.

For $\mathrm{cl}(W^u(P))$ to deserve the name of attractor, its basin of attraction should have nonempty interior. This is known to be the case when the map is invertible: see [PT93] Appendix 3. However, the same argument does not hold when singularities exist. Meanwhile, Benedicks personally communicated to us that he has a new argument which holds even if singularities exist.

Adapting [BY93] [BY00] to our setting, one can prove the existence of physical measures with nice statistical properties, under the same assumption as in Theorem A. Other known properties in[WY01] would follow, in the invertible case.

In view of the history in one-dimensional dynamics, it would be interesting to look for an weaker condition which guarantees the existence of physical measures. However, before doing this we must pause on any intrinsic meaning of the critical points we are going to introduce.

A hidden aim of this paper is to lay a ground work for possible further developments. What we have in mind is the *basin problem* for the case of non-invertible maps with fold singularities, posed by Tsujii more than three years ago. It is a question on the coincidence of the asymtotic distribution of Lebesgue almost every point in the basin of attraction. A new difficulty here is how to eventually control Lebesgue almost every orbit in the basin of attraction, in defiance of singularities. Based on the present paper, we shall give a positive solution to this problem. A positive solution to the same problem for invertible case was initially given by Benedicks and Viana [BV01], and then by Wang and Young [WY01], under certain regularity condition on the Jacobian of the map. While this condition has been removed in [T06], invertibility remains crucial.

1.2. **Overview of the paper.** The rest of this paper consists of seven sections. Section 2 provides basic estimates and constructions which will be frequently used later. Some are new and some are old, already appearing in [BC91] [MV93] [WY01] in one form or another. Building on some of them we define (pre) critical points (Sect.2.9, Sect.2.11). Intuitively, they are points of tangencies between stable and unstable directions having regular backward orbits.

One important problem is the analysis of the growth of orbits starting from neighborhoods of critical points. Assuming *strong regularity condition* on critical orbits and *admissible position* (Sect.3.1), we prove that an exponential growth of derivatives prevails (Lemma 3.3.2). At this point, a precise distortion estimate in Lemma 2.4.1 is crucial in order to faithfully copy the growth of the critical orbit.

In Section 4, we introduce the exponential growth condition $(EG)_n$ on the orbit of critical points of order n. This condition is sufficient to develop a capture argument which systematically assigns suitable critical points (binding points) to every free return. As a by-product, we conclude a proof of Theorem A.

Sections from 5 to 8 deal with parameter issues. The goal is the construction of the parameter set in Theorem B. Parameters which satisfy $(EG)_{n-1}$ but not $(EG)_n$ are discarded at step n. The condition $(EG)_n$ is not well-adapted to our inductive scheme. Hence, we introduce in Sect.6.2 a stronger condition, called *reluctant recurrence condition*[1] $(RR)_n$. Parameters have to satisfy this condition to be selected.

We pay attention to the complement of good parameter sets. This idea has been borrowed from the work of Tsujii [Tsu93a] [Tsu93b] on the Benedicks-Carleson-Jakobson theorem in one dimension. He proved that parameters discarded at step n are contained in a finite union of well-structured sets the measures of which are uniformly exponentially small in n. We show that essentially the same thing prevails in two dimension. In doing this, two issues intrinsic to two-dimension need to be considered and remedies are made accordingly, as explained in the next two paragraphs.

Critical points disappear when parameters are varied. Hence we work with *quasi critical points* (Sect.5.1) rather than the critical points itself. Proposition 5.4.1 guarantees the existence of smooth continuations of quasi critical points in a sufficiently large interval. This sets the stage for considering the dynamics of critical curves, in section 7. Under the assumption of $(RR)_{n-1}$, we manage to recover three consequences which are known to hold in one-dimension [Tsu93a] [Tsu93b] : good distortion and curvature estimates (Proposition 7.1.2); a large amount of expansion in parameter space at essential returns (Proposition 7.4.1); existence of binding points for critical curves (Proposition 7.5.1).

By definition, there are uncountably many critical points of the same order. Nevertheless, the total number of analytically distinguishable critical points at step n is finite and not too large. Here, we regard two distinct critical points of the same order as indistinguishable, if their backward and forward orbits are characterized by the same set of discrete data, called *sample points* (Sect.5.2), *essential return times* (Sect.6.1), *essential return depths* (Sect.7.4). Each indistinguishable class of critical points makes holes in good parameter sets. It turns out that these sets are well-structured and the total measure of parameters discarded at step n is smaller than the total number of indistinguishable classes times some exponentially small number in n. Consequently, a positive measure set is left over (Proposition 8.1.2).

I am grateful to Masato Tsujii for having brought this problem to my attention. I have to say his notes [Tsu03] is very important for the existence of this paper. Most of this work has been done while I was at Instituto de Matemàtica Pura e Aplicada. Above all, I am grateful to Paulo Varandas and Samuel Senti for useful conversations, to Jacob Palis and Marcelo Viana for their hospitality and providing a stimulating atmosphere. Research supported by CNPq.

[1]This terminology appears in one-dimensional dynamics and designates a different object.

2. Basic estimates and constructions

This section is mostly devoted to basic estimates and constructions which will be frequently used later.

2.1. Trapping region.

To begin with, we identify a positively invariant region in which strange attractors potentially exist. This is done up to certain unimportant restriction on parameters. See [MV93] Proposition 4.1 for related discussions in the invertible case. We need to employ a different argument for our purpose.

Let Q denote the hyperbolic fixed point which is not P. For $b>0$ small, two straight lines $[-2,2]\times\{\pm 1/10\}$ cut two curves S_1 and S_2 in the stable set of Q, such that $Q\in S_1$ and $H(S_2)\subset S_1$. Define D to be the closed region surrounded by these two lines and two curves. Clearly, $P\in \mathrm{Int}\, D$ holds.

Let e_a denote the f_a-preimage of Q which is not Q.The number $e_a-f_a(0)$ is positive and strictly monotone decreasing to zero as $a\to 2$. This implies $f_a([Q,e_a])\subsetneq [Q,e_a]$, and hence $H_{a,0}(D)\subsetneq D$. Put $a_n=2-n^{-1}$. Define a sequence $\{b_n\}_{n=0}^{+\infty}$ as follows: choose $b_0>0$ such that $H(D)\subset D$ for all $(a,b)\in[a_0,a_1]\times[0,b_0]$; given b_{n-1}, choose $b_n\leq b_{n-1}$ such that $H(D)\subset D$ for all $(a,b)\in[a_n,a_{n+1}]\times[0,b_n]$. Define Ω' to be the set of $(a,b)\in\Omega$ such that $a\in[a_n,a_{n+1}]$ and $b\in[0,b_n]$ for some $n\geq 0$. The following holds by construction:

Proposition 2.1.1. *For any open neighborhood U of $(2,0)$, $\Omega'\cap U$ contains an open set. Moreover, $H(D)\subset D$ and $\mathrm{cl}(W^u(P))\subset D$ holds for all $(a,b)\in\Omega'$.*

2.2. Constants and notation.

We introduce *absolute constants* which are definitely fixed throughout the argument. They are

$$\Delta=3, \sigma=100, \ell=49/100, \hat\lambda\approx\log 2.$$

The choice of Δ ensures that the norms of all the partial derivatives of $(a,z)\to H_a(z)$ are bounded from above by e^{Δ}. The constants entirely determined by the family $(H_{a,b})$ are mostly denoted by K. Keep in mind that the values of K are different in different places in general. Hence we have the liberty to write $2K\leq K$ and so on. We reserve the letters K_0, K_1 for special use as follows:

K_0 concerns hyperbolic behaviors away from the critical region[2] (Lemma 2.5.1);

K_1 determines the angle of vertical cones in which the mostly contracting directions reside (Lemma 2.7.4).

On the other hand, we introduce *system constants* which are allowed to change, provided that a finite number of conditions are satisfied. They are

$$\alpha,\, M,\, \beta,\, \delta,\, \theta,\, b,$$

chosen in this order. We have $\alpha,\,\delta,\,\theta,\,b\ll 1$ and $M,\,\beta\gg 1$.

We frequently use the following notation: $A_i=H^i(A)$ for a set $A\subset D$ and $i\geq 0$; $v_i(z_i)=DH^i(z_0)v_0$ for $v_0(z_0)\in T\mathbb{R}^2|D$ and $i\geq 0$. The sequence $\{v_i(z_i)\}_{i=0}^n$ is called a *vector orbit* of H. We only consider vector orbits which consist of nonzero vectors.

---

[2]In the present context, $K_0\approx 1$ holds.

2.3. Curvature.

Lemma 2.3.1. *Let* $\mathbf{v} = \{v_i(z_i)\}_{i=0}^n$ *be a vector orbit, and* $\gamma_0 \subset D$ *a* C^2 *curve which is tangent to* $v_0(z_0)$. *Let* $\kappa_j(z_j)$ *denote the curvature of* γ_j *at* z_j. *Then for every* $1 \leq j \leq n$,

$$\kappa_j(z_j) \leq (Kb)^j \frac{\|v_0\|^3}{\|v_j\|^3}\kappa_0(z_0) + \sum_{\ell=1}^{j}(Kb)^\ell \frac{\|v_{j-\ell}\|^3}{\|v_j\|^3}.$$

Proof. Parametrize γ_0 by $s \in [0,1]$, and suppose that $z_0 = \gamma_0(s_0)$. Let $\gamma_i(s) = H^i(\gamma_0(s))$ for $i \geq 0$. By the chain rule,

$$\gamma_i'(s) = DH(\gamma_{i-1}(s))\gamma_{i-1}'(s)$$

and

$$\gamma_i''(s) = D^2H(\gamma_{i-1}(s))\gamma_{i-1}'(s) + DH(\gamma_{i-1}(s))\gamma_{i-1}''(s),$$

where

$$DH = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and

$$D^2H(\gamma_{i-1}(s)) = \begin{pmatrix} \langle \nabla A, \gamma_{i-1}'(s)\rangle & \langle \nabla B, \gamma_{i-1}'(s)\rangle \\ \langle \nabla C, \gamma_{i-1}'(s)\rangle & \langle \nabla D, \gamma_{i-1}'(s)\rangle \end{pmatrix}.$$

Then

$$\kappa_i(z_i) = \frac{\|\gamma_i'(s_0) \times \gamma_i''(s_0)\|}{\|\gamma_i'(s_0)\|^3} \leq I + II,$$

where

$$I = Kb \cdot \frac{\|\gamma_{i-1}'(s_0)\|^3}{\|\gamma_i'(s_0)\|^3}\kappa_{i-1}(z_{i-1})$$

and

$$II = \|\gamma_i'(s_0)\|^{-3}\|DH(\gamma_{i-1}(s_0))\gamma_{i-1}'(s_0) \times D^2H(\gamma_{i-1}(s_0))\gamma_{i-1}'(s_0)\|.$$

The vector product in II is degree three homogeneous in $\|\gamma_{i-1}'(s_0)\|$. Moreover, since the C^1-norms of B, C, D are bounded by Kb, the second components of the two vectors in the product have a factor b. Therefore

$$\kappa_i(z_i) \leq \frac{\|v_{i-1}\|^3}{\|v_i\|^3}(Kb + Kb \cdot \kappa_{i-1}(z_{i-1})).$$

A recursive use of this inequality gives the desired one. □

2.4. Distortion.

For a vector orbit $\mathbf{v} = \{v_i(z_i)\}_{i=0}^n$, define

$$\Theta(\mathbf{v}, i) = \min_{i \leq j \leq n} \frac{\|v_0\|}{\|v_i\|}\frac{\|v_j\|^2}{\|v_i\|^2}$$

and

$$\Xi(\mathbf{v}) = e^{-\alpha\sigma n} \cdot \min_{0 \leq i \leq n} \Theta(\mathbf{v}, i).$$

We say $\mathbf{v}$ is κ-*expanding*, or simply *expanding*, if there exists $\kappa \geq b^{1/4}$ such that

$$\|v_i\| \geq \kappa^i\|v_0\| \quad 1 \leq \forall i \leq n.$$

Choose a large integer $M > 0$ such that $ne^{-\alpha\sigma n} \leq 1/2$ holds for every $n \geq M$. For a C^1 curve $\gamma_0 \subset D$ and $z_0 \in \gamma_0$, let $t_{\gamma_0}(z_0)$ denote the unit vector tangent at z_0 to γ_0.

Lemma 2.4.1. *Let $n \geq M$, and suppose that $\mathbf{v} = \{v_i(z_i)\}_{i=0}^n$ is expanding. Let $\gamma_0 \subset D$ be a C^2 curve which is tangent to v_0, $\mathrm{length}(\gamma_0) \leq \Xi(\mathbf{v})$, and curvature ≤ 1 everywhere. For every $1 \leq i \leq n$ and $z_0' \in \gamma_0$,*

$$\left|\log \frac{\|DH^i(z_0)t_{\gamma_0}(z_0)\|}{\|DH^i(z_0')t_{\gamma_0}(z_0')\|}\right| \leq \frac{1}{2}.$$

Proof. Let κ_i denote the maximum of the curvature of γ_i. Then

$$\begin{aligned}(e^\Delta + \kappa_0) \cdot \mathrm{length}(\gamma_0) &\leq e^{2\Delta}\Xi(\mathbf{v})\Theta(\mathbf{v},0)^{-1}\Theta(\mathbf{v},0)\\ &\leq e^{2\Delta}\Xi(\mathbf{v})\Theta(\mathbf{v},0)^{-1}\frac{\|v_1\|^2}{\|v_0\|^2}\\ &\leq e^{3\Delta-\alpha\sigma n}\frac{\|v_1\|}{\|v_0\|}.\end{aligned}$$

Since $n \geq M$, it is enough to prove the following by induction on $i \in [0, n-1]$:

$$(e^\Delta + \kappa_i) \cdot \mathrm{length}(\gamma_i) \leq e^{3\Delta-\alpha\sigma n}\frac{\|v_{i+1}\|}{\|v_i\|}; \tag{3}$$

$$\left|\log \frac{\|DH^{i+1}(z_0)t_{\gamma_0}(z_0)\|}{\|DH^{i+1}(z_0')t_{\gamma_0}(z_0')\|}\right| \leq (i+1)e^{3\Delta-\alpha\sigma n/2} \;\forall z_0' \in \gamma_0. \tag{4}$$

(3)$\Longrightarrow$(4). Let $0 \leq j \leq i$ and $z_0' \in \gamma_0$ be arbitrary. Put $v_j' = DH^j(z_0')t_{\gamma_0}(z_0')$ to ease notation. Using (3),

$$\left\|\frac{v_{j+1}}{\|v_j\|} - \frac{v_{j+1}'}{\|v_j'\|}\right\| \leq (e^\Delta + \kappa_j)\mathrm{length}(\gamma_j) \leq e^{3\Delta-\alpha\sigma n}\frac{\|v_{j+1}\|}{\|v_j\|},$$

and thus

$$\frac{\|v_{j+1}'\|}{\|v_j'\|} \geq \frac{\|v_{j+1}\|}{\|v_j\|} - \left\|\frac{v_{j+1}'}{\|v_j'\|} - \frac{v_{j+1}}{\|v_j\|}\right\| \geq (1 - e^{3\Delta-\alpha\sigma n})\frac{\|v_{j+1}\|}{\|v_j\|}.$$

Taking logs,

$$\left|\log\frac{\|v_{j+1}\|}{\|v_j\|} - \log\frac{\|v_{j+1}'\|}{\|v_j'\|}\right| \leq e^{3\Delta-\alpha\sigma n/2}.$$

Using this for every $0 \leq j \leq i$, we obtain (4).

(4)$\Longrightarrow$(3) with $i = i+1$. Using (4),

$$\mathrm{length}(\gamma_{i+1}) \leq e \cdot \frac{\|v_{i+1}\|}{\|v_0\|}\mathrm{length}(\gamma_0) \leq e \cdot \Xi(\mathbf{v})\frac{\|v_{i+1}\|}{\|v_0\|}.$$

Using Lemma 2.3.1 and $\kappa_0 \leq 1$,

$$(e^\Delta + \kappa_{i+1}) \cdot \mathrm{length}(\gamma_{i+1}) \leq \Xi(\mathbf{v})\left(I + II + III\right),$$

where

$$I = e\frac{\|v_{i+1}\|}{\|v_0\|},$$
$$II = e^4(Kb)^{i+1}\frac{\|v_0\|^2}{\|v_{i+1}\|^2},$$
$$III = e^4\frac{\|v_{i+1}\|}{\|v_0\|}\sum_{j=1}^{i+1}(Kb)^j\frac{\|v_{i+1-j}\|^3}{\|v_{i+1}\|^3}.$$

By the definition of $\Theta(\mathbf{v}, i+1)$,

$$\Theta(\mathbf{v}, i+1)^{-1}\Theta(\mathbf{v}, i+1) \leq \Theta(\mathbf{v}, i+1)^{-1}\frac{\|v_0\|}{\|v_{i+1}\|}\frac{\|v_{i+2}\|^2}{\|v_{i+1}\|^2}, \tag{5}$$

and therefore

$$I \leq e^{\Delta}\Theta(\mathbf{v}, i+1)^{-1}\frac{\|v_{i+2}\|}{\|v_{i+1}\|}.$$

Using (5) and the expansivity of $\mathbf{v}$,

$$\begin{aligned} II &\leq e^4(Kb)^{i+1}\Theta(\mathbf{v}, i+1)^{-1}\frac{\|v_0\|^3}{\|v_{i+1}\|^3}\left(\frac{\|v_{i+2}\|}{\|v_{i+1}\|}\right)^2 \\ &\leq (Kb)^{i+1}b^{\frac{-3(i+1)}{4}}e^{4+\Delta}\Theta(\mathbf{v}, i+1)^{-1}\frac{\|v_{i+2}\|}{\|v_{i+1}\|} \\ &\leq \Theta(\mathbf{v}, i+1)^{-1}\frac{\|v_{i+2}\|}{\|v_{i+1}\|}. \end{aligned}$$

Regarding III, for every $0 \leq k \leq n$ we have

$$\begin{aligned} \frac{\|v_{i+1}\|}{\|v_0\|}\frac{\|v_{i+1-j}\|^3}{\|v_{i+1}\|^3} &= \Theta(\mathbf{v}, k)^{-1}\Theta(\mathbf{v}, k)\frac{\|v_{i+1}\|}{\|v_0\|}\frac{\|v_{i+1-j}\|^3}{\|v_{i+1}\|^3} \\ &= \Theta(\mathbf{v}, k)^{-1}\min_{k\leq \ell\leq n}\frac{\|v_{i+1}\|}{\|v_k\|}\frac{\|v_\ell\|^2}{\|v_k\|^2}\frac{\|v_{i+1-j}\|^3}{\|v_{i+1}\|^3}. \end{aligned}$$

Substituting $k = i+1-j \leq n-1$ into the right hand side and then using $\min_{i+1-j\leq \ell\leq n}\|v_\ell\|^2 \leq \|v_{i+1}\|\|v_{i+2}\|$, we have

$$\frac{\|v_{i+1}\|}{\|v_0\|}\frac{\|v_{i+1-j}\|^3}{\|v_{i+1}\|^3} \leq \Theta(\mathbf{v}, i+1-j)^{-1}\frac{\|v_{i+2}\|}{\|v_{i+1}\|}.$$

Consequently,

$$III \leq \frac{\|v_{i+2}\|}{\|v_{i+1}\|}\sum_{j=1}^{i+1}(Kb)^j\cdot\Theta(\mathbf{v}, i+1-j)^{-1}.$$

Altogether these three inequalities and the definition of $\Xi(\mathbf{v})$ yield (3) with $i+1$ in the place of i. □

2.5. **Hyperbolicity and regularity.** The following lemma ensures certain amount of hyperbolicity outside of the critical region $\mathcal{C}_\delta = (-\delta, \delta) \times [-1/10, 1/10]$.

Lemma 2.5.1. *For all $\hat{\lambda} < \log 2$, $\alpha > 0$, $\delta > 0$, there exists $K_0 > 0$ and the following holds for all $H = H_{a,b}$ with (a, b) close to $(2, 0)$ and $\lambda = \hat{\lambda} - \alpha > 0$: suppose that $\{v_i(z_i)\}_{i=0}^n$, $n \geq 1$ is a vector orbit of H such that* $\text{slope}(v_0) \leq K_0 b$.
(i) *If $z_0, z_1, \cdots, z_{n-1} \notin \mathcal{C}_\delta$, then for every $0 \leq i \leq j \leq n$,*

$$\text{slope}(v_i) \leq K_0 b \text{ and } \|v_j\| \geq K_0 \delta e^{\lambda(j-i)} \|v_i\|.$$

(ii) *Let x_i denote the x-coordinate of z_i. If moreover $|x_n| \leq 2|x_0|$, then*

$$\|v_n\| \geq K_0 e^{\lambda n} \|v_0\|.$$

(iii) *If $\|v_n\| \geq e^{-2} K_0 \delta \|v_{n-1}\|$, then* $\text{slope}(v_n) \leq K_0 b$.

We omit a proof because it is well-known.

A vector orbit $\{v_i(z_i)\}_{i=0}^n$ is called *r-regular* ($r > 0$) if

$$\|v_n\| \geq K_0 r \delta \|v_i\| \quad 0 \leq \forall i \leq m.$$

It is easy to see that the following holds.

Corollary 2.5.2. *Let $r \geq e^{-2}$, and suppose that $\{v_i(z_i)\}_{i=0}^n$ is an r-regular vector orbit of H as in Lemma 2.5.1. Then* $\text{slope}(v_n) \leq K_0 b$. *Let $m = \min\{i \geq n \colon z_i \in \mathcal{C}_\delta\}$. Then $\{v_i(z_i)\}_{i=0}^m$ is r-regular.*

2.6. **Admissible curves.** A C^2 curve γ is called *admissible* if $\text{slope}(t_\gamma(z)) \leq K_0 b$ for all $z \in \gamma$ and the curvature is ≤ 1 everywhere on γ.

Lemma 2.6.1. *Let $n \geq M$. Suppose that a vector orbit $\mathbf{v} = \{v_i(z_i)\}_{i=0}^n$ is κ-expanding and e^{-4}-regular. Let γ_0 be a C^2 curve which is tangent to $v_0(z_0)$,* $\text{length}(\gamma_0) = \Xi(\mathbf{v})$, *curvature ≤ 1 everywhere. Then γ_n is an admissible curve and*

$$\text{length}(\gamma_n) \geq e^{-3\Delta n} \kappa^{3n}.$$

Proof. Using Lemma 2.4.1, $\|v_j\| \geq \kappa^j \|v_0\| \geq \kappa^n \|v_0\|$, and $\|v_i\| \leq e^{\Delta i} \|v_0\| \leq e^{\Delta n} \|v_0\|$,

$$\begin{aligned}\text{length}(\gamma_n) &\geq e^{-5\alpha\sigma n - 1/2} \frac{\|v_n\|}{\|v_0\|} \cdot \min_{0 \leq i \leq n} \left(\min_{i \leq j \leq n} \frac{\|v_0\|}{\|v_i\|} \frac{\|v_j\|^2}{\|v_i\|^2} \right) \\ &\geq e^{-\alpha\sigma n} \frac{\|v_n\|}{\|v_0\|} \cdot \min_{0 \leq i \leq n} \frac{\|v_0\|}{\|v_i\|} \cdot \min_{i \leq j \leq n} \frac{\|v_j\|^2}{\|v_i\|^2} \\ &\geq e^{-3\Delta n} \kappa^{3n}.\end{aligned}$$

By Lemma 2.3.1 and the regularity of $\mathbf{v}$, the curvature of γ_n is ≤ 1 everywhere. The slope estimate follows from (iii) in Lemma 2.5.1. □

2.7. **Mostly contracting directions.** Let M be a 2×2 matrix. Denote by $e(M)$ a unit vector (up to sign) such that $\|Me(M)\| \leq \|Mu\|$ holds for any unit vector u. We call $e(M)$, when it exists, the *mostly contracting direction* of M. We analogously define a unit vector $f(M)$ which is mostly expanded by M. Clearly $Me(M) \perp Mf(M)$, and moreover $e(M) \perp f(M)$ holds[3].

[3]Consider the dual M^*. Then $e(M^*)$, $f(M^*)$ is well-defined and $M^* e(M^*) \perp M^* f(M^*)$. Since $Me(M) \in \ker f(M^*)$ and $Mf(M) \in \ker e(M^*)$ we have $M^*Me(M) \in \ker M^* f(M^*)$ and $M^*Mf(M) \in \ker M^* e(M^*)$. This implies $e(M) \perp f(M)$.

For a sequence of matrices M_1, $M_2\cdots$, we use $M^{(i)}$ to denote the matrix product $M_i\cdots M_2M_1$, and e_i to denote the mostly contracting direction of $M^{(i)}$. We assume $|\det M_i|\leq Kb$ and $\|M_i\|$, $\|DM_i\|\leq e^{\Delta}$. We quote some results in [WY01] without proofs.

Lemma 2.7.1. ([WY01] Lemma 2.1) Suppose that $\|M^{(i)}\|\geq\kappa^i$ and $\|M^{(i-1)}\|\geq\kappa^{i-1}$ for some $\kappa\geq b^{1/4}$. Then e_i and e_{i-1} are well-defined, and satisfy

$$\|e_i\times e_{i-1}\|\leq\left(\frac{Kb}{\kappa}\right)^{i-1}.$$

Corollary 2.7.2. ([WY01] Corollary 2.1) *If $\|M^{(i)}\|\geq\kappa^i$ for every $1\leq i\leq n$, then $\|e_n-e_1\|\leq\kappa^{-1}Kb$, and $\|M^{(i)}e_n\|\leq(Kb)^i$ holds for every $1\leq i\leq n$.*

Next we consider parametrized matrices $M_i(s_1,s_2,s_3)$ such that $\|\partial M_i(s_1,s_2,s_3)\|\leq e^{\Delta}$ and $|\det M_i(s_1,s_2,s_3)|\leq e^{\Delta}$, where ∂ denotes any first order partial derivatives.

Corollary 2.7.3. ([WY01] Corollary 2.2) Suppose that $\|M^{(i)}(s_1,s_2,s_3)\|\geq\kappa^i$ for every $1\leq i\leq n$. Then for every $2\leq i\leq n$,

$$|\partial(e_i\times e_{i-1})|\leq\left(\frac{Kb}{\kappa^3}\right)^{i-1}.$$

Let us come back to the original setting. For $z\in D$ and $n\geq 1$, define $e_n(z)=e(DH^n(z))$ and $f_n(z)=f(DH^n(z))$ when they make sense.

Lemma 2.7.4. *There exists K_1 such that if $z=(x,y)\notin\mathcal{C}_\delta$ then $e_1(z)$ is well-defined and*

$$\mathrm{slope}(e_1(z))\geq K_1^{-1}\delta b^{-1}\ \text{and}\ \|\partial e_1(z)\|\leq K_1|x|^{-2}.$$

If moreover $\|DH^i(z)\|\geq\kappa^i$ for every $1\leq i\leq n$ then

$$\mathrm{slope}(e_n(z))\geq K_1^{-1}\delta b^{-1}\ \text{and}\ \|\partial e_n\|\leq K_1|x|^{-2}.$$

Proof. The well-definedness of $e_1(z)$ follows from $|\det DH(z)|\leq Kb$ and $\|DH(z)\|\geq 2\delta\gg\sqrt{Kb/\pi}$. The Lagrange method of undetermined coefficients gives

$$e_1=\rho^{-1}(C^2+D^2-\lambda^2,-(AC+BD)),$$

where $\rho>0$ is the normalizing constant,

$$DH=\begin{pmatrix}A&B\\C&D\end{pmatrix},\ \|DHe_1\|=\lambda,\ \lambda=\frac{I-\sqrt{I^2-4II}}{2},$$

and $I=A^2+B^2+C^2+D^2$, $II=A^2D^2+B^2C^2-2ABCD$. By (1), B, C, D are $\mathcal{O}(b)$, and $|A|\leq K|x|$. Hence the slope estimate follows.

We now estimate the partial derivatives of e_1. By (1), all partial derivatives of B, C, D are $\mathcal{O}(b)$, and $\|\partial A\|\leq K$, $\rho\geq Kb|x|$. This gives $\sqrt{I^2-4II}\geq K|x|$, I, $\|\partial I\|\leq K|x|$, $\|\partial II\|=\mathcal{O}(b)$, and in particular $\|\partial\rho\|\leq Kb$ and $\|\partial\lambda\|\leq K|x|$. Putting altogether these we obtain the desired inequality. The rest of the assertion readily follows from Corollary 2.7.2 and Corollary 2.7.3. □

2.8. **Long stable leaves.** A *long stable leaf* of order k is an integral curve of e_k having the form

$$\Gamma=\{(x(y),y)\in D\colon |y|\leq 1/10\}, |x'(y)|, |x''(y)|\leq Kb.$$

For a long stable leaf Γ and $r>0$, define a strip

$$\Gamma(r)=\{(x,y)\in D\colon |x-x(y)|\leq r\}.$$

The following proposition asserts the existence of long stable leaves around expanding orbits. While similar constructions have already appeared in [BC91] [MV93] [WY01], we work with the distortion estimate in Lemma 2.4.1 rather than the so-called matrix perturbation Lemma ([BC91] Lemma 5.5). This yields a more intuitive construction and better bounds on the width of the strip, which plays a crucial role later.

Proposition 2.8.1. *Let $n\geq M$, $z_0\notin\mathcal{C}_\delta$, and define a vector orbit $\mathbf{w}=\{w_i(z_i)\}_{i=0}^n$ by $w_i=DH^i(z_0)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. If $\mathbf{w}$ is expanding, then*

(i) *for every $1\leq k\leq n$, the maximal integral curve $\Gamma^{(k)}$ of e_k through z_0 is a long stable leaf of order k.*

(ii) *for $1\leq k\leq n-1$, e_i $(1\leq i\leq k+1)$ is well-defined on $\Gamma^{(k)}(\Pi_0^{\max\{M,k+1\}}\mathbf{w})$. Moreover, for all $z_0'\in\Gamma^{(k)}(\Pi_0^{\max\{M,k+1\}}\mathbf{w})$ and $1\leq i\leq k+1$,*

$$\left|\log\frac{\|DH^i(z_0)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\|}{\|DH^i(z_0')\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\|}\right|\leq 1. \tag{6}$$

Proof. It is easy to see that $\Gamma^{(1)}$ is a long stable leaf of order 1. We prove (ii) for $k=j\leq n-1$, assuming that $\Gamma^{(j)}$ is a long stable leaf, and that $\Gamma^{(j)}$ is contained in $\Gamma^{(j-1)}(\Pi_0^{\max\{M,j\}}\mathbf{w})$ when $j\geq 2$. For $z_0'\in\Gamma^{(j)}$ and $i\in[0,j]$, define $w_i'=DH^i(z_0')\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. Put $A=DH(z_{i-1})$, $A'=DH(z_{i-1}')$. Then for $i\geq 1$,

$$\begin{aligned}
\text{angle}(w_i,w_i')&=\frac{\|w_i\times w_i'\|}{\|w_i\|\cdot\|w_i'\|}\\
&=\frac{\|A'w_{i-1}\times A'w_{i-1}'+(A-A')w_{i-1}\times A'w_{i-1}'\|}{\|w_i\|\cdot\|w_i'\|}\\
&\leq\frac{\|w_{i-1}\|}{\|w_i\|}\frac{\|w_{i-1}'\|}{\|w_i'\|}(|\det A'|\cdot\text{angle}(w_{i-1},w_{i-1}')+K|z_{i-1}-z_{i-1}'|)\\
&\leq\frac{\|w_{i-1}\|}{\|w_i\|}\frac{\|w_{i-1}'\|}{\|w_i'\|}(Kb\cdot\text{angle}(w_{i-1},w_{i-1}')+(Kb)^{i-1}).
\end{aligned}$$

Using this recursively and then $\|w_i\|\geq\kappa^i$, $\|w_i'\|\geq e^{-1}\kappa^i$,

$$\text{angle}(w_i,w_i')\leq(Kb)^{i-1}\sum_{j=0}^{i}\frac{\|w_j\|}{\|w_i\|}\frac{\|w_j'\|}{\|w_i'\|}\leq b^{\frac{i-1}{2}}.$$

This and $\|w_i\|\approx\|w_i'\|$ imply $\|w_i-w_i'\|\leq\sqrt{3}\|w_i\|\cdot\text{angle}(w_i,w_i')\leq b^{\frac{i-1}{3}}$. Thus

$$\big|\|w_{i+1}\|-\|w_{i+1}'\|\big|\leq\|w_{i+1}-w_{i+1}'\|\leq\|A\|\|w_i-w_i'\|+\|A-B\|\|w_i'\|\leq b^{\frac{i-1}{4}}.$$

This and $\|w_{i+1}\| \geq \kappa^{i+1} \gg b^{\frac{i-1}{4}}$ yield

$$\left|\log \frac{\|w_{i+1}\|}{\|w'_{i+1}\|}\right| \leq b^{\frac{i}{5}}.$$

We now choose arbitrary $z''_0 \in \Gamma^{(j)}(\Pi_0^{\max\{M,j+1\}}\mathbf{w})$, and take $z'_0 \in \Gamma^{(j)}$ whose y-coordinate coincides with that of z''_0. Then $|z'_0 - z''_0| \leq \Xi(\Pi_0^{\max\{M,j+1\}}\mathbf{w})$ holds. For $i \in [0, j]$, define $DH^i(z''_0)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = w''_i$. Using the previous estimate and Lemma 2.4.1,

$$\left|\log \frac{\|w_{i+1}\|}{\|w''_{i+1}\|}\right| \leq \left|\log \frac{\|w_{i+1}\|}{\|w'_{i+1}\|}\right| + \left|\log \frac{\|w'_{i+1}\|}{\|w''_{i+1}\|}\right| \leq b^{\frac{i}{4}} + \frac{1}{2} \leq 1.$$

This yields $\|w''_{i+1}\| \geq e^{-1}\kappa^{i+1}\|w''_0\|$, and hence $e_{i+1}(z''_0)$ is well-defined. Consequently, (ii) holds with $k = j$.

We show that $\Gamma^{(j+1)}$ is a long stable leaf. Parametrize $\Gamma^{(j+1)}$ and $\Gamma^{(j)}$ by arc length and assume that $z_0 = \Gamma^{(j+1)}(0) = \Gamma^{(j)}(0)$. Suppose that $\Gamma^{(j+1)}(s)$ is well-defined for $s \in [0, s_0]$. For any such s, using Lemma 2.7.1 and Corollary 2.7.4,

$$\begin{aligned}\|e_{j+1}(\Gamma^{(j+1)}(s)) - e_j(\Gamma^{(j)}(s))\| \leq &\|e_{j+1}(\Gamma^{(j+1)}(s)) - e_j(\Gamma^{(j+1)}(s))\| \\ &+ \|e_j(\Gamma^{(j+1)}(s)) - e_j(\Gamma^{(j)}(s))\| \\ \leq &\left(\frac{Kb}{\kappa}\right)^j + K|\Gamma^{(j+1)}(s) - \Gamma^{(j)}(s)|.\end{aligned}$$

Therefore

$$\begin{aligned}|\Gamma^{(j+1)}(s) - \Gamma^{(j)}(s)| &= \left|\int_0^s \frac{d\Gamma^{(j+1)}(s)}{ds} - \frac{d\Gamma^{(j)}(s)}{ds}ds\right| \\ &\leq \int_0^s \|e_{j+1}(\Gamma^{(j+1)}(s)) - e_j(\Gamma^{(j)}(s))\|ds \\ &\leq \left(\frac{Kb}{\kappa}\right)^j s + K\int_0^s |\Gamma^{(j+1)}(s) - \Gamma^{(j)}(s)|ds \\ &\leq Ks + \left(\frac{Kb}{\kappa}\right)^j s \\ &\vdots \\ &\leq \frac{(Ks)^m}{m!} + \left(\frac{Kb}{\kappa}\right)^j \sum_{k=1}^m \frac{(Ks)^k}{k!}.\end{aligned}$$

Notice that the third inequality follows from substituting $|\Gamma^{(j+1)}(s) - \Gamma^{(j)}(s)| \leq 1$ into the inside of the integral. Similarly, the m-th inequality ($m \geq 4$) follows from substituting the $m-1$-th one into the same place. We now Put $s = s_0$ and pass m to the limit $\to +\infty$. Then it follows that $\Gamma^{(j+1)}(z_0)$ hits neither the left nor right boundary of $\Gamma^{(j)}(\Pi_0^{\max M,j+1}\mathbf{w})$. Hence $\Gamma^{(j+1)}(s)$ defined for all $s \in [-1/10, 1/10]$. Corollary 2.7.4 implies that $\Gamma^{(j+1)}$ is indeed a long stable leaf with the desired derivative estimate.

It is left to prove $\Gamma^{(j+1)} \subset \Gamma^{(j)}(\Pi_0^{\max\{M,j+1\}}\mathbf{w})$. For $(x_0', y_0') \in \Gamma^{(j+1)}$, choose $(x_0'', y_0') \in \Gamma^{(j)}$. Then $|x_0' - x_0''| \leq (Kb)^j \leq \Xi(\Pi_0^{\max\{M,j+1\}}\mathbf{w})$ holds, regardless of whether $j+1 \geq M$ or not. This implies the inclusion. □

2.9. **Precritical points.** Suppose that γ_0 is an admissible curve in $\mathcal{C}_\delta$. We say $\zeta_0 \in \gamma_0$ is a *precritical point of order* n on γ_0, if
(*a*) $\|DH^i(\zeta_1)\| \geq e^{-1}$ for every $1 \leq i \leq n$;
(*b*) $e_n(\zeta_1)$ is tangent to $DH(\zeta_0)t_{\gamma_0}(\zeta_0)$.

Remark 2.9.1. By Lemma 2.7.1 and Lemma 2.7.4, we have

$$\text{slope}(DHt_\gamma(\zeta)) \geq K_1^{-1}\delta b^{-1}. \tag{7}$$

This implies that all precritical points are contained in a small neighborhood of $(0,0)$, provided that b is sufficiently small.

Remark 2.9.2. Every admissible curve admits no more than two precritical points of the same order. Let us see why this is so. By definition, precritical points are points of tangencies between the images of admissible curves and long stable leaves. For any long stable leaf and any admissible curve, there is at most one point of tangency between them. Meanwhile, by the uniqueness of solutions in ordinary differential equations, two distinct long stable leaves do not intersect each other. These two facts together imply the claim.

2.10. **Creation of new precritical points.** The following two lemmas are used to create new precritical points around the existing ones. For related discussions, see: [BC91] p.113, Lemma 6.1; [MV93] sect.7A, 7B; [WY01] Lemma 2.10, 2.11. Our argument that follows is a slight adaptation of them. In this subsection, all admissible curves are assumed to be parametrized by arc length.

Lemma 2.10.1. *Let γ_0 be an admissible curve in $\mathcal{C}_\delta$, where $\gamma_0(0) = \zeta_0$ is a precritical point of order m. Let $\varepsilon \in [Kb, e^{-40\beta}]$, and suppose that $\gamma_0(s)$ is defined for $s \in [-\varepsilon^{m/2}, \varepsilon^{m/2}]$. Suppose that there exists $j \in [\beta^{-1}m, \beta m]$ such that $\|DH^i(\zeta_1)\| \geq 1$ holds for every $1 \leq i \leq j$. Then there exists a precritical point $\hat{\zeta}_0$ of order j on γ_0 such that $|\zeta_0 - \hat{\zeta}_0| \leq \varepsilon^{m/2}$.*

Proof. Let $\Gamma^{(j-1)}$ denote the long stable leaf of order $j-1$ through ζ_1. Let $\mathbf{w}$ denote the forward vector orbit of ζ_0. Since $\Pi_0^j\mathbf{w} \geq e^{-20\Delta j} \geq e^{-20\beta m} \gg 2e^\Delta\varepsilon^{m/2}$ and $\text{diam}(\gamma_1) \leq e^\Delta \cdot \text{length}(\gamma_0) \leq 2e^\Delta\varepsilon^{m/2}$, we have $\gamma_1 \subset \Gamma^{(j-1)}(\Pi_0^j\mathbf{w})$. Hence it makes sense for $z_0 \in \gamma_0$ to consider the splitting

$$DH(z_0)t_{\gamma_0}(z_0) = \tilde{\xi}e_j(z_1) + \tilde{\eta}f_j(z_1) \quad \text{and} \quad DH(z_0)t_{\gamma_0}(z_0) = \xi e_m(\zeta_1) + \eta f_m(\zeta_1).$$

Put $\psi(z_0) = \text{angle}(e_m(\zeta_1), e_j(z_1))$. We clearly have $\tilde{\eta} = \eta\cos\psi \pm \xi\sin\psi$, the sign being chosen as the case may be. By Lemma 2.8.1,

$$\begin{aligned}\psi(z_0) &\leq \text{angle}(e_m(\zeta_1), e_m(z_1)) + \text{angle}(e_m(z_1), e_j(z_1)) \\ &\leq K|\zeta_0 - z_0| + (Kb)^m.\end{aligned}$$

In particular, we have $\psi(z_0) \ll 1$. Suppose that z_0 is the endpoint of γ_0. Then the above inequality implies $\psi(z_0) \leq K|\zeta_0 - z_0|$. Lemma 3.3.3 implies $|\eta(z_0)| = |\zeta_0 - z_0|$,

$|\xi(z_0)| \le K_1\delta^{-1}b$, and $\eta(z_0)\eta(z') < 0$, where z' is the other endpoint of γ_0. Without loss of generality we may assume $\eta(z_0) > 0$. Then

$$\tilde\eta(z_0) \ge |\zeta_0 - z_0|\left(\frac{1}{2} - K\delta^{-1}b\right) > 0,$$

and $\tilde\eta(z') < 0$ on the other hand. By the intermediate value theorem, there exists $\hat\zeta \in \gamma_0$ such that $\tilde\eta(\hat\zeta_0) = 0$. In other words, $\hat\zeta_0$ is a critical point of order j. □

Lemma 2.10.2. *Let $\varepsilon \in (0, e^{-10\Delta}]$ and $m \ge \log\delta^4/\log\varepsilon$. Let γ and $\tilde\gamma$ be two admissible curves in $\mathcal{C}_\delta$. Suppose that:*
(a) $\gamma(s)$, $\tilde\gamma(s)$ are defined for $s \in [-\varepsilon^{m/2}, \varepsilon^{m/2}]$.
(b) $\gamma(0)$ is a precritical point of order m and $\|DH^i(\gamma(0))\| \ge e$ for $1 \le i \le m$;
(c) the x-coordinates of $\gamma(0)$ and $\tilde\gamma(0)$ coincide,
(d) $|\gamma(0) - \tilde\gamma(0)| \le \min(Kb, \varepsilon^m)$ and $\mathrm{angle}(\gamma'(0), \tilde\gamma'(0)) \le \varepsilon^m$.
Then there exists $s_0 \in [-\varepsilon^{m/2}, \varepsilon^{m/2}]$ such that $\tilde\gamma(s_0)$ is a precritical point of order m.

Remark 2.10.3. In [BC91] [MV93] [WY01], γ and $\tilde\gamma$ are assumed to be disjoint, which is crucial. The smallness of the angle between $\gamma'(0)$ and $\tilde\gamma'(0)$ automatically follows from this, for them to avoid intersecting each other. In the present context, we need to allow γ to intersect $\tilde\gamma$, and thus the smallness of the angle needs to be taken as an independent assumption as in (d).

Proof. Let $z \in \tilde\gamma_1$. Since $\mathrm{diam}(\tilde\gamma_1) \le \varepsilon^{m/2}$ and $\varepsilon \le e^{-10\Delta}$, we have

$$|H(\gamma(0)) - z| \le |H(\gamma(0)) - H(\tilde\gamma(0))| + \varepsilon^{m/2} \le 2\varepsilon^{m/2} \le e^{-10\Delta m}.$$

By the same reasoning as in the proof of Lemma 2.10.1, e_m is well-defined on a neighborhood of $\tilde\gamma_1$. Hence, it makes sense for $z_0 \in \tilde\gamma$ to consider the splitting

$$DHt_{\tilde\gamma}(z_0) = \xi t_{\tilde\gamma_1}(\tilde z_1) + \eta t_{\tilde\gamma_1}(\tilde z_1)^{\perp} \text{ and } DHt_{\tilde\gamma}(z_0) = \tilde\xi e_m(z_1) + \tilde\eta f_m(z_1).$$

Then $\tilde\eta = \eta\cos\psi \pm \xi\sin\psi$ holds, where $\psi = \mathrm{angle}(DH\tilde\gamma'(0), e_m(z_1))$. Since $\gamma(0)$ is a precritical point, we have $\mathrm{slope}(DH\tilde\gamma'(0)) \ge K_1^{-1}\delta b^{-1}$. Thus Lemma 3.3.3 gives $\eta = L|\tilde\gamma(0) - z_0|$ and $|\xi| \le K_1\delta^{-1}b$. Suppose that z_0 is one of the endpoints of $\tilde\gamma$. Using the fact that $DH\gamma'(0)$ is collinear to $e_m(H(\gamma(0)))$ we have

$$\begin{aligned}\psi &\le \mathrm{angle}(DH\gamma'(0), DH\tilde\gamma'(0)) + \mathrm{angle}(e_m(H(\gamma(0))), e_m(z_1))\\ &\le (Kb^{-1}\varepsilon^{m/2} + 1)e^{\Delta}|\gamma(0) - z_0|.\end{aligned}$$

In particular, $\psi \ll 1$ holds. For the same reason as in the proof of Lemma 2.10.1, we may assume $\tilde\eta(z_0) > 0$. Moreover we assume

$$\mathrm{angle}(DH\gamma'(0), DH\tilde\gamma'(0)) \le Kb^{-1}e^{\Delta}(|\gamma(0) - \tilde\gamma(0)| + \|\gamma'(0) - \tilde\gamma'(0)\|). \tag{8}$$

Then

$$\begin{aligned}\tilde\eta(z_0) &\ge L|\tilde\gamma(0) - z_0|\cos\psi - K\delta^{-1}b\sin\psi\\ &\ge |\tilde\gamma(0) - z_0|\left(1 - K\delta^{-1}b - K\delta^{-1}\varepsilon^{m/2}\right) > 0,\end{aligned}$$

where the last inequality follows from the assumption on m, ε, δ. On the other hand we have $\tilde\eta(z') < 0$, where z' is the other endpoint of $\tilde\gamma$. By the intermediate value theorem there exists $s_0 \in [-\varepsilon^{m/2}, \varepsilon^{m/2}]$ such that $\tilde\eta(\tilde\gamma(s_0)) = 0$. In other words, $\tilde\gamma(s_0)$ is a critical point of order m.

It is left to prove (8). For this it is enough to prove angle$(DH\gamma'(0), DH\tilde{\gamma}'(0)) \ll 1$. Let us see why this is so. This inequality implies

$$\mathrm{angle}(DH\gamma'(0), DH\tilde{\gamma}(0)) \leq \frac{\|DH\gamma'(0) - DH\tilde{\gamma}'(0)\|}{\min\left(\|DH\gamma'(0)\|, \|DH\tilde{\gamma}'(0)\|\right)}.$$

The denominator is $\geq Kb$, by (1) (2) and the fact that the slopes of $\gamma'(0)$ and $\tilde{\gamma}'(0)$ are $\leq K_0 b$. Hence (8) follows.

Put $DH\gamma'(0) = (\xi, \eta)$, $DH\tilde{\gamma}'(0) = (\tilde{\xi}, \tilde{\eta})$. It is enough to show slope$(DH\tilde{\gamma}'(0)) \leq K_1^{-1}\delta b^{-1}$, or equivalently $|\tilde{\xi}| \leq 2K_1\delta^{-1}b|\tilde{\eta}|$. Put $\gamma'(0) = \rho\cdot(1, \theta)$ and $\tilde{\gamma}'(0) = \tilde{\rho}\cdot(1, \tilde{\theta})$, where $\rho, \tilde{\rho} \approx 1$ are the normalizing constants. By (2) and the fact that $|\theta|$, $|\tilde{\theta}| \leq K_0 b \ll 1$, $|\eta|, |\tilde{\eta}|$ have the order b. Thus

$$\begin{aligned}\frac{|\tilde{\xi}|}{|\tilde{\eta}|} \leq &(Kb)^{-1}|\tilde{\xi}| \leq (Kb)^{-1}|\xi| \\ &+ K^{-1}(|\partial_x u(\gamma(0)) - \partial_x u(\tilde{\gamma}(0))| + \tilde{\theta}|\partial_y u(\gamma(0)) - \partial_y u(\tilde{\gamma}(0))|) \\ &+ K^{-1}|\theta - \tilde{\theta}||\partial_y u(\gamma(0))|.\end{aligned}$$

Using $|\gamma(0) - \tilde{\gamma}(0)| \leq Kb$ and $|\theta - \tilde{\theta}| \leq Kb$,

$$\frac{|\tilde{\xi}|}{|\tilde{\eta}|} \leq (Kb)^{-1}|\xi| + Kb \leq K\frac{|\xi|}{|\eta|} + Kb \leq K_1\delta^{-1}b + Kb \leq 2K_1\delta^{-1}b.$$

This completes the proof (8) and hence that of Lemma 2.10.2. □

2.11. **Critical points.** Put

$$N = -\Delta^{-1}\log\delta.$$

We say a precritical point ζ_0 of order $n \geq N$ on an admissible curve γ is a *critical point of order* n, if:

(a) $\|DH^i(H(\zeta_0))\| \geq 1$ for every $1 \leq i \leq n$;

(b) there exists an e^{-2}-regular and $e^{-10\Delta}$-expanding orbit $\{w_i(\zeta_i)\}_{i=-n}^{0} \subset D$ such that $\zeta_{-n} \notin \mathcal{C}_\delta$ and $w_0(z_0)$ is tangent to γ_0 at ζ_0.

Remark 2.11.1. Suppose that ζ_0 is a critical point of order n. Then $\zeta_0 \in D_n$ holds. In other words, critical points dig deeper inside as their orders increase.

Remark 2.11.2. By (b), the long stable leaf of order n through ζ_{-n} is well-defined. It intersects the top side of D. Hence, ζ_0 is approximated by the H^n-image of it. We shall take advantage of this fact in later sections. For example, we construct secondary quasi critical points on the forward iterates of the top side.

2.12. **Hyperbolic times.** Let $\mathbf{v} = \{v_i(z_i)\}_{i=0}^{n}$ be a vector orbit. An integer $h \in [0, n]$ is called a *hyperbolic time* if $z_{n-h} \notin \mathcal{C}_\delta$ and the vector orbit $\Pi_{n-h}^{n}\mathbf{v}$ is $e^{-9\Delta}$-expanding. This notion is a reminiscent of *favorable iterates*, introduced in [BC91] Lemma 6.6, [MV93] Lemma 9.1. The next lemma asserts that there exist plenty of hyperbolic times in regular orbits and that they are nicely distributed.

Lemma 2.12.1. *Let $m \geq N$. Suppose that a vector orbit $\mathbf{v} = \{v_i(z_i)\}_{i=0}^{m}$ is e^{-3}-regular and $z_0 \notin \mathcal{C}_\delta$. Then there exists a sequence of hyperbolic times $h_1 < h_2 < \cdots < h_s = m$ such that $h_{i+1} \leq 4h_i$ for every $1 \leq i \leq s-1$.*

Proof. The following is a slight modification of [[WY01], Claim 5.1].

Sublemma 2.12.2. *For every $i \in [N, m]$, there exists a hyperbolic time $i' \in [[i/2], i]$.*

Proof. Suppose that $i \neq m$. Consider the graph, denoted by $\mathcal{G}$, of the function $k \to \log \|v_k\|$ defined on $[m-i, m]$. Let L be the infinite line through $(m, \log \|v_m\|)$ with slope Δ. Clearly, all points of $\mathcal{G}$ lies above L. Let P be the point of intersection between L and the line $\{x = m - [i/2]\}$. Let L be pivoted at P and rotate it clockwise until it hits $\mathcal{G}$. With L in its final position, $\mathcal{G}$ still lies above L. For $i \neq m$, define an integer i'' so that $(m - i'', \log \|v_{m-i''}\|)$ belongs to the set of the first hit. Define $m' = h_s$. We clearly have $i'' \in [[i/2], i]$. Since $\|v_m\| \geq K_0 \delta e^{-3} \|v_{m-i''}\|$ and $i \geq N$, the slope of L in its final position is bigger than

$$-\Delta + \frac{\log \|v_m\| - \log \|v_{m-i''}\|}{[i/2]} \geq -\Delta + 2i^{-1} \log(K_0 e^{-3} \delta) \geq -4\Delta.$$

This implies that $\Pi_{m-i''}^{m} \mathbf{v}$ is $e^{-4\Delta}$ - expanding. Define $i' = i'' - 1$ if $z_{m-i''} \in \mathcal{C}_\delta$, and $i' = i''$ otherwise. Then $z_{m-i'} \notin \mathcal{C}_\delta$ and $i' \in [[i/2], i]$ hold. Moreover, for every $1 \leq j \leq i'$ we have

$$\|v_{m-i'+j}\| = \|v_{m-i''+j+1}\| \geq e^{-4\Delta(j+1)} \|v_{m-i''}\| \geq e^{-4\Delta j - 5\Delta} \|v_{m-i'}\| \geq e^{-9\Delta j} \|v_{m-i'}\|,$$

where the second inequality follows from $\|v_{m-i''}\| \geq e^{-\Delta} \|v_{m-i'}\|$.

It is left to define m'. We define $m' = h_s$. It is easy to see that m' satisfies the desired properties. □

We now complete the proof of the lemma. Align the sequence $\{i'\}_{i=N}^{m}$ of hyperbolic times in an increasing order and define a new sequence $\mathcal{I}$. Define $\{h_i\}_{i=1}^{s}$ to be the subsequence of $\mathcal{I}$ which is strictly monotone and maximal with respect to inclusion. It is enough to prove $h_{i+1} \leq 4h_i$. Suppose that $h_{i+1} = j'$ for some $j \in [N, m]$. If $j \leq 2N$, then $h_{i+1} \leq 2N$ holds, by Sublemma 2.12.2. On the other hand we have $h_i \geq N/2$, and therefore $h_{i+1} \leq 4h_i$. Suppose that $j > 2N$. Then

$$h_{i+1} \geq \left[\frac{j+1}{2}\right] > \left[\frac{j-1}{2}\right] \geq \left[\frac{j-1}{2}\right]'.$$

Since h_i and h_{i+1} are two consecutive hyperbolic times, we have

$$h_i \geq \left[\frac{j-1}{2}\right]' \geq \frac{1}{2} \left[\frac{j-1}{2}\right].$$

This and $h_{i+1} \leq j$ yields $h_{i+1} \leq 4h_i$. □

3. Critical dynamics

In this section we study the dynamics along the orbit of a precritical point which has exponentially growing derivatives.

3.1. **Strong regularity.** Let ζ_0 be a precritical point of order $n \geq M$ on an admissible curve γ_0. A vector orbit $\mathbf{w} = \{w_i(\zeta_{i+1})\}_{i=0}^{\beta n}$ defined by $w_i = DH^i(\zeta_1) \binom{1}{0}$ is called *a forward vector orbit* of ζ_0. We say $\mathbf{w}$ is *strongly regular* if:

(a) $$\|w_j\| \geq e^{(\lambda-\alpha)(j-i)-\alpha\sigma i} \|w_i\| \quad 0 \leq \forall i \leq \forall j \leq \beta n;$$

(b) for every $k \in [0, \beta n]$ there exists $\chi(k) \in [(1-\alpha\sigma)k, k]$ such that $\Pi_0^{\chi(k)} \mathbf{w}$ is 1-regular.

We say ζ_0 is *good* if $\mathbf{w}$ is strongly regular.

Remark 3.1.1. By Remark 2.9.1 and $f_2(0) = -1 = f(-1)$, it follows that for an arbitrarily large integer N, one may assume that all precritical points of order $\le N$ are good, shrinking Ω' close to $(2,0)$ if necessary.

3.2. **Admissible position.** Suppose that ζ_0 is a good precritical point of order $n \ge M$ on an admissible curve γ_0. A nonzero vector $v_0(z_0)$ is in *admissible position* relative to ζ_0 if it is tangent to γ_0 and

$$\left(\frac{\|w_0\|}{\|w_{\chi(\beta n)}\|}\right)^{1-\ell} \le |\zeta_0 - z_0| \le \left(L^{-1}\Xi(\Pi_0^{\chi(\beta n)}\mathbf{w})\right)^{\frac{1}{2}}, \tag{9}$$

where $L = |f_2''(0)| = 4$. We say $v_0(z_0)$ is in *critical position* relative to ζ_0 if

$$|\zeta_0 - z_0| \le \left(\frac{\|w_0\|}{\|w_{\chi(\beta n)}\|}\right)^{1-\ell}.$$

We say $v_0(z_0)$ is *related* to ζ_0 if it is either in critical position or in admissible position relative to ζ_0. The definition of admissible position makes sense by the next lemma.

Lemma 3.2.1. *For the above* $\mathbf{w}$*, we have*

$$\Xi(\Pi_0^{\chi(\beta n)}\mathbf{w}) \cdot \left(\frac{\|w_{\chi(\beta n)}\|}{\|w_0\|}\right)^{2-2\ell} \ge e^{(1-2\ell)\lambda\beta n/2}.$$

Proof. Suppose that $K_0\delta e^{\alpha\sigma\beta n} \ge 1$. The strong regularity of $\mathbf{w}$ gives

$$\|w_j\| \ge e^{-\alpha\sigma\beta n}\|w_i\| \quad 0 \le \forall i \le \forall j \le \chi(\beta n)$$

and

$$\frac{\|w_0\|}{\|w_i\|} = \frac{\|w_{\chi(\beta n)}\|}{\|w_i\|}\frac{\|w_0\|}{\|w_{\chi(\beta n)}\|} \ge K_0\delta\frac{\|w_0\|}{\|w_{\chi(\beta n)}\|} \ge e^{-\alpha\beta\sigma n}\frac{\|w_0\|}{\|w_{\chi(\beta n)}\|} \quad 0 \le \forall i \le \chi(\beta n).$$

Substituting these into the definition of $\Theta(\Pi_0^{\chi(\beta n)}\mathbf{w}, i)$ and rearranging gives

$$\Theta(\Pi_0^{\chi(\beta n)}\mathbf{w}, i)\frac{\|w_{\chi(\beta n)}\|}{\|w_0\|} \ge e^{-2\alpha\sigma\beta n},$$

and thus

$$\Theta(\Pi_0^{\chi(\beta n)}\mathbf{w}, i)\left(\frac{\|w_{\chi(\beta n)}\|}{\|w_0\|}\right)^{2-2\ell} \ge e^{-2\alpha\sigma\beta n}\left(\frac{\|w_{\chi(\beta n)}\|}{\|w_0\|}\right)^{1-2\ell} \ge e^{(1-2\ell)\lambda\beta n/2}.$$

Since $i \in [0, \chi(\beta n)]$ is arbitrary, we obtain the desired inequality.

Suppose that $K_0\delta e^{\alpha\sigma\beta n} < 1$. Since ζ_0 is a precritical point, $\|w_j\| \ge K_0\|w_i\|$ holds for every $0 \le i \le j \le \chi(\beta n)$. Hence

$$\Theta(\Pi_0^{\chi(\beta n)}\mathbf{w}, i)\frac{\|w_{\chi(\beta n)}\|}{\|w_0\|} \ge K_0^3 e^{-\alpha\beta\sigma n}.$$

The rest of the reasoning is almost the same as the previous case. □

3.3. **Derivative recovery.** Define

$$p=\left[\frac{(1-\ell)\beta\Delta n}{-\log\sqrt{b}}\right]+1,$$

and

$$q=\chi(\beta n),$$

where $[\cdot]$ is the Gauss symbol. We call p the *folding period*, and q the *binding period.*

Remark 3.3.1. Dynamical meanings of there two periods are the following. The binding period is the time of duration in which the orbit of the point in admissible position shadows the critical orbit in a sufficiently regular way. During this time we compare the growth of these two orbits. The folding period is a time at which the corresponding two vectors become sufficiently parallel to each other.

Proposition 3.3.2. *Suppose that a nonzero vector $v_0(z_0)$ is in admissible position relative to a good precritical point ζ_0 of order $n\geq M$. Then*

$$\|v_i\|\leq\|v_0\|e^{-\beta i}\quad 0\leq\forall i\leq p; \tag{10}$$

$$L|\zeta_0-z_0|^{1+\tilde{\alpha}}\|v_0\|\leq\|v_p\|\leq L|\zeta_0-z_0|^{1-\tilde{\alpha}}\|v_0\|, \tag{11}$$

where $\tilde{\alpha}$ is a constant which can be made arbitrarily small by choosing small b;

$$\|v_{q+1}\|\geq\|v_0\|e^{(\lambda-\alpha-2\alpha\sigma)\ell(q+1)}; \tag{12}$$

$$\|v_0\||\zeta_0-z_0|^{-1+3(1-2\ell)}\leq\|v_{q+1}\|\leq\|v_0\||\zeta_0-z_0|^{-1-\tilde{\alpha}+\frac{3\alpha\sigma}{\Delta(2-2\ell)}}; \tag{13}$$

$$\log|\zeta_0-z_0|^{-\frac{3}{\Delta(2-2\ell)}}\leq q\leq\log|\zeta_0-z_0|^{-\frac{3}{\lambda}}; \tag{14}$$

$$|\zeta_i-z_i|\leq e^{-\alpha\sigma q/2}\quad 1\leq\forall i\leq q+1; \tag{15}$$

$$\|v_{q+1}\|\geq e^{-1}K_0\delta\|v_i\|\quad 0\leq i\leq q+1; \tag{16}$$

$$\frac{\|v_j\|}{\|v_i\|}\geq\left(\frac{\|v_p\|}{\|v_0\|}\right)^{1+\frac{3\alpha\sigma}{\lambda(1+\tilde{\alpha})}}\quad 0\leq\forall i\leq\forall j\leq q+1. \tag{17}$$

Proof. We begin by studying the action of H on admissible curves containing precritical points.

Lemma 3.3.3. *Let γ_0 be an admissible curve in $\mathcal{C}_\delta$. Suppose that there exists $\zeta_0\in\gamma_0$ such that* $\mathrm{slope}(DHt_{\gamma_0}(\zeta_0))\geq K_1^{-1}\delta b^{-1}$. *For $z\in\gamma_0$, split $DH(z)t_{\gamma_0}(z)=\xi(z)t_{\gamma_1}(\zeta_1)+\eta(z)t_{\gamma_1}(\zeta_1)^{\perp}$. Then*

$$|\xi|\leq 2K_1\delta^{-1}b$$

and

$$(1-\theta)L|\zeta_0-z|\leq|\eta|\leq(1+\theta)L|\zeta_0-z|.$$

Proof. Put $\psi = \mathrm{angle}(DHt_{\gamma_0}(\zeta_0), (\begin{smallmatrix}0\\1\end{smallmatrix}))$. Define two matrices $T_0^{-1} = (t_{\gamma_0}(z), t_{\gamma_0}(z)^\perp)$ and $T_1^{-1} = (t_{\gamma_1}(\zeta_1)^\perp, t_{\gamma_1}(\zeta_1))$. Since γ_0 is an admissible curve, there exists a closed interval $I \subset [-\delta, \delta]$ and a function $\hat{\gamma}_0$ on I such that $\gamma_0 = \mathrm{graph}(\hat{\gamma}_0)$. Hence, any $z \in \gamma_0$ is written as $z = (x, \hat{\gamma}_0(x))$. The matrix T_0 is the rotation by $\theta(x) = \mathrm{angle}(t_{\gamma_0}(z), (\begin{smallmatrix}1\\0\end{smallmatrix}))$, where $|\theta(x)| \leq Kb$ and $|\theta'(x)| \leq K$. The matrix T_1 is the rotation with angle ψ. We have the identity

$$DH(z)(t_{\gamma_0}(z), t_{\gamma_0}(z)^\perp) = (t_{\gamma_1}(\zeta_1)^\perp, t_{\gamma_1}(\zeta_1))T_1 DH(z)T_0^{-1}.$$

The number $\xi(z)$ corresponds to the $(2, 1)$-entry of $T_1 DH(z)T_0^{-1}$, and hence the desired inequality follows. The number $\eta(z)$ corresponds to the $(1, 1)$-entry of the same matrix. A direct computation using $b \ll \delta$ gives

$$(1 - \theta/2)|f_a''(0)| \leq \left|\frac{d\eta(z)}{dx}\right| \leq (1 + \theta/2)|f_a''(0)|.$$

Using the Taylor expansion around $\zeta_0 = (x_0, y_0)$ and $\eta(\zeta_0) = 0$,

$$(1 - \theta)L|x_0 - x| \leq |\eta(z)| \leq (1 + \theta)L|x_0 - x|,$$

for small δ and a close to 2. This implies the desired inequality because $|x_0 - x| \approx |\zeta_0 - z|$ holds. □

Claim 3.3.4. *Let $\Gamma^{(q-1)}$ denote the long stable leaf of order $q - 1$ through ζ_1. Then we have $z_1 \in \Gamma^{(q-1)}(\Xi(\Pi_0^q \mathbf{w}))$.*

Proof. Suppose that $\zeta_1 - z_1 = (\xi, \eta)$. Let z' (resp. z'') denote the unique point in $\Gamma^{(n)}$ (resp. $\Gamma^{(q-1)}$) whose y-coodinate coincides with that of z_1. Then $\zeta_1 - z' = (\xi', \eta)$ and $\zeta_1 - z'' = (\xi'', \eta)$ hold for some ξ', ξ''. Parametrize $\Gamma^{(n)}$ by arc length and assume that $\Gamma^{(n)}(0) = \zeta_1$. Define $\varphi(s) = \mathrm{angle}(e_n(\Gamma^{(n)}(s)), e_n(\zeta_1))$. Then we have $\varphi(0) = 0$ and $|\varphi'(s)| \leq K$. Thus

$$|\xi'| \leq K\int_0^{|\eta|} |\varphi(s)|ds \leq K\int_0^{|\eta|} sds \leq K\eta^2.$$

By Lemma 3.3.3 we have $\eta^2 \leq K_1\delta^{-1}b|\xi|$, and thus $|\xi'| \leq KK_1\delta^{-1}b|\xi|$. Hence $|\xi - \xi'| \leq |\xi| + |\xi'| \leq 2|\xi|$, and by Lemma 3.3.3 again,

$$|\xi| \leq (1 + 2\theta)\int L|\zeta_0 - z|dz,$$

where z ranges over all $z \in \gamma_0$ in between ζ_0 and z_0. Integrating this and using (9) we obtain $|\xi| \leq \Xi(\Pi_0^q \mathbf{w})/3$. Using the proof of Proposition 2.8.1 to bound $|\xi' - \xi''|$ by $(Kb)^n$, we obtain

$$|\xi - \xi''| \leq |\xi - \xi'| + |\xi' - \xi''| \leq 2\Xi(\Pi_0^q \mathbf{w})/3 + (Kb)^n \leq \Xi(\Pi_0^q \mathbf{w}).$$

This implies the claim. □

Split $v_1(z_1) = \xi e_q(z_1) + \eta f_q(z_1)$. We estimate $|\eta|$. The idea is to compare this decomposition with the one in Lemma 3.3.3. By Lemma 2.7.4,

$$\mathrm{angle}(e_q(\zeta_1), e_q(z_1)) \leq \|De_q\||\zeta_1 - z_1| \leq KK_1\delta|\zeta_0 - z_0|.$$

By Lemma 2.7.1 and the left hand side of (9),

$$\mathrm{angle}(e_q(\zeta_1), e_n(\zeta_1)) \leq (Kb)^n \leq |\zeta_0 - z_0|^2.$$

Thus $\mathrm{angle}(e_n(\zeta_1), e_q(z_1)) \leq K\delta|\zeta_0 - z_0|$, and this implies

$$|\eta| \simeq L|\zeta_0 - z_0|\|v_0\|. \tag{18}$$

We prove (10). Using (18), for every $0 \leq i \leq p$ we have

$$\|DH^i \eta f_q(z_1)\| \leq \|DH^i(z_1)\||\eta| \leq e^{-\alpha\sigma q}\|v_0\|.$$

Using $2p \leq \alpha\sigma n$,

$$\|DH^i \eta f_q(z_1)\| \leq e^{-2\beta p}\|v_0\| \leq e^{-2\beta i}\|v_0\|.$$

This and $\|DH^i \xi e_q(z_1)\| \leq (Kb)^i\|v_0\|$ yield (10).

We prove (11). For every $0 \leq i \leq q$,

$$\|DH^i f_q(z_1)\| \geq e^{-1} \cdot \frac{\|w_i\|}{\|w_0\|} \geq e^{(\lambda-\alpha)i-1}.$$

Since z_0 is in admissible position, (18) implies

$$|\eta| \geq \left(\frac{\|w_0\|}{\|w_q\|}\right)^{1-\ell} \|v_0\| \geq e^{(\ell-1)\Delta q}\|v_0\|.$$

Using the definition of p, for every $p \leq i \leq q$ we have

$$\frac{\|DH^i \xi e_q(z_1)\|}{\|DH^i \eta f_q(z_1)\|} \leq \frac{(Kb)^i}{e^{-1}e^{(\ell-1)\Delta q}e^{(\lambda-\alpha)i}} \leq b^{i/2} \leq \theta.$$

This implies

$$(1-\theta)\|DH^i \eta f_q(z_1)\| \leq \|v_{i+1}\| \leq (1+\theta)\|DH^i \eta f_q(z_1)\|. \tag{19}$$

Take small $\tilde{\alpha} > 0$ such that $\Delta p/n - \alpha\tilde{\alpha}\beta\sigma < 0$ holds. Then

$$\begin{aligned}\frac{\|v_p\|}{\|v_0\|} &\leq (1+\theta)L \cdot |\zeta_0 - z_0|\|DH^p f_p(z_1)\| \\ &\leq L|\zeta_0 - z_0|^{1-\tilde{\alpha}}e^{\Delta p - \alpha\tilde{\alpha}\beta\sigma n} \\ &\leq |\zeta_0 - z_0|^{1-\tilde{\alpha}}.\end{aligned}$$

This yields the upper estimate in (11). On the other hand, $p \geq 1$ and $\|DH^p f_q(z_1)\| \geq e^{-1}\|w_p\| \geq e^{\lambda-\alpha-1}$ gives

$$\frac{\|v_p\|}{\|v_0\|} \geq Le^{\lambda-\alpha-1}|\zeta_0 - z_0| \geq L|\zeta_0 - z_0|^{1+\tilde{\alpha}}.$$

We prove (12). Using (19), for all $p-1 \leq i \leq j \leq q$ we have

$$\left|\log\frac{\|v_{j+1}\|}{\|v_{i+1}\|} - \log\frac{\|w_j\|}{\|w_i\|}\right| \leq 1. \tag{20}$$

Therefore

$$\|v_{q+1}\| \geq \|DH^q \eta f_q(z_1)\| \geq e^{-1}\left(\frac{\|w_q\|}{\|w_0\|}\right)^{\ell} \|v_0\| \geq e^{(\lambda-\alpha-2\alpha\sigma)\ell(q+1)}\|v_0\|.$$

We prove (13). Let τ_0 denote the straight segment whose endpoints are z_1 and z''. By Lemma 2.4.1, for every $0 \leq i \leq q$ we have

$$e^{-1}\frac{\|w_i\|}{\|w_0\|} \leq \frac{\mathrm{length}(\tau_i)}{\mathrm{length}(\tau_0)} \leq e\frac{\|w_i\|}{\|w_0\|}. \tag{21}$$

Integrating (18) and using $\eta^2 \le K_1\delta^{-1}b$, we have $\mathrm{length}(\tau_0) \simeq |\zeta_0 - z_0|^2$. Hence

$$\mathrm{length}(\tau_q) \ge e^{-1}\frac{\|w_q\|}{\|w_0\|}|\zeta_0 - z_0|^2 \ge e^{-3}\left(\frac{\|w_q\|}{\|w_0\|}\right)^{2\ell-1} \ge e^{-(1-2\ell)\lambda q}.$$

Rearranging this and using the upper estimate of q,

$$\frac{\|DH^q\eta f_q(z_1)\|}{|\eta|} \ge e^{-1}\frac{\|w_q\|}{\|w_0\|} \ge |\zeta_0 - z_0|^{-2}e^{-(1-2\ell)\lambda q} \ge |\zeta_0 - z_0|^{-2+3(1-2\ell)}.$$

This yields the lower estimate. On the other hand, using (21) for $i = p$ and q,

$$\frac{\|v_p\|}{\|v_0\|}\frac{\|v_{q+1}\|}{\|v_p\|} \le e|\zeta_0 - z_0|^{1-\tilde{\alpha}}\frac{\|w_q\|}{\|w_{p-1}\|} \le e^2|\zeta_0 - z_0|^{1-\tilde{\alpha}}\frac{\mathrm{length}(\tau_q)}{\mathrm{length}(\tau_{p-1})}.$$

To estimate the right hand side, we use (14) and (15) to yield $\mathrm{length}(\tau_q) \le e^{-\alpha\sigma q} \le |\zeta_0 - z_0|^{\frac{3\alpha\sigma}{\Delta(2-2\ell)}}$. Moreover, by (20) we have $\mathrm{length}(\tau_{p-1}) \ge \mathrm{length}(\tau_0) \ge |\zeta_0 - z_0|^2$. Substituting these into the right hand side we obtain the upper estimate.

We prove (14). Using (20),

$$\mathrm{length}(\tau_q) \le e\cdot|\zeta_0 - z_0|^2\cdot\frac{\|w_q\|}{\|w_0\|} \le e\cdot\Xi(\Pi_0^q\mathbf{w})\cdot\frac{\|w_q\|}{\|w_0\|} \le e^{1-\alpha\sigma q}.$$

On the other hand,

$$\mathrm{length}(\tau_q) \ge e^{-1}|\zeta_0 - z_0|^2\cdot\frac{\|w_q\|}{\|w_0\|} \ge |\zeta_0 - z_0|^2 e^{-1+(\lambda-\alpha-\alpha\sigma)q}.$$

These two inequalities together imply the upper estimate of q. On the other hand, we have

$$e^{-1}\left(\frac{\|w_q\|}{\|w_0\|}\right)^{2\ell-1} \le e^{-1}|\zeta_0 - z_0|^2\frac{\|w_q\|}{\|w_0\|} \le \mathrm{length}(\tau_q) \le e\cdot|\zeta_0 - z_0|^2\frac{\|w_q\|}{\|w_0\|},$$

and thus

$$|\zeta_0 - z_0|^2 \ge e^{-2}\left(\frac{\|w_q\|}{\|w_0\|}\right)^{2\ell-2} \ge e^{-\Delta(2-2\ell)q-4}.$$

Taking logs and rearranging we obtain the lower estimate of q.

We prove (15). We have

$$|\zeta_i - z_i| \le |\zeta_i - z''_{i-1}| + |z_i - z''_{i-1}|.$$

We clearly have

$$|z_i - z''_{i-1}| \le \frac{\|w_i\|}{\|w_0\|}\cdot\Xi(\Pi_0^q\mathbf{w}) \le e^{-\alpha\sigma q}.$$

Since $z''_0 \in \Gamma^{(q-1)}$ we have $|\zeta_i - z''_{i-1}| \le |\zeta_1 - z''_0|$ for $1 \le i \le q$. Moreover, Lemma 3.3.3 gives $|\zeta_1 - z''_0| \le K_1\delta^{-1}b|\zeta_0 - z_0|^2 \le e^{-\alpha\sigma q}$. Altogether these imply the desired inequality.

We prove (16). Using (10) (12), for every $0 \le i \le p$ we have

$$\frac{\|v_{q+1}\|}{\|v_i\|} \ge \frac{\|v_{q+1}\|}{\|v_0\|} \ge e^{(\lambda-\alpha-2\alpha\sigma)\ell q} \ge e^{-1}K_0\delta.$$

Using (20), for every $p+1 \le i \le q$ we have

$$\frac{\|v_{q+1}\|}{\|v_i\|} \ge e^{-1} \cdot \frac{\|w_q\|}{\|w_{i-1}\|} \ge e^{-1} K_0 \delta.$$

Thus (16) follows.

We prove (17). There are three cases: $i \le j \le p$; $i \le p \le j$; $p \le i \le j$. In the first case, using $\|v_j\| \ge e^{-\Delta p}\|v_p\|$ and (10) (11),

$$\frac{\|v_j\|}{\|v_i\|} \ge \frac{\|v_p\|}{\|v_i\|}\frac{\|v_j\|}{\|v_p\|} \ge \frac{\|v_p\|}{\|v_0\|} e^{-\Delta p} \ge \left(\frac{\|v_p\|}{\|v_0\|}\right)^{1+\Delta\tilde{\alpha}} \ge \left(\frac{\|v_p\|}{\|v_0\|}\right)^{1+\frac{3\alpha\sigma}{\lambda(1+\tilde{\alpha})}}.$$

The remaining cases have similar proofs. Using (20), for all $p \le i \le j \le q$ we have

$$\frac{\|v_j\|}{\|v_i\|} \ge e^{-2}\frac{\|w_j\|}{\|w_i\|} \ge e^{-\alpha\sigma j} \ge e^{-\alpha\sigma q}.$$

Substituting (11) (14) into this we obtain

$$\frac{\|v_j\|}{\|v_i\|} \ge \left(\frac{\|v_p\|}{\|v_0\|}\right)^{\frac{3\alpha\sigma}{\lambda(1+\tilde{\alpha})}} \ge \left(\frac{\|v_p\|}{\|v_0\|}\right)^{1+\frac{3\alpha\sigma}{\lambda(1+\tilde{\alpha})}}.$$

This finishes the proof in the last case. In the second case, the above inequality with $i=p$ and $\|v_i\| \le \|v_0\|$ in (10) yields the desired one. □

4. Global dynamics

The aim of this section is to study global behaviors of generic orbits. We begin by introducing the exponential growth condition $(EG)_n$ in Theorem A. Assuming this condition, we develop an argument to find a suitable precritical points to which the results in Section 3 apply. Consequently we obtain a proof of Theorem A.

4.1. **Exponential growth condition.** Let $n \ge N$. We say H satisfies $(EG)_n$ if all critical points of order $\le n$ on any admissible curves are good.

4.2. **Capture argument.** The following proposition guarantees that under the assumption $(EG)_n$, one can associate suitable critical points (binding points) to all e^{-1}-regular orbits which fall inside $\mathcal{C}_\delta$.

Proposition 4.2.1. *Suppose that H satisfies $(EG)_n$ for some $n \ge N$.Let $\{v_i(z_i)\}_{i=0}^m$ be a e^{-1}-regular vector orbit of H such that $m \ge N$ and $z_m \in \mathcal{C}_\delta$. Let $\{h_i\}_{i=1}^s$ denote the sequence of hyperbolic times associated with $\{v_i(z_i)\}_{i=0}^m$. Let i_0 denote the largest integer such that $h_{i_0} \le n$. Then there exists a good precritical point of order $\le h_{i_0}$ relative to which $v_m(z_m)$ is in admissible position, or else there exists a good critical point of order h_{i_0} relative to which $v_m(z_m)$ is in critical position. In the first case, $\{v_i(z_i)\}_{i=0}^{m+q+1}$ is e^{-1}-regular, where q is the binding period.*

Remark 4.2.2. It is important that no relation between m and n is assumed. In particular m is allowed to be larger than n. If $m \le n$, then $h_{i_0} = h_s$ by definition.

Proof of Proposition 4.2.1. We fix some notation. For a nonzero vector $v(z)$ and $r > 0$, let $\gamma(v(z), r)$ denote the straight line of length r which is centered at z and tangent to $v(z)$. Put $\rho = e^{-10\Delta}$. For every $1 \le i \le s$, put $\gamma^{(i)} = H^{h_i}(\gamma(v_{m-h_i}, \rho^{h_i}))$. Since h_i is a hyperbolic time, $\rho^{h_i} \le \Xi(\{v_j\}_{j=m-h_i}^m)$ holds. Thus, by Lemma 2.6.1,

$\gamma^{(i)}$ is an admissible curve with length $\geq \rho^{2h_i}$. In particular, it makes sense to speak about the existence of precritical points on $\gamma^{(i)}$.

Lemma 4.2.3. *Let $i \leq i_0 - 1$, and suppose that there exists a good critical point of order h_i on $\gamma^{(i)}$ relative to which $v_m(z_m)$ is in critical position. Then there exists a good precritical point of order $\in [h_i+1, h_{i+1}]$ on $\gamma^{(i+1)}$ relative to which $v_m(z_m)$ is in admissible position, or else there exists a good critical point of order h_{i+1} on $\gamma^{(i+1)}$ relative to which $v_m(z_m)$ is in critical position.*

Proof. Let $\zeta_0^{(h_i,i)}$ denote the good critical point of order h_i on $\gamma^{(i)}$ relative to which v_m is in critical position. Take $\hat{z} \in \gamma^{(i+1)}$ whose x-coordinate coincides with that of $\zeta_0^{(h_i,i)}$. Such $\hat{z}$ uniquely exists because of the lower bound on the length of $\gamma^{(i+1)}$ and the assumption that $v_m(z_m)$ is in critical position relative to $\zeta_0^{(h_i,i)}$. Let $\mathbf{w} = \{w_i\}_{i=0}^{\beta h_i}$ denote the forward vector orbit of $\zeta_0^{(h_i,i)}$.

Claim 4.2.4. *We have*

$$|\zeta_0^{(h_i,i)} - \hat{z}| \leq K \left(\frac{\|w_0\|}{\|w_{\beta h_i}\|} \right)^{2-2\ell}$$

and

$$\mathrm{angle}(t_{\gamma^{[i]}}\zeta_0^{(h_i,i)}, t_{\gamma^{[i+1]}}(\hat{z})) \leq K \left(\frac{\|w_0\|}{\|w_{\beta h_i}\|} \right)^{2-2\ell}.$$

Proof. Parametrize $\gamma^{(i)}$ and $\gamma^{(i+1)}$ by arc length so that $\gamma^{(i)}(0) = z_m = \gamma^{(i+1)}(0)$ and the x-components of the derivatives have the same sign. Then

$$|\gamma^{(i)}(s) - \gamma^{(i+1)}(s)| \leq K \int_0^s \|\dot{\gamma}^{(i)}(t) - \dot{\gamma}^{(i+1)}(t)\| dt.$$

Since $\gamma^{(i)}$ and $\gamma^{(i+1)}$ are admissible curves which are tangent to $v_m(z_m)$, we have $\dot{\gamma}^{(i)}(0) = \dot{\gamma}^{(i+1)}(0)$ and $\|\ddot{\gamma}^{(i)}(0)\|$, $\|\ddot{\gamma}^{(i+1)}(0)\| \leq 1$. Thus

$$\int_0^s \|\dot{\gamma}^{(i)}(t) - \dot{\gamma}^{(i+1)}(t)\| dt \leq K \int_0^s t dt \leq K s^2.$$

This implies the first inequality. The second one follows from the bound on the curvatures of $\gamma^{(i)}$ and $\gamma^{(i+1)}$. $\square$

Claim 4.2.5. *For every $1 \leq k \leq \beta h_i$,*

$$\left| \log \frac{\|DH^k(H(\zeta_0^{(h_i,i+1)}))\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\|}{\|DH^k(H(\zeta_0^{(h_i,i)}))\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\|} \right| \leq 1.$$

Proof. Since $\beta \gg 1$, $\gamma^{(i)}$ (resp. $\gamma^{(i+1)}$) contains a curve of length $\gg \left(\frac{\|w_0\|}{\|w_{\beta h_i}\|}\right)^{1-\ell}$ centered at $\zeta_0^{(h_i,i)}$ (resp. $\hat{z}$). Hence, by Lemma 2.10.2, there exists a precritical point of order h_i on $\gamma^{(i+1)}$, called $\zeta_0^{(h_i,i+1)}$, such that $|\hat{z} - \zeta_0^{(h_i,i+1)}| \leq K \left(\frac{\|w_0\|}{\|w_{\beta h_i}\|}\right)^{1-\ell}$. Combining this with the first inequality in Claim 4.2.4,

$$|\zeta_0^{(h_i,i)} - \zeta_0^{(h_i,i+1)}| \leq K \left(\frac{\|w_0\|}{\|w_{\beta h_i}\|} \right)^{1-\ell}.$$

Using Lemma 3.3.3 and Lemma 3.2.1, we obtain $\zeta_0^{(h_i,i+1)} \in \Gamma^{(\chi(\beta h_i)-1)}(\Xi\Pi_0^{\chi(\beta h_i)}\mathbf{w})$. Hence the inequality follows. $\square$

For every $k \in [h_i+1, h_{i+1}]$, Lemma 2.10.1 yields a precritical point of order k on $\gamma^{(i+1)}$, called $\zeta_0^{(k,i+1)}$. In fact, $\zeta_0^{(h_{i+1},i+1)}$ is a good critical point of order h_{i+1}, because of $(EG)_n$, $h_{i+1} \le n$, and the fact that there exists a e^{-2}-regular backward orbit of length h_{i+1}, by Lemma 2.4.1. Hence all $\zeta_0^{(k,i+1)}$ is a good precritical point for every $h_i+1 \le k \le h_{i+1}-1$.

Sublemma 4.2.6. *Suppose that ζ_0, ζ_0' are good precritical points of order m and $m+1$ on an admissible curve γ_0 such that $|\zeta_0-\zeta_0'| \le (Kb)^{m/2}$. Let $\mathbf{w} = \{w_i(\zeta_{i+1})\}_{i=0}^{\beta m}$, $\mathbf{w}' = \{w_i'(\zeta_{i+1}')\}_{i=0}^{\beta(m+1)}$ denote the respective forward vector orbtits. Denote by $\chi'(\cdot)$ the function $\chi(\cdot)$ for $\mathbf{w}'$. Then*

$$\Xi(\Pi_0^{\chi'(\beta(m+1))}\mathbf{w}') \cdot \left(\frac{\|w_{\chi(\beta m)}\|}{\|w_0\|}\right)^{2-2\ell} \ge e^{(1-2\ell)\lambda\beta m/2}.$$

Proof. First of all, recall that by (6) we have

$$\left|\log\frac{\|w_i'\|}{\|w_i\|}\right| \le 1 \quad 1 \le \forall i \le \beta m. \tag{22}$$

Suppose that $\chi'(\beta(m+1)) < \chi(\beta m)$. Then the inequality immediately follows from (22) and $\Xi(\Pi_0^{\chi'(\beta(m+1))}\mathbf{w}') \ge \Xi(\Pi_0^{\chi(\beta m)}\mathbf{w}')$.

Suppose that $\chi'(\beta(m+1)) \ge \chi(\beta m)$, and moreover $K_0 e^{-2}\delta \ge e^{-2\Delta\alpha\beta\sigma m}$. Using(22), for every $0 \le i \le \chi(\beta m)$ we have

$$\frac{\|w_0'\|}{\|w_i'\|} = \frac{\|w_0'\|}{\|w_{\chi(\beta m)}'\|}\frac{\|w_{\chi(\beta m)}'\|}{\|w_i'\|} \ge K_0 e^{-2}\delta\frac{\|w_0'\|}{\|w_{\chi(\beta m)}'\|}.$$

It is straightforward to check that $|\chi'(\beta(m+1)) - \chi(\beta m)| \le 2\alpha\beta\sigma m$. Using this, for every $\chi(\beta m) \le i \le \chi'(\beta(m+1))$ we have $\|w_{\chi(\beta m)}'\| \ge e^{-2\Delta\alpha\beta\sigma m}\|w_i'\|$. Thus, for every $0 \le i \le \chi'(\beta(m+1))$ we obtain

$$\frac{\|w_0'\|}{\|w_i'\|} \ge e^{-2\Delta\alpha\beta\sigma m}\frac{\|w_0'\|}{\|w_{\chi(\beta m)}'\|}.$$

Using $\|w_j'\| \ge e^{-\alpha\sigma\beta m}\|w_i'\|$ and (22),

$$\Xi(\Pi_0^{\chi'(\beta(m+1))}\mathbf{w}') \ge e^{-3\Delta\alpha\beta\sigma m}\frac{\|w_0\|}{\|w_{\chi(\beta m)}\|}.$$

This implies the desired inequality. The case $K_0\delta < e^{-2\Delta\alpha\beta\sigma m}$ can be handled similarly to the last part of the proof of Lemma 3.2.1. $\square$

Claim 4.2.5 implies that $v_m(z_m)$ is related to $\zeta_0^{(h_i,i+1)}$. Suppose that $v_m(z_m)$ is in critical position relative to $\zeta_0^{(h_i,i+1)}$. In this case, it follows from Sublemma 4.2.6 that $v_m(z_m)$ is related to $\zeta_0^{(h_i+1,i+1)}$. If $v_m(z_m)$ is in admissible position relative to $\zeta_0^{(h_i+1,i+1)}$, then it is done. Otherwise, we again use Sublemma 4.2.6 and repeat the same argument. Eventually, only two possibilities are left: there exists $k \in [h_i+1, h_{i+1}]$ such that $v_m(z_m)$ is in admissible position relative to $\zeta_0^{(k,i+1)}$, or else

$v_m(z_m)$ is in critical position relative to $\zeta_0^{(h_{i+1},i+1)}$. This completes the proof of Lemma 4.2.3. □

Let us come back to the proof of Proposition 4.2.1. We firstly consider the case $z_m \notin \mathcal{C}_{\delta^{10}}$. Choose a large integer R which do not depend on δ, and consider $H = H_{a,b}$ such that (a, b) is close enough to $(2, 0)$ so that all precritical points of order $\leq R$ are good. Take a straight segment γ_0 which is tangent at z_m to v_m and intersects both $\{\delta\} \times \mathbb{R}$ and $\{-\delta\} \times \mathbb{R}$. Clearly, γ_0 is an admissible curve, and there exists a good precritical point of order M on γ_0 to which $v_m(z_m)$ is related. Since all precritical points of order $\leq R$ are good, we can successively apply Lemma 2.10.2 to create good precritical points of higher order on γ_0.

We claim that there exists a precritical point of order $\leq R$ on γ_0 relative to which $v_m(z_m)$ is in admissible position. Let us see why this is so. Sublemma 4.2.6 implies that if $v_m(z_m)$ is in critical position relative to a precritical point z_0 of order $j < R$ on γ, then $v_m(z_m)$ is related to the precritical point of order $j+1$ on γ_0. This leaves out only two possibilities: either there exists a precritical point of order $\leq R$ on γ_0 relative to which $v_m(z_m)$ is in admissible position, or $v_m(z_m)$ is in critical position relative to the precritical point of order R on γ_0. However, the second possibility is eliminated by the fact that all precritical points are contained in $\mathcal{C}_{\delta^{10}}$, and R can be made arbitrarily large after δ is fixed. Hence the claim follows.

Next, we consider the case $z_m \in \mathcal{C}_{\delta^{10}}$. Since length$(\gamma^{[1]}) \geq \rho^{2h_1} \geq \rho^N$, the admissible curve $\gamma_{h_1}^{(h_1)}$ intersects both $\{\delta^{10}\} \times \mathbb{R}$ and $\{-\delta^{10}\} \times \mathbb{R}$. Hence there exists a good precritical point of order N on $\gamma^{(1)}$ to which $v_m(z_m)$ is related. If $v_m(z_m)$ is related to it then it is done. If not, we appeal to Lemma 4.2.3. This finishes the proof of the first half of the assertion of the proposition.

It is left to prove that $\mathbf{v}'$ is e^{-1}-regular. This follows from $\|v_{m+q+1}\| \geq \|v_m\|$ and (16). □

4.3. **Controlled vector orbits.** Suppose that H satisfies $(EG)_n$. Consider a vector orbit $\mathbf{v} = \{v_i(z_i)\}_{i=0}^m$. We say an integer $i \in [0, m]$ is a *return time* if $z_i \in \mathcal{C}_\delta$ holds. We say $\mathbf{v}$ is *controlled up to time* m, if slope$(v_0) \leq K_0 b$, and no return takes place up to time m, or else there exists a sequence of return times $m_0 < m_1 \cdots < m_t \leq m$ such that:

(a) m_0 is the first return time and $m_0 \geq N$;

(b) $\Pi_0^{m_0}\mathbf{v}$ is e^{-1}-regular;

(b) for every $0 \leq s \leq t$, there exists a binding point of order $\leq \min(m_s, n)$ relative to which $v_{m_s}(z_{m_s})$ is in admissible position;

(c) for every $0 \leq s \leq t-1$, $m_{s+1} = \min\{i\colon i \geq m_s + q_s + 1, z_i \in \mathcal{C}_\delta\}$, where q_s is the corresponding binding period;

(d) $m_t \leq m \leq m_t + q_t + 1$, or $m > m_t + q_t + 1$ and no return takes place from $m_t + q_t + 1$ to $m - 1$.

We call i *bound* if $i \in [m_s + 1, m_s + q_s]$ for some $s \in [0, t]$. We call i *free* if it is not bound.

Lemma 4.3.1. *If* $\mathbf{v} = \{v_i\}_{i=0}^m$ *is controlled, then for every free iterate* $0 \leq i \leq m$,

$$\|v_i\| \geq K_0 \delta e^{\lambda i/3} \|v_0\|.$$

Proof. By Lemma 2.5.1, for every $i \le m_0$ we have $\|v_i\| \ge K_0\delta e^{\lambda i}\|v_0\|$. Since $m_0 \ge N$, we have $\|v_{m_0}\| \ge e^{\lambda i/3}\|v_0\|$. Suppose that for every $i \in \cup_{s=0}^t\{m_s+q_s+1\}$,

$$\|v_i\| \ge e^{\lambda i/3}\|v_0\|. \tag{23}$$

Using slope$(v_{m_s+q_s+1}) \le K_0 b$ and Lemma 2.5.1, for every $m_s+q_s+1 \le i \le m_{s+1}$ we have

$$\frac{\|v_i\|}{\|v_0\|} \ge \frac{\|v_i\|}{\|v_{m_s+q_s+1}\|}\frac{\|v_{m_s+q_s+1}\|}{\|v_0\|} \ge K_0\delta e^{\lambda(i-m_s-q_s-1)}e^{\frac{\lambda}{3}(m_s+q_s+1)} \ge K_0\delta e^{\frac{\lambda}{3}i}.$$

By the same reasoning we have $\|v_i\| \ge K_0\delta e^{\lambda i/3}\|v_0\|$ for every free iterate in between m_t and m.

It is left to prove (23) for $i \in \cup_{s=0}^t\{m_s+q_s+1\}$. By (12), (23) holds for $i = m_0+q_0+1$. Suppose that (23) holds for some $i = m_s+q_s+1$. Then the better estimate in Lemma 2.5.1 and (12) together yield (23) for $i = m_{s+1}+q_{s+1}+1$. This completes the proof. □

4.4. **Proof of Theorem A.** We are in position to prove Theorem A. We fix α, M, β, δ, one and for all. For small $b > 0$, let $\Omega^{(0)}$ denote a small a-interval such that $\{(a,b)\colon a \in \Omega^{(0)}\} \subset \Omega'$, where Ω' is the one appearing in Proposition 2.1.1. We moreover assume that $\{(a,b)\colon a \in \Omega^{(0)}\}$ is close enough to $(2,0)$ so that all the previous estimates and arguments hold. In what follows we only consider $H = H_{a,b}$ such that $a \in \Omega^{(0)}$.

Suppose that H satisfies $(EG)_n$ for every $n \ge N$. For $z_0 \in W^u(P)$, take an integer $k_0 \ge 0$ such that the set of preimages $H^{-k_0}(z_0)$ intersects $W^u_{\rm loc}(P)$. Pick one point from $H^{-k_0}(z_0) \cap W^u_{\rm loc}(P)$ and denote it by z_{-k_0}. Notice that $z_i = H^{i+k_0}z_{-k_0}$ is uniquely determined for $i \le -k_0$. For an arbitrary $j \le \min\{-k_0, N\}$, define a vecor orbit $\{v_i(z_i)\}_{i=j}^{-k_0}$ by $v_i = DH^{i+k_0}t_{W^u_{\rm loc}(P)}(z_{-k_0})$. Since P is a hyperbolic fixed point, we have $\|v_{-k_0}\| \ge \|v_i\|$ for $j \le i \le -k_0$. Let $m_0 = \min\{i : H^i(z_{-k_0}) \in \mathcal{C}_\delta\}$. By Lemma 2.5.1 and slope$(v_{-k_0}) \le K_0 b$, we have $\|v_{m_0-k_0}\| \ge K_0\delta\|v_i\|$ for $j \le i \le m_0-k_0$. Since $m_0-k_0-j \ge -j \ge N$, the necessary conditions are satisfied for the capture argument to work. Moreover, since j is arbitrary, we can successively apply the capture argument and end up with either of the following two cases: obtain a good precritical point relative to which $v_m(z_m)$ is in admissible position; not so, namely, $v_m(z_m)$ is in critical position relative to all the precritical points assigned by the capture argument. In the first case, we iterate further. When the next free return takes place, we apply the capture argument again. By the same reasoning, two possibilities are left.

By now it is clear how to define $\mathcal{C}$. Define $\mathcal{C}$ to be the set of all $z_0 \in W^u(P)$ such that there exists a controlled vector orbit $\{v_i(z_i)\}_{i=-j}^0$ such that: (i) z_{-j} is near P and v_{-j} is tangent to $W^u_{\rm loc}(P)$; (ii) z_0 is a free return; (iii) $v_0(z_0)$ is in critical position relative to any critical point which is assigned by the capture argument. Let us see $\mathcal{C}$ satisfies the desired properties.

First of all, by Lemma 2.6.1 and the fact that $W^u_{\rm loc}(P)$ is an admissible curve, any $z_0 \in \mathcal{C}$ is contained in the interior of an admissible curve, say γ, which is contained in $W^u(P)$. For now let us suppose that there is no self intersection of $W^u(P)$. Lemma 2.10.2 and the definition of $\mathcal{C}$ implies the existence of a sequence of infinitely many good precritical points of arbitrarily high order on γ, converging on z_0. This implies

$\mathcal{C} \cap \gamma = \{z_0\}$. Let us see why this is so. Suppose that $z_0' \in \mathcal{C} \cap \gamma$. Then, by the same reasoning, there exists a sequence of infinitely many precritical points of arbitrarily high order on γ which converges on z_0'. Since γ is an admissible curve, there exists no more than two distinct critical points on γ of the same prder. This implies that the two sequences must converge on the same point. Hence $z_0' = z_0$, and the claim follows. Let us now suppose that there is a self intersection of $W^u(P)$. In this case, the above argument is slightly incomplete because there may exist two distinct critical points on two distinct admissible curves which intersect each other. To deal with this, consider an immersion $\iota \colon \mathbb{R} \to W^u(P)$. Then the above argument shows that $\iota^{-1}(\gamma \cap \mathcal{C})$ contains exactly one point. Consequently, $\mathcal{C}$ is a countable set regardless of whether $W^u(P)$ intersects itself or not.

For $z_0 \in \mathcal{C}$. let y_n denote the good precritical point of order n which belongs to the sequence converging on z_0. Since the speed of this convergence is exponential which does not depend on z_0, (iia) follows. Let $\Gamma^{(n)}$ denote the long stable leaf of order n through $H(y_n)$. It follows from the proof of Proposition 2.8.1 that $\{\Gamma^{(n)}\}_{n=1}^{\infty}$ forms a Cauchy sequence in the C^2 topology. Let $\Gamma^{(\infty)}$ denote its C^2 limit. Since $\Gamma^{(n)}$ is tangent to $H(\gamma)$ at $H(y_n)$ and $H(y_n) \to z_1$, $\Gamma^{(\infty)}$ is tangent at z_1 to $H(\gamma)$. This yields (iib). (iic) automatically follows from the definition of $\mathcal{C}$.

It is left to prove prove (iii). Since the Lyapunov exponents of all periodic points of f_2 are $\log 2$, we may assume that the largest Lyapunov exponents of all periodic points of H with period $\leq N$ are $\geq \log 2/3$. For a periodic orbit $\mathcal{O}$ with period $p \geq N$, there exists a sub-orbit of length N which stays outside of $\mathcal{C}_\delta$. Along this orbit we construct an e^{-1}-regular vector orbit of length N and then apply the capture argument. If the vector orbit is always in admissible position, then the largest Lyapunov exponent of $\mathcal{O}$ is $\geq \log 2/3$, by Lemma 4.3.1. Otherwise, there exists a vector orbit of length $\geq \sqrt{\beta} N$ which shadows the orbit of the critical point. In particular it is e^{-1}-regular and grows exponentially fast in norm. If $\sqrt{\beta} N \geq p$, then the largest Lyapunov exponent of $\mathcal{O}$ is $\geq \lambda - \alpha$. If $\sqrt{\beta} N \leq p$, then we apply the capture argument to this longer vector orbit and repeat the same argument. Since p is finite, this argument stops sooner or later. Consequently, the largest Lyapunov exponents of all periodic points are $\geq \log 2/3$. □

5. Smooth continuation of critical points

In this section we deal with parameter dependence of critical points. We prove that quasi critical points continue to exist in a sufficiently large parameter interval. Besides, we prove that their dependence on parameter is rather small.

5.1. **Quasi critical points.** We say a precritical point ζ_0 of order $n \geq N$ on an admissible curve γ_0 is a *primary quasi critical point* if there exists an e^{-3}-regular and $e^{-11\Delta}$-expanding orbit $\{w_i(\zeta_i)\}_{i=-n}^{0}$ such that $\zeta_{-n} \notin \mathcal{C}_\delta$ and $w_0(\zeta_0) \in T_{\zeta_0}\gamma_0$. We say ζ_0 is a *secondary quasi critical point* if there exists an $e^{-12\Delta}$-expanding vector orbit $\{w_i(\zeta_i)\}_{i=-n}^{0}$ such that $\zeta_{-n} \notin \mathcal{C}_\delta$ and $w_0(\zeta_0) \in T_{\zeta_0}\gamma_0$.

The following lemma states that near critical points there exists a stack of primary quasi critical points of lower order.

Lemma 5.1.1. *Let $\hat{\zeta}_0^{(j)}$ be a critical point of order h_j on γ_0, with $\{w_i\}_{i=-h_j}^{0}$ its backward orbit and $\{h_i\}_{i=1}^{j}$ the corresponding sequence of hyperbolic times. For every*

$1 \le i \le j$ there exists a primary quasi critical point $\hat\zeta_0^{(i)}$ of order h_i on an admissible curve $\gamma^{(i)} := H^{h_i}\gamma(w_{-h_i}, \rho^{h_i})$ such that

$$|\hat\zeta_0^{(i)} - \hat\zeta_0^{(j)}| \le \sum_{k=i}^{j} (Kb)^{h_k/3}. \tag{24}$$

Proof. Clearly, the assertion with $i = j$ holds, because γ_0 and $\gamma^{(j)}$ are tangent at $\hat\zeta_0^{(j)}$. Let $i \in [1, j-1]$, and suppose that there exists a primary quasi critical point $\hat\zeta_0^{(i+1)}$ of order h_{i+1} on $\gamma^{(i+1)}$ with $|\hat\zeta_0^{(i+1)} - \hat\zeta_0^{(j)}| \le \sum_{k=i+1}^{j}(Kb)^{h_k/3}$. Then the lower bound on the length of $\gamma^{(i+1)}$ implies that $\hat\zeta_0^{(i+1)}$ is located around the middle of $\gamma^{(i+1)}$. This permits us to use Lemma 2.10.2 to yield a precritical point of order h_i on $\gamma^{(i+1)}$, called $\hat\zeta_0^{(h_i,i+1)}$, such that $|\hat\zeta_0^{(i+1)} - \hat\zeta_0^{(h_i,i+1)}| \le (Kb)^{h_{i+1}/2}$. Let $z_0 \in \gamma^{(i)}$ denote the point whose x-coordinate coincides with that of $\hat\zeta_0^{(h_i,i+1)}$. Such z_0 uniquely exists because length$(\gamma^{(i)}) \gg |\hat\zeta_0^{(j)} - \hat\zeta_0^{(h_i,i+1)}|$ holds.

Claim 5.1.2. *We have*

$$|\hat\zeta_0^{(h_i,i+1)} - z_0| \le (Kb)^{h_i/2}$$

and

$$\mathrm{angle}(t_{\gamma^{(i+1)}}(\hat\zeta_0^{(h_i,i+1)}), t_{\gamma^{(i)}}(z_0)) \le (Kb)^{h_i/2}.$$

Proof. Since h_i is a hyperbolic time, we have

$$|z_{-h_i} - \hat\zeta_{-h_i}^{(h_i,i+1)}| \le e|z_0 - \hat\zeta_0^{(h_i,i+1)}| \frac{\|w_{-h_i}\|}{\|w_0\|}.$$

Since $\gamma^{(i+1)}$ and $\gamma^{(i)}$ are admissible curves which are tantent to w_0, we have $|z_0 - \hat\zeta_0^{(h_i,i+1)}| \le |\hat\zeta_0^{(h_i,i+1)} - \hat\zeta_0^{(h_i,i+1)}|$. Using the assumption of the induction,

$$|z_{-h_i} - \hat\zeta_{-h_i}^{(h_i,i+1)}| \le e^{10\Delta h_i}\left((Kb)^{h_{i+1}/2} + \sum_{k=i+1}^{j}(Kb)^{h_k/3}\right) \le (Kb)^{h_i/4}.$$

Thus the long stable leaf $\Gamma^{(h_i)}$ of order h_i through $\hat\zeta_{-h_i}^{(h_i,i+1)}$ is well-defined. In view of the proof of Proposition 5.3.1, the desired inequality follows if $\Gamma^{(h_i)}$ intersects $\gamma(w_{-h_i}, \rho^{h_i})$. This follows from Sublemma 5.3.3 and the fact that $\gamma(w_{-h_i}, \rho^{h_i})$ is a straight segment. ∎

By the above claim and Lemma 2.10.2, there exists a precritical point $\hat{\zeta}_0^{(i)}$ of order h_i on $\gamma^{(i)}$ such that $|\hat{\zeta}_0^{(i)} - z_0| \le (Kb)^{h_i/2}$. Consequently,

$$\begin{aligned}
|\hat{\zeta}_0^{(i)} - \hat{\zeta}_0^{(j)}| &\le |\hat{\zeta}_0^{(i)} - z_0| + |z_0 - \hat{\zeta}_0^{(h_i,i+1)}| + |\hat{\zeta}_0^{(h_i,i+1)} - \hat{\zeta}_0^{(i+1)}| + |\hat{\zeta}_0^{(i+1)} - \hat{\zeta}_0^{(j)}| \\
&\le 2(Kb)^{h_i/2} + (Kb)^{h_{i+1}/2} + \sum_{k=i+1}^{j} (Kb)^{h_k/3} \\
&\le 3(Kb)^{h_i/2} + \sum_{k=i+1}^{j} (Kb)^{h_k/3} \\
&\le \sum_{k=i}^{j} (Kb)^{h_k/3}.
\end{aligned}$$

This restores the assumption of the induction and completes the proof. □

5.2. **Sample points.** Let $n \ge N$. Cut the segment $\mathcal{I} = \{(x, 1/10) : \delta^2 \le |x| \le 2\}$ into $e^{100\Delta n}$ subsegments of equal length. The mid points of these subsegments are called *sample points*. Let $S(n)$ denote the set of all sample points. Clearly we have

$$\operatorname{Card}(S(n)) = e^{100\Delta n}. \tag{25}$$

We say a vector orbit $\mathbf{w} = \{w_i(z_i)\}_{i=-h}^{0}$ is *linked* to a sample point $\tilde{z} \in S(n)$ if $w_{-h}(z_{-h})$ is tangent to $\mathcal{I}$ and $|z_{-h} - \tilde{z}| \le \Xi(\mathbf{w})$ holds.

5.3. **Existence of smooth continuations.** Suppose that ζ_0 is a secondary quasi critical point of H_{a_*} of order $h \in [N, n]$, whose backward orbit is linked to $\tilde{z} \in S(n)$. We say ζ_0 has a *smooth continuation* on an interval J containing a_*, if there exists a C^3 map $\zeta_0(\cdot)\colon J \to \mathbb{R}^2$ such that $\zeta_0(a_*) = \zeta_0$ and $\zeta_0(a)$ is a secondary quasi critical point of order h of H_a which is linked to $\tilde{z}$.

For $a \in \Omega^{(0)}$ and $h > 0$, define

$$\hat{J}(a, h) = [a - e^{-\lambda h/2}, a + e^{-\lambda h/2}] \cap \Omega^{(0)}.$$

The following proposition asserts the existence of smooth continuations.

Proposition 5.3.1. *Let $a_* \in \Omega^{(0)}$, and suppose that $\hat{\zeta}_0$ is a good primary quasi critical point of order $h \in [N, 2n]$ of H_{a_*}. There exists a secondary quasi critical point ζ_0 of order h such that*

$$|\hat{\zeta}_0 - \zeta_0| \le (Kb)^{h/2}. \tag{26}$$

The backward vector orbit of ζ_0 is linked to some $\tilde{z} \in S(n)$. Moreover, ζ_0 has a smooth continuation $a \in \hat{J}(a_*, h) \to \zeta_0^{(j)}(a)$ such that $\|\ddot{\zeta}_0(a)\|, \|\dddot{\zeta}_0(a)\| \le e^{100\Delta h}$ holds for all $a \in \hat{J}(a_*, h)$.

Proof of Proposition 5.3.1. We divide the argument into three parts: proof of the existence of ζ_0; the existence of smooth continuations of ζ_0; the derivative estiamte of smooth continuations.

Existence of ζ_0. Since the backward vector orbit $\{w_i\}_{i=-h}^0$ of $\hat\zeta_0$ is expanding and $\hat\zeta_{-h}\notin\mathcal{C}_\delta$, the long stable leaf $\Gamma^{(h)}$ of order h through $\hat\zeta_{-h}$ is well-defined. Let $z_0=\Gamma^{(h)}\cap\mathcal{I}$, and define $\mathbf{v}=\{v_i(z_i)\}_{i=0}^h$ by $v_i(z_i)=DH^i(z_0)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ for $0\le i\le h$. Take a straight segment $\tilde\gamma\subset\mathcal{I}$ of length ρ^h which is centered at z_0. Then length$(\tilde\gamma)\ge e^{-100\Delta n}$ holds, and thus there exists a sample point $\tilde z\in\tilde\gamma$. Since $\{DH^i(\hat\zeta_{-h})f_h\}_{i=-h}^0$ is expanding, $\mathbf{v}$ is $e^{-12\Delta}$- expanding, by Lemma 2.7.1. Hence length$(\tilde\gamma)\le\Xi(\mathbf{v})$ holds. This and Lemma 2.4.1 give length$(\tilde\gamma_h)\ge\rho^{2h}$.

Claim 5.3.2. *$\tilde\gamma_h$ is an admissible curve.*

Proof. A different argument from that of Lemma 2.6.1 is needed because $\mathbf{v}$ is not regular in general. The statement is not affected even if we assume that w_{-h} is a unit vector, and we do so. Since $\tilde\gamma$ is a straight segment, the curvature is smaller than

$$e^3\sum_{\ell=1}^h(Kb)^\ell\frac{\|v_{h-\ell}\|^3}{\|v_h\|^3}.$$

To bound the sum, we argue as follows.

Claim 5.3.3. angle$(e_h,w_{-h})\ge e^{-12\Delta h}$.

Proof. Put $\psi=\mathrm{angle}(e_h,w_{-h})$. Split $w_{-h}=\|w_{-h}\|(\cos\psi\cdot e_h+\sin\psi\cdot f_h)$. Then

$$e^{-20\Delta h}\le\frac{\|w_0\|^2}{\|w_{-h}\|^2}\le(Kb)^{2h}\cos^2\psi+e^{2\Delta h}\sin^2\psi\le(Kb)^{2h}+e^{2\Delta h}\sin^2\psi.$$

Taking the both sides of the inequality and rearranging gives the inequality. ☐

Split $w_{-h}=\xi e_h+\eta f_h$. By Claim 5.3.3, we have $|\eta|\ge e^{-10\Delta h}$ and thus $\|w_{i-h}\|\approx\|DH^i\eta f_h\|$ for $i\ge h/10$. For $\ell\in[1,9h/10]$, by Lemma 2.8.1 we obtain

$$\frac{\|v_{h-\ell}\|}{\|v_h\|}\le e\frac{\|DH^{h-\ell}\eta f_h\|}{\|DH^h\eta f_h\|}\le e\cdot\frac{\|w_{-\ell}\|}{\|w_0\|}\le K_0^{-1}\delta^{-1}e^4. \tag{27}$$

For $\ell\in[9h/10,h]$ we have

$$\frac{\|v_{h-\ell}\|}{\|v_h\|}=\frac{\|v_{h-\ell}\|}{\|v_0\|}\frac{\|v_0\|}{\|v_h\|}\le e^{\Delta(h-\ell)}e^{12\Delta h}\le e^{13\Delta\ell}. \tag{28}$$

Substituting (27) (28) into the sum we obtain the bound on the curvature. (27) with $\ell=1$ and Lemma 2.5.1 yields that the slopes of tangent directions of $\tilde\gamma_h$ are $\le K_0b$. ☐

In the same spirit as the beginning of the proof of Proposition 2.8.1, we have

$$\mathrm{angle}(v_h,w_0)\le(Kb)^{h-1}\sum_{i=0}^h\frac{\|v_i\|}{\|v_h\|}\frac{\|w_{i-h}\|}{\|w_0\|}.$$

To bound the sum, we use (27) (28) and $\|w_{i-h}\|\le K_0^{-1}e^3\delta^{-1}\|w_0\|$. This yields angle$(v_h,w_0)\le(Kb)^{h/2}$. Take a straight segment γ_0 of length ρ^h which is centered at $\hat\zeta_{-h}$ and tangent to w_{-h}. Then γ_h is an admissible curve of length $\ge\rho^{2h}\ge(Kb)^{h/2}$ by Lemma 2.6.1. Applying Lemma 2.10.2 to the pair of admissible curves $\gamma_h,\tilde\gamma_h$, we conclude the existence of a precritical point ζ_0 of order h on $\tilde\gamma_h$. Since the distortion estimate in Lemma 2.4.1 holds on $\tilde\gamma$, ζ_0 has an $e^{-12\Delta}$-expanding backward orbit of length h, which in addition is linked to $\tilde{z}$, by constuction. Hence ζ_0 is a secondary quasi critical point of order h.

Existence of a smooth continuation of ζ_0.

Claim 5.3.4. *For every $a \in \hat{J}$, $H_a^h \tilde{\gamma}$ is an admissible curve of length $\geq \rho^{3h}$.*

Proof. Define $\mathbf{v}(a) = \{v_i(a)\}_{i=0}^h$ by $v_i(a) = DH_a^i(z_0)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. The chain rule gives $\|\partial_a DH_a^i(z_0)\| \leq he^{\Delta h}$ for $1 \leq i \leq h$, and therefore

$$\|v_i(a_*) - v_i(a)\| \leq he^{\Delta h}|a_* - a| \leq e^{-\lambda\beta h/3}, \tag{29}$$

where the last inequality follows from $a \in \hat{J}$. By Lemma 2.8.1, $\|v_i(a_*)\| \geq e^{-12\Delta i}$ holds. This and (29) yields $\|v_i(a)\| \geq e^{-13\Delta i}$. Hence the claim follows. Let us record

$$|\log\|v_i(a_*)\| - \log\|v_i(a)\|| \leq 1. \tag{30}$$

□

Put $z_i(a) = H_a^i z_0$. Since $\|\dot{z}_i(a)\| \leq ie^{\Delta i}$, we have

$$|z_h(a_*) - z_h(a)| \leq he^{\Delta h}|a_* - a| \leq e^{-\lambda\beta h/3}. \tag{31}$$

Let $\zeta(a) \in \tilde{\gamma}$ denote the point such that the x-coordinate of $H_a^h\zeta(a)$ coincides with that of $z_h(a_*)$. Such $\zeta(a)$ uniquely exists for all $a \in \hat{J}$, because length$(H_a^h\tilde{\gamma}) \gg |z_h(a_*) - z_h(a)|$ holds. Using the fact that $H_a^h\tilde{\gamma}$ is an admissible curve and the "Pythagoras theorem", we have

$$|z_h(a) - H_a^h\zeta(a)| \leq |z_h(a_*) - z_h(a)| \leq e^{-\lambda\beta h/2},$$

and thus angle$(v_h(a), DH_a^h(\zeta(a))\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)) \leq e^{-\lambda\beta h/2}$. Using this and (29) we have

$$\text{angle}(DH_a^h(\zeta(a))\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), v_h(a_*)) \leq e^{-\lambda\beta h/2}.$$

Put $\tilde{\gamma}_h(a) = H_a^h(\tilde{\gamma})$, and parametrize $\tilde{\gamma}_h(a)$ so that $\tilde{\gamma}_h(a)(0) = H_a^h(\zeta(a))$ holds. Then $\gamma_h(a)(s)$ is well-defined on $[-e^{-\lambda\beta h/4}, e^{-\lambda\beta h/4}]$. This and the above two inequalities permits us to apply Lemma 2.10.2 to conclude that there exists $s \in [-e^{-\lambda\beta h/4}, e^{-\lambda\beta h/4}]$ such that $\gamma_h(a)(s)$ is a precritical point of order h of H_{a^*}.

Sublemma 5.3.5. *Let γ be an admissible curve in $\mathcal{C}_\delta$, where $\gamma(0) = \zeta$ is a precritical point of order m of H_{a_*}. Assume that $\varepsilon \geq e^{-\lambda\beta/2}$, and $\gamma(s)$ is defined for $s \in [-\varepsilon^{m/2}, \varepsilon^{m/2}]$. Then for all $a \in [a_* - e^{-\lambda\beta m/2}, a_* + e^{-\lambda\beta m/2}]$ there exists $\hat{s}(a) \in [-\varepsilon^{m/2}, \varepsilon^{m/2}]$ such that $\tilde{\gamma}(\hat{s}(a))$ is a precritical point of order m of H_a.*

By this sublemma, there exists a precritical point of order h of H_a on $H_a^h\gamma$. By construction, it is a secondary quasi critical point of order h which is linked to $\tilde{z} \in S(n)$.

Proof of Sublemma 2.10.1. Let $\mathbf{w} = \{w_i\}_{i=0}^{\beta m}$ denote the forward vector orbit of ζ, and let $\Gamma^{(m-1)}$ denote the long stable leaf of order $m-1$ through z_1. Then $H_a\gamma \subset \Gamma^{(m-1)}(\Xi(\Pi_0^m\mathbf{w}))$ holds, because $|H_a\zeta - H_{a_*}\zeta| \leq e^{\Delta}|a - a_*| \ll \Xi(\Pi_0^m\mathbf{w})$, and diam$(H_a\gamma) \leq$ length$(\gamma) \leq e^{-49\Delta n} \ll \Xi(\Pi_0^m\mathbf{w})$. Hence, for every $1 \leq i \leq m$, the contractive field under the iteration of DH_{a_*}, denoted by $e_i(a_*)$, is well-defined on a neighborhood of $H_a(\gamma)$. Define $\mathbf{w}(a) = \{w_i(a)\}_{i=0}^m$ by $w_i(a) = DH_a^i(H_{a_*}\zeta)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $\mathbf{w}'(a) = \{w_i'(a)\}_{i=0}^m$ by $w_i'(a) = DH_a^i(H_a\zeta)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. The same estimate as in (30) applies and for every $1 \leq i \leq m$ we have $|\log\|w_i(a_*)\| - \|w_i(a)\|| \leq 1$. In particular,

$\mathbf{w}(a)$ is expanding up to time m. On the other hand, by $|H_{a_*}\zeta - H_a\zeta| \ll \Xi(\mathbf{w}(a))$ and (6), for every $1 \le i \le m$ we have $|\log \|w_i(a)\| - \|w_i'(a)\|| \le 1$. Hence $\mathbf{w}'(a)$ is expanding up to time m. By a similar reasoning to before, $e_i(a)$ is well-defined on a neighborhood of $H_a\gamma$ for every $1 \le i \le m$.

The rest of the argument goes similarly to that of Lemma 2.10.1, with parameter dependence in mind. For $z \in \gamma$, split

$$DH_{a_*}t_\gamma(z) = \xi e_n(a_*)(H_a\zeta) + \eta f_n(a_*)(H_a\zeta)$$

and

$$DH_a t_\gamma(z) = \tilde{\xi} e_n(a)(H_a z) + \tilde{\eta} f_n(a)(H_a z).$$

By Lemma 3.3.3 we have $\eta(z) = |\zeta - z|$ and $|\xi(z)| \le K_1\delta^{-1}b$. Put

$$\psi = \mathrm{angle}(e_m(a_*)(H_{a_*}\zeta), e_m(a)(H_a z)).$$

Comparing the coefficients of the both sides of the identity $DH_a t_\gamma(z) = DH_{a_*}t_\gamma(z) + (DH_a - DH_{a_*})t_\gamma(z)$, we have

$$\tilde{\eta}(z) = \eta(z)\cos\psi \pm \xi(z)\sin\psi + R,$$

where $|R| \le \|DH_{a_*} - DH_a\| \le e^{-\lambda\beta m/2}$. By Lemma 2.8.1,

$$\begin{aligned}\psi &\le \mathrm{angle}(e_m(a_*)(H_{a_*}\zeta), e_m(a_*)(H_a z)) + \mathrm{angle}(e_m(a_*)(H_a z), e_m(a)(H_a z))\\ &\le K|H_{a_*}\zeta - H_a z| + K|a_* - a|\\ &\le Ke^{\Delta}|\zeta - z| + K|a_* - a| \ll 1.\end{aligned}$$

Suppose that z is one of the two endpoints of γ. Then $\psi \le Ke^{\Delta}|\zeta - z|$ holds. Without loss of generality we may assume $\eta(z) > 0$. Then

$$\tilde{\eta}(z) \ge |\zeta - z|(1 - 2K\delta^{-1}b) - |R| > 0.$$

In the same way we have $\tilde{\eta}(z') < 0$, where z' is the other endpoint of γ. Hence there exists $\hat{s}(a) \in [-\varepsilon^{m/2}, \varepsilon^{m/2}]$ such that $\tilde{\eta}(\tilde{\gamma}(\hat{s}(a))) = 0$. In other words, $H_a\tilde{\gamma}(\hat{s}(a))$ is a critical point of H_a of order m. □

Derivative estimates. We consider an implicit representation of $\zeta_0(a)$. Parametrize $\tilde{\gamma}$ by arc length and let $s(a)$ be the one such that $\zeta_0(a) = H_a^h(\tilde{\gamma}(s(a)))$. We estimate the derivatives of $s(a)$. For $(s,a) \in \tilde{\gamma} \times \hat{J}$, define

$$v(s,a) = \frac{DH_a^{h+1}(\tilde{\gamma}(s))\binom{1}{0}}{\|DH_a^{h+1}(\tilde{\gamma}(s))\binom{1}{0}\|} \quad \text{and} \quad w(s,a) = e_h(a)(H_a^{h+1}(\tilde{\gamma}(s))).$$

Notice that $v(s(a),a) - w(s(a),a) \equiv 0$. Let κ denote the curvature of $H_a^{h+1}\tilde{\gamma}$ at $\zeta_{h+1}(a)$. It is easy to see that $\kappa = \mathcal{O}(b^{-2})$. Let $\{w_i(a)\}_{i=-h}^{0}$ denote the backward vector orbit of $\zeta_0(a)$. Using (2), for small variance ds we have $\|v(s+ds,a) - v(s,a)\| \ge Kb\kappa ds\|w_{-h}(a)\|^{-1}$. Taking limit $ds \to 0$ we have $\|\partial_s v(s,a)\| \ge Kb^{-1}\|w_{-h}(a)\|^{-1}$. On the other hand, by Lemma 2.7.4 we have $\|\partial_s w\| \le K\|w_{-h}(a)\|^{-1}$. Hence we obtain

$$\|\partial_s v(s,a) - \partial_s w(s,a)\| \ge K\|w_{-h}(a)\|^{-1} \ge Ke^{-15\Delta h}.$$

In particular, one of the component of the difference is $\ge Ke^{-20\Delta h}$. By the implicit function theorem we obtain

$$|\dot{s}(a)|, |\ddot{s}(a)|, |\dddot{s}(a)| \le Ke^{70\Delta h}. \tag{32}$$

Put $A_i(a)=H_a^i(\tilde\gamma(s(a)))$. Then $A_h(a)=\zeta_0(a)$ holds. Since $A_i(a)=\mathcal{H}(a,A_{i-1}(a))$, we have $\dot A_i=\partial_a\mathcal{H}(a,A_{i-1})+DH_a(A_{i-1})\dot A_{i-1}$. Using this for ℓ-times ($\ell\le i$),

$$\dot A_i=DH_a^\ell(A_{i-\ell})\dot A_{i-\ell}+\sum_{s=0}^{\ell-1}DH_a^s(A_{i-s})\partial_a\mathcal{H}(a,A_{i-s-1}). \tag{33}$$

Substituting $\ell=i$, and then $i=h$, and using (32) we obtain

$$\|\dot A_h\|\le he^{\Delta h}+e^{\Delta h}\|\dot s(a)\|\le e^{100\Delta h}. \tag{34}$$

To estimate $\|\ddot A_h\|$, we differentiate (33) use the second order derivative estimate in (32). The estimate of $\|\dddot A_h\|$ is analogous. The details are left as an elementary excersise of computation. This completes the proof of Proposition 5.3.1. □

5.4. **Derivative estimates of smooth continuations.** The derivative estimates of smooth continuations in Proposition 5.3.1 are too coarse to be adapted to our argument. To rectify this, we derive much finer derivative estimates.

We make clear a link between hyperbolic times and sample points. Fix $n\ge N$, and suppose that ζ_0 is a critical point of order $h_s\ge n$, with $\{h_j\}_{j=1}^s$ the corresponding sequence of hyperbolic times. Let j_0 denote the miminum integer such that $n\le h_{j_0}$. It can be read out from the proof of Proposition 5.3.1 that for every $1\le j\le j_0$, there exists $z^{(j)}\in S(n)$ such that $z^{(j)}\in\gamma\left(t_{\mathcal{I}}(\Gamma^{(h_j)}\cap\mathcal{I}),\rho^{h_j}\right)$ holds, where $\Gamma^{(h_j)}$ is the long stable leaf of order h_j through ζ_{-h_j} and $\mathcal{I}$ is the one appearing in the definition of sample points (Sect. 5.2). We say $z^{(j)}$ is a sample point corresponding to the hyperbolic time h_j. Multiple sample points may correspond to one hyperbolic time and it does not matter.

Proposition 5.4.1. *Suppose that ζ_0 is a critical point of H_{a_*} of order $\xi(=h_s)\ge n$. Let $\{h_j\}_{j=1}^s$ and denote the sequence of hyperbolic times associated with the backward orbit of ζ_0. Let j_0 denote the miminum integer such that $n\le h_{j_0}$. Let $\{z^{(j)}\}_{j=1}^{j_0}$ denote the sequence of corresponding sample points in $S(n)$. For every $1\le j\le j_0$, there exists a secondary quasi critical point $\zeta_0^{(j)}$ which is linked to $z^{(j)}$, and has a smooth continuation $a\in\hat J(a_*,h_j)\to\zeta_0^{(j)}(a)$ such that*

$$\|\dot\zeta_0^{(j)}(a)\|,\|\ddot\zeta_0^{(j)}(a)\|\le\delta. \tag{35}$$

Moreover, if the forward vector orbit of ζ_0 is strongly regular up to time $m\in[M,\beta\xi]$, then for every $1\le j\le j_0$ and $1\le i\le\min\{m,\beta h_j\}$,

$$\left|\log\frac{\|DH^i(\zeta_1)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\|}{\|DH^i(H(\zeta_0^{(j)}))\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\|}\right|\le 1. \tag{36}$$

Proof. By Lemma 5.1.1, there exists a primary quasi critical point $\hat\zeta_0^{(j)}$ of order h_j such that $|\zeta_0-\hat\zeta_0^{(j)}|\le\sum_{k=j}^s(Kb)^{h_k/3}$. For every $1\le j\le j_0$, applying Proposition 5.3.1 to $\hat\zeta_0^{(j)}$, we obtain a secondary quasi critical point $\zeta_0^{(j)}$ of order h_j which has a smooth continuation $\zeta_0^{(j)}(a)$ on $\hat J(a_*,\zeta_0,\beta h_j/2)$. By construction, it is linked to $z^{(j)}$. By (24) and (26) we have $|\zeta_0-\zeta_0^{(j)}|\le|\zeta_0-\hat\zeta_0^{(j)}|+|\hat\zeta_0^{(j)}-\zeta_0^{(j)}|\le(Kb)^{h_j/4}$. Hence (36) follows.

We now estimate $\|\dot{\zeta}_0^{(j)}(a)\|$. A basic idea is to apply the Hadamard lemma to $\zeta_0^{(i+1)}(a)-\zeta_0^{(i)}(a)$ for $1\leq i\leq j-1$, together with the coarse second order derivative estimate in Proposition 5.3.1.

Lemma 5.4.2. (Hadamard) *Let $g\in C^2[0,L]$ be such that $|g|\leq M_0$ and $|g''|<M_2$. If $4M_0<L^2$ then $|g'|\leq\sqrt{M_0}(1+M_2)$.*

Unfortunately, the construction of smooth continuations in itself does not imply any correlation between $\zeta_0^{(i+1)}(a)$ and $\zeta_0^{(i)}(a)$. Therefore, in order to bound the norm of the difference $\zeta_0^{(i+1)}(a)-\zeta_0^{(i)}(a)$, we consider another expression of smooth continuations. Let us explain how to do this. We begin by constructing for all $a\in\hat{J}(a_*,h_j)$ a primary quasi critical point $\hat{\zeta}_{0,a}^{(j)}$ of order h_j of H_a which smoothly (C^3 sense) depends on a and whose backward orbits share the same combinatorial type. Then, applying Lemma 5.1.1 to $\hat{\zeta}_{0,a}^{(j)}$, we obtain for every $1\leq i\leq j$ a primary quasi critical point $\hat{\zeta}_{0,a}^{(i)}$ of order h_i of H_a. By Proposition 5.3.1, we obtain an associated secondary quasi critical point $\zeta_{0,a}^{(i)}$ of order h_i. By construction it follows that $\zeta_{0,a}^{(i)}$ is linked to $z^{(i)}$. It turns out that $\zeta_{0,a}^{(i)}=\zeta_0^{(i)}(a)$ holds. Hence, it is enough to consider $|\zeta_{0,a}^{(i+1)}-\zeta_{0,a}^{(i)}|$. This can be bounded by (24) and (26).

Let $\hat{\zeta}_{0,a_*}^{(j)}$ denote the primary quasi critical point of order h_j which is constructed from ζ_0 by Lemma 5.1.1. Let $\{w_i(a_*)\}_{i=-h_j}^0$ denote its backward vector orbit. It can be read out from the proof of Proposition 5.3.1 that $H_a^{h_j}\gamma(w_{-h_j}(a_*),\rho^{h_j})$ is an admissible curve for all $a\in\hat{J}(a_*,h_j)$. Comparing the two admissible curves $H_{a_*}^{h_j}\gamma(w_{-h_j}(a_*),\rho^{h_j})$ and $H_a^{h_j}\gamma(w_{-h_j}(a_*),\rho^{h_j})$ as in the proof of Proposition 5.3.1 and using Sublemma 5.3.5, we can construct a primary quasi critical point $\hat{\zeta}_{0,a}^{(j)}$ of order h_j of H_a on $H_a^{h_j}\gamma(w_{-h_j}(a_*),\rho^{h_j})$. By construction, the backward vector orbit of $\hat{\zeta}_{0,a}^{(j)}$ satisfies the following for all $a\in\hat{J}(a_*,h_j)$:

(i) $e^{-11\Delta}$-expanding and e^{-3}-regular (slightly better than the mere primary quasi critical case);

(ii) the associated sequence of hyperbolic times is $\{h_i\}_{i=1}^j$

(iii) the associated sequence of sample points in $S(n)$ is $\{z^{(i)}\}_{i=1}^j$.

(i) allows us to apply Lemma 5.1.1 to $\hat{\zeta}_{0,a}^{(j)}$ to yield a primary quasi critical point $\hat{\zeta}_{0,a}^{(i)}$ of order h_i for every $1\leq i\leq j$. By Proposition 5.3.1 and (iii), to each $\hat{\zeta}_{0,a}^{(i)}$ there exists an associated secondary quasi critical point $\zeta_{0,a}^{(i)}$ which is linked $z^{(i)}$. On the other hand, there exists a smooth continuation $a\in\hat{J}(a_*,h_i)\to\zeta_0^{(i)}(a)$. Since $\hat{J}(a_*,h_i)\supset\hat{J}(a_*,h_j)$, $\zeta_0^{(i)}(a)$ is well-defined. In fact, the construction of $\zeta_{0,a}^{(i)}$, $\zeta_0^{(i)}(\cdot)$, (ii) (iii), and Remark 2.9.2 together imply $\zeta_{0,a}^{(i)}=\zeta_0^{(i)}(a)$. Using this, (24) and (26),

$$\begin{aligned}\|\zeta_0^{(i+1)}(a)-\zeta_0^{(i)}(a)\|&\leq\|\zeta_{0,a}^{(i+1)}-\hat{\zeta}_{0,a}^{(i+1)}\|+\|\hat{\zeta}_{0,a}^{(i+1)}-\hat{\zeta}_{0,a}^{(i)}\|+\|\hat{\zeta}_{0,a}^{(i)}-\zeta_{0,a}^{(i)}\|\\&\leq 4(Kb)^{h_i}.\end{aligned}$$

The second order derivative estimate in Proposition 5.3.1 permits us to apply Lemma 5.4.2 to yield $\|\dot{\zeta}_0^{(i+1)}(a)-\dot{\zeta}_0^{(i)}(a)\|\leq(Kb)^{h_i}$. Meanwhile we clearly have $\|\dot{\zeta}_0^{(1)}(a)\|\leq\delta$, because b is chosen to be small after δ. Consequently,

$$\|\dot{\zeta}_0^{(j)}(a)\| \leq \|\dot{\zeta}_0^{(1)}(a)\| + \sum_{i=1}^{j-1} \|\dot{\zeta}_0^{(i+1)}(a) - \dot{\zeta}_0^{(i)}(a)\| \leq Kb + \delta/2 \leq \delta.$$

The second order derivative estimate is done in the same way. We use Lemma 5.4.2 with respect to $\dot{\zeta}_0^{(i+1)}(a) - \dot{\zeta}_0^{(i)}(a)$ together with the third order derivative estimate in Proposition 5.3.1. This completes the proof of Proposition 5.4.1. □

6. Inductive assumption

In this section we introduce reluctant recurrence condition $(RR)_n$. It is shown to be stronger than $(EG)_{n+1}$.

6.1. **Essential returns.** Suppose that H satisfies $(EG)_n$ for some $n \geq N$. Let $\mathbf{w} = \{w_i\}_{i=0}^m$ be a controlled vector orbit of H. Let $0 < m_1 < m_2 < \cdots < m_t \leq m$ denote the set of all free returns up to time m. Denote by p_i and q_i the folding and binding periods associated with the free return m_i. Suppose that $m_i < m_j$. We say m_j is *subject to* m_i if

$$\sum_{i+1 \leq k \leq j} \log \frac{\|w_{m_k+p_k}\|}{\|w_{m_k}\|} \geq 10 \cdot \log \frac{\|w_{m_i+p_i}\|}{\|w_{m_i}\|}. \tag{37}$$

A free return m_i is called *essential* if $i = 1$, or else it is not subject to any previous free return. We say $\mathbf{w}$ is *reluctantly recurrent* up to time m if

$$\sum_{m_i \leq j:\,\text{essential}} \log \frac{\|w_{m_i+p_i}\|}{\|w_{m_i}\|} \geq -\frac{\alpha j}{100} \tag{38}$$

holds for every $0 \leq j \leq m$, where the sum runs over all essential returns which take place before j.

6.2. **Reluctant recurrence condition.** Suppose that H satisfies $(EG)_n$ for some $n \geq N$. We say H satisfies $(RR)_n$ if the forward orbit of every critical point is controlled and reluctantly recurrent up to time $\min(\beta(n+1), \beta\xi) - 1$, where ξ is the order of the critical point. To simplify formalism, we say $H_{a,b}$ satisfies $(RR)_{N-1}$ if $a \in \Omega^{(0)}$.

Remark 6.2.1. The condition $(RR)_n$ is a condition on all critical points.

Remark 6.2.2. An inductive nature lurks behind the definition of $(RR)_n$, concerning the relation between the order of binding points and that of controlled critical points. No contradiction arises at this point because of the following two facts: forward orbits of critical points of order N are obviously controlled; to control forward orbits of critical points at most up to time $\beta(n+1)$, only those critical points of order $\leq \alpha(n+1)$ are used. This follows from (38).

Proposition 6.2.3. *Suppose that H satisfies $(EG)_n$, and ζ_0 is a critical point of order m. If the forward orbit of ζ_0 is reluctantly recurrent up to time $k \leq \beta m - 1$, then it is strongly regular up to time $k+1$. In particular, if H satisfies $(RR)_n$ then $(EG)_{n+1}$ holds.*

Proof. We firstly prove $\|w_j\| \geq e^{(\lambda-\alpha)(j-i)-\alpha\sigma i}\|w_i\|$ for all $0 \leq i \leq j \leq k+1$. Then we define a function $\chi(\cdot)$.

Case I: no free return takes place in (i, j), and i is free. It is easy to see that the inequality holds if $K_0 e^{\alpha j}\delta \leq 1$, because ζ_0 is a critical point and thus no return takes place up to time j. If $K_0 e^{\alpha j}\delta \geq 1$, Lemma 2.5.1 and $\sigma \geq 1$ gives

$$\|w_j\| \geq K_0\delta e^{\lambda(j-i)}\|w_i\| = K_0\delta e^{(\lambda-\alpha)(j-i)}e^{-\alpha i}e^{\alpha j}\|w_i\| \geq e^{(\lambda-\alpha)(j-i)-\alpha\sigma i}\|w_i\|.$$

Case II: some free returns take place in (i, j) and both i, j are free. Let $i < m_{i_0} < m_{i_0+1} \cdots < m_{j_0} < j$ denote all such free returns. Then

$$\frac{\|w_j\|}{\|w_i\|} = \frac{\|w_j\|}{\|w_{m_{j_0}+q_{j_0}+1}\|} \cdot \prod_{i=i_0}^{j_0-1} \frac{\|w_{m_{i+1}}\|}{\|w_{m_i+q_i+1}\|} \cdot \prod_{i=i_0}^{j_0} \frac{\|w_{m_i+q_i+1}\|}{\|w_{m_i}\|} \cdot \frac{\|w_{m_{i_0}}\|}{\|w_i\|}.$$

Using $\|w_{m_i+q_i+1}\| \geq \|w_{m_i}\|$ for every $i_0 \leq i \leq j_0$ and Lemma 2.5.1 with respect to the first and last fractions, we have

$$\frac{\|w_j\|}{\|w_i\|} \geq K_0^{j_0-i_0+1}\delta \exp\left[\lambda\left(j - i - \sum_{i=i_0}^{j_0} q_i\right)\right].$$

Since ζ_0 is a critical point and some return takes place before j, we have $K_0\delta e^{\alpha j/10} \geq 1$. Thus

$$\frac{\|w_j\|}{\|w_i\|} \geq K_0^{j_0-i_0} \exp\left[\lambda\left(j - i - \sum_{i=i_0}^{j_0} q_i\right) - \alpha j/10\right].$$

To bound the sum of the binding periods we argue as follows. Using (12),

$$\sum_{i=i_0}^{j_0} q_i \leq -\frac{3}{\lambda(1-\tilde{\alpha})} \sum_{i=i_0}^{j_0} \log \frac{\|w_{m_i+p_i}\|}{\|w_{m_i}\|}.$$

Since each m_i is an essential return, or else is subject to some previous essential return, we have

$$\sum_{i=i_0}^{j_0} q_i \leq -\frac{33}{\lambda(1-\tilde{\alpha})} \sum_{\substack{m_i<j \\ \text{essential}}} \log \frac{\|v_{m_i+p_i}\|}{\|v_{m_i}\|} \leq \frac{\alpha j}{10},$$

where the last inequality follows from (38). To bound $K_0^{j_0-i_0}$, we use the next elementary sublemma and obtain $j_0 - i_0 \leq \frac{\Delta(j-i)}{-\log\delta}$. A proof of the sublemma is left as an exercise. Consider a perturbation from $H_{2,0}$.

Sublemma 6.2.4. $\max\{i \in \mathbb{N} \colon H^i(\mathcal{C}_\delta) \cap \mathcal{C}_\delta = \emptyset\} \geq -\Delta^{-1}\log\delta$.

Substituting these two inequalities into the above one we have

$$\|w_j\| \geq e^{(\lambda-\alpha(\lambda+1)/10)j-\lambda i} \geq e^{(\lambda-\alpha(\lambda+1)/20)(j-i)}\|w_i\| \geq e^{(\lambda-\alpha)(j-i)-\alpha\sigma i}\|w_i\|. \quad (39)$$

Case III: some free returns take place in (i,j), i is free, j is bound. Let m_{j_0} denote the free return such that $m_{j_0} < j \le m_{j_0} + q_{j_0} + 1$. Then

$$\frac{\|w_j\|}{\|w_i\|} = \frac{\|w_j\|}{\|w_{m_{j_0}+q_{j_0}+1}\|}\frac{\|w_{m_{j_0}+q_{j_0}+1}\|}{\|w_i\|}.$$

Regarding the first term, we have

$$\|w_j\| \ge e^{-\Delta(m_{j_0}+q_{j_0}+1-j)}\|w_{m_{j_0}+q_{j_0}+1}\| \ge e^{-\Delta q_{j_0}}\|w_{m_{j_0}+q_{j_0}+1}\| \ge e^{-\Delta\alpha j/10}\|w_{m_{j_0}+q_{j_0}+1}\|.$$

Using this and applying (39) to the second term, we obtain

$$\|w_j\| \ge e^{(\lambda-\alpha(\lambda+1)/20)(j-i)-\alpha\Delta j/10}\|w_i\| \ge e^{(\lambda-\alpha)(j-i)-\alpha\sigma i}\|w_i\|.$$

Case IV: some free returns take place in (i,j), i is bound, j is free. Let m_{i_0} denote the free return such that $m_{i_0} < i \le m_{i_0} + q_{i_0} + 1$. Suppose that $i \le m_{i_0} + p_{i_0}$. By (10) we have

$$\frac{\|w_j\|}{\|w_i\|} = \frac{\|w_j\|}{\|w_{m_{i_0}}\|}\frac{\|w_{m_{i_0}}\|}{\|w_i\|} \ge \frac{\|w_j\|}{\|w_{m_{i_0}}\|}. \tag{40}$$

Since m_{i_0} and j are free, (39) applies to the right hand side. Since $m_{i_0} < i$, we obtain the desired inequality. Suppose that $i > m_{i_0} + p_{i_0}$. (19) implies

$$\|w_i\| \le (1+\theta)L|\tilde{\zeta}_0 - \zeta_{m_{i_0}+1}|e^{\Delta(i-m_{i_0})}\|w_{m_{i_0}}\|,$$

where $\tilde{\zeta}_0$ is a critical point relative to which $w_{m_{i_0}}$ is in admissible position. Since $i - m_{i_0} \le q_{i_0} \le \alpha m_{i_0}/10$ and $|\tilde{\zeta}_0 - \zeta_{m_{i_0}+1}| \le \delta$ we have $\|w_i\| \le \sqrt{\delta}e^{\Delta\alpha m_{i_0}}\|w_{m_{i_0}}\|$. Using this and (39),

$$\frac{\|w_j\|}{\|w_i\|} = \frac{\|w_j\|}{\|w_{m_{i_0}}\|}\frac{\|w_{m_{i_0}}\|}{\|w_i\|} \ge e^{(\lambda-\alpha(\lambda+1)/20)(j-m_{i_0})}e^{-\Delta\alpha m_{i_0}/10} \ge e^{(\lambda-\alpha)(j-i)-\alpha\sigma i}.$$

6.2.5. *Case V: both i and j are bound.* Suppose that i and j are bound to different free returns. In this case, there exists a free return m_{i_0} such that $i < m_{i_0} < j$. Using the estimates in III and IV we have

$$\frac{\|w_j\|}{\|w_i\|} = \frac{\|w_j\|}{\|w_{m_{i_0}}\|}\frac{\|w_{m_{i_0}}\|}{\|w_i\|} \ge e^{(\lambda-\alpha(\lambda+1)/20)(j-i)-\alpha\Delta(j+m_{i_0})/10} \ge e^{(\lambda-\alpha)(j-i)-\alpha\sigma i}.$$

Suppose that i and j are bound to the same free return m_{i_0}. Let $\tilde{\zeta}_0$ denote the critical point of order k relative to which $w_{m_{i_0}}$ is in admissible position. Let $\tilde{\mathbf{w}} = \{\tilde{w}_i\}_{i=0}^{\beta k}$ denote the forward vector orbit of $\tilde{\zeta}_0$. By $(EG)_n$, $\tilde{\mathbf{w}}$ is strongly regular. Three cases need to be considered separately:

(*i*) $m_{i_0} + p_{i_0} \le i < j$. Using (20) we have

$$\frac{\|w_j\|}{\|w_i\|} \ge e^{-2}\frac{\|\tilde{w}_{j-m_{i_0}-1}\|}{\|\tilde{w}_{i-m_{i_0}-1}\|} \ge e^{-2}e^{(\lambda-\alpha)(j-i)-\alpha\sigma(i-m_{i_0}-1)} \ge e^{(\lambda-\alpha)(j-i)-\alpha\sigma i}.$$

(ii) $m_{i_0} \le i \le m_{i_0} + p_{i_0} \le j$. Using (20) we have

$$|\tilde{\zeta}_0 - \zeta_{m_{i_0}+1}|^{-1} \frac{\|w_j\|}{\|w_{m_{i_0}}\|} \ge e^{-2} \frac{\|\tilde{w}_{j-m_{i_0}-1}\|}{\|\tilde{w}_0\|}.$$

Rearranging this and using $|\tilde{\zeta}_0 - \zeta_{m_{i_0}+1}| \ge e^{-\alpha m_{i_0}/10}$ which follows from (12) and $(RR)_n$, we have

$$\|w_j\| \ge e^{-2-2\alpha m_0} e^{(\lambda-\alpha)(j-m_0-1)} \|w_{m_{i_0}}\| \ge e^{(\lambda-\alpha)(j-i)-\alpha\sigma i/2} \|w_{m_{i_0}}\|.$$

This and $\|w_i\| \le \|v_{m_{i_0}}\|$ yield the desired inequality.

(iii) $m_{i_0} \le i < j \le m_{i_0} + p_{i_0}$. Using the estimate in *(ii)* and $p_0 \ll \alpha m_0$ we have

$$\|w_{m_{i_0}+p_{i_0}}\| \ge e^{(\lambda-\alpha)(j-i)-\alpha\sigma i/2} \|w_i\|.$$

On the other hand, the definition of the folding period gives

$$\|w_{m_{i_0}+p_{i_0}}\| \le e^{\Delta(m_{i_0}+p_{i_0}-j)} \|w_j\| \le e^{\Delta p_{i_0}} \|w_j\| \le e^{\alpha m_{i_0}} \le e^{\alpha i} \|w_j\|.$$

Combining these two inequalities we obtain the desired one.

It is left to define a function $\chi(\cdot)$. For convenience we introduce the following terminology. We say $j \in [0, m+1]$ is *isolated* if (1) it is free, and (2) there is no return before j, or else $j \ge j' + q - \lambda^{-1} e \log(K_0\delta)$ holds for the last free return j' before j with the binding period q. Define $\chi(j)$ to be the largest integer in $[0, j]$ which is isolated.

Let us see $\chi(\cdot)$ indeed satisfies the desired properties. They are clearly satisfied when there is no return before j, by Lemma 2.5.1 and $\chi(j) = j$ in this case. Suppose that that j' is the last free return before $\chi(j)$. Since there is no return in between $j'+q$ and $\chi(j)$, and by Lemma 2.5.1, we have $\|w_{\chi(j)}\| \ge K_0\delta \|w_i\|$ for every $j'+q+1 \le i \le \chi(j)$. On the other hand, by Proposition 3.3.2 we have $\|w_{j'+q+1}\| \ge e^{-1} K_0 \delta \|w_i\|$ for every $0 \le i \le j' + q + 1$, and therefore

$$\frac{\|w_{\chi(j)}\|}{\|w_i\|} = \frac{\|w_{\chi(j)}\|}{\|w_{j'+q+1}\|} \frac{\|w_{j'+q+1}\|}{\|w_i\|} \ge K_0 \delta e^{\lambda(\chi(j)-j'-q)} \cdot e^{-1} K_0 \delta \ge K_0 \delta.$$

It is left to prove $\chi(j) \in [(1-\alpha\sigma)j, j]$. If j is isolated then it is done because $\chi(j) = j$ by definition. Suppose the contrary, and let $\psi(j)$ denote the last free return which takes place before j. We derive a contradiction assuming that there exists $k \ge 1$ such that $\psi(j), \cdots, \psi^k(j) = \psi \circ \cdots \circ \psi(j)$ (k-composite) are not isolated and $\psi^k(j) \le (1-\alpha\sigma)j$. By the definition of isolated iterates, two consecutive free returns in $[(1-\alpha\sigma)j, j]$ are close to each other. More precisely, one free return takes place right after $-\lambda^{-1}\log(K_0\delta)$ iterates of the end of the binding period of another at the latest. Meanwhile, any binding period is $\ge -\frac{3}{\Delta(2-2\ell)} \log \delta$, by Lemma 3.3.2. This implies that the proportion of total bound iterates in $[j - \alpha\sigma, j]$ is bigger than certain uniform constant which only depends on Δ and λ. On the other hand, the total number of bound iterates in $[(1-\alpha\sigma)j, j]$ is clearly smaller than the sum of the binding periods of free returns which take place before j, which is $\le \alpha j$ as was already proved. These two estimates yield a contradiction. This completes the proof of Proposition 6.2.3. □

7. Dynamics of critical curves

The aim of this section is to study the growth of curves of secondary quasi critical points under the assumption $(RR)_{n-1}$.

7.1. **Distortion with respect to smooth continuations.**

Let RR_{n-1} denote the set of $a \in \Omega^{(0)}$ such that $H_{a,b}$ satisfies $(RR)_{n-1}$. Let $a_* \in RR_{n-1}$, and suppose that ζ_0 is a critical point of order $\xi > n$ of H_{a_*}. Let $m \le \beta(n+1)-1$ denote the largest integer up to which the forward vector orbit $\mathbf{w} = \{w_i(\zeta_{i+1})\}_{i=1}^{\beta\xi}$ of ζ_0 is reluctantly recurrrent. Recall that $(RR)_{n-1}$ implies $m \ge \beta n - 1$. Define

$$\Phi(\Pi_0^\nu \mathbf{w}) = e^{-10\Delta} \cdot \left[\sum_{\substack{0 \le i \le \min(\nu-1,m) \\ \text{free}}} \Theta(\Pi_0^\nu \mathbf{w}, i)^{-1} \right]^{-1} .$$

Let $\{h_i\}_{i=1}^s$ denote the sequence of hyperbolic times associated with the backward orbit of ζ_0. Put $\alpha_0 = \frac{\alpha\lambda\sigma}{200\Delta}$, and define

$$J(a_*, \zeta_0, \nu, d) = [a_* - e^{-\alpha_0 d/2}\Phi(\Pi_0^\nu \mathbf{w}), a_* - e^{-\alpha_0 d/2}\Phi(\Pi_0^\nu \mathbf{w})] \cap \Omega^{(0)}.$$

Lemma 7.1.1. *For all $h > 0$ such that $\beta h/2 \le \nu \le \beta h$, we have*

$$J(a_*, \zeta_0, \nu, 0) \subset \hat{J}(a_*, h).$$

Proof. Let us recall from the proof of Proposition 6.2.3 that $\chi(\cdot)$ is a free iterate. Thus $\Phi(\Pi_0^\nu \mathbf{w}) \le \Theta(\Pi_0^\nu \mathbf{w}, \chi(\nu-1))$ holds. By the strong regularity of $\mathbf{w}$ and the assumption on ν, we have

$$\Theta(\Pi_0^\nu \mathbf{w}, \chi(\nu-1)) \le \frac{\|w_0\|}{\|w_{\chi(\nu-1)}\|} \le e^{-\lambda\beta h/2}.$$

This implies the inclusion. □

Proposition 7.1.2. *Let $a_* \in RR_{n-1}$, and suppose that ζ_0 is a critical point of H_{a_*} of order $\xi > n$, with $\{h_j\}_{j=1}^s$ the associated sequence of hyperbolic times. Let j_0 be the mimimum integer such that $n \le h_{j_0}$. For every $1 \le j \le j_0$, $a \in \hat{J}(a_*, h_j)$, and $i \ge 0$, define $\zeta_i^{(j)}(a) = H_a^i(\zeta_0^{(j)}(a))$ and $w_i(a) = DH_a^i(\zeta_1^{(j)}(a))\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, where $\zeta_0^{(j)}(a)$ is the smooth continuation of order h_j in Proposition 5.4.1. For every $\nu \in [\beta h_j/2, \beta h_j]$ and every free iterate $1 \le i \le \min\{\nu, m+1\}$, $\mathcal{J}_i := \{\zeta_{i+1}^{(j)}(a) \colon a \in J(a_*, \zeta_0, \nu, 0)\}$ is an admissible curve. Moreover, for all $a \in J(a_*, \zeta_0, \nu, 0)$,*

$$\left| \log \frac{\|\dot{\zeta}_{i+1}^{(j)}(a_*)\|}{\|\dot{\zeta}_{i+1}^{(j)}(a)\|} \right| \le \frac{1}{2} + \sum_{\substack{0 \le k \le i-1 \\ \text{free}}} \left[2\Phi(\Pi_0^\nu \mathbf{w})\Theta(\Pi_0^\nu \mathbf{w}, k)^{-1} + \left(\frac{\|w_0\|}{\|w_k\|} \right)^{\frac{1}{2}} \right]; \tag{41}$$

$$\left| \log \frac{\|w_i(a_*)\|}{\|w_i(a)\|} \right| \le \frac{1}{2} + \sum_{\substack{0 \le k \le i-1 \\ \text{free}}} \left[2\Phi(\Pi_0^\nu \mathbf{w})\Theta(\Pi_0^\nu \mathbf{w}, k)^{-1} + \left(\frac{\|w_0\|}{\|w_k\|} \right)^{\frac{1}{2}} \right]; \tag{42}$$

$$\|\ddot{\zeta}_{i+1-k}^{(j)}(a)\| \le \|\dot{\zeta}_{i+1}^{(j)}(a)\|^3 \quad 0 \le \forall k \le i+1. \tag{43}$$

Remark 7.1.3. The number in the right hand side of (41) (42) is ≤ 1.

Remark 7.1.4. It is worth to call attention to subtleties behind the proof of the proposition. In the first place, it involves a double induction with respect to n and i. When considering the case for general n, it is necessary that binding structures for $\mathbf{w}$ are available uniformly on $J(a_*, \zeta_0, \nu, 0)$. To be more precise, let $k < n$ denote the order of a binding point $\tilde{\zeta}_0$ at a free return $i \in [0, m]$ of $\mathbf{w}$. We need that the secondary quasi critical point of order k associated with $\tilde{\zeta}_0$ has a smooth continuation on $J(a_*, \zeta_0, \nu, 0)$ whose forward orbits obey a uniform distortion estimate in the form of (42). This follows if $\Phi(\mathbf{w}) \leq \Phi(\tilde{\mathbf{w}})$, where $\tilde{\mathbf{w}}$ is the forward orbit of $\tilde{\zeta}_0$. Let us see this. The condition $(RR)_k$ implies $\chi(\beta k) \leq \alpha i$, and hence $\beta k \leq \alpha i \leq \alpha n \ll n$, and in particular

$$\Phi(\mathbf{w}) \leq \frac{\|w_0\|}{\|w_n\|} \leq e^{-2\alpha\sigma\beta k - \Delta\beta k} \leq \frac{1}{\beta k} \min_{1 \leq i \leq \beta k} \Theta(\tilde{\mathbf{w}}, i) \leq \Phi(\tilde{\mathbf{w}}).$$

7.2. **Wang-Young's inequality.** Before entering the proof of the proposition, we prove a very useful inequality which will be used later. It is an adaptation of [[WY01] Lemma 6.2] to our context.

Lemma 7.2.1. *Suppose that H satisfies $(RR)_{n-1}$, and that $\{w_j(z_s)\}_{j=0}^{i}$ is reluctantly recurrent up to time $i-1$. Then for every $0 \leq s \leq i$,*

$$\|DH^{i-s}(z_0)\| \leq Ke^{-\lambda s/2} \frac{\|w_i\|}{\|w_0\|}.$$

Proof. Let q_t denote the binding period of a free return $t \leq i$, and define $I_t = [t - q_t, t + q_t]$. These intervals are not necessarily two by two disjoint and it does not matter.

Claim 7.2.2. *For every $s \notin \cup I_t$ and $j \in [1, i-s]$,*

$$\|w_{s+j}\| \geq e^{-2\Delta j}\|w_s\|.$$

Proof. Fix s, and then fix j. Let r be the last free return between s and $s+j$. If no such r exists, then the inequality follows because s is free. Let $j' \geq j$ be the smallest integer such that $z_{s+j'}$ is free. Notice that j' may be bigger than i and it does not matter. Using the fact that s is free,

$$\|w_{s+j}\| \geq e^{-\Delta(j'-j)}\|w_{s+j'}\| \geq e^{-\Delta(j'-j)}|\tilde{\zeta}_0 - z_r|\|w_s\| \geq e^{-\Delta(j'-j)}e^{-\lambda q_r/3}\|w_s\|,$$

where $\tilde{\zeta}_0$ is the binding point for z_r. Since r is the last free return, $s + j' \leq r + q_r$ holds, and thus $j' \leq j + q_r$. Since $s < r - q_r \leq r \leq s + j$, we have $q_r \leq j$. This yields the desired inequality. □

Suppose that $s \notin \cup I_t$. Then $e_k(z_s)$ is well-defined for $1 \leq k \leq i - s$. Since s is free, $\mathrm{slope}(w_s) \leq K_0 b$. Hence we obtain

$$\|DH^{i-s}(z_s)\| \leq K\frac{\|w_i\|}{\|w_s\|} \leq Ke^{-\lambda s}\frac{\|w_i\|}{\|w_0\|},$$

where the last inequality follows from the strong regularity of $\mathbf{w}$.

Suppose that $s \in \cup I_t$. Let r denote the last return such that $s \in I_r$. Since $\mathbf{w}$ is reluctantly recurrent, we have $q_r \leq 10\alpha s$. If $i \in I_r$, then

$$\|DH^{i-s}(z_s)\| \leq e^{\Delta q_r} \leq e^{10\alpha\Delta s} \leq e^{-\lambda s/2}\frac{\|w_i\|}{\|w_0\|}.$$

Suppose that $i \notin I_r$. Suppose that $s \geq (1-10\alpha)i$. Then we have

$$\|DH^{i-s}(z_s)\| \leq e^{\Delta\alpha i} \leq e^{\frac{\Delta\alpha s}{1-\alpha}} \leq e^{-\lambda s/2}\frac{\|w_i\|}{\|w_0\|}.$$

It is left to consider the case $s < (1-10\alpha)i$. We consider the following operation. Put $s_1 = r_0 + 10q_{r_0}$. Ask whether $s_1 \notin \cup I_t$ or not. If so, then stop the operation. If not, then let r_1 denote the last return such that $s_1 \in I_{r_1}$. Put $s_2 = r_1 + 10q_{r_1}$, and ask whether $s_2 \notin \cup I_t$ or not. If so, then stop the operation. If not, then let $r_2 \leq i$ denote the last return such that $s_2 \in I_{r_2}$. Put $s_3 = r_2 + 10q_2$. Repeat this. This operation defines an increasing sequence of integers. Denote by $\{s_i\}_{i=0}^{\ell}$ such a sequence which is maximal with respect to inclusion as a set. Suppose that $s_\ell \in \cup I_t$. This implies $s_\ell \geq i$. By construction, $s_{i+1} - s_i \leq 2q_{r_i}$. This implies

$$\sum_{i=0}^{\ell} q_{r_i} \geq s_\ell - s_0 \geq i - s_0 \geq 10\alpha i.$$

On the other hand, since $\mathbf{w}$ is reluctantly recurrent, $\sum_{i=0}^{\ell} q_{r_i} \leq \alpha s_\ell \leq \alpha i$ holds. This yields a contradiction. Consequently, $s_\ell \notin \cup I_t$ holds. Then

$$\begin{aligned}\|DH^{i-s}(z_s)\| &\leq \|DH^{i-s_\ell}(z_{s_\ell})\| \prod_{i=0}^{\ell-1} \|DH^{s_{i+1}-s_i}(z_{s_i})\| \\ &\leq Ke^{-\lambda s_\ell/2}e^{-\lambda s_\ell/2}\frac{\|w_i\|}{\|w_0\|}e^{\alpha s_\ell} \\ &\leq Ke^{-\lambda s/2}\frac{\|w_i\|}{\|w_0\|}.\end{aligned}$$

□

The following lemma is a slight adaptation of [WY01] Proposition 6.1 to our context. The proof is almost the same and we omit it. Here, Lemma 7.2.1 plays an important role.

Lemma 7.2.3. *There exists D_1, $D_2 > 0$ such that for every $i \leq \min\{\nu, m+1\}$,*

$$D_1 \leq \|\dot{\zeta}_{i+1}^{(j)}(a_*)\| \cdot \frac{\|w_0(a_*)\|}{\|w_i(a_*)\|} \leq D_2.$$

7.3. **Proof of Proposition 7.1.2.** An idea in the proof is the same as the classical ones. We show that the critical orbits share essentially the same itineraries up to ν, that is, they return and then become free simultaneously, being bound to essentially the same binding points. Hence it makes sense to speak abount bound and free states uniformly on the parameter interval in question. We split the time interval $[0, \nu]$ accordingly and estimate the contribution to distortion bounds one by one.

Put $Z_i(a) = \zeta_{i+1}^{(j)}(a)$.

Claim 7.3.1. *For all $a \in J(a_*, \zeta_0, \nu, 0)$,*

$$\|\dot{Z}_0(a)\| \leq K\delta, \|\ddot{Z}_0(a)\| \leq K\delta,\ \|\dot{Z}_1(a)\| \approx 2a \textit{ and } \mathrm{slope}(Z_1'(a)) \leq K_0 b.$$

Proof. The first and the second inequalities follows from (35). The third one follows from the first one and the fact that $Z_0(a) \in H(\mathcal{C}_\delta)$. The last one follows from the third one and (1). □

By the claim and the fact that $\partial_a^2 f_a \equiv 0$, we have $\|\ddot{Z}_1(a)\| \leq K\delta$. Consequently, $\mathcal{J}_1$ is an admissible curve. (41) and (43) for $i = 0$ follow from the claim. (42) for $i = 0$ follows from $|Z_0(a_*) - Z_0(a)| \leq K|a_* - a|$ and Lemma 2.5.1. This completes the proof of the assertion for $i = 1$.

Let $i \in [1, \min\{\nu, m+1\}]$ be a free iterate. If i is a return, then let q denote the corresponding binding period. Otherwise, let $q = 0$. We prove the assertion for $i = i + q + 1$, assuming that they hold for i.

We prove (41). Define

$$D(a,i) = \left|\log \frac{\|Z'_{i+q+1}(a)\|}{\|Z'_i(a)\|} - \log \frac{\|w_{i+q+1}\|}{\|w_i\|}\right|.$$

If $Z'_{i+q+1}(a) = 0$ (as it really never does), we define $D(a,i) = +\infty$. By the chain rule it is enough to prove thefollowing for all $a \in J(a_*, \zeta_0, \nu, 0)$:

$$2D(a,i) \leq \Phi(\Pi_0^\nu \mathbf{w}) \cdot \Theta(\Pi_0^\nu \mathbf{w}, i)^{-1} + \left(\frac{\|w_0\|}{\|w_i\|}\right)^{\frac{1}{2}}. \tag{44}$$

Split $D(a,j) \leq A + B$, where

$$A = \left|\log \frac{\|DH_a^{q+1}(Z_i(a))Z'_i(a)\|}{\|Z'_i(a)\|} - \log \frac{\|w_{i+q+1}\|}{\|w_i\|}\right|,$$
$$B = \left|\log \frac{\|DH_a^{q+1}(Z_i(a))Z'_i(a)\|}{\|Z'_i(a)\|} - \log \frac{\|Z'_{i+q+1}(a)\|}{\|Z'_i(a)\|}\right|.$$

7.3.2. *Estimate of A.* Split

$$\mathcal{A} := \left|\frac{\|DH_a^{q+1}(Z_i(a))Z'_i(a)\|}{\|Z'_i(a)\|} - \frac{\|w_{i+q+1}\|}{\|w_i\|}\right| \leq I + II + III + IV + V + VI,$$

where

$$I = \left|\frac{\|DH_{a_*}^{q+1}(Z_i(a))Z'_i(a)\|}{\|Z'_i(a)\|} - \frac{\|DH_{a_*}^{q+1}(Z_i(a_*))Z'_i(a_*)\|}{\|Z'_i(a_*)\|}\right|,$$
$$II = 2 \cdot \|DH_{a_*}^{q+1}(Z_i(a))\| \left\|\frac{Z'_i(a_*)}{\|Z'_i(a_*)\|} - \frac{Z'_i(a)}{\|Z'_i(a)\|}\right\|,$$
$$III = \|DH_{a_*}^{q+1}(Z_i(a_*))\| \left\|\frac{Z'_i(a_*)}{\|Z'_i(a_*)\|} - \frac{Z'_i(a)}{\|Z'_i(a)\|}\right\|,$$
$$IV = 2 \cdot \|DH_{a_*}^{q+1}(Z_i(a))\| \left\|\frac{Z'_i(a_*)}{\|Z'_i(a_*)\|} - \frac{w_i}{\|w_i\|}\right\|,$$
$$V = \|DH_{a_*}^{q+1}(Z_i(a_*))\| \left\|\frac{Z'_i(a_*)}{\|Z'_i(a_*)\|} - \frac{w_i}{\|w_i\|}\right\|,$$
$$VI = \|DH_{a_*}^{q+1}(Z_i(a)) - DH_a^{q+1}(Z_i(a))\|.$$

Suppose that $q=0$. Using (41),

$$I, II, III, VI \leq K|Z_i(a_*)-Z_i(a)| \leq e|a_*-a||Z_i'(a_*)|.$$

Using Lemma 7.2.3 and (36),

$$\begin{aligned} I, II, III, VI &\leq \Phi(\Pi_0^\nu \mathbf{w})\Theta(\Pi_0^\nu \mathbf{w}, i)^{-1}\Theta(\Pi_0^\nu \mathbf{w}, i)\frac{\|w_i\|}{\|w_0\|} \\ &\leq \Phi(\Pi_0^\nu \mathbf{w})\Theta(\Pi_0^\nu \mathbf{w})^{-1}\frac{\|w_{i+q+1}\|}{\|w_i\|}. \end{aligned}$$

Lemma 7.3.3. ([WY01] Lemma 6.3) *We have* $z_i'(a_*) \neq 0$, $\forall i \geq 0$.

$$\text{angle}(\dot{\zeta}_{i+1}^{(j)}(a), w_i(a)) \leq \frac{\|w_0(a)\|}{\|w_i(a)\|}\left(\sum_{s=1}^{i}\frac{\|w_s(a)\|}{\|w_i(a)\|}b^{i-s}+\frac{\|w_0(a)\|}{\|w_i(a)\|}b^i\right). \tag{45}$$

Using (45),

$$IV, V \leq K\frac{\|w_0\|}{\|w_i\|} \leq \frac{\|w_{i+q+1}\|}{\|w_i\|}\left(\frac{\|w_0\|}{\|w_i\|}\right)^{\frac{1}{2}}.$$

Hence we obtain

$$\mathcal{A} \leq \frac{\|w_{i+q+1}\|}{\|w_i\|}\left[\Phi(\Pi_0^\nu \mathbf{w})\Theta(\Pi_0^\nu \mathbf{w})^{-1}+\left(\frac{\|w_0\|}{\|w_i\|}\right)^{\frac{1}{2}}\right]. \tag{46}$$

Suppose that $q \neq 0$. Let $\tilde{\zeta}_0$ denote a binding point at the free return i and $\tilde{\mathbf{w}} = \{\tilde{w}_i\}_{i=0}^q$ the corresponding foward vector orbit. Let p denote the folding period. By Remark 7.1.4, there exists a smooth continuation $a \in J(a_*, \zeta_0, \nu, 0) \to \tilde{\zeta}_0(a)$ such that the corresponding forward vector orbits $\tilde{\mathbf{w}}(a)$ obey (42).

Lemma 7.3.4. *Let* a, $b \in J(a_*, \zeta_0, \nu, 0)$. *The tangent vector* $(Z_i(a), Z_i'(a))$ *is in admissible position relative to* $\tilde{\zeta}_0(b)$. *In particular,* $H_b Z_i(a) \subset \Gamma^{(q-1)}(\tilde{\mathbf{w}}(b))$ *holds.*

Proof. The second half of the assertion follows from the definition of admissible position and Lemma 3.3.3. To show the first half, we begin by claiming that $(Z_i(a), Z_i'(a))$ is in admissible position relative to $\tilde{\zeta}_0$. Using Lemma 7.2.3,

$$\begin{aligned} |Z_i(a_*)-Z_i(a)| &\leq \frac{\|w_i\|}{\|w_0\|}\Phi(\Pi_0^\nu \mathbf{w}) \leq \frac{\|w_i\|}{\|w_0\|}\Theta(\Pi_0^\nu \mathbf{w}, i) \\ &\leq \left(\frac{\|w_i\|}{\|w_{i+p}\|}\right)^2 \leq L^2|\tilde{\zeta}_0-\zeta_{i+1}|^{2(1-\tilde{\alpha})} \ll |\tilde{\zeta}_0-\zeta_{i+1}|. \end{aligned}$$

This and the fact that $\mathcal{J}_i$ is an admissible curve together imply the claim, provided that $(Z_i(a_*), Z_i'(a_*))$ is in admissible position relative to $\tilde{\zeta}_0$. This is indeed the case by (45). On the other hand, by (26) and (35),

$$|\tilde{\zeta}_0-\tilde{\zeta}_0(b)| \leq |\tilde{\zeta}_0-\tilde{\zeta}_0(a_*)|+|\tilde{\zeta}_0(a_*)-\tilde{\zeta}_0(b)| \ll |\tilde{\zeta}_0-\zeta_{i+1}|.$$

This yields the claim. □

Lemma 7.3.5. *For all* $a \in J(a_*, \zeta_0, \nu, 0)$ *we have*

$$I \leq \frac{\|w_{i+q+1}\|}{\|w_i\|}\left(\frac{\|w_i\|}{\|w_{i+p}\|}\right)^2|Z_i(a_*)-Z_i(a)|$$

Proof. By Lemma 7.3.4 we have $H_{a_*}(Z_i(a)) \in \Gamma^{(q-1)}(\tilde{\mathbf{w}}(a_*))$, and hence the contractive directions e_i $(i = 1, \cdots, q)$ under the interations of H_{a_*} are well-defined at $H_{a_*}(Z_i(a))$. Split

$$\frac{DH_{a_*}(Z_i(a_*))Z_i'(a_*)}{\|Z_i'(a_*)\|} = \xi e_q(Z_{i+1}(a_*)) + \eta f_q(Z_{i+1}(a_*))$$

and

$$\frac{DH_{a_*}(Z_i(a))Z_i'(a)}{\|Z_i'(a)\|} = \tilde{\xi} e_q(H_{a_*}Z_i(a)) + \tilde{\eta} f_q(H_{a_*}Z_i(a)).$$

Then $I \leq A + B + C + D$, where

$$\begin{aligned}
A &= |\xi - \tilde{\xi}| \|DH^q_{a_*} e_q(H_{a_*}Z_i(a))\|,\\
B &= |\eta - \tilde{\eta}| \|DH^q_{a_*} f_q(Z_{i+1}(a_*))\|,\\
C &= |\xi| \|DH^q_{a_*} e_q(Z_{i+1}(a_*)) - DH^q_{a_*} e_q(H_{a_*}Z_i(a))\|,\\
D &= |\eta| \|DH^q_{a_*} f_q(Z_{i+1}(a_*)) - DH^q_{a_*} f_q(H_{a_*}Z_i(a))\|.
\end{aligned}$$

We estimate A, B, C, D one by one. It can be read out from the proof of Lemma 3.3.3 that the Lipschitz continuity of the first order derivatives of H and the fact that $\mathcal{J}_i$ is an admissible curve together imply

$$A \leq |\xi - \tilde{\xi}| \leq K|Z_i(a_*) - Z_i(a)|.$$

Applying the capture argument, we can find an admissible curve γ which contains $Z_i(a_*)$ and a critical point in its boundary. Applying the argument in the proof of Lemma 3.3.3 to $\gamma \cup \mathcal{J}_i$, we have $|\eta - \tilde{\eta}| \leq K|Z_i(a_*) - Z_i(a)|$, and thus

$$B \leq K|Z_i(a_*) - Z_i(a)| \frac{\|\tilde{w}_q\|}{\|\tilde{w}_0\|}.$$

Let $z \in S(\tilde{\mathbf{w}})$. By the chain rule and Lemma 7.2.1,

$$\|D(DH^q_{a_*}(z)) \cdot e_q(z)\| \leq e^{\Delta} \sum_{s=1}^{q} \|DH^{q-s}_{a_*}(z_s)\| \|DH^{s-1}_{a_*}(z) e_q(z)\| \leq \frac{\|\tilde{w}_q\|}{\|\tilde{w}_0\|}.$$

$$\|DH^q_{a_*}(z) \cdot De_q(z)\| = \|DH^q_{a_*}(z) f_q(z)\| \leq K \frac{\|\tilde{w}_q\|}{\|\tilde{w}_0\|}.$$

Using these and the mean value theorem,

$$C \leq K|Z_i(a_*) - Z_i(a)| \frac{\|\tilde{w}_q\|}{\|\tilde{w}_0\|}.$$

Claim 7.3.6. *Let $\partial = \partial_x, \partial_y$, or ∂_a. For every $1 \leq k \leq q$, $a \in J(a_*, \zeta_0, \nu, 0)$, and $z \in \Gamma^{(q-1)}(\mathbf{w}(a))$, we have*

$$\|\partial(DH^q_a(z))\| \leq K e^{\alpha\sigma k} \frac{\|\tilde{w}_k\|^2}{\|\tilde{w}_0\|^2}.$$

Proof. By the strong regularity of $\tilde{\mathbf{w}}(a)$ and that q is free, we have

$$\|DH^{s-1}_a(z)\| \leq K \frac{\|\tilde{w}_{s-1}(a)\|}{\|\tilde{w}_0(a)\|} \leq e^{-(\lambda-\alpha)(q-s+1)} e^{\alpha\sigma q} \frac{\|\tilde{w}_q(a)\|}{\|\tilde{w}_0(a)\|} \leq e^{-(\lambda-\alpha)(q-s+1)} e^{\alpha\sigma q} \frac{\|\tilde{w}_q\|}{\|\tilde{w}_0\|}$$

for $1 \le s \le k$. Using this and Lemma 7.2.1,

$$\begin{aligned}\|\partial(DH_a^k(z))\| &\le K\sum_{s=1}^{k}\|DH_a^{k-s}(z_s)\|\|DH_a^{s-1}(z)\| \\ &\le K\sum_{s=1}^{k}\|DH_a^{k-s}(z_s)\|\cdot\sum_{s=1}^{k}\|DH_a^{s-1}(z)\| \\ &\le Ke^{\alpha\sigma k}\frac{\|\tilde{w}_k\|^2}{\|\tilde{w}_0\|^2}.\end{aligned}$$

□

(11) in Proposition 3.3.2 implies

$$\frac{\|\tilde{w}_q\|}{\|\tilde{w}_0\|} \le K|\tilde{\zeta}_0-\zeta_{i+1}|^{-1}\frac{\|w_{i+q+1}\|}{\|w_i\|}. \tag{47}$$

Using this and Claim 7.3.6 for $k=q$, and then (47) and (11) (13) in Proposition 3.3.2,

$$\|\partial(DH_a^q(z))\| \le |\zeta_{i+1}-\tilde{\zeta}_0|^{-1}\frac{\|w_{i+q+1}\|}{\|w_i\|}\left(\frac{\|w_i\|}{\|w_{i+p}\|}\right)^{2-10\tilde{\alpha}}.$$

By the mean value theorem and $|\eta|=|\zeta_{i+1}-\tilde{\zeta}_0|$,

$$\begin{aligned}D &\le |\eta|\|D(DH_{a_*}^q f_q(\cdot))\||Z_{i+1}(a_*)-H_{a_*}Z_i(a)| \\ &\le |\eta|\left(\|D(DH_{a_*}^q)(\cdot)\|+\|DH_{a_*}^q e_q(\cdot)\|\right)e^{\Delta}|Z_i(a_*)-Z_i(a)| \\ &\le \frac{\|w_{i+q+1}\|}{\|w_i\|}\frac{\|w_i\|^2}{\|w_{i+p}\|^2}|Z_i(a_*)-Z_i(a)|.\end{aligned}$$

Consequently we obtain the desired upper estimate of I. □

Lemma 7.3.5 gives

$$\begin{aligned}I &\le \frac{\|w_{i+q+1}\|}{\|w_i\|}\frac{\|w_i\|^2}{\|w_{i+p}\|^2}\frac{\|w_i\|}{\|w_0\|}\Phi(\Pi_0^\nu\mathbf{w})\Theta(\Pi_0^\nu\mathbf{w},i)\Theta(\Pi_0^\nu\mathbf{w},i)^{-1} \\ &\le \frac{\|w_{i+q+1}\|}{\|w_i\|}\Phi(\Pi_0^\nu\mathbf{w})\Theta(\Pi_0^\nu\mathbf{w},i)^{-1}.\end{aligned}$$

Regarding II and III, we have $\|DH_{a_*}^{q+1}(Z_i(a))\| \le \|DH_{a_*}^q(Z_{i+1}(a))\| \le \|\tilde{w}_q\|$ for all $a\in J(a_*,\zeta_0,\nu,0)$, by Lemma 7.3.4. This yields

$$\begin{aligned}II, III &\le \left(\frac{\|w_i\|}{\|w_{i+p}\|}\right)^{1+\tilde{\alpha}}\frac{\|w_{i+q+1}\|}{\|w_i\|}\|w_i\|\Phi(\Pi_0^\nu\mathbf{w})\Theta(\Pi_0^\nu\mathbf{w},i)\Theta(\Pi_0^\nu\mathbf{w},i)^{-1} \\ &\le \frac{\|w_{i+q+1}\|}{\|w_i\|}\Phi(\Pi_0^\nu\mathbf{w})\Theta(\Pi_0^\nu\mathbf{w},i)^{-1}.\end{aligned}$$

Moreover, using Lemma 7.3.3,

$$IV, V \le \frac{\|w_{i+q+1}\|}{\|w_i\|}\left(\frac{\|w_i\|}{\|w_{i+p}\|}\right)^{1+\tilde{\alpha}}\frac{\|w_0\|}{\|w_i\|} \le \frac{\|w_{i+q+1}\|}{\|w_i\|}\left(\frac{\|w_0\|}{\|w_i\|}\right)^{\frac{1}{2}}.$$

Now it is left to consider VI. Fix a, and consider the matrix valued function $\varphi\colon b \to DH_b^{q+1}(Z_i(a))$. Denote by D_b the b-derivative. The chain rule gives

$$\begin{aligned}\|D_b\varphi(b)\| &= \|D_b(DH_b^q(H_b(Z_i(a))) \cdot DH_b(Z_i(a)))\| \\ &\leq K\|(D_bDH_b^q)(H_b(Z_i(a)))\| + e^{\Delta}\|DH_b^q(H_b(Z_i(a)))\|.\end{aligned}$$

Let $z \in S(\tilde{w}(b))$. Using Claim 7.3.6,

$$\|D_b(DH_b^q)(z)\| \leq \frac{\|w_{i+q+1}\|}{\|w_i\|}\frac{\|w_i\|^3}{\|w_{i+p}\|^3}.$$

By the mean value theorem,

$$\begin{aligned}VI &\leq \frac{\|w_{i+q+1}\|}{\|w_i\|}\frac{\|w_i\|^3}{\|w_{i+p}\|^3}\Phi(\Pi_0^\nu\mathbf{w})\Theta(\Pi_0^\nu\mathbf{w},i)\Theta(\mathbf{w},i)^{-1} \\ &\leq \frac{\|w_{i+q+1}\|}{\|w_i\|}\Phi(\Pi_0^\nu\mathbf{w})\Theta(\Pi_0^\nu\mathbf{w},i)^{-1},\end{aligned}$$

where the last inequality follows from $\|w_0\| \leq \|w_{i+p}\|$. Consequently, (46) follows in this case as well.

We are in position to complete the estimate of A. Since the number in the biggest parenthesis in (46) is much smaller than 1, we have

$$\frac{\|DH_a^{q+1}(Z_i(a))Z_i'(a)\|}{\|Z_i'(a)\|} \geq \frac{9}{10}\frac{\|w_{i+q+1}\|}{\|w_i\|} > 0. \tag{48}$$

Hence we obtain

$$A \leq \frac{10}{9}\left(\Phi(\Pi_0^\nu\mathbf{w})\Theta(\Pi_0^\nu\mathbf{w},i)^{-1} + \left(\frac{\|w_0\|}{\|w_i\|}\right)^{\frac{1}{2}}\right). \tag{49}$$

Estimate of B. In view of (33) we have

$$\|Z_{i+q+1}'(a) - DH_a^{q+1}(Z_i(a))Z_i'(a)\| \leq e^{\Delta q}. \tag{50}$$

Dividing both sides by $\|Z_i'(a)\|$ and using $q \leq \alpha i$ the inductive assumption,

$$\left|\frac{\|Z_{i+q+1}'(a)\|}{\|Z_i'(a)\|} - \frac{\|DH_a^{q+1}(Z_i(a))Z_i'(a)\|}{\|Z_i'(a)\|}\right| \leq e^{\Delta q}\frac{\|w_0\|}{\|w_i\|} \leq \left(\frac{\|w_0\|}{\|w_i\|}\right)^{1/2}.$$

This and (48), and the strong regularity of $\mathbf{w}$ together imply

$$\frac{\|Z_{i+q+1}'(a)\|}{\|Z_i'(a)\|} \geq \frac{\|w_{i+q+1}\|}{\|w_i\|} - \left(\frac{\|w_0\|}{\|w_i\|}\right)^{\frac{1}{2}} \geq \frac{1}{2}\frac{\|w_{i+q+1}\|}{\|w_i\|}.$$

Taking logs and rearranging gives

$$B \leq \frac{\|w_i\|}{\|w_{i+q+1}\|}\left(\frac{\|w_0\|}{\|w_i\|}\right)^{1/2} \leq \left(\frac{\|w_0\|}{\|w_i\|}\right)^{\frac{1}{2}}. \tag{51}$$

Since $a \in J(a_*,\zeta_0,\nu,0)$ is arbitrary, (49) (51) yield (44). This completes the proof of (41).

A proof of (42) for $i = i+q+1$ goes analogously, with

$$\tilde{D}(a,i) = \left|\frac{\|w_{i+q+1}(a)\|}{\|w_i(a)\|} - \frac{\|w_{i+q+1}\|}{\|w_i\|}\right|$$

in the place of $D(a,i)$. It holds that

$$\tilde{D}(a,i) \le \left| \frac{\|DH_{a_*}(Z_i(a))w_i(a)\|}{\|w_i(a)\|} - \frac{\|w_{i+q+1}\|}{\|w_0\|} \right| + VI,$$

and the first term can be estimated similarly to the case of I.

We now prove (43) for $i = i+q+1$. Let $1 \le k \le i$. Then by 41 and Lemma 7.2.3, $\|\dot{Z}_{i+q+1}(a)\| \ge \|\dot{Z}_i(a)\|$. Hence, it is enough to prove $\|\ddot{Z}_j(a)\| \le \|\dot{Z}_{i+q+1}(a)\|$ for $i+1 \le j \le i+q+1$. Let $k \in [1, q+1]$. We compute Z''_{i+k} in view of (33), and split $\|Z''_{i+k}\|/\|Z'_{i+q+1}\|^3 \le A+B+C+D$, where

$$\begin{aligned}
A &= \|Z'_{i+q+1}\|^{-3}\|DH_a^k(Z_i)Z''_i\|,\\
B &= \|Z'_{i+q+1}\|^{-3}\left\|\sum_{s=0}^{k-1} DH_a^s(Z_{i+k-s})\left(\partial_a^2\mathcal{H} + \partial_a(\partial_a\mathcal{H})Z'_{i+k-s-1}\right)\right\|,\\
C &= \|Z'_{i+q+1}\|^{-3}\|\partial_a(DH_a^k(Z_i))Z'_i\|\\
D &= \|Z'_{i+q+1}\|^{-3}\left\|\sum_{s=0}^{k-1} \partial_a(DH_a^s(Z_{i+k-s}))\partial_a\mathcal{H}\right\|
\end{aligned}$$

where all the partial derivatives of $\mathcal{H}$ inside the two sums are taken at $(a, Z_{i+k-s-1})$.

Using the previous inequality and the strong regularity of $\tilde{\mathbf{w}}$ gives

$$\|DH_a^{k-1}(Z_{i+1})\| \le K\frac{\|\tilde{w}_{k-1}\|}{\|\tilde{w}_0\|} \le \frac{\|\tilde{w}_q\|^{1+1/3}}{\|\tilde{w}_0\|^{1+1/3}} \le \left[\frac{\|w_{i+q+1}\|}{\|w_i\|}\frac{\|w_i\|^{1+\tilde{\alpha}}}{\|w_{i+p}\|^{1+\tilde{\alpha}}}\right]^{1+1/3}.$$

Using this and $\|Z''_i\| \le \|Z'_i\|^3$, which is part of the assumption of the induction,

$$\begin{aligned}
A &\le \frac{\|DH_a^{k-1}(Z_{i+1})Z''_i\|}{\|Z'_{i+q+1}\|^3}\\
&\le \frac{\|Z'_i\|^3\|DH_a^{k-1}(Z_{i+1})\|}{\|Z'_{i+q+1}\|^3}\\
&\le \frac{\|Z'_i\|^3}{\|Z'_{i+q+1}\|^3}\left[\frac{\|w_{i+q+1}\|}{\|w_i\|}\frac{\|w_i\|^{1+\tilde{\alpha}}}{\|w_{i+p}\|^{1+\tilde{\alpha}}}\right]^{1+1/3}\\
&\le 1/4.
\end{aligned}$$

Claim 7.3.7. *For every $\ell \in [1, q+1]$, we have*

$$\|Z'_{i+\ell}\| \le \|w_i\|^{1+\frac{1}{3}}.$$

Proof. We have

$$\|Z'_{i+\ell}\| \le \|Z'_{i+\ell} - DH_a^{i+\ell-\chi(i+\ell)}Z'_{\chi(i+\ell)}\| + \|DH_a^{i+\ell-\chi(i+\ell)}Z'_{\chi(i+\ell)}\|.$$

By (50), the first term is $\le e^{\Delta\alpha\sigma(i+\ell)}$. To estimate the second term, we use the fact that $\chi(i+\ell)$ is a free iterate before i, (41), and Lemma 7.2.3. Then

$$\|Z'_{i+\ell}\| \le e^{\Delta\alpha\sigma(i+\ell)}(1 + \|Z'_{\chi(i+\ell)}\|) \le \|w_{\chi(i+\ell)}\|^{1+\frac{1}{10}} \le e^{\alpha\sigma i}\|w_i\| \le \|w_i\|^{1+\frac{1}{3}}.$$

□

Using this claim,

$$B \leq e^{\Delta q}\frac{\|Z_i'\|^{2+2/3}}{\|Z_{i+q+1}'\|^3} \leq e^{\Delta q}\frac{1}{\|Z_{i+q+1}'\|^{1/3}} \leq \frac{1}{4}.$$

We estimate C. By the chain rule,

$$\|\partial_a(DH_a^k(Z_i(a)))\| \leq K\|DH_a^{k-1}(Z_{i+1})\| + K\|\partial_a(DH_a^{k-1}(Z_{i+1}))\|.$$

Using Claim 7.3.6 and $\|\dot{Z}_{i+1}\| \leq K\|\dot{Z}_i\|$,

$$\begin{aligned}\|\partial_a(DH_a^k(Z_i(a)))\| &\leq \frac{\|\tilde{w}_k\|}{\|\tilde{w}_0\|} + e^{\alpha\sigma q}\frac{\|\tilde{w}_k\|^2}{\|\tilde{w}_0\|^2}\|Z_i'\| \\ &\leq e^{-(\lambda-\alpha)(q-k)+\alpha\sigma q}(1+e^{\alpha\sigma q}\|Z_i'\|)\frac{\|\tilde{w}_q\|^2}{\|\tilde{w}_0\|^2}.\end{aligned}$$

Using (47) and $q \leq \alpha i$, we obtain

$$\|\partial_a(DH_a^k(Z_i(a)))\| \leq e^{3\alpha i}\|Z_i'\|\frac{\|w_{i+q+1}\|^2}{\|w_i\|^2} \leq \|w_{i+q+1}\|^2,$$

and therefore $C \leq 1/4$.

We estimate D. By the chain rule and Claim 7.3.7, we have

$$\|\partial_a DH_a^s(Z_{i+k-s})\| \leq Kse^{\Delta s}\|Z_{i+k-s}'\| \leq e^{2\Delta s}\|w_i\|^{1+1/3}.$$

This yields

$$D \leq e^{\Delta q}\frac{\|w_i\|^{1+1/3}}{\|w_{i+q+1}\|^3} \leq \frac{1}{4}.$$

Altogether these yield (41) for $i = i+q+1$.

It is left to prove that $\mathcal{J}_{i+q+1}$ is an admissible curve. For an arbitrary i and $a \in J(a_*, \zeta_0, h_j, 0)$, let $\kappa_i(a)$ denote the curvature of $\mathcal{J}_i$ at $Z_i(a)$. Split $\kappa_{i+1}(a) \leq \kappa_{i+1}'(a) + \kappa_{i+1}''(a)$, where

$$\begin{aligned}\kappa_{i+1}' &= \frac{\|DH_a(Z_i(a))Z_i'(a) \times Z_i''(a)\|}{\|Z_{i+1}'(a)\|^3}, \\ \kappa_{i+1}'' &= \frac{\|Z_{i+1}''(a)\|}{\|Z_{i+1}'(a)\|^3}.\end{aligned}$$

Sublemma 7.3.8. *For every $i \geq 0$,*

$$\kappa_{i+1}' \leq Kb \cdot \frac{\|Z_i'\|^3}{\|Z_{i+1}'\|^3}(\kappa_i' + \kappa_i'' + 1).$$

Proof. Split $\kappa_{i+1}' \leq I + II + III$, where

$$\begin{aligned}I &= \|Z_{i+1}'\|^{-3}\|DH_a(Z_i)Z_i' \times \partial_a^2\mathcal{H}\|, \\ II &= \|Z_{i+1}'\|^{-3}\|DH_a(Z_i)Z_i' \times \partial_a(\partial_a\mathcal{H}) \cdot Z_i'\|, \\ III &= \|Z_{i+1}'\|^{-3}\|DH_a(Z_i)Z_i' \times DH_a(Z_i)Z_i''\|.\end{aligned}$$

where all the partial derivatives are taken at (a, Z_i). Since H is a small perturbation of $(x, y) \to (1 - ax^2, 0)$, the C^0 norm of $\partial_a^2 \mathcal{H}(a, Z_i)$ is close to zero. In particluar we have

$$I \leq Kb \frac{\|Z_i'\|}{\|Z_{i+1}'\|^3}.$$

Clearly, $\|\partial_a(\partial_a \mathcal{H}(a, Z_i))\| \leq K\|Z_i'\|$ holds, and thus the numerator of II is defree three homogeneour in $\|Z_i'(a)\|$. Moreover, it is easy to see that the second components of the two vectors involved in the product is smaller than Kb in norm. Hence we obtain

$$II \leq Kb \frac{\|Z_i'\|^3}{\|Z_{i+1}'\|^3}.$$

Meanwhile we have

$$III \leq Kb \left(\frac{\|Z_i'(a)\|}{\|Z_{i+1}'(a)\|} \right)^3 (\kappa_i' + \kappa_i'').$$

Putting these three inequalities together we obtain the desired one.

□

A recursive use of this inequality in Sublemma 7.3.8, we have

$$\kappa_{i+q+1}' \leq (Kb)^{i+q} \frac{\|Z_0'\|^3}{\|Z_{i+q+1}'\|^3} \kappa_0' + \sum_{\ell=0}^{i+q} (Kb)^{\ell+1} \frac{\|Z_{i+q-\ell}'\|^3}{\|Z_{i+q+1}'\|^3} (\kappa_{i+q-\ell}'' + 1).$$

Using 43 for $i = i + q + 1$,

$$\frac{\|Z_{i+q-\ell}'\|^3}{\|Z_{i+q+1}'\|^3} \kappa_{i+q-\ell}'' \leq 1.$$

The inductive assumption, Lemma 7.2.3, gives

$$\frac{\|Z_0'(a)\|}{\|Z_{i+q+1}'(a)\|} \leq e \frac{\|w_0\|}{\|w_{i+q+1}\|} \leq e^2 K_0^{-1} \delta^{-1}.$$

Substituting these into the above inequality, we obtain $\kappa_{i+q+1}' \ll 1$. Hence we obtain $\kappa_{i+q+1} \leq 1$. Regarding the slope, recall that q is a free iterate of $\mathbf{w}(a)$ for all $a \in J(a_*, \zeta_0, h_j, 0)$. Thus slope$(w_q(a)) \leq K_0 b$ holds. This and Lemma 7.3.3 together yield slope$(Z_{i+q+1}'(a)) \leq K_0 b$. Hence $\mathcal{J}_{i+q+1}$ is an admissible curve. This completes the proof of Proposition 7.1.2. □

7.4. **Expansion at essential returns.** We fix some assumptions and notation for the rest of this section. Let $a_* \in RR_{n-1}$, and suppose that ζ_0 is a critical point of order $\xi \geq n$ of H_{a_*}. Let $\{h_i\}_{i=1}^s$ denote the sequence of hyperbolic times associated with the backward orbit of ζ_0. Let $0 < \nu_1 < \nu_2 < \cdots < \nu_t \leq \beta n$ denote the maximal sequence of essential returns. For $i \in [0, t]$, let $s(i) \in [1, s]$ denote the smallest integer such that $\nu_i \leq \beta h_{s(i)}$ holds. Let $\tilde{\zeta}_0$ denote a binding point to which $w_{\nu_i}(\zeta_{\nu_i+1})$ is in admissible position. Define

$$d(\nu_i) = -\log|\tilde{\zeta}_0 - \zeta_{\nu_i+1}|.$$

We call $d(\nu_i)$ an *essential return depth.*

Proposition 7.4.1. *The secondary quasi critical point* $\zeta_0^{(s(i))}$ *has a smooth continuation on* $J(a_*,\zeta_0,\nu_i,0)$. *Moreover, for all* $a\in J(a_*,\zeta_0,\nu_i,0)\setminus J(a_*,\zeta_0,\nu_i,d(\nu_i))$,

$$|\zeta_{\nu_i+1}^{(s(i))}(a_*)-\zeta_{\nu_i+1}^{(s(i))}(a)|\geq|\tilde{\zeta}_0-\zeta_{\nu_i+1}|^{1-\alpha_0/2}.$$

Proof. We prove the first half of the assertion. By Proposition 7.1.2 it is enough to prove $\beta h_{s(i)}/4\leq\nu_i\leq\beta h_{s(i)}$. The right hand side is obvious by definition. Regarding the left hand side, since $\nu_i\geq\nu_1>\beta h_1$ we have $s(i)\geq 2$. Thus $\beta h_{s(i)}/4\leq\beta h_{s(i)-1}<\nu_i$ holds, by Lemma 2.12.1.

Lemma 7.4.2. *We have*

$$\Phi(\Pi_0^{\nu_i}\mathbf{w})\cdot\frac{\|w_{\nu_i}\|}{\|w_0\|}\geq|\tilde{\zeta}_0-\zeta_{\nu_i+1}|^{1-\alpha_0}.$$

The second half of the assertion is an immediate consequence of this lemma. To see this, recall that ν_i is an essential return and hence it is free. Thus $\mathcal{J}_{\nu_i}$ is an admissible curve. By (41) and Lemma 7.2.3,

$$|\zeta_{\nu_i+1}^{(s(i))}(a_*)-\zeta_{\nu_i+1}^{(s(i))}(a)|\geq e^{-3}\frac{\|w_{\nu_i}\|}{\|w_0\|}|a_*-a|\geq e^{-3}\frac{\|w_{\nu_i}\|}{\|w_0\|}\Phi(\Pi_0^{\nu_i}\mathbf{w})e^{-\alpha_0 d(\nu_i)}.$$

Therefore, Lemma 7.4.2 yields the desired inequality.

Proof of Lemma 7.4.2. Put $\nu_i=\nu$. Let $0<m_0<m_1<\cdots<m_t<\nu$ denote the set of all free returns which take place before ν. Let p_s, q_s $(0\leq s\leq t)$ denote the corresponding folding and binding periods.

Sublemma 7.4.3. *For every* $0\leq s\leq t$ *and* $m_s\leq i\leq m_s+q_s+1$,

$$\min_{i\leq j\leq\nu}\frac{\|w_j\|}{\|w_i\|}\geq\min_{s\leq u\leq t}\frac{\|w_{m_u+p_u}\|^3}{\|w_{m_u}\|^3}.$$

Proof. There are three cases: $j\leq m_s+q_s+1$; $j>m_s+q_s+1$ and j is free; $j>m_s+q_s+1$ and j is bound. In the first case, the desired inequality immediately follows from (17). In the second case, split

$$\frac{\|w_j\|}{\|w_i\|}=\frac{\|w_j\|}{\|w_{m_s+q_s+1}\|}\frac{\|w_{m_s+q_s+1}\|}{\|w_i\|}.$$

The first term is $\geq K_0\delta$, because m_s+q_s+1 and j are free. Applying (17) to the second term,

$$\frac{\|w_j\|}{\|w_i\|}\geq K_0\delta\left(\frac{\|w_{m_s+p_s}\|}{\|w_{m_s}\|}\right)^{1+\frac{3\alpha\sigma}{\lambda(1+\tilde{\alpha})}}\geq\frac{\|w_{m_s+p_s}\|^3}{\|w_{m_s}\|^3}.$$

In the last case, there exists $u\in[s+1,t]$ such that $j\in[m_u+1,m_u+q_u+1]$. Split

$$\frac{\|w_j\|}{\|w_i\|}=\frac{\|w_j\|}{\|w_{m_u}\|}\frac{\|w_{m_u}\|}{\|w_i\|}.$$

Using (17) again and $\|w_{m_u}\|\geq K_0e^{-1}\delta\|w_i\|$,

$$\frac{\|w_j\|}{\|w_i\|}\geq K_0e^{-1}\delta\left(\frac{\|w_{m_u+p_u}\|}{\|w_{m_u}\|}\right)^{1+\frac{3\alpha\sigma}{\lambda(1+\tilde{\alpha})}}\geq\frac{\|w_{m_u+p_u}\|^3}{\|w_{m_u}\|^3}.$$

□

Sublemma 7.4.4. *For every $0 \le s \le t$,*

$$\sum_{i=m_{s-1}+q_{s-1}+1}^{m_s} \Theta(\Pi_0^\nu \mathbf{w}, i)^{-1} \le \frac{1}{1-e^{-\lambda}} \frac{\|w_{m_s}\|}{\|w_0\|} \max_{s \le u \le t} \frac{\|w_{m_s}\|^6}{\|w_{m_s+p_s}\|^6}.$$

Proof. Let $j \in [i, \nu]$. Suppose that $j \ge m_s$. By Sublemma 7.4.3,

$$\frac{\|w_j\|}{\|w_i\|} = \frac{\|w_{m_s}\|}{\|w_i\|} \frac{\|w_j\|}{\|w_{m_s}\|} \ge \frac{\|w_j\|}{\|w_{m_s}\|} \ge \min_{s \le u \le t} \frac{\|w_{m_u+p_u}\|^3}{\|w_{m_u}\|^3}. \tag{52}$$

Suppose that $j < m_s$. Then $\|w_j\| \ge K_0\delta\|w_i\|$ holds because i is free and no return takes place until j. Hence the inequality in (52) holds in this case as well. Substituting (52) and $\|w_i\| \le \|w_{m_s}\|e^{-\lambda(m_s-i)}$ into the definition of $\Theta(\Pi_0^\nu \mathbf{w}, i)$,

$$\Theta(\Pi_0^\nu \mathbf{w}, i)^{-1} \le e^{-\lambda(m_s-i)} \frac{\|w_{m_s}\|}{\|w_0\|} \max_{s \le u \le t} \frac{\|w_{m_u}\|^6}{\|w_{m_u+p_u}\|^6}.$$

Summing up this for every $i \in [m_{s-1}+q_{s-1}+q, m_s]$ yields the desired inequality. □

Sublemma 7.4.5. *We have*

$$\sum_{i=\mu(t)+q(t)+1}^{\nu-1} \frac{\|w_0\|}{\|w_\nu\|} \cdot \Theta(\Pi_0^\nu \mathbf{w}, i)^{-1} \le -\lambda^{-1}\log(K_0\delta) \cdot \delta^{\frac{\alpha\lambda\sigma}{100\Delta}-1}.$$

Proof. Put $s_0 = -2\lambda^{-1}\log(K_0\delta) \gg 1$. Since no return takes place from i to ν,

$$\frac{\|w_\nu\|}{\|w_0\|}\Theta(\Pi_0^\nu \mathbf{w}, i) = \min_{i \le j \le \nu} \frac{\|w_\nu\|}{\|w_i\|} \left(\frac{\|w_j\|}{\|w_i\|}\right)^2 \ge (K_0\delta)^2 e^{\lambda(\nu-i)} \ge e^{\lambda(\nu-i-s_0)},$$

and thus

$$\sum_{\substack{m_t+q_t+1 \le i \le \nu-1 \\ i \le \nu-s_0}} \frac{\|w_0\|}{\|w_\nu\|} \cdot \Theta(\Pi_0^\nu \mathbf{w}, i)^{-1} \le \sum_{i=0}^{\infty} e^{-\lambda i} = \frac{1}{1-e^{-\lambda}}. \tag{53}$$

Suppose that $i \ge \nu - s_0$. Let $j \in [i, \nu]$ denote an integer such that

$$\Theta(\Pi_0^\nu \mathbf{w}, i) = \frac{\|w_0\|}{\|w_i\|} \frac{\|w_j\|^2}{\|w_i\|^2}.$$

Let x_i denote the x-coordinate of z_i, and suppose that $|x_{j_0}| = \min_{i \le k \le j-1}|x_k| \ge \delta^{1/100}$. Using (ii) in Lemma 2.5.1 successively we have

$$\frac{\|w_\nu\|}{\|w_0\|} \cdot \Theta(\Pi_0^\nu \mathbf{w}, i) \ge |x_{j_0}|^2 \frac{\|w_\nu\|}{\|w_{j_0}\|} \ge |x_{j_0}|^2 \ge \delta^{1/50}.$$

Suppose that $\delta \le |x_{j_0}| \le \delta^{1/100}$. In this case, although j_0 is not a return time, we can consider a binding period q initiated at j_0, and it is easy to show that the same estimates as in Lemma 3.3.2 holds. In particular, $|x_{j_0}|\|w_{j_0+q+1}\| \ge \|w_{j_0}\|$ holds. Moreover, $|x_{j_0+q+2}+1| \le \delta^{\frac{\alpha\sigma}{100}}$ holds, by (15) in Proposition 3.3.2 and the fact that $f_2^2(0) = -1 = f_2(-1)$. Since $x_\nu \in (-\delta, \delta)$, we have $\nu - j_0 - q - 1 \ge -\frac{\alpha\sigma}{100\Delta}\log\delta$. This yields $|x_{j_0}|\|w_\nu\| \ge \delta^{1-\frac{\alpha\lambda\sigma}{100\Delta}}\|w_{j_0+q+1}\|$, and thus

$$\frac{\|w_\nu\|}{\|w_0\|} \cdot \Theta(\Pi_0^\nu \mathbf{w}, i) \ge |x_{j_0}|^2 \frac{\|w_\nu\|}{\|w_{j_0+q+1}\|} \frac{\|w_{j_0+q+1}\|}{\|w_{j_0}\|} \ge \delta^{1-2\alpha_0}.$$

Therefore

$$\sum_{\substack{m_t+q_t+1\leq i\leq \nu-1\\ i\geq \nu-s_0}} \frac{\|w_0\|}{\|w_\nu\|}\Theta(\Pi_0^\nu \mathbf{w}, i)^{-1} \leq s_0\delta^{2\alpha_0-1}.$$

This and (53) yield the desired inequality because $s_0\delta^{2\alpha_0-1}\to+\infty$ as $\delta\to 0$. □

We are now in position to conclude a proof of the lemma. It is enough to show that for every $0\leq s\leq t$,

$$\frac{\|w_\nu\|}{\|w_0\|}\left[\sum_{i=m_{s-1}+q_{s-1}+1}^{m_s}\Theta(\Pi_0^\nu\mathbf{w},i)^{-1}\right]^{-1}\geq|\tilde{\zeta}_0-\zeta_{\nu+1}|^{3/5}\delta^{-(t-s)/1000}. \tag{54}$$

Indeed, taking reciprocals of both sides and summing up for all $0\leq s\leq t$ we obtain

$$\begin{aligned}\frac{\|w_0\|}{\|w_\nu\|}\Phi(\Pi_0^\nu\mathbf{w})^{-1} &= \sum_{\substack{1\leq i\leq \nu-1\\ \text{free}}}\frac{\|w_0\|}{\|w_\nu\|}\Theta(\Pi_0^\nu\mathbf{w},i)^{-1}\\ &=\sum_{s=0}^{t}\left(\sum_{i=m_{s-1}+q_{s-1}+1}^{m_s}\right)+\sum_{i=m_t+q_t+1}^{\nu-1}\\ &\leq -\log\delta\cdot\delta^{2\alpha_0-1}+|c_0-z_{\nu+1}|^{-3/5}\cdot\sum_{s=0}^{t}\delta^{(t-s)/1000}\\ &\leq|\tilde{\zeta}_0-\zeta_{\nu+1}|^{\alpha_0-1}.\end{aligned}$$

Taking the reciprocals of both sides we obtain the desired inequality.

It is left to prove (54). Using this and Sublemma 7.4.4,

$$\frac{\|w_\nu\|}{\|w_0\|}\left[\sum_{i=m_{s-1}+q_{s-1}+1}^{m_s}\Theta(\Pi_0^\nu\mathbf{w},i)^{-1}\right]^{-1}\geq\frac{\|w_\nu\|}{\|w_{m_s}\|}\frac{\|w_{m_s+p_s}\|^6}{\|w_{m_s}\|^6}. \tag{55}$$

Suppose that $t=s$. Since ν is an essential return, we have

$$\log|\tilde{\zeta}_0-\zeta_{\nu+1}|\leq 10\cdot\log\frac{\|w_{m_t+p_t}\|}{\|w_{m_t}\|}.$$

Substituting this into (55) we obtain (54).

Suppose that $0\leq s\leq t-1$. On the first term of the right hand side of (55),

$$\begin{aligned}\frac{\|w_\nu\|}{\|w_{m_s}\|}&=\frac{\|w_\nu\|}{\|w_{m_t+q_t+1}\|}\frac{\|w_{m_t+q_t+1}\|}{\|w_{\mu_t}\|}\cdots\frac{\|w_{m_{s+1}}\|}{\|w_{m_s+q_s+1}\|}\frac{\|w_{m_s+q_s+1}\|}{\|w_{m_s}\|}\\ &\geq\prod_{s\leq u\leq t}\frac{\|w_{m_u+q_u+1}\|}{\|w_{m_u}\|}.\end{aligned}$$

Since ν is an essential return, for every $0\leq s\leq t-1$,

$$\log|\tilde{\zeta}_0-z_{\nu+1}|+\sum_{s+1\leq u\leq t}\log\frac{\|w_{m_u+p_u}\|}{\|w_{m_u}\|}\leq 10\cdot\log\frac{\|w_{m_s+p_s}\|}{\|w_{m_s}\|}.$$

Therefore

$$\frac{\|w_\nu\|}{\|w_0\|}\left[\sum_{i=m_{s-1}+q_{s-1}+1}^{m_s}\Theta(\Pi_0^\nu\mathbf{w},i)^{-1}\right]^{-1}\geq\prod_{s+1\leq u\leq t}\frac{\|w_{m_u+q_u+1}\|}{\|w_{m_u}\|}\left(\frac{\|w_{m_u+p_u}\|}{\|w_{m_u}\|}\right)^{3/5}\times|\tilde\zeta_0-\zeta_{\nu+1}|^{\frac{3}{5}}.$$

By Lemma 3.3.2,

$$\frac{\|w_{m_u+q_u+1}\|}{\|w_{m_u}\|}\left(\frac{\|w_{m_u+p_u}\|}{\|w_{m_u}\|}\right)^{3/5}\geq\left(\frac{\|w_{m_u+p_u}\|}{\|w_{m_u}\|}\right)^{-1/100}\geq\delta^{-\frac{1}{100}}.$$

Sustituting this into the right hand side we obtain (54). This completes the proof of Lemma 7.4.2 and hence that of Proposition 7.4.1. □

7.5. **Binding points for critical values.** We keep the same assumptions and notations as in Sect. 7.4. The following lemma asserts that one can find binding points for all critical values at any essential return or at the last free return $m+1$ at which the reluctant recurrence condition is violated.

Lemma 7.5.1. *Suppose that ν_i is an essential return and $w_{\nu_i}(\zeta_{\nu_i+1})$ is in admissible position relative to a critical point $\tilde\zeta_0$. For all $a\in J(a_*,\zeta_0,\nu_i,0)\setminus J(a_*,\zeta_0,\nu_i,d(\nu_i))$ such that $\zeta_{\nu_i+1}^{(s(i))}(a)\in\mathcal{C}_\delta$, there exists a precritical point $\zeta_0(a)$ of H_a relative to which $\zeta_{\nu_i+1}^{(s(i))}(a)$ is in admissible position. Moreover we have*

$$-\log|\zeta_0(a)-\zeta_{\nu_i+1}^{(s(i))}(a)|\leq(1-\alpha_0)d(\nu_i). \tag{56}$$

If w_{m+1} is in critical position, then the same thing holds with ν_i and $d(\nu_i)$ replaced by $m+1$, $\alpha(m+1)$.

Proof. Let $\{\tilde h_i\}_{i=1}^{\tilde s}$ denote the sequence of hyperbolic times associated with the backward orbit of $\tilde\zeta_0$. Let $\tilde\zeta_0^{(j)}$ $(j=1,\cdots,\tilde s)$ denote the associated secondary quasi critical points, with smooth continuations $a\in J(a_*,\tilde\zeta_0,\beta\tilde h_j/2,0)\to\tilde\zeta_0^{(j)}(a)$. By Remark 7.1.4, $J(a_*,\zeta_0,\nu_i,0)\subset J(a_*,\tilde\zeta_0,\beta\tilde h_j/2,0)$ holds for every j. Corollary 7.4.1 permits us to apply Lemma 2.10.2 to yield a precritical point $\zeta_0^{[\tilde h_{\tilde s}]}(a)$ of H_a of order $\tilde h_{\tilde s}$ on $\mathcal{J}_{\nu_i}$, whose x-coordinate is roughly equal to that of $\zeta_0^{(\tilde h_{\tilde s})}(a_*)$. We apply Lemma 2.10.1 to construct a sequence of precritical points of lower order. There are two cases: $\zeta_0^{[\tilde h_{\tilde s}-1]}(a),\cdots,\zeta_0^{[\beta^{-1}\tilde h_{\tilde s}]}(a)$ are constructed on $\mathcal{J}_{\nu_i}$, or else there exists some $k\in[\beta^{-1}\tilde h_{\tilde s}+1,\tilde h_{\tilde s}]$ such that $\zeta_0^{[k]}(a)$ is so close to the boundary of $\mathcal{J}_{\nu_i}$ that there is no room on $\mathcal{J}_{\nu_i}$ for $\zeta_0^{[k-1]}(a)$ to be created. In the first case, we stop further construction. In the second case, take s' to be the smallest integer such that $h_{s'}\geq\beta^{-1}\tilde s$, and apply Lemma 2.10.2 with respect to $\zeta_0^{(s')}(a)$ to create a precritical point of order $h_{s'}$ on $\mathcal{J}_{\nu_i}$. Since any admissible curve admits only one precritical point of the same order, $\zeta_0^{[s']}(a)$ coincides with the one which was constructed at the previous step. We repeat the same construction using $\zeta_0^{(s')}(a)$ instead of $\zeta_0^{(\tilde s)}(a)$.

Sublemma 7.5.2. *Suppose that $Z_{\nu_i}(a)\in\mathcal{C}_\delta$. If $|\zeta_0^{[k]}(a)-\zeta_0^{[\tilde h_{\tilde s}]}(a)|\geq 1/3\cdot$length$(\mathcal{J}_{\nu_i}\cap\mathcal{C}_\delta)$, then $(Z'_{\nu_i}(a),Z_{\nu_i}(a))$ is related to $\zeta_0^{[k]}(a)$.*

Proof. By Lemma 2.10.1 we have $|\zeta_0^{[k]}(a)-\zeta_0^{[\tilde{h}_{\tilde{s}}]}(a)|\leq (Kb)^k$. Thus the assmuption implies

$$k\leq \frac{\log(1/3\cdot \mathrm{length}(\mathcal{J}_{\nu_i}\cap\mathcal{C}_\delta))}{\log(Kb)}=:c.$$

Suppose that $(Z'_{\nu_i}(a),Z_{\nu_i}(a))$ is not related to $\zeta_0^{[k]}(a)$. Then

$$|Z_{\nu_i}(a)-\zeta_0^{[k]}(a)|\geq e^{-c\Delta\beta}\geq K\cdot(\mathrm{length}(\mathcal{J}_{\nu_i}\cap\mathcal{C}_\delta))^{\frac{1}{2}}.$$

This yields a contradiction because $Z_{\nu_i}(a)$, $\zeta_0^{[k]}(a)\in\mathcal{J}_{\nu_i}\cap\mathcal{C}_\delta$ and $\mathrm{length}(\mathcal{J}_{\nu_i}\cap\mathcal{C}_\delta)<1$. □

Let $k_0=k_0(a)<\tilde{h}_{\tilde{s}}$ denote the largest integer such that $\zeta_0^{[k_0]}$ is well-defined and $(Z'_{\nu_i}(a),Z_{\nu_i}(a))$ is related to $\zeta_0^{[k_0]}$. We claim that k_0 exists. To see this it is enough to show that there exists a precritical point to which $(Z'_{\nu_i}(a),Z_{\nu_i}(a))$ is related. This is indeed the case when the sequence of all precritical points are contained in the $1/3\cdot\mathrm{length}(\mathcal{J}_{\nu_i})$-neighborhood of $\zeta_0^{[\tilde{h}_{\tilde{s}}]}(a)$. Otherwise, we appeal to the sublemma, and the claim follows.

Suppose that $(Z'_{\nu_i}(a),Z_{\nu_i}(a))$ is in critical position relative to $\zeta_0^{[k_0]}$. Then it is related to $\zeta_0^{[k_0+1]}$, by Sublemma 4.2.6. By the maximality of of k_0, it follows that $k_0=\tilde{h}_{\tilde{s}}-1$. On the other hand, by Corollary 7.4.1, $(Z'_{\nu_i}(a),Z_{\nu_i}(a))$ is not related to $\zeta_0^{[\tilde{h}_{\tilde{s}}]}$ for all $a\in J(a_*,\zeta_0,\nu_i,0)\setminus J(a_*,\zeta_0,\nu_i,d(\nu_i))$. This yields a contradiction. Therefore, $(Z'_{\nu_i}(a),Z_{\nu_i}(a))$ is in admissible position relative to $\zeta_0^{[k_0]}$. (56) readily follows from Proposition 7.4.1.

The argument in the other case is almost identical, with $m+1$ in the place of ν_i. The only one difference is the way to show that $(Z'_{m+1}(a),Z_{m+1}(a))$ is not related to $\zeta_0^{[\tilde{h}_{\tilde{s}}]}$ for all $a\in J(a_*,\zeta_0,m+1,0)\setminus J(a_*,\zeta_0,m+1,\alpha(m+1))$. This follows from the fact that the order of $\tilde{\zeta}_0$ is $m+1$ in this case and

$$\begin{aligned}|Z_{\nu_i}(a_*)-Z_{\nu_i}(a)|&\geq\frac{\|w_\nu\|}{\|w_0\|}\Phi(\Pi_0^\nu\mathbf{w})e^{-d(\nu_i)}\\&\geq\frac{\|w_m\|}{\|w_0\|}e^{-d(\nu_i)}\Xi(\Pi_0^\nu\mathbf{w})\geq K_0e^{-1}\delta e^{-3\alpha\sigma\nu}\geq e^{-4\alpha\sigma\nu}.\end{aligned}$$

□

8. Proof of Theorem B

In this last section we prove that the set of $a\in\Omega^{(0)}$ such that H_a satisfies $(EG)_n$ for all $n\geq N$ has positive Lebesgue measure.

8.1. **Definition of bad parameter sets.** Let $n\geq N$. We define a subset of $\Omega^{(0)}$ which contains $RR_{n-1}-RR_n$. Fix two integers r, R such that $r\in[1,-\Delta\beta n/\log\delta]$ and $R\geq\alpha\beta(n-1)/100$. Define $\mathcal{N}_r$ to be the set of all strictly monotone sequences of integers $\mathbf{n}=\{\nu_i\}_{i=1}^r$ in $[0,\beta n]$. Define $\mathcal{D}_R$ to be the set of all sequences of integers $\mathbf{d}=\{d_i\}_{i=1}^r$ such that

$$-\log\delta\leq d_i\ \text{and}\ \sum_{i=1}^r d_i=R.$$

For a triple $(\mathbf{n}, \mathbf{d}, u) \in \mathcal{N}_r \times \mathcal{D}_R \times S(n)$, define $\Omega^{(n)}(\mathbf{n}, \mathbf{d}, u)$ to be the set of all $a \in RR_{n-1}$ such that:

(a) there exists a critical point ζ_0 of order $> n$ whose forward orbit is not reluctantly recurrent up to time $\beta(n+1)-1$.

(b) the forward orbit of ζ_0 makes essential returns exactly at $\nu_1 < \nu_2 < \cdots < \nu_r < \beta n$, and $\nu_r - 1$ is the largest integer up to which it is reluctantly recurrent. For every $1 \le i \le r$,

$$d_i = d(\nu_i),$$

where $d(\nu_i)$ is the essential return depth at ν_i.

(c) the associated secondary quasi critical point of order ξ is linked to $u \in S(n)$.

Fix $m \in [\beta n, \beta(n+1)-1]$. Define $\tilde{\Omega}^{(n)}(\mathbf{h}, \mathbf{u}, m)$ to be the set of all $a \in RR_{n-1}$ such that

(d) there exists a critical point ζ_0 of order $\xi \ge n$ such that $m-1$ is the largest integer up to which the forward orbit $\mathbf{w} = \{w_i(\zeta_{i+1})\}_{i=0}^{\beta\xi}$ of ζ_0 is reluctantly recurrent.

(c) the associated secondary quasi critical point of order ξ is linked to $u \in S(n)$.

(e) there is NO precritical point of order $\le n-1$ relative to which w_m is in admissible position.

Define

$$\Omega^{(n)} = \bigcup_{R,r} \bigcup_{\mathbf{n},\mathbf{d},u} \Omega^{(n)}(\mathbf{n}, \mathbf{d}, u),$$

where, the unions run over all possible combinations of the subscripts. Analogously we define

$$\tilde{\Omega}^{(n)} = \bigcup_{u,m} \tilde{\Omega}^{(n)}(u, m).$$

The following lemma is more or less automatic from the above definition.

Lemma 8.1.1. *For every $n \ge N+1$,*

$$RR_{n-1} - RR_n \subset \Omega^{(n)} \cup \tilde{\Omega}^{(n)}.$$

Proof. Suppose that $a \in RR_{n-1} - RR_n$. By definition, there exists a critical point ζ_0 of H_a of order $\xi \ge n$ whose forward orbit $\mathbf{w} = \{w_i(\zeta_{i+1})\}_{i=0}^{\beta\xi}$ is not reluctantly recurrent up to time $\beta(n+1)-1$. Take $u \in S(n)$ so that the condition (c) is met with respect to ζ_0. Let $m-1$ denote the largest integer up to which $\mathbf{w}$ is reluctantly recurrent. By $(RR)_{n-1}$, we have $\beta n - 1 \le m-1$. Clearly, m is a free return. There are two cases: w_m in critical position, or in admissible position. In the first case, it is straightforward to see $a \in \tilde{\Omega}^{(n)}(u, m)$. In the second case, m is clearly an essential return. Let $\mathbf{n} = \{\nu_1 < \nu_2 < \cdots < \nu_r = m\}$ denote all the essential returns up to time m, with $\mathbf{d} = \{d_i\}_{i=1}^r$ the corresponding sequence of essential return depths. By Sublemma 6.2.4, two consecutive essential returns are separated by at least $\Delta^{-1}\log\delta^{-1}$ iterates. Hence $r \le \Delta\beta n/\log\delta^{-1}$ holds. Since $\mathbf{w}$ is not reluctantly recurrent up to time m, we have

$$R := \sum_{i=1}^{r} d_i \ge \frac{\alpha m}{100}.$$

Hence we obtain $a \in \Omega^{(n)}(\mathbf{n}, \mathbf{d}, u)$. □

Let $|\cdot|$ denote the one-dimensional Lebesgue measure.

Proposition 8.1.2. *There exists a large integer R such that*

$$\Omega^{(n)} \cup \tilde{\Omega}^{(n)} = \emptyset$$

holds for every $N \leq n \leq R$, and

$$|\Omega^{(n)} \cup \tilde{\Omega}^{(n)}| \leq |\Omega^{(0)}| \cdot e^{-\alpha_0 \alpha \beta n/3}$$

holds for every $n \geq R$.

As a corollary we obtain

$$\left| \bigcup_{n \geq N} (\Omega^{(n)} \cup \tilde{\Omega}^{(n)}) \right| < |\Omega^{(0)}| \sum_{n \geq R} e^{-\alpha_0 \alpha \beta n/3} < |\Omega^{(0)}|.$$

Hence, the set $\bigcap_{n \geq N} RR_n$ contains a positive measure subset. By Proposition 6.2.3, this implies Theorem B.

A proof of Proposition 8.1.2 needs some preliminary considerations and thus we postpone it to the end of this section. In what follows, we write $\Omega^{(n)}(\mathbf{n}, \mathbf{d}, u) = \Omega^{(n)}(\cdot)$, once $(\mathbf{n}, \mathbf{d}, u)$ is fixed. The meaning of $\tilde{\Omega}^{(n)}(\cdot)$ is analogous.

8.2. **Structure in parameter space.** Let $a \in \Omega^{(n)}(\cdot)$ (resp. $a \in \tilde{\Omega}^{(n)}(\cdot)$). We say a critical point ζ_0 of H_a of order $\geq n$ is *responsible* for a if ζ_0 satisfies (a), (b), (c) (resp. (d), (c), (e)).

Lemma 8.2.1. *Suppose that ζ_0, $\tilde{\zeta}_0$ are critical points of order n. If their backward orbits are linked to the same sample point in $S(n)$, then they share the same sequence of hyperbolic times and the associated sequence of sample points in $S(n)$.*

Lemma 8.2.2. *Let a, $\tilde{a} \in \Omega^{(n)}(\mathbf{n}, \mathbf{d}, u)$, or a, $\tilde{a} \in \tilde{\Omega}^{(n)}(u, m)$. Suppose that ζ_0, $\tilde{\zeta}_0$ are critical points of the same order which are responsible for a and $\tilde{a}$ respectively. Let $\{h_i\}_{i=1}^s$ denote the associated sequence of hyperbolic times, and let $\zeta_0^{(i)}(\cdot)$, $\tilde{\zeta}_0^{(i)}(\cdot)$ denote the smooth continuations of order h_i of ζ_0 and $\tilde{\zeta}_0$. If $J(a, \zeta_0, \nu, 0) \cap J(\tilde{a}, \tilde{\zeta}_0, \nu, 0) \neq \emptyset$ holds for some $\nu \in [\beta h_i/2, \beta h_i]$, then $\zeta_0^{(i)}(b) = \tilde{\zeta}_0^{(i)}(b)$ holds for all $b \in J(a, \zeta_0, \nu, 0) \cap J(\tilde{a}, \tilde{\zeta}, \nu, 0)$.*

Proof. Recall the construction of smooth continuations in Section 5 and use the fact that one admissible curve does not admit more than two precritical points of the same order (Remark 2.9.2). □

Lemma 8.2.3. *Let $a_* \in \Omega^{(n)}(\mathbf{n}, \mathbf{d}, u)$, and let ζ_0 denote a critical point which is responsible for a_*. For every $i \in [1, r]$, the set $J(a_*, \zeta_0, \nu_i, 0) - J(a_*, \zeta_0, \nu_i, d_i)$ does not intersect $\Omega^{(n)}(\mathbf{n}, \mathbf{d}, u)$.*

Proof. Consider the smooth continuation $\zeta_0^{(s(i))}(\cdot)$ of the quasi critical point of order $s(i)$ associated with ζ_0. Take $a \in J(a_*, \zeta_0, \nu_i, 0) - J(a_*, \zeta_0, \nu_i, d_i)$, and suppose that $a \in \Omega^{(n)}(\cdot)$. Let $\tilde{\zeta}_0$ denote **any** critical point which is responsible for a. Consider the smooth continuation $\tilde{\zeta}_0^{(s(i))}(\cdot)$ of the quasi critical point of order $s(i)$ associated with $\tilde{\zeta}_0$. By Lemma 8.2.2, $\zeta_0^{(s(i))}(a)$ coincides with $\tilde{\zeta}_0^{(s(i))}(a)$, which is exactly the secondary quasi critical point of order $s(i)$ associated with $\tilde{\zeta}_0$. By Lemma 7.5.1 and the assumption on a, $\zeta_{\nu_i+1}^{(s(i))}(a) = \tilde{\zeta}_{\nu_i+1}^{(s(i))}(a)$ is in admissible position. By (24) and (26), it follows that $\tilde{\zeta}_{\nu_i+1}$ is in admissible position as well.

Sublemma 8.2.4. *Suppose that $v_0(z_0)$ is in admissible position relative to two critical points ζ_0 and $\tilde{\zeta}_0$. Then*

$$-\log|\zeta_0 - z_0| \leq -(1+\alpha_0)\log|\tilde{\zeta}_0 - z_0|.$$

Proof. Let η denote the x-component of $DH(z_0)v_0$. By Lemma 3.3.3, we have $(1-\theta)|\eta| \leq |\zeta_0 - z_0| \leq (1+\theta)|\eta|$, and the same for $|\tilde{\zeta}_0 - z_0|$. Therefore $|\zeta_0 - z_0| \geq (1-\theta)(1+\theta)^{-1}|\tilde{\zeta}_0 - z_0| \geq |\tilde{\zeta}_0 - z_0|^{1+\alpha_0}$. □

By Sublemma 8.2.4 and Proposition 7.4.1, the essential return depth $d(\nu_i)$ of the forward orbit of $\tilde{\zeta}_0$ at time ν_i is strictly smaller than d_i. Thus (b) does not hold. This yields a contradiction to the assumption that $\tilde{\zeta}_0$ is responsible for a. □

Lemma 8.2.5. *Let $a_* \in \tilde{\Omega}^{(n)}(u,m)$, and let ζ_0 denote a critical point which is responsible for a_*. Then the set $J(a_*,\zeta_0,m,0) - J(a_*,\zeta_0,m,\alpha m)$ does not intersect $\tilde{\Omega}^{(n)}(\mathbf{h},\mathbf{u},m)$.*

Proof. Consider the smooth continuation $\zeta_0^{(s)}(\cdot)$ of the secondary quasi critical point of order n associated with ζ_0. Take $a \in J(a_*,\zeta_0,m,0) - J(a_*,\zeta_0,m,\alpha m)$, and suppose that $a \in \tilde{\Omega}^{(n)}(\cdot)$. Let $\tilde{\zeta}_0$ denote **any** critical point which is responsible for a. Consider the smooth continuation $\tilde{\zeta}_0^{(s)}(\cdot)$ of the quasi critical point of order n associated with $\tilde{\zeta}_0$. By Lemma 8.2.2, $\zeta_0^{(s)}(a)$ coincides with $\tilde{\zeta}_0^{(s)}(a)$, which is exactly the secondary quasi critical point of order n associated with $\tilde{\zeta}_0$. By Lemma 7.5.1 and the assumption on a, $\tilde{\zeta}_{m+1}^{(s)}(a)$ is in admissible position. By (24) and (26), $\tilde{\zeta}_{m+1}$ is in admissible position as well, and thus (e) does not hold. This yields a contradiction to the assumption that $\tilde{\zeta}_0$ is responsible for a. □

Lemma 8.2.6. *Let a, $\tilde{a} \in \Omega^{(n)}(\mathbf{n},\mathbf{d},u)$. Suppose that ζ_0, $\tilde{\zeta}_0$ are critical points of the same order which are responsible for a and $\tilde{a}$ respectively. Let ν_i, $\nu_j \in \mathbf{n}$ and suppose that $\nu_i < \nu_j$. If $J(a,\zeta_0,\nu_i,d_i) \cap J(\tilde{a},\tilde{\zeta}_0,\nu_j,d) \neq \emptyset$ holds for some $d \geq -\log\delta$, then $J(\tilde{a},\tilde{\zeta}_0,\nu_j,0) \subset J(a,\zeta_0,\nu_i,d_i - \alpha_0^{-1})$.*

Proof. By Proposition 7.1.2, the critical curve $\{\zeta_{\nu_i+1}^{(s(i))}(b) \colon b \in J(a,\zeta_0,\nu_i,d_i)\}$ is an admissible curve. By Lemma 7.4.2, there exists $\hat{a} \subset J(a,\zeta_0,\nu_i,d_i)$ such that $\zeta_{\nu_i+1}^{(s(i))}(\hat{a})$ is a critical point of order ν_i of $H_{\hat{a}}$. We claim that $\hat{a} \notin J(\tilde{a},\tilde{\zeta}_0,\nu_j,0)$ holds. This implies that one of the connected components of $J(\tilde{a},\tilde{\zeta}_0,\nu_j,0) - J(\tilde{a},\tilde{\zeta}_0,\nu_j,d)$ is contained in $J(a,z_0,\nu_i,d_i)$. This implies

$$2^{-1}(1-e^{-\alpha_0 d/2})|J(\tilde{a},\tilde{\zeta}_0,\nu_j,0)| \leq |J(a,\zeta_0,\nu_i,d_i)|.$$

Using $d \geq -\log\delta$ and the fact that δ is chosen after α_0, we obtain the inclusion.

It is left to prove the claim. Suppose that $\hat{a} \in J(\tilde{a},\tilde{\zeta}_0,\nu_j,0)$. Consider the smooth continuation $\tilde{\zeta}_0^{(s(i))}(\cdot)$ of order $s(i)$ of the secondary quasi critical point associated with $\tilde{\zeta}_0$. By Lemma 8.2.2, we have $\zeta_0^{(s(i))}(\hat{a}) = \tilde{\zeta}_0^{(s(i))}(\hat{a})$, and thus $\tilde{\zeta}_{\nu_i+1}^{(s(i))}(\hat{a})$ is a critical point of order $s(i)$. This yields a contradiction to the fact that points on the critical curve is in admissible position relative to some critical point, which was already proved in the proof of Lemma 7.3.5. □

Lemma 8.2.7. *Let a, $\tilde{a} \in \Omega^{(n)}(\mathbf{n}, \mathbf{d}, u)$. Suppose that ζ_0, $\tilde{\zeta}_0$ are critical points of the same order which are respectively responsible for a and $\tilde{a}$. If $J(a, \zeta_0, \nu_i, 0) \cap J(\tilde{a}, \tilde{\zeta}_0, \nu_i, 0) \neq \emptyset$ and $\tilde{a} \notin J(a, \zeta_0, \nu_i, 0)$, then $J(a, \zeta_0, \nu_i, \alpha_0^{-1}) \cap J(a', \tilde{\zeta}_0, \nu_i, d_i) = \emptyset$.*

Proof. Without loss of generality we may assume $a < \tilde{a}$. Put $2\ell = \text{length}(J(a, \zeta_0, \nu_i, 0))$ and $2\tilde{\ell} = \text{length}(J(\tilde{a}, \tilde{\zeta}_0, \nu_i, 0))$. Since $\tilde{a} \notin J(a, \zeta_0, \nu_i, 0)$, we have $h = \tilde{a} - a - \ell > 0$. By Lemma 8.2.3 we have $a \notin J(\tilde{a}, \tilde{\zeta}_0, \nu_i, 0)$, that is $\tilde{a} - a \geq \tilde{\ell}$, or equivalently $h + \ell \geq \tilde{\ell}$. Suppose that $h \geq \ell$. Then $\tilde{a} - \tilde{\ell} e^{-\alpha_0 d_i} \geq \tilde{a} - (\ell + h) e^{-\alpha_0 d_i} > \tilde{a} - (\tilde{a} - a)/2 > a + \ell$. This implies $J(a, \zeta_0, \nu_i, 0) \cap J(\tilde{a}, \tilde{\zeta}_0, \nu_i, d_i) = \emptyset$. Next, suppose that $h \leq \ell$. Then $\tilde{\ell} \leq 2\ell$, and therefore $\tilde{a} - \tilde{\ell} e^{-\alpha_0 d_i} \geq a + \ell - \tilde{\ell} e^{-\alpha_0 d_i} \geq a + \ell(1 - 2e^{-\alpha_0 d_i}) \geq a + \ell e^{-1}$. This implies $J(a, \zeta_0, \nu_i, \alpha_0^{-1}) \cap J(\tilde{a}, \tilde{\zeta}_0, \nu_i, d_i) = \emptyset$. □

Analogously one can prove the following, which is left as an exercise.

Lemma 8.2.8. *Let a, $\tilde{a} \in \tilde{\Omega}^{(n)}(u, m)$. Suppose that ζ_0, $\tilde{\zeta}_0$ are critical points of the same order which are respectively responsible for a and $\tilde{a}$. If $J(a, \zeta_0, m, 0) \cap J(\tilde{a}, \tilde{\zeta}_0, m, 0) \neq \emptyset$ and $\tilde{a} \notin J(a, \zeta_0, m, 0)$, then $J(a, \zeta_0, m, \alpha m) \cap J(\tilde{a}, \tilde{\zeta}_0, m, \alpha m) = \emptyset$.*

8.3. **Total number of combinations.**

Lemma 8.3.1.

$$\text{card}(\mathcal{D}_R) \leq e^{\frac{\alpha_0 R}{10}} \text{ and } \text{card}(\mathcal{N}_r) \leq e^{\frac{\alpha_0 \beta n}{10}}.$$

Proof. The cardinality of $\mathcal{D}_R$ is smaller than the total number of combinations of dividing R balls into r groups, which is smaller than the total number of combinations of aligning $R + r$ balls in a row, $\text{Card}(\mathcal{D}_R) \leq \binom{R+r}{r}$. The same argument applies to $\mathcal{N}_r$ and we have $\text{Card}(\mathcal{N}_r) \leq \binom{\beta n + r}{r}$

Sublemma 8.3.2. *For every $c > 0$, there exists $s_0 > 0$ such that*

$$\binom{n+s}{s} \leq e^{3cn}$$

holds for all positive integers n, s such that $s \leq s_0 n$.

Proof. Choose $s_0 > 0$ such that $s_0 \leq c$, ${s_0}^{-s_0} \leq e^c$, and $(1 + s_0)^{s_0} \leq e^c$. The Stirling formula for factorials $k! \in [1 + 1/4k]\sqrt{2\pi k} k^k e^{-k}$ gives

$$\binom{n+s}{s} = \frac{(n+s)!}{n!s!} \leq \frac{(n+s)^{n+s}}{n^n s^s} \leq \left(\frac{n+s}{n}\right)^n \left(\frac{n+s}{s}\right)^s.$$

Regarding the first term,

$$\left(\frac{n+s}{n}\right)^n = \left(1 + \frac{s}{n}\right)^n = e^{n \log\left(1 + \frac{s}{n}\right)} \leq e^s \leq e^{s_0 n} \leq e^{cn}.$$

Regarding the second term,

$$\left(\frac{n+s}{s}\right)^s = \left[\left(\frac{s}{n(1+s/n)}\right)^{-s/n}\right]^n \leq \left[\left(\frac{s}{n}\right)^{-s/n} \left(1 + \frac{s}{n}\right)^{s/n}\right]^n \leq e^{2cn}.$$

□

We use this sublemma with $c = \alpha_0/10$. Since s_0 depends only on α_0, and by the relation $1 \le r \le R/\log\delta^{-1}$ and $1 \le r \le \Delta\beta n/\log\delta^{-1}$, the requirement in the sublemma is satisfied for sufficiently small δ. Hence we obtain the desired inequalities. □

8.4. **Proof of Proposition 8.1.2.** The first half of the assertion follows from $f_2(0) = -1 = f_2(-1)$, because this implies that the first time at which parameter selection takes place can be made arbitrarily large.

We prove the second half of the assertion. We firstly estimate the measure of $\Omega^{(n)}$. For $a \in \Omega^{(n)}(\cdot)$ and $d \ge 0$, denote by $J(a,\nu_i,d)$ any parameter interval of the form $J(a,\zeta_0,\nu_i,d)$, where ζ_0 is a critical point which is responsible for a.

Consider the following operation. Choose some $a_1 \in \Omega^{(n)}(\cdot)$. If $\Omega^{(n)}(\cdot) \subset J(a_1,\nu_1,0)$, then stop the operation. If not, which can occur due to the presence of multiple critical points, choose $a_2 \in \Omega^{(n)}(\cdot) - J(a_1,\nu_1,0)$ and ask whether $\Omega^{(n)}(\cdot) \subset J(a_1,\nu_1,0) \cup J(a_2,\nu_1,0)$ or not. If so, then stop the operation. If not, choose $a_3 \in \Omega^{(n)}(\cdot) - J(a_1,\nu_1,0) - J(a_2,\nu_1,0)$ and ask whether $\Omega^{(n)}(\cdot) \subset J(a_1,\nu_1,0) \cup J(a_2,\nu_1,0) \cup J(a_3,\nu_1,0)$ or not. Repeat this. Since the length of intervals of the form $J(a_*,\nu_1,0)$ are bounded from below, this operation stops sooner or later and we end up with a finite set of parameters $S_1 = \{a_1,\cdots,a_{\ell_1}\} \subset \Omega^{(n)}(\cdot)$ such that

$$\Omega^{(n)}(\cdot) \subset \cup_{j_1=1}^{\ell_1} J(a_{j_1},\nu_1,0).$$

We extend this operation in the following way. Let $i \ge 1$, and denote by $\mathbf{j}(i) = (j_1, j_2 \cdots, j_i)$ the multi index. Suppose that we are given a finite set of parameters $S_i = \{a_{\mathbf{j}(i)}\} \subset \Omega^{(n)}(\cdot)$ which are indexed by $\mathbf{j}(i)$ and satisfy $\Omega^{(n)}(\cdot) \subset \cup_{\mathbf{j}(i)} J(a_{\mathbf{j}(i)},\nu_i,0)$. Take $a_{\mathbf{j}(i)} \in S_i$. Applying the above operation to $J(a_{\mathbf{j}(i)},\nu_i,0) \cap \Omega^{(n)}(\cdot)$ in the place of $\Omega^{(n)}(\cdot)$, we define a finite set of parameters $S_{i+1} = \{a_{\mathbf{j}(i),1}, a_{\mathbf{j}(i),2},\cdots,a_{\mathbf{j}(i),\ell_{i+1}}\} \subset \Omega^{(n)}(\cdot)$ such that

$$J(a_{\mathbf{j}(i)},\nu_i,0) \cap \Omega^{(n)}(\cdot) \subset \cup_{j_{i+1}=1}^{\ell_{i+1}} J(a_{\mathbf{j}(i),j_{i+1}},\nu_{i+1},0).$$

In particular, $\Omega^{(n)}(\cdot) \subset \cup_{\mathbf{j}(i+1)} J(a_{\mathbf{j}(i+1)},\nu_{i+1},0)$ holds. We repeat this construction up to $i = r$.

Claim 8.4.1.

$$\sum_{j_1=1}^{\ell_1} |J(a_{j_1},\nu_1,d_1)| \le |\Omega^{(0)}| \cdot e^{-\alpha_0 d_1/2}.$$

Proof. It holds that

$$\sum_{j_1=1}^{\ell_1} |J(a_{j_1},\nu_1,d_1)| \le e^{-\alpha_0 d_1+1} \sum_{j_1=1}^{\ell_1} |J(a_{j_1},\nu_1,\alpha_0^{-1})|.$$

By Lemma 8.2.7, the intervals $\{J(a_{j_1},\nu_1,\alpha_0^{-1})\}_{j_1=1}^{\ell_1}$ are two by two disjoint. Since they are contained in $\Omega^{(0)}$, we get the claim. □

Claim 8.4.2. *For every $1 \le i \le r-1$ and $a_{\mathbf{j}(i)} \in S_i$,*

$$\sum_{j_{i+1}=1}^{\ell_{i+1}} |J(a_{\mathbf{j}(i),j_{i+1}},\nu_{i+1},d_{i+1})| \le e^{-\alpha_0 d_{i+1}/2} \cdot |J(a_{\mathbf{j}(i)},\nu_i,d_i)|.$$

Proof. It holds that

$$\sum_{j_{i+1}=1}^{\ell_{i+1}} |J(a_{\mathbf{j}(i),j_{i+1}}, \nu_{i+1}, d_{i+1})| \le e^{-\alpha_0 d_{i+1}+1} \sum_{j_{i+1}=1}^{\ell_{i+1}} |J(a_{\mathbf{j}(i),j_{i+1}}, \nu_{i+1}, \alpha_0^{-1})|.$$

By Lemma 8.2.7, the intervals $\{J(a_{\mathbf{j}(i),j_{i+1}}, \nu_{i+1}, \alpha_0^{-1})\}_{j_{i+1}=1}^{\ell_{i+1}}$ are two by two disjoint. Hence it is enough to show that they are contained in $J(a_{\mathbf{j}(i)}, \nu_i, d_i - \alpha_0^{-1})$. This follows from Lemma 8.2.6 and $J(a_{\mathbf{j}(i),j_{i+1}}, \nu_{i+1}, d_{i+1}) \cap J(a_{\mathbf{j}(i)}, \nu_i, d_i) \neq \emptyset$ for every j_{i+1}, by construction. □

We are now in position to estimate the measure of $\Omega^{(n)}(\cdot)$. Lemma 8.2.3 gives $\Omega^{(n)}(\cdot) \subset \cup_{\mathbf{j}(r)} J(a_{\mathbf{j}(r)}, \nu_r, d_r)$, and thus

$$|\Omega^{(n)}(\cdot)| \le \sum_{\mathbf{j}(r)} |J(a_{\mathbf{j}(r)}, \nu_r, d_r)| = \sum_{\mathbf{j}(r-1)} \sum_{j_r=1}^{\ell_r} |J(a_{\mathbf{j}(r-1),j_r}, \nu_r, d_r)|.$$

Notice the nested nature of the expression of the right hand side: ℓ_r depends on $\mathbf{j}(r-1)$. Using Lemma 8.4.2,

$$\sum_{\mathbf{j}(r)} |J(a_{\mathbf{j}(r)}, \nu_r, d_r)| \le e^{-\alpha_0 d_r} \sum_{\mathbf{j}(r-1)} |J(a_{\mathbf{j}(r-1)}, \nu_{r-1}, d_{r-1})|.$$

Using this recursively we obtain

$$\sum_{\mathbf{j}(r)} |J(a_{\mathbf{j}(r)}, \nu_r, d_r)| \le |\Omega^{(0)}| e^{-\alpha_0 R/2}.$$

Using Lemma 8.3.1 and $r \le R$,

$$\begin{aligned} |\Omega^{(n)}| &\le \sum_{R,r} \sum_{\mathbf{n},\mathbf{d},u} |\Omega^{(n)}(\mathbf{n}, \mathbf{d}, u)| \\ &\le \mathrm{Card}(\mathcal{N}_r \times \mathcal{D}_R \times S(n)) \cdot |\Omega^{(0)}| \cdot \sum_{R \ge \frac{\alpha\beta(n-1)}{100}} R e^{-\alpha_0 R/2} \\ &\le |\Omega^{(0)}| e^{1000\Delta n} \sum_{R \ge \frac{\alpha\beta(n-1)}{100}} e^{-\alpha_0 R/3} \\ &\le |\Omega^{(0)}| e^{-\alpha\alpha_0\beta n/4}. \end{aligned}$$

The estimate of the measure of $\tilde{\Omega}^{(n)}$ is analogous and much simpler. Fix $u \in S(n)$ and $m \in [\beta(n-1), \beta n]$. By the same reasoning as before, one can find a finite number of parameters $a_1, \cdots, a_\ell \in \tilde{\Omega}^{(n)}(\cdot)$ such that $\tilde{\Omega}^{(n)}(\cdot) \subset \cup_{j=1}^{\ell} J(a_j, m, 0)$. The intervals $\{J(a_j, m, \alpha_0^{-1})\}$ are two by two disjoint, and $\tilde{\Omega}^{(n)}(\cdot) \subset \cup_{j=1}^{\ell} J(a_j, m, \alpha m)$. Hence we obtain

$$|\tilde{\Omega}^{(n)}(\cdot)| \le e^{-\alpha_0 \alpha m} \sum_{j=1}^{\ell} |J(a_j, m, \alpha_0^{-1})| \le e^{-\alpha_0 \alpha m} |\Omega^{(0)}| \le e^{-\alpha_0 \alpha \beta(n-1)}.$$

Therefore

$$|\tilde{\Omega}^{(n)}| \le |\Omega^{(0)}| m \cdot \mathrm{Card}(S(n)) \cdot e^{-\alpha_0 \alpha \beta (n-1)}$$
$$\le |\Omega^{(0)}| \beta n e^{100\Delta n} e^{-\alpha \alpha_0 \beta (n-1)} \le |\Omega^{(0)}| e^{-\alpha \alpha_0 \beta n/2}.$$

This finishes the proof of Proposition 8.1.2, and hence that of Theorem B. □

References

[BC85] M. Benedicks and L. Carleson - On iterations of $1 - ax^2$ on $(-1, 1)$. Ann. of Math. (2) 122 (1985), no. 1, 1–25.

[BC91] M. Benedicks and L. Carleson - The dynamics of the Hénon map. *Ann of Math. (2)* **133**, 73–169, (1991)

[BV01] M. Benedicks and M. Viana - Solution of the basin problem for Hénon-like attractors. *Invent math.* **143**, 375–434, (2001)

[BV06] M. Benedicks and M. Viana - Random perturbations and statistical properties of Hénon-like maps. Ann. Inst. H. Poincare Anal. Non Lineaire 23 (2006), no. 5, 713–752.

[BY93] M. Benedicks and L-S. Young - Sinai-Bowen-Ruelle measures for certain Hénon maps. *Invent Math.* **112**, 541–576, (1993)

[BY00] M. Benedicks and L-S. Young - Markov extensions and decay of correlations for certain Hénon maps. *Asterisque* No. 261 (2000), xi, 13–56.

[CE83] P. Collet and J. P. Eckmann - Positive Lyapunov exponents and absolute continuity for maps of the interval. *Ergod. Th. & Dynam. Sys.* (1983), **3**, 13-46

[DRV96] L. Díaz, J. Rocha, and M. Viana - Strange attractors in saddle-node cycles: prevalence and globality. *Invent. Math.* 125 (1996), no. 1, 37–74.

[J81] M. Jakobson - Absolutely continuous invariant measures for one-parameter families of one-dimensional maps. *Comm. Math. Phys.* **81** (1981), 39–88.

[MV93] L. Mora and M. Viana - Abundance of strange attractors. *Acta Math.* **171** 1–71. (1993)

[OP00] D. R. Orendovici and Y. B. Pesin - Chaos in traveling waves of lattice systems of unbounded media. Numerical methods for bifurcation problems and large-scale dynamical systems (Minneapolis, MN, 1997), 327–358, IMA Vol. Math. Appl., 119, Springer, New York, 2000.

[PH] E. Pujals and F. R. Hertz - Critical points for surface diffeomorphisms. preprint IMPA.

[PT93] J. Palis and F. Takens - *Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations. Fractal dimensions and infinitely many attractors.* Cambridge Studies in Advanced Mathematics, 35. Cambridge University Press. (1993)

[San00] E. Sander - Homoclinic tangles for noninvertible maps. Nonlinear Anal. 41 (2000), no. 1-2, Ser. A: Theory Methods, 259–276.

[T06] H. Takahasi - On the basin problem for Hénon-like attractors, *Journal of Math Kyoto Univ.* **46** no.2 (2006) 303-348

[Tsu93a] M. Tsujii - A proof of Benedicks-Carleson-Jakobson Theorem. Tokyo J. Math. 16 (1993), no. 2, 295–310.

[Tsu93b] M. Tsujii Positive Lyapunov exponents in families of one-dimensional dynamical systems. Invent. Math. 111 (1993), no. 1, 113–137

[Tsu03] M. Tsujii - Strange attractors in Hénon maps, in Japanese.

[Via95] M. Viana - Strange attractors in higher dimensions. *Bol. Soc. Brasil. Mat.* **24** (1993), no. 1, 13–62.

[WY01] Q. Wang and L-S. Young - Strange attractors with one direction of instability. *Comm. Math. Phys.* **218** (2001), 1–97.

[WY] Q. Wang and L-S. Young - Toward a theory of rank one attractors.

ESTRADA DONA CASTORINA 110 22460-320 IMPA RIO DE JANEIRO BRAZIL
E-mail address: takahash@impa.br