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Normality and Modulability Indices.
Part II: Convex Cones in Hilbert Spaces

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Abstract. Let K be a closed convex cone in a Hilbert space X. Let B_X be the closed unit ball of X and $K_\bullet = (B_X + K) \cap (B_X - K)$. The normality index

$$\nu(K) = \sup\{r \geq 0 : \; rK_\bullet \subset B_X\}$$

is a coefficient that measures to which extent the cone K is normal. We establish a formula that relates $\nu(K)$ to the maximal angle of K. A concept dual to normality is that of modulability. As a by-product one obtains a formula for computing the modulability index

$$\mu(K) = \sup\{r \geq 0 : \; rB_X \subset K^\bullet\}$$

of K. The symbol $K^\bullet$ stands for the absolutely convex hull of $K \cap B_X$. We show that $\mu(K)$ can be expressed in terms of the smallest critical angle of K.

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1 Introduction

This is a continuation of our paper [11] and therefore we keep the same notation and terminology. Given a normed space X, the symbols B_X and S_X refer, respectively, to the closed unit ball and the unit sphere. There are several metrics that serve to measure distances between elements of the set

$$\Xi(X) \equiv \text{nontrivial closed convex cones in } X.$$

In this work we consider the standard choice

$$\varrho(K_1, K_2) = \text{haus}(K_1 \cap B_X, K_2 \cap B_X), \tag{1}$$

where

$$\text{haus}(C_1, C_2) = \max\left\{ \sup_{z \in C_1} \text{dist}[z, C_2], \sup_{z \in C_2} \text{dist}[z, C_1] \right\}$$

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stands for the classical Pompeiu-Hausdorff distance between two bounded closed nonempty sets C_1, C_2, and $\mathrm{dist}[x, C]$ refers to the distance from x to the set C. By a convex cone we understand a nonempty set K satisfying $K + K \subset K$ and $\mathbb{R}_+ K \subset K$. Saying that a convex cone K is nontrivial simply means that K is different from $\{0\}$ and different from the whole space X. For the sake of completeness, we recall also the following two concepts:

Definition 1. *Let K be a convex cone K in a normed space X. One says that*

(a) K is normal if there is a constant $\beta > 0$ such that

$$\beta\,(\|u\| + \|v\|) \leq \|u + v\| \quad \textit{for all } u, v \in K. \tag{2}$$

(b) K is modulable if there is a constant $\gamma > 0$ such that

$$\begin{cases} \textit{any } x \in X \textit{ is expressible in the form } x = u - v \\ \textit{with } u, v \in K \textit{ satisfying } \gamma\,\|(u, v)\| \leq \|x\|. \end{cases}$$

We are taking $\|(u, v)\| = \left\{\|u\|^2 + \|v\|^2\right\}^{1/2}$, but any equivalent norm in the Cartesian product $X \times X$ is acceptable. Modulability is a fundamental concept of the theory of convex cones. Unfortunately, there is no universal agreement with respect to the terminology. Other names for the concept of modulability can be found for instance in [12, 14, 19, 22]. The concept of normality is also classical and doesn't need further justification; see for instance the books [4, 19, 22] or the pioneering works by Krein and collaborators [3, 15].

The next theorem is a combination of several sources. We take this result for granted.

Theorem 1. *Let K be a convex cone K in a normed space X. Then,*

(a) K is normal if and only if $K_\bullet = (B_X + K) \cap (B_X - K)$ is bounded.

(b) K is modulable if and only if $K^\bullet = \mathrm{co}[(K \cup -K) \cap B_X]$ is a neighborhood of the origin.

One of the main issues of this paper concerns the practical computation of the normality index

$$\nu(K) = \sup\{r \geq 0 : \; rK_\bullet \subset B_X\} \tag{3}$$

and the modulability index

$$\mu(K) = \sup\{r \geq 0 : \; rB_X \subset K^\bullet\} \tag{4}$$

of a nontrivial closed convex cone K. The meanings of these indices are explained in full length in our previous work [11]. It is not always easy to compute the right-hand sides in (3) and (4). The evaluation of the sets $K_\bullet$ and $K^\bullet$ is already a difficult task by itself. The main merit of the present paper is showing that $\nu(K)$ and $\mu(K)$ admit nice and simple characterization formulas when the underlying space X is Hilbert. To be more precise, we show that $\nu(K)$ can be expressed in terms of the maximal angle of K, whereas $\mu(K)$ is expressible in terms of the smallest critical angle of K.

Another theme discussed in this paper has to do with the link existing between pointedness and normality. The purpose of Section 4 is establishing the following topological results in the context of an infinite dimensional Hilbert space:

i) With respect to the truncated Pompeiu-Hausdorff metric ϱ, the pointed elements of $\Xi(X)$ don't form open set. Equivalently, the unpointed elements of $\Xi(X)$ don't form a closed set.

ii) By contrast, the abnormal elements of $\Xi(X)$ do form a closed set. In fact, the set of abnormal elements is precisely the ϱ-closure of the set of unpointed elements of $\Xi(X)$.

To the best of our knowledge, the above density result is new. By using a duality argument, we establish also a link between almost reproducibility and modulability.

2 A Preliminary Formula for Computing $\nu(K)$

As mentioned in the introduction, the practical computation of the coefficient $\nu(K)$ offers sometimes a serious challenge. All efforts in characterizing $\nu(K)$ by means of alternative formulas are not to be despised. The next theorem suggests considering the expression

$$\chi(K) = \inf_{\|z\|=1} \max\{\mathrm{dist}[z,K], \mathrm{dist}[-z,K]\} \tag{5}$$

as tool for evaluating $\nu(K)$. Of course, the computation of the infimum (5) is greatly simplified if one knows in advance the distance function $\mathrm{dist}[\cdot,K]$.

The term $\chi(K)$ appears already in references [5] and [9], but no connection with $\nu(K)$ has been made insofar.

Theorem 2. *For a nontrivial convex cone K in a normed space X, one has*

$$\nu(K) = \chi(K). \tag{6}$$

Proof. We start by proving the inequality $\chi(K) \leq \nu(K)$. Assume that $\chi(K) > 0$, otherwise we are done. We claim that

$$(B_X + K) \cap (B_X - K) \subset [\chi(K)]^{-1} B_X. \tag{7}$$

Take a nonzero vector x in the above intersection. Since

$$\begin{aligned} x \in B_X + K &\implies \mathrm{dist}[x,K] \leq 1, \\ x \in B_X - K &\implies \mathrm{dist}[-x,K] \leq 1, \end{aligned}$$

one has $\max\{\mathrm{dist}[x,K], \mathrm{dist}[-x,K]\} \leq 1$. In view of the positive homogeneity of the distance function $\mathrm{dist}[\cdot,K]$, one gets

$$\chi(K) \leq \max\left\{\mathrm{dist}\left[\frac{x}{\|x\|}, K\right], \mathrm{dist}\left[-\frac{x}{\|x\|}, K\right]\right\} \leq \frac{1}{\|x\|}.$$

This proves that x belongs to the right-hand side of (7) as needed. We now take care of the reverse inequality $\nu(K) \leq \chi(K)$. Ab absurdo, suppose that

$$(B_X + K) \cap (B_X - K) \subset [\chi(K) + s]^{-1} B_X \tag{8}$$

for some $s > 0$. We must arrive to a contradiction. Take $\varepsilon \in]0, s[$ and find a vector $z_\varepsilon \in X$ such that

$$\|z_\varepsilon\| = 1, \tag{9}$$

$$\max\{\mathrm{dist}[z_\varepsilon,K], \mathrm{dist}[-z_\varepsilon,K]\} < \chi(K) + \varepsilon. \tag{10}$$

The condition (10) implies that

$$\begin{aligned} z_\varepsilon &\in (\chi(K)+\varepsilon)B_X + K, \\ z_\varepsilon &\in (\chi(K)+\varepsilon)B_X - K. \end{aligned}$$

Since K is a convex cone, an elementary rearrangement yields

$$[\chi(K)+\varepsilon]^{-1}\, z_\varepsilon \in (B_X+K)\cap(B_X-K).$$

In view of (8) and (9), one gets in such a case

$$[\chi(K)+\varepsilon]^{-1} \leq [\chi(K)+s]^{-1},$$

contradicting the fact that $\varepsilon < s$. □

An important merit of Theorem 2 is its great generality: X is any normed space and K is not necessarily closed. Since $\chi(\mathrm{cl}(K)) = \chi(K)$, Theorem 2 confirms that the concept of normality is blind with respect to topological closure, i.e., a nontrivial convex cone K is normal if and only if $\mathrm{cl}(K)$ is normal.

We state below some additional by-products of the representation formula (6). We start with a Lipschitz continuity result for the function $\nu(\cdot)$. The notation

$$d(K_1,K_2) = \inf_{\|z\|\leq 1} |\mathrm{dist}[z,K_1] - \mathrm{dist}[z,K_2]|$$

indicates another metric on $\Xi(X)$ that is popular among convex analysts [21].

Corollary 1. *Let K_1 and K_2 be nontrivial closed convex cones in a normed space X. Then,*

$$|\nu(K_1)-\nu(K_2)| \leq d(K_1,K_2).$$

Proof. It is immediate from (6) and the definition of the uniform metric d. □

We continue with a topological result taking place in our usual metric space $(\Xi(X),\varrho)$.

Corollary 2. *Let X be a normed space. Then,*

$$\mathrm{Nor}(X) = \{K\in\Xi(X) : K \textit{ is normal}\}$$

is an open set in $(\Xi(X),\varrho)$.

Proof. A small adjustment in the proof of [1, Proposition 1.2] show that

$$d(K_1,K_2) \leq 2\,\varrho(K_1,K_2) \quad \forall K_1,K_2\in\Xi(X). \tag{11}$$

Thanks to (11) and Corollary 1, the function $\nu : (\Xi(X),\varrho)\to\mathbb{R}$ turns out to be Lipschitz continuous. This proves the announced result. □

In the same spirit as the radiuses of modulability, solidity, and sharpness, considered in our previous work [11], we introduce now the *radius of normality*

$$\rho_{\mathrm{nor}}(K) = \inf_{\substack{Q\in\Xi(X)\\ Q\ abnormal}} \varrho(K,Q) \tag{12}$$

of a given of $K \in \Xi(X)$. The least-distance problem (12) is of interest for itself and will be studied in detail in Section 4.

The next corollary is recorded for the sake of later use. Notice, parenthetically, that $\rho_{\rm nor} : (\Xi(X), \varrho) \to \mathbb{R}$ is the largest nonexpansive map that vanishes exactly over the abnormal elements of $\Xi(X)$.

Corollary 3. *If K is a nontrivial closed convex cone in a Hilbert space X, then $\nu(K) \leq \rho_{\rm nor}(K)$.*

Proof. If X is a Hilbert space, then d and ϱ are in fact identical. It suffices then to apply Corollary 1 and the pointwise maximality of $\rho_{\rm nor}(\cdot)$ among all the nonexpansive maps on $(\Xi(X), \varrho)$ that vanish exactly over the abnormal elements of $\Xi(X)$. □

3 The Best Normality Constant and the Angular Coefficient

3.1 Comparing $\beta(K)$ and $\sigma(K)$

A number $\beta > 0$ satisfying the inequality (2) is called a *normality constant* of K. The best possible normality constant of K is of course

$$\beta(K) = \inf_{\substack{u,v \in K \\ (u,v) \neq (0,0)}} \frac{\|u+v\|}{\|u\| + \|v\|}\,. \tag{13}$$

The number $\beta(K) \in [0,1]$ can be used as a tool for quantifying the degree of normality of K.

The purpose of this section is to compare (13) with the much simpler expression

$$\sigma(K) = \inf_{u,v \in K \cap S_X} \left\| \frac{u+v}{2} \right\|. \tag{14}$$

We have divided by 2 in the right-hand side of (14) just to make sure that $\sigma(K) \in [0,1]$. We shall come in a moment to the interpretation of the coefficient $\sigma(K)$. First, we state:

Proposition 1. *Let K be a nontrivial convex cone in a normed space X. Then*

$$\frac{1}{2}\,\sigma(K) \leq \beta(K) \leq \sigma(K). \tag{15}$$

Proof. If one adds the constraints $\|u\| = 1$ and $\|v\| = 1$ in the feasible set of (13), then one arrives at the minimization problem (14). This simple observation shows the second inequality in (15). The relation $\sigma(K) \leq 2\beta(K)$ is obtained as a consequence of the Massera-Schäffer inequality [16] which asserts that

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\,\|x-y\|}{\max\{\|x\|, \|y\|\}}$$

for all nonzero vectors x, y in a normed space X. □

In view of Proposition 1, the term $\sigma(K)$ is also an acceptable candidate as tool for measuring the degree of normality of K. By the way, are we sure that $\beta(K)$ and $\sigma(K)$ are different numbers? The answer is yes, but a difference can be observed only in a non-Hilbertian setting. We start with an easy example showing that $\beta(K)$ and $\sigma(K)$ may differ.

Example 1. Let the plane $\mathbb{R}^2$ be equipped with the Manhattan norm $\|x\| = |x_1| + |x_2|$, and let K be the nontrivial closed convex cone in $\mathbb{R}^2$ given by $K = \{x \in \mathbb{R}^2 : x_2 \geq 0,\ x_1 + x_2 \geq 0\}$. The set $K \cap S_X$ is a union of two segments, namely

$$K \cap S_X = \underbrace{\mathrm{co}\,\{(-1/2, 1/2), (0,1)\}}_{\Gamma_1} \bigcup \underbrace{\mathrm{co}\,\{(1,0), (0,1)\}}_{\Gamma_2}.$$

If u and v are on the same segment, say Γ_i, then the midpoint $(u+v)/2$ has unit length because it remains in Γ_i. Thus, for solving the minimization problem (14) one may suppose that u and v are on different segments, say $u \in \Gamma_1$ and $v \in \Gamma_2$. If one writes

$$\begin{aligned} u &= t\,(-1/2, 1/2) + (1-t)\,(0,1), \\ v &= s\,(1,0) + (1-s)\,(0,1), \end{aligned}$$

then one is lead to minimize

$$\left\| \frac{u+v}{2} \right\| = \frac{\left|s - \frac{t}{2}\right| + 2 - \frac{t}{2} - s}{2}$$

with respect to $t, s \in [0,1]$. The infimum is attained with $t = 1$ and arbitrary $s \geq 1/2$. Thus, $\sigma(K) = 1/2$. On the other hand, by choosing $u = (-1/3, 1/3)$ and $v = (1/3, 0)$, one gets

$$\beta(K) \leq \frac{\|u+v\|}{\|u\| + \|v\|} = \frac{1/3}{1} \ < \ \sigma(K).$$

Incidentally, Example 1 shows that the components of a pair (u,v) solving (14) don't need to be in the boundary of K, even if K is pointed and has nonempty interior. This phenomenon cannot occur in a Hilbert space setting (cf. [7, Lemma 2.1]).

One can sharpen the lower estimate for $\beta(K)$ if one has additional information on the geometry of the normed space X. Let c_X denote the *sphericity defect* of X, i.e., the infimum of all $c \geq 0$ such that

$$\frac{1}{2} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq (1+c)\, \frac{\|x-y\|}{\|x\| + \|y\|}$$

for all $x, y \in X \backslash \{0\}$. The Massera-Schäffer inequality implies that that $0 \leq c_X \leq 1$. This observation and the very definition of c_X leads to

$$\frac{1}{2}\, \sigma(K) \leq \frac{1}{1 + c_X}\ \sigma(K) \leq \beta(K). \tag{16}$$

Corollary 4. *If K is a nontrivial convex cone in a Hilbert space X, then $\sigma(K) = \beta(K)$.*

Proof. It remains to check that $\sigma(K) \leq \beta(K)$, but this is a consequence of (16) and the fact that $c_X = 0$ whenever X is a Hilbert space. That a Hilbert space has no sphericity defect follows from the Dunkl-Williams inequality [2] which asserts that

$$\frac{1}{2} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{\|x-y\|}{\|x\| + \|y\|}$$

for all $x, y \in X \backslash \{0\}$. □

Remark 1. The Dunkl-Williams inequality doesn't hold in a general normed space. In fact, the Dunkl-Williams inequality characterizes the norms that derive from an inner product (cf. [13]).

3.2 Angular Interpretation of $\sigma(K)$

If the norm $\|\cdot\|$ derives from an inner product $\langle\cdot,\cdot\rangle$, then the coefficient $\sigma(K)$ admits an interesting angular interpretation. Indeed, one can write

$$\sigma(K) = \cos\left(\frac{\theta_{\max}(K)}{2}\right) \tag{17}$$

with

$$\theta_{\max}(K) = \sup_{u,v\in K\cap S_X} \arccos\langle u,v\rangle \tag{18}$$

denoting the maximal angle of K. The angle maximization problem (18) it of interest for itself and has been extensively studied in [7] and [8]. The function $\theta_{\max}(\cdot)$ has found a large variety of applications as one can see in [6, 9, 18], among other references.

By obvious reasons, we refer to $\sigma(K)$ as the *angular coefficient*[3] of K. Notice, incidentally, that

$$\sigma(K) = 0 \quad \text{if and only if} \quad \theta_{\max}(K) = \pi.$$

Recall that a convex cone K in a normed space is said to be *pointed* if it doesn't contain a line, that is to say, $K\cap -K = \{0\}$. In a finite dimensional Hilbert space, pointedness of a nontrivial closed convex cone K is equivalent to the condition $\theta_{\max}(K) = \pi$. This fact is no longer true if the Hilbert space is infinite dimensional.

Example 2. Let $\ell_2(\mathbb{R})$ be the Hilbert space of square-summable real sequences. Notice that

$$K = \{x\in\ell_2(\mathbb{R}) : \sum_{k=1}^{n} x_k \geq 0, \forall n\geq 1\}$$

is a closed convex cone because it is expressible as intersection of closed half-spaces. One can easily check that K is pointed. Now, for each $n\geq 1$, we construct

$$\begin{aligned} u_n &= \frac{1}{\sqrt{2n}}\,(\underbrace{1,-1,1,-1,\ldots,1,-1}_{2n\text{ terms}},0,0,\ldots),\\ v_n &= \frac{1}{\sqrt{2n}}\,(0,\underbrace{1,-1,1,-1,\ldots,1,-1}_{2n\text{ terms}},0,0\ldots). \end{aligned}$$

Notice that (u_n,v_n) is a pair of unit vectors in K and

$$0\leq\sigma(K)\leq\left\|\frac{u_n+v_n}{2}\right\| = \frac{1}{2\sqrt{n}}\,.$$

By letting $n\to\infty$ one gets $\sigma(K)=0$. This infimum is not attained because otherwise K should contain a unit vector and its opposite.

Example 2 shows that a pointed closed convex cone may well have a maximal angle equal to π. This is what we call the *degeneracy phenomenon*. We insist on the fact that the degeneracy phenomenon cannot occur in a finite dimensional setting.

[3] Inspired by the relation (17), one could use the equality $\theta_{\max}(K) = 2\arccos[\sigma(K)]$ as definition of the maximal angle of a nontrivial convex cone K contained in a general normed space. Such definition is however purely formal.

3.3 Lipschitz Behavior of the Angular Coefficient

We start with a useful lemma on the nonexpansiveness of the angular coefficient $\sigma(\cdot)$ with respect to the metric

$$\vartheta(K_1, K_2) = \mathrm{haus}(K_1 \cap S_X, K_2 \cap S_X).$$

The definition of ϑ bears a strong resemblance with the definition (1) that we gave of ϱ. Be aware, however, that ϑ and ϱ are not the same metric.

Lemma 1. *Let K_1 and K_2 be nontrivial closed convex cones in a normed space X. Then,*

$$|\sigma(K_1) - \sigma(K_2)| \leq \vartheta(K_1, K_2).$$

Proof. Consider an arbitrary $\varepsilon > 0$. Pick up $u_\varepsilon, v_\varepsilon \in K_2 \cap S_X$ such that

$$\left\| \frac{u_\varepsilon + v_\varepsilon}{2} \right\| \leq \sigma(K_2) + \varepsilon.$$

Select then a couple of vectors $\hat{u}_\varepsilon, \hat{v}_\varepsilon$ in $K_1 \cap S_X$ such that

$$\begin{aligned} \|u_\varepsilon - \hat{u}_\varepsilon\| &\leq \mathrm{dist}[u_\varepsilon, K_1 \cap S_X] + \varepsilon, \\ \|v_\varepsilon - \hat{v}_\varepsilon\| &\leq \mathrm{dist}[v_\varepsilon, K_1 \cap S_X] + \varepsilon. \end{aligned}$$

One gets

$$\begin{aligned} 2[\sigma(K_1) - \sigma(K_2)] &\leq \|\hat{u}_\varepsilon + \hat{v}_\varepsilon\| - \|u_\varepsilon + v_\varepsilon\| + 2\varepsilon \\ &\leq \|\hat{u}_\varepsilon - u_\varepsilon\| + \|\hat{v}_\varepsilon - v_\varepsilon\| + 2\varepsilon \\ &\leq \mathrm{dist}[u_\varepsilon, K_1 \cap S_X] + \mathrm{dist}[v_\varepsilon, K_1 \cap S_X] + 4\varepsilon \\ &\leq 2\left[\sup_{w \in K_2 \cap S_X} \mathrm{dist}[w, K_1 \cap S_X]\right] + 4\varepsilon \end{aligned}$$

By letting $\varepsilon \to 0$ one arrives at

$$\sigma(K_1) - \sigma(K_2) \leq \sup_{w \in K_2 \cap S_X} \mathrm{dist}[w, K_1 \cap S_X] \ \leq \ \vartheta(K_1, K_2).$$

The proof of the inequality $\sigma(K_2) - \sigma(K_1) \ \leq \ \vartheta(K_1, K_2)$ is analogous. □

Remark 2. The metric ϑ is majorized by 2ϱ, so the angular coefficient $\sigma(\cdot)$ varies in a Lipschitz continuous manner also with respect to the metric ϱ.

4 Antipodality and Distance to Abnormality

We are interested in better understanding the minimization problem (12) that defines the radius of normality of a given nontrivial closed convex cone K.

This section takes place in a Hilbert space setting. Recall that in such a context, the truncated Pompeiu-Hausdorff distance ϱ admits the equivalent formulation

$$\varrho(K_1, K_2) = \max\left\{ \sup_{x \in K_1 \cap S_X} \mathrm{dist}[x, K_2], \ \sup_{x \in K_2 \cap S_X} \mathrm{dist}[x, K_1] \right\}. \tag{19}$$

The expression on the right-hand side of (19) is sometimes referred to as the gap distance between K_1 and K_2. For computational purposes, it is preferable to work with (19) and not with the original definition (1) of ϱ.

As explained next, the least-distance problem (12) turns out to be related to the the angle maximization problem (18). For the sake of convenience, we reformulate (18) in the equivalent form

$$\cos[\theta_{\max}(K)] = \inf_{u,v\in K\cap S_X} \langle u, v\rangle. \tag{20}$$

As in reference [7], one says that $(u_0, v_0) \in X \times X$ is an *antipodal pair* of K if

$$u_0, v_0 \in K \cap S_X \quad \text{and} \quad \langle u_0, v_0\rangle = \cos[\theta_{\max}(K)]. \tag{21}$$

Since the infimum in (20) is not necessarily attained, it is helpful to introduce also a suitable concept of approximate antipodality.

Definition 2. *Let K be a nontrivial convex cone in a Hilbert space X. An antipodal pair of K within a tolerance level $\varepsilon \geq 0$ is a pair $(u_\varepsilon, v_\varepsilon) \in X \times X$ such that*

$$u_\varepsilon, v_\varepsilon \in K \cap S_X \quad \textit{and} \quad \langle u_\varepsilon, v_\varepsilon\rangle \leq \cos[\theta_{\max}(K)] + \varepsilon. \tag{22}$$

Antipodality in the sense (21) is recovered by setting $\varepsilon = 0$. Example 2 nicely illustrates the fact that antipodal pairs may not exist for nontrivial closed convex cones in infinite dimensional Hilbert spaces.

In the sequel, $\mathbb{R}w = \{tw : t \in \mathbb{R}\}$ denotes the line generated by a nonzero vector $w \in X$ and $w^\perp$ indicates the hyperplane that is orthogonal to this line.

Lemma 2. *Let K be a nontrivial closed convex cone in a Hilbert space X. Let $(u_\varepsilon, v_\varepsilon)$ be an antipodal pair of K within a tolerance level $\varepsilon \geq 0$. Assume that $u_\varepsilon \neq v_\varepsilon$ and define*

$$Q_\varepsilon = (K \cap (u_\varepsilon - v_\varepsilon)^\perp) + \mathbb{R}(u_\varepsilon - v_\varepsilon).$$

Then, Q_ε is an unpointed closed convex cone such that

$$\begin{aligned} \text{dist}[x, K] &\leq \sqrt{\frac{1 + \langle u_\varepsilon, v_\varepsilon\rangle}{2}} && \forall x \in Q_\varepsilon \cap S_X, && (23)\\ \text{dist}[x, Q_\varepsilon] &\leq \left[1 + \frac{\varepsilon}{1 - \langle u_\varepsilon, v_\varepsilon\rangle}\right] \sqrt{\frac{1 + \langle u_\varepsilon, v_\varepsilon\rangle}{2}} && \forall x \in K \cap S_X. && (24)\end{aligned}$$

Proof. This result was obtained in [10] for the particular case $\varepsilon = 0$ and under the additional assumption that X is finite dimensional. Important adjustments in the proof are needed in order to take care of the general case. For convenience, we introduce the notation

$$w_\varepsilon = \|u_\varepsilon - v_\varepsilon\|^{-1}(u_\varepsilon - v_\varepsilon)$$

and divide the proof in three steps.

Step 1. The convex cone Q_ε is closed because it is expressible as sum of a line $\mathbb{R}w_\varepsilon$ and a closed set contained in $w_\varepsilon^\perp$. Observe that Q_ε is unpointed because $Q_\varepsilon \cap -Q_\varepsilon$ contains the unit vector w_ε.

Step 2. We establish the inequality (23). Take any $x \in Q_\varepsilon \cap S_X$, so that $x = z + tw_\varepsilon$, with $z \in K \cap w_\varepsilon^\perp$ and $t \in \mathbb{R}$. Clearly $t = \langle x, w_\varepsilon \rangle$, and therefore $|t| \leq 1$ by the Cauchy-Schwarz inequality. Consider the point y defined as

$$y = \begin{cases} z + (t/2)\, \|u_\varepsilon - v_\varepsilon\|\, u_\varepsilon & \text{if } t \geq 0, \\ z - (t/2)\, \|u_\varepsilon - v_\varepsilon\|\, v_\varepsilon & \text{if } t \leq 0. \end{cases}$$

Note that in both cases y belongs to K, because $z, u_\varepsilon, v_\varepsilon \in K$. We proceed to estimate the distance between x and y. Consider first the case of $t \geq 0$. One has

$$\begin{aligned} \|x - y\|^2 &= t^2 \left\| w_\varepsilon - \sqrt{\frac{1 - \langle u_\varepsilon, v_\varepsilon \rangle}{2}}\; u_\varepsilon \right\|^2 \\ &= t^2 \left[1 + \frac{1 - \langle u_\varepsilon, v_\varepsilon \rangle}{2} - 2\sqrt{\frac{1 - \langle u_\varepsilon, v_\varepsilon \rangle}{2}}\; \langle u_\varepsilon, w_\varepsilon \rangle \right]. \end{aligned}$$

Plugging in the above line the definition of w_ε, one ends up with

$$\|x - y\|^2 = t^2 \left[\frac{1 + \langle u_\varepsilon, v_\varepsilon \rangle}{2} \right].$$

Hence, $\mathrm{dist}[x, K] \leq \|x - y\| \leq \sqrt{(1 + \langle u_\varepsilon, v_\varepsilon \rangle)/2}$. The case of $t \leq 0$ is dealt in a similar way.

Step 3. We prove the inequality (24). Take $x \in K \cap S_X$ and consider the vector

$$y = x + \frac{|\langle x, w_\varepsilon \rangle|}{\|u_\varepsilon - v_\varepsilon\|}\, (u_\varepsilon + v_\varepsilon). \tag{25}$$

We decompose y in the form

$$y = \underbrace{x - \langle x, w_\varepsilon \rangle w_\varepsilon + \frac{|\langle x, w_\varepsilon \rangle|}{\|u_\varepsilon - v_\varepsilon\|}(u_\varepsilon + v_\varepsilon)}_{\tilde{y}} + \underbrace{\langle x, w_\varepsilon \rangle w_\varepsilon}_{\hat{y}}.$$

Clearly $\hat{y} \in \mathbb{R} w_\varepsilon$. We claim that $\tilde{y} \in K \cap w_\varepsilon^\perp$. For checking that $\tilde{y} \in w_\varepsilon^\perp$, note that

$$\begin{aligned} \langle \tilde{y}, w_\varepsilon \rangle &= \langle x, w_\varepsilon \rangle - \langle x, w_\varepsilon \rangle \|w_\varepsilon\|^2 + \frac{|\langle x, w_\varepsilon \rangle|}{\|u_\varepsilon - v_\varepsilon\|^2} \langle u_\varepsilon + v_\varepsilon, u_\varepsilon - v_\varepsilon \rangle \\ &= \frac{|\langle x, w_\varepsilon \rangle|}{\|u_\varepsilon - v_\varepsilon\|^2} \langle u_\varepsilon + v_\varepsilon, u_\varepsilon - v_\varepsilon \rangle = 0, \end{aligned}$$

using the fact that $\|w_\varepsilon\| = \|u_\varepsilon\| = \|v_\varepsilon\| = 1$. For checking that $\tilde{y} \in K$, rewrite $\tilde{y}$ as

$$\begin{aligned} \tilde{y} &= x - \frac{\langle x, w_\varepsilon \rangle}{\|u_\varepsilon - v_\varepsilon\|}(u_\varepsilon - v_\varepsilon) + \frac{|\langle x, w_\varepsilon \rangle|}{\|u_\varepsilon - v_\varepsilon\|}(u_\varepsilon + v_\varepsilon) \\ &= \begin{cases} x + 2\, \|u_\varepsilon - v_\varepsilon\|^{-1}\, |\langle x, w_\varepsilon \rangle|\, v_\varepsilon & \text{if } \langle x, w_\varepsilon \rangle \geq 0, \\ x + 2\, \|u_\varepsilon - v_\varepsilon\|^{-1}\, |\langle x, w_\varepsilon \rangle|\, u_\varepsilon & \text{if } \langle x, w_\varepsilon \rangle \leq 0. \end{cases} \end{aligned}$$

In both cases, $\tilde{y} \in K$ because x, u_ε and v_ε belong to K. We conclude that $y = \tilde{y} + \hat{y}$ belongs to Q_ε. We estimate next the distance between x and y. Directly from (25) one gets

$$\|x - y\| = \frac{|\langle x, u_\varepsilon - v_\varepsilon \rangle|}{\|u_\varepsilon - v_\varepsilon\|^2} \|u_\varepsilon + v_\varepsilon\| = \frac{|\langle x, u_\varepsilon - v_\varepsilon \rangle|}{2(1 - \langle u_\varepsilon, v_\varepsilon \rangle)} \sqrt{2(1 + \langle u_\varepsilon, v_\varepsilon \rangle)}.$$

In other words,

$$\|x - y\| = \eta \sqrt{\frac{1 + \langle u_\varepsilon, v_\varepsilon \rangle}{2}} \quad \text{with} \quad \eta = \frac{|\langle x, u_\varepsilon - v_\varepsilon \rangle|}{1 - \langle u_\varepsilon, v_\varepsilon \rangle} \geq 0.$$

To complete the proof of (24) we must check that $\eta \leq 1 + (1 - \langle u_\varepsilon, v_\varepsilon \rangle)^{-1} \varepsilon$, i.e.,

$$|\langle x, u_\varepsilon - v_\varepsilon \rangle| \leq 1 - \langle u_\varepsilon, v_\varepsilon \rangle + \varepsilon. \tag{26}$$

Since $(u_\varepsilon, v_\varepsilon)$ satisfies the approximate antipodality condition (22), it is clear that

$$\begin{aligned} \langle u_\varepsilon, v_\varepsilon \rangle &\leq \langle x, v_\varepsilon \rangle + \varepsilon, \\ \langle u_\varepsilon, v_\varepsilon \rangle &\leq \langle x, u_\varepsilon \rangle + \varepsilon. \end{aligned}$$

and, a posteriori,

$$\begin{aligned} \langle u_\varepsilon, v_\varepsilon \rangle &\leq \langle x, v_\varepsilon \rangle + \varepsilon + (1 - \langle x, u_\varepsilon \rangle), \\ \langle u_\varepsilon, v_\varepsilon \rangle &\leq \langle x, u_\varepsilon \rangle + \varepsilon + (1 - \langle x, v_\varepsilon \rangle). \end{aligned}$$

The combination of the last two inequalities can be written in the compact form

$$\max\{\langle x, u_\varepsilon \rangle - \langle x, v_\varepsilon \rangle, \langle x, v_\varepsilon \rangle - \langle x, u_\varepsilon \rangle\} \leq 1 - \langle u_\varepsilon, v_\varepsilon \rangle + \varepsilon,$$

but this is precisely (26). □

Keeping in mind the characterization (19) of ϱ, one sees that (23) and (24) produce

$$\varrho(K, Q_\varepsilon) \leq \left[1 + \frac{\varepsilon}{1 - \langle u_\varepsilon, v_\varepsilon \rangle}\right] \sqrt{\frac{1 + \langle u_\varepsilon, v_\varepsilon \rangle}{2}},$$

i.e., one gets an upper estimate for the truncated Pompeiu-Hausdorff distance between K and Q_ε. This observation has some noteworthy consequences. For example, it allows us to establish the following topological result relating the sets

$$\begin{aligned} \mathrm{Abn}(X) &= \{K \in \Xi(X) : K \text{ is abnormal}\}, \\ \mathcal{U}(X) &= \{K \in \Xi(X) : K \text{ is unpointed}\}. \end{aligned}$$

Theorem 3. *Let X be a Hilbert space. Then,*

$$\rho_{\mathrm{nor}}(K) = \inf_{\substack{Q \in \Xi(X) \\ Q \text{ unpointed}}} \varrho(K, Q) \qquad \forall K \in \Xi(X).$$

In particular, $\mathrm{Abn}(X)$ *is the closure with respect to the metric ϱ of the set $\mathcal{U}(X)$.*

Proof. The set $\mathrm{Abn}(X)$ is closed because its complement is open (cf. Corollary 2). Since $\mathcal{U}(X)$ is contained in $\mathrm{Abn}(X)$, one gets $\mathrm{cl}_\varrho[\mathcal{U}(X)] \subset \mathrm{Abn}(X)$ and

$$\rho_{\mathrm{nor}}(K) \leq \inf_{\substack{Q \in \Xi(X) \\ Q \text{ unpointed}}} \varrho(K, Q). \tag{27}$$

We claim that

$$\inf_{\substack{Q \in \Xi(X) \\ Q \text{ unpointed}}} \varrho(K, Q) \leq \sigma(K). \tag{28}$$

If K is a ray, then $\sigma(K) = 1$ and (28) holds trivially. Assume then that K is not a ray. Consider a minimizing sequence $\{(u_n, v_n)\}_{n \geq 1}$ for the antipodality problem (20), i.e., $u_n, v_n \in K \cap S_X$ are such that

$$\varepsilon_n = \langle u_n, v_n \rangle - \cos[\theta_{\max}(K)]$$

goes to 0 as $n \to \infty$. Since K is not a ray, there is no loss of generality in assuming that $u_n \neq v_n$. In view of Lemma 2, the set

$$Q_n = (K \cap (u_n - v_n)^\perp) + \mathbb{R}(u_n - v_n) \tag{29}$$

belongs to $\mathcal{U}(X)$ and

$$\inf_{\substack{Q \in \Xi(X) \\ Q \text{ unpointed}}} \varrho(K, Q) \leq \varrho(K, Q_n) \leq \left[1 + \frac{\varepsilon_n}{1 - \langle u_n, v_n \rangle}\right] \sqrt{\frac{1 + \langle u_n, v_n \rangle}{2}}\,.$$

But $\varepsilon_n \to 0$ and $\langle u_n, v_n \rangle \to \cos[\theta_{\max}(K)] \neq 1$. Hence,

$$\left[1 + \frac{\varepsilon_n}{1 - \langle u_n, v_n \rangle}\right] \sqrt{\frac{1 + \langle u_n, v_n \rangle}{2}} \;\to\; \sqrt{\frac{1 + \cos[\theta_{\max}(K)]}{2}} \;=\; \sigma(K).$$

This takes care of (28). By-the-way, if K is abnormal, then $\sigma(K) = 0$ and $\varrho(K, Q_n) \to 0$. In other words, $\mathrm{Abn}(X) \subset \mathrm{cl}_\varrho[\mathcal{U}(X)]$. Hence, $\mathcal{U}(X)$ is dense in $\mathrm{Abn}(X)$ and (27) is in fact an equality. □

Remark 3. Let K and $\{(u_n, v_n)\}_{n \in \mathbb{N}}$ be as in Example 2. We know that K is abnormal. Let Q_n be defined by (29). Since $\{(u_n, v_n)\}_{n \in \mathbb{N}}$ is a minimizing sequence for (20), it follows that $\varrho(K, Q_n) \to 0$. Observe that each Q_n is unpointed but the limit K is pointed! This confirms that $\mathcal{U}(X)$ is not a closed set in the metric space $(\Xi(X), \varrho)$.

5 Relating $\nu(K)$ to the Maximal Angle of K

The computation of $\chi(K)$ is not always as easy as one may thing at first sight, so we consider now an alternative characterization of the normality index $\nu(K)$. This time our analysis is restricted to a Hilbert space setting.

Theorem 4. *For a nontrivial closed convex cone K in a Hilbert space X, one has*

$$\nu(K) = \sigma(K) = \rho_{\mathrm{nor}}(K). \tag{30}$$

Proof. Most of the heavy work has been done already. From the proof of Theorem 3 we know already that $\rho_{\rm nor}(K) \le \sigma(K)$. On the other hand, Theorem 2 and Corollary 3 yield $\nu(K) = \chi(K)$ and $\nu(K) \le \rho_{\rm nor}(K)$, respectively. So, we just need to prove that

$$\sigma(K) \le \chi(K). \tag{31}$$

The inequality (31) is established in [9, Proposition 1], but only in a finite dimensional Hilbert space setting. In a general Hilbert space the proof follows a similar pattern, except that now the infimum in the definition of $\chi(K)$ is not necessarily attained. One works then with an approximate solution z_ε as in (9)-(10), and, of course, at the very end of the proof one lets $\varepsilon \to 0$. It is not worthwhile writing down all the details again. □

That $\|\cdot\|$ derives from an inner product is an essential assumption in Theorem 4. The following two examples show that the equality $\nu(K) = \sigma(K)$ is not necessarily true in a general normed space.

Example 3. Consider the space $\mathcal{C}([a,b],\mathbb{R})$ of continuous functions $x:[a,b]\to\mathbb{R}$ equipped with the uniform (or Chebyshev) norm $\|x\| = \max_{a\le t\le b}|x(t)|$. As indicated in [11, Example 6], for the closed convex cone

$$K = \{u \in \mathcal{C}([a,b],\mathbb{R}) : u(t) \ge 0 \ \forall t \in [a,b]\}$$

one has $\nu(K) = 1$. We claim that $\sigma(K) = 1/2$. To see this, take any pair of vectors $u, v \in K \cap S_X$. Let $t^* \in [a,b]$ be such that $u(t^*) = 1$. Then, $\|u+v\| \ge u(t^*) + v(t^*) \ge 1$, getting in this way $\sigma(K) \ge 1/2$. This lower bound is attained by choosing for instance

$$u(t) = (b-a)^{-1}(t-a)\ , \quad v(t) = (b-a)^{-1}(b-t).$$

In the above example one gets $\nu(K) > \sigma(K)$, but obtaining the reverse inequality $\nu(K) < \sigma(K)$ is also possible.

Example 4. Suppose now that $\mathcal{C}([a,b],\mathbb{R})$ is equipped with the norm $\|x\| = \int_a^b |x(t)|dt$. Let K be the same cone as in Example 3. This time one has $\sigma(K) = 1$ because $\|u+v\| = \|u\| + \|v\|$ for all $u, v \in K$. On the other hand, $\nu(K) = 1/2$ (cf. [11, Example 7]).

6 Dualization

We continue working in a Hilbert space setting and "dualize" some of the results established in the previous sections. What we do basically is exploiting the involutory relation between a closed convex cone $K \subset X$ and its dual cone

$$K^+ = \{y \in X : \langle y, x\rangle \ge 0 \ \forall x \in K\}.$$

Of course, one can recover the original cone K starting from K^+, to wit

$$K = \{x \in X : \langle y, x\rangle \ge 0 \ \forall y \in K^+\}.$$

Modulability and normality are dual concepts. There are two ways of expressing this fact in a more precise manner: either in terms of the indices $\mu(\cdot)$ and $\nu(\cdot)$, or in terms of the radiuses $\rho_{\rm nor}(\cdot)$ and $\rho_{\rm mod}(\cdot)$.

Theorem 5. *Let K be a closed convex cone in a Hilbert space X. Then,* *(a)* $\mu(K) = \nu(K^+)$ *and* $\nu(K) = \mu(K^+)$.

(b) $\rho_{\rm mod}(K) = \rho_{\rm nor}(K^+)$ *and* $\rho_{\rm nor}(K) = \rho_{\rm mod}(K^+)$.

Proof. Part (a) is established in [11, Theorem 6]. Part (b) is obtained by exploiting the Walkup-Wets Isometry Theorem (cf. [23]) which asserts that $Q \mapsto Q^+$ is a distance-preserving operation on $(\Xi(X), \varrho)$. The proof of the first formula in (b) runs as follows:

$$\begin{aligned}\rho_{\rm mod}(K) &= \inf_{\substack{Q\in\Xi(X)\\ Q \text{ not modulable}}} \varrho(K,Q) = \inf_{\substack{Q\in\Xi(X)\\ Q \text{ not modulable}}} \varrho(K^+,Q^+)\\ &= \inf_{\substack{P\in\Xi(X)\\ P \text{ abnormal}}} \varrho(K^+,P) = \rho_{\rm nor}(K^+).\end{aligned}$$

The second formula in (b) is proven in a similar way. □

A convex cone K in a normed space X is called *almost reproducing* if $\text{span}(K) = K - K$ is a dense subspace of X. Recall that a closed convex cone K in a reflexive Banach space X is almost reproducing if and only if K^+ is pointed.

The next theorem relates the sets

$$\begin{aligned}\text{Nar}(X) &= \{K \in \Xi(X) : \ K \text{ is not almost reproducing}\},\\ \text{Nmod}(X) &= \{K \in \Xi(X) : \ K \text{ is not modulable}\},\end{aligned}$$

as well as the functions $\rho_{\rm mod}(\cdot)$ and $\mu(\cdot)$.

Theorem 6. *Let X be a Hilbert space. Then,*

(a) For all $K \in \Xi(X)$ one has

$$\rho_{\rm mod}(K) = \inf_{Q\in\text{Nar}(X)} \varrho(K,Q).$$

In particular, $\text{Nmod}(X)$ *is the closure with respect to the metric ϱ of the set* $\text{Nar}(X)$.

(b) The radius of modulability of $K \in \Xi(X)$ admits also the representation

$$\rho_{\rm mod}(K) = \mu(K).$$

Proof. Part (a) is a matter of rephrasing Theorem 3. We just need to keep in mind the duality formulas of Theorem 5(b) and the fact that

$$\begin{aligned}\text{Nar}(X) &= \{K \in \Xi(X) : \ K^+ \text{ is not pointed}\},\\ \text{Nmod}(X) &= \{K \in \Xi(X) : \ K^+ \text{ is abnormal}\}.\end{aligned}$$

Part (b) is also a matter of using duality arguments. By recalling Theorems 5(b) and 4, in that order, one gets the equalities

$$\rho_{\rm mod}(K) = \rho_{\rm nor}(K^+) = \nu(K^+).$$

Theorem 5(a) yields $\nu(K^+) = \mu(K)$ and completes the proof of the theorem. □

6.1 Relating $\mu(K)$ to the Smallest Critical Angle of K

What means such a thing as the smallest critical angle of K? What is a critical angle anyway? These questions need to be clarified before formulating the dual version of Theorem 4.

The concept of critical angle derives from the first-order stationarity (or criticality) conditions for the antipodality problem (20). It reads as follows (cf. [7, 8]):

Definition 3. *Let K be a nontrivial closed convex cone in a Hilbert space X. A critical pair of K is any pair $(u,v) \in X \times X$ of vectors satisfying*

$$\begin{aligned} u, v &\in K \cap S_X, \\ v - \langle u, v\rangle u &\in K^+, \\ u - \langle u, v\rangle v &\in K^+. \end{aligned}$$

The angle $\theta(u,v) = \arccos\langle u,v\rangle$ formed by a critical pair is called a critical angle. The adjective proper is added when u and v are not collinear, that is to say, $|\langle u,v\rangle| \neq 1$.

Improper critical angles are irrelevant and usually left aside from the discussion. The proper critical angles of K and those of K^+ are related by a certain reflexion principle whose formulation is astonishingly simple:

$$\theta \text{ is a proper critical angle of } K \iff \pi - \theta \text{ is a proper critical angle of } K^+.$$

This principle was established in [8, Theorem 3] in a finite dimensional Hilbert space, but the finite dimensionality asssumption can be dropped.

In an infinite dimensional context there is no guarantee about the attainability of critical angles because the unit sphere S_X is no longer compact. Despite this fact, it makes sense to refer to the number

$$\theta_{\min}(K) = \pi - \theta_{\max}(K^+)$$

as the smallest (proper) critical angle of K. We are perhaps abusing of language, but not too seriously.

Theorem 7. *For a nontrivial convex cone K in a Hilbert space X, one has*

$$\mu(K) = \sin\left(\frac{\theta_{\min}(K)}{2}\right). \tag{32}$$

Proof. By combining Theorems 4 and 5(a), one gets

$$\mu(K) = \nu(K^+) = \sigma(K^+) = \cos\left(\frac{\theta_{\max}(K^+)}{2}\right) = \cos\left(\frac{\pi - \theta_{\min}(K)}{2}\right) = \sin\left(\frac{\theta_{\min}(K)}{2}\right).$$

□

Formula (32) applies even if K doesn't admit a critical pair (u,v) forming the angle $\theta_{\min}(K)$. The lack of attainability of the smallest critical angle is not a problem at all.

7 By Way of Conclusion

Theorems 3 and 4, and their dual counterparts, are perhaps the most significant contributions of this paper. It is not the intention here to display a full list of conclusions that can be drawn from these results, but at least two corollaries deserve to be properly recorded.

The first corollary concerns the modulability and normality indices of infra-dual cones. Recall that a closed convex cone K in a Hilbert space X is said to be

$$\begin{cases} \text{infra-dual} & \text{if} \quad K \subset K^+, \\ \text{supra-dual} & \text{if} \quad K \supset K^+, \\ \text{self-dual} & \text{if} \quad K = K^+. \end{cases}$$

That K is infra-dual amounts to saying that $\langle x, y \rangle \geq 0$ for all $x, y \in K$.

Corollary 5. *Let K be a nontrivial closed convex cone K in a Hilbert space X.*

(a) If K is infra-dual, then $\mu(K) \leq \sqrt{2}/2 \leq \nu(K)$.

(b) If K is supra-dual, then $\nu(K) \leq \sqrt{2}/2 \leq \mu(K)$.

(c) If K is self-dual, then $\mu(K) = \nu(K) = \sqrt{2}/2$.

Proof. Part (a). Infra-duality of K implies that $\theta_{\max}(K) \leq \pi/2$. Formula (30) yields then $\nu(K) \geq \sqrt{2}/2$. The proof of the inequality $\mu(K) \leq \sqrt{2}/2$ is a bit more delicate. We suppose that K is modulable, otherwise we are done. According to [11, Proposition 2], one has

$$\frac{1}{\mu(K)} = \underbrace{\sup_{\|x\| \leq 1} \inf_{(u,v) \in D_K(x)} \{\|u\| + \|v\|\}}_{\zeta(K)} \tag{33}$$

with $D_K(x) = \{(u, v) \in X \times X : u, v \in K, u - v = x\}$ denoting the set of all decompositions of a given x as difference of two vectors in K. We claim that

$$(u, v) \in D_K(x) \quad \Longrightarrow \quad \|u\| \geq \text{dist}[x, -K] \;\text{ and }\; \|v\| \geq \text{dist}[x, K].$$

Indeed, if $x = u - v$ with $u, v \in K$, then

$$\begin{aligned} \|u\| = \|x - (-v)\| &\geq \text{dist}[x, -K], \\ \|v\| = \|x - u\| &\geq \text{dist}[x, K]. \end{aligned}$$

Since K is assumed to be infra-dual, one has $-K \subset K^-$, with $K^- = -K^+$ indicating the negative dual of K. Hence, $\text{dist}[x, -K] \geq \text{dist}[x, K^-]$ and

$$\inf_{(u,v) \in D_K(x)} \{\|u\| + \|v\|\} \geq \text{dist}[x, K] + \text{dist}[x, K^-].$$

We now take on both sides the supremum with respect to $x \in B_X$. By positive homogeneity, this produces the same result as taking the supremum over the unit sphere S_X. Hence,

$$\zeta(K) \geq \sup_{\|x\|=1} \{\text{dist}[x, K] + \text{dist}[x, K^-]\}.$$

In view of (33), we just need to prove that

$$\sup_{\|x\|=1} \{\mathrm{dist}[x,K] + \mathrm{dist}[x,K^-]\} \geq \sqrt{2}\,.$$

To check this inequality it suffices to guarantee the existence of a vector x such that

$$\mathrm{dist}[x,K] = \frac{\sqrt{2}}{2}, \quad \mathrm{dist}[x,K^-] = \frac{\sqrt{2}}{2}, \quad \|x\| = 1. \tag{34}$$

To see that this system is solvable, we start with a vector w of length $\sqrt{2}/2$ lying in the boundary of K. We take then a unit vector $h \in X$ such that

$$\langle h, w' - w\rangle \leq 0 \quad \forall w' \in K. \tag{35}$$

Geometrically speaking, the inequality (35) means that h is normal to K at w. The collection of all such h is usually refered to as the normal cone to K at w (cf. [20, Section 2]). By taking $w' = 2w$ and then $w' = (1/2)w$, one sees that $\langle h, w\rangle = 0$, i.e., h is orthogonal to w. Another useful observation is this: w is a point in K at minimal distance from $w + (\sqrt{2}/2)\,h$. So, it is not difficult to check that $x = w + (\sqrt{2}/2)\,h$ solves the system (34). Indeed,

$$\begin{aligned}
&\|x\|^2 = \|w + (\sqrt{2}/2)\,h\|^2 = \|w\|^2 + (\sqrt{2}/2)^2\,\|h\|^2 = 1,\\
&\mathrm{dist}[x,K] = \|x - w\| = \|(\sqrt{2}/2)\,h\| = \sqrt{2}/2,\\
&\mathrm{dist}[x,K^-] = \sqrt{\|x\|^2 - (\mathrm{dist}[x,K])^2} = \sqrt{1 - (\sqrt{2}/2)^2} = \sqrt{2}/2,
\end{aligned} \tag{36}$$

the first equality in (36) being a known Pythagorean formula that relates the distance functions $\mathrm{dist}[\cdot,K]$ and $\mathrm{dist}[\cdot,K^-]$ (cf. [17]).

Part (b). If K is supra-dual, then K^+ is infra-dual. Part (a) yields $\mu(K^+) \leq \sqrt{2}/2 \leq \nu(K^+)$. Theorem 5(a) does the rest of the job.

Part (c). It is obtained by combining (a) and (b). ☐

Our last corollary concerns a practical algorithm for solving the least-distance problem (12).

Corollary 6. *Suppose that $K \in \Xi(X)$ is not a ray. The following statements hold true:*

(a) If $\{(u_n, v_n)\}_{n\in\mathbb{N}}$, with $u_n \neq v_n$, is a minimizing sequence for the antipodality problem (20), then the sequence $\{Q_n\}_{n\in\mathbb{N}}$ defined by (29) is minimizing for the least-distance problem (12).

(b) If K admits an antipodal pair, say (u_0, v_0), then

$$Q_0 = (K \cap (u_0 - v_0)^\perp) + \mathbb{R}(u_0 - v_0)$$

is an "exact" solution to (12), i.e., an abnormal element of $\Xi(X)$ lying at minimal distance from K.

Proof. Combine Theorem 4 and the argument developed in the proof of Theorem 3. ☐

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