Special ramification loci on the double product of a general curve *

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Abstract

Let C be a general connected, smooth, projective curve of positive genus g. For each integer $i \geq 0$ we give formulas for the number of pairs $(P, Q) \in C \times C$ off the diagonal such that $(g+i-1)Q-(i+1)P$ is linearly equivalent to an effective divisor, and the number of pairs $(P, Q) \in C \times C$ off the diagonal such that $(g+i+1)Q-(i+1)P$ is linearly equivalent to a moving effective divisor.

1 Introduction

Let C be a general connected, smooth, projective curve of genus $g > 0$. Put $C^2 := C \times C$, and let $\Delta \subset C^2$ be the diagonal. For each integer $i \geq 0$ consider

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the following loci on C^2:

$$D_i := \{(P,Q) \in C^2 - \Delta \mid h^0(\mathcal{O}_C((g+i-1)Q-(i+1)P)) > 0\},$$
$$E_i := \{(P,Q) \in C^2 - \Delta \mid h^0(\mathcal{O}_C((g+i+1)Q-(i+1)P)) > 1\}.$$

Our Proposition 5.4 claims that D_i and E_i are finite, and our main result, Theorem 5.6, gives formulas for the number of points in D_i and E_i.

A formula for the number of points in D_i appeared already as Lemma 6.3 on page 24 of the seminal work by Diaz [6], where the unnecessary extra hypotheses that $g \geq 2$ and $i \geq 2$ are made. Diaz used this formula to compute the class in the moduli space of genus-g stable curves $\overline{\mathcal{M}}_g$ of the closure $\overline{\mathcal{D}}_g$ of the locus of smooth curves C having a Weierstrass point P of type $g-1$, i.e. such that $h^0(\mathcal{O}_C((g-1)P)) \geq 2$.

Later on, Cukierman [4] gave a formula for the class in $\overline{\mathcal{M}}_g$ of the closure $\overline{\mathcal{E}}_g$ of the locus of smooth curves C containing a Weierstrass point P of type $g+1$, i.e. such that $h^0(\mathcal{O}_C((g+1)P)) \geq 3$. He did not follow in Diaz's footsteps for this formula, but rather observed that the union $\overline{\mathcal{D}}_g \cup \overline{\mathcal{E}}_g$ is the branch locus of the Weierstrass divisor on the "universal" curve over $\overline{\mathcal{M}}_g$, and used a Hurwitz formula with singularities to compute the class of this branch divisor.

Had Cukierman followed in Diaz's footsteps, he would probably have found he needed a formula for the number of points in E_i. We give this formula here.

In fact, in a sense to be explained below, it is slightly easier to obtain the number of points in E_i than in D_i, though we obtain both in a quite integrated form here. To obtain these numbers, the natural procedure is to use Porteous formula to compute the virtual classes of certain natural ramification schemes D_i^+ and E_i^+ of maps of vector bundles on C^2; see Subsection 5.2. Set-theoretically, D_i^+ and E_i^+ are given exactly as D_i and E_i, but without the restriction that the pair (P,Q) lies off Δ.

The problem is that D_i^+ and E_i^+ are both larger than D_i and E_i. Indeed, E_i^+ is the union of E_i with the set of points (P,P) such that P is a Weierstrass point of C and, worse, D_i^+ is the union of D_i and the whole diagonal Δ. Since E_i^+ is finite, Porteous formula does give an expression for the number of points in E_i^+, with weights, and thus at least an upper bound for the number of points in E_i. But it does not a priori give any information on D_i.

To compute the number of points in D_i and E_i, we use the fact that, by the Riemann-Roch Theorem, the union of D_i and E_i is the locus SW_i of pairs

$(P,Q) \in C^2 - \Delta$ such that Q is a special ramification point of the complete linear system $H^0(\omega_C((i+1)P))$, where ω_C is the canonical bundle of C.

We give SW_i a scheme structure as follows. First, we consider the ramification divisor $Z_i \subset C^2$ of the family of linear systems $H^0(\omega_C((i+1)P))$ parameterized by $P \in C$. Our Proposition 2.2 implies that Z_i contains Δ with multiplicity exactly g. Set $W_i := Z_i - g\Delta$. Our Proposition 4.3 gives an expression for the cycle $[W_i]$, and our Proposition 4.4 claims that W_i is nonsingular. Furthermore, in Subsection 5.2 we observe that the branch divisor of W_i with respect to the projection $p_1\colon C^2 \to C$ over the first factor has support SW_i. We give SW_i the structure of this branch divisor.

The advantage of considering SW_i is that it is quite easy to compute its degree. Indeed, $[SW_i]$ is the second Chern class of the bundle of first-order relative jets of p_1 with coefficients in $\mathcal{O}_{C^2}(W_i)$. Having an expression for $[W_i]$ we derive very quickly an expression for $\int_{C^2}[SW_i]$ in Proposition 5.5

Now, giving D_i and E_i the subscheme structures induced from D_i^+ and E_i^+, our Proposition 5.4 shows that, as 0-cycles,

$$[D_i] + [E_i] = [SW_i].$$

Actually, D_i and E_i are reduced. Indeed, in the proof of Theorem 5.6 we show that the weight of (P,Q) in $[SW_i]$ is at most 2, and the maximum weight is achieved if and only if $(P,Q) \in D_i \cap E_i$.

Now, as we already know $\int_{C^2}[SW_i]$, it is enough to compute either $\int_{C^2}[D_i]$ or $\int_{C^2}[E_i]$. As mentioned above, we compute the latter. In fact, we can get $\int_{C^2}[E_i^+]$ using Porteous formula, and a local analysis, done in Proposition 5.4, shows that the weight of (P,P) in $[E_i^+]$ is equal to $g+1$ for each Weierstrass point of C. Thus $\int_{C^2}[E_i]$ follows.

In a second article [5], we show how the knowledge of the number of points in E_i can be used to compute the class of $\overline{\mathcal{E}}_g$ in $\overline{\mathcal{M}}_g$. This computation is not straightforward as, following in Diaz's footsteps, we have to determine the limits of special Weierstrass points of type $g+1$ on stable curves with just one node. This is the main result of [5].

The limits of special Weierstrass points of type $g-1$ were computed by Diaz, using admissible covers. However, the same method does not apply to points of type $g+1$. For those we apply in [5] the theory of limit linear series in a rather new way, using 2-parameter families. Actually, as in the present article, we use an integrated approach in [5] that yields simultaneously the limits of special Weierstrass points of both types, and also formulas for the classes of both $\overline{\mathcal{D}}_g$ and $\overline{\mathcal{E}}_g$.

Here is a layout of the article. In Section 2 we review the theory of linear systems and ramification on a smooth curve C, introduce the linear systems we will consider in the remainder of the article, $H^0(\omega_C((i+1)P))$ for $P \in C$, and prove a preliminary result about them. In Section 3, assuming C is general, we obtain through degeneration methods results that bound the order sequence of $H^0(\omega_C((i+1)P))$ at any point of C. In Section 4, we describe the structure of the ramification divisor $Z_i \subset C^2$ of the family of linear systems $H^0(\omega_C((i+1)P))$ parameterized by $P \in C$. Finally, in Section 5 we define the loci D_i and E_i and compute their number of points, through the study of the locus SW_i of special ramification points of the family $H^0(\omega_C((i+1)P))$ for $P \in C$.

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2 Setup

2.1 (*Linear systems and ramification*) Let C be a *smooth curve*, that is, a projective, connected, smooth scheme of dimension 1 over $\mathbb{C}$. Denote by ω_C its canonical sheaf. Let $g := h^0(C, \omega_C)$, the genus of C.

Let V be a $\mathbb{C}$-vector space of sections of a line bundle $\mathcal{L}$ on C. We call V a *linear system*. The linear system is called *complete* if $V = H^0(C, \mathcal{L})$. Let $r := \dim V - 1$ and $d := \deg \mathcal{L}$. We call r the *rank* of V and d its *degree*. We say as well that $\dim V$ is the *dimension* of V.

For each point P of C, and each integer $j \geq 0$, let $V(-jP)$ denote the vector subspace of V of sections of $\mathcal{L}$ that vanish with order at least j at P. We say that j is an *order* of V at P if $V(-jP) \neq V(-(j+1)P)$. There are $r+1$ orders, which, in an increasing sequence, will be denoted by

$$\epsilon_0(V, P),\ \epsilon_1(V, P),\ \ldots,\ \epsilon_r(V, P).$$

For each integer $\ell \geq 0$ and each line bundle $\mathcal{M}$ on C, let $\mathcal{J}^\ell_C(\mathcal{M})$ be the bundle of *jets*, or *principal parts*, of $\mathcal{M}$ truncated in order ℓ. Consider the map of rank-r bundles,

$$V \otimes \mathcal{O}_C \longrightarrow \mathcal{J}^r_C(\mathcal{L}),$$

locally obtained by differentiating up to order r the sections of $\mathcal{L}$ in V. The *wronskian* w_V of V is the (nonzero) section of

$$\mathcal{L}^{\otimes r+1} \otimes \omega_C^{\otimes r(r+1)/2}$$

induced by taking determinants in the above map of bundles.

For each point P of C, the *weight* $\mathrm{wt}_V(P)$ of P in V is the order of vanishing of w_V at P. We call P a *ramification point* of V if $\mathrm{wt}_V(P) > 0$; otherwise we call P *ordinary*. A local analysis yields the formula

$$\mathrm{wt}_V(P) = \sum_{j=0}^{r} (\epsilon_j(V, P) - j).$$

We call P a *simple* ramification point if $\mathrm{wt}_V(P) = 1$; otherwise we call P *special*. The point P is special if and only if the section

$$Dw_V \in H^0\Big(C, \mathcal{J}^1_C(\mathcal{L}^{\otimes r+1} \otimes \omega_C^{\otimes r(r+1)/2})\Big),$$

locally obtained from w_V by differentiating, vanishes at P.

The *total weight* of the ramification points of V is the (finite) sum

$$\mathrm{wt}_V := \sum_{P \in C} \mathrm{wt}_V(P).$$

It is equal to the degree of the line bundle of which w_V is a section, that is,

$$\mathrm{wt}_V = (r+1)(d + (g-1)r),$$

a formula usually referred to as the *Brill–Segre* or *Plücker formula*.

The *canonical system* is the complete linear system of sections of ω_C. Its rank is $g-1$, and its degree is $2(g-1)$. For each point P of C, its *Weierstrass weight* $\mathrm{wt}(P)$ is its weight in the canonical system, and the *Weierstrass order sequence* at P is the increasing sequence of orders at P of the canonical system.

For each integer $i \geq -1$ and each $P \in C$, let $V_C(i, P)$ denote the complete linear system of sections of $\omega_C((i+1)P)$.

2.2 Proposition. *Let C be a smooth curve of genus g. For each integer $i \geq 0$ and each $P \in C$, the following two statements hold for $V := V_C(i, P)$:*

1. *The weight $\mathrm{wt}_V(P)$ of P as a ramification point of V satisfies*

$$\mathrm{wt}_V(P) = g + \mathrm{wt}(P), \tag{1}$$

where $\mathrm{wt}(P)$ is the Weierstrass weight of P.

2. *The total weight* wt_V *of the ramification points of* V *satisfies*

$$\mathrm{wt}_V = g(g+i)^2. \tag{2}$$

Proof. From the Riemann–Roch theorem, for each $j = 0, \ldots, i$,

$$\dim V(-jP) = g + i - j.$$

In particular, comparing dimensions, we get

$$V(-iP) = V(-(i+1)P) = H^0(C, \omega_C).$$

Hence, the order sequence of V at P is

$$0,\ 1,\ \ldots,\ i-1,\ (i+1)+\epsilon_0,\ (i+1)+\epsilon_1,\ \ldots,\ (i+1)+\epsilon_{g-1},$$

where ϵ_0, ϵ_1, ..., ϵ_{g-1} is the Weierstrass order sequence at P. Thus

$$\mathrm{wt}_V(P) = \sum_{k=0}^{g-1}(i+1+\epsilon_k - i - k) = g + \mathrm{wt}(P).$$

The second statement is a direct application of the Brill–Segre formula, using that the rank of V is $g+i-1$ and its degree is $2g-2+(i+1)$. ∎

3 The general curve

3.1 Proposition. *Fix an integer* $i_0 \geq 0$*. Let* C *be a general smooth curve of genus* $g \geq 1$*. Then the following two statements hold for each nonnegative integer* $i \leq i_0$*:*

1. *For a general point* P *of* C*, the linear system* $V_C(i,P)$ *ramifies at* P *with weight* g*, and has otherwise at most simple ramification points.*

2. *For any two points* P *and* R *of* C*,*

$$h^0(C, \omega_C((i+1)P - (g+i)R)) \leq 1 \tag{3}$$

$$h^0(C, \omega_C((i+1)P - (g+i+2)R) = 0. \tag{4}$$

Proof. Let us first observe that the property required of C is open. Indeed, let $f\colon X \to S$ be any family of smooth curves, that is, a projective, smooth map with connected fibers of dimension 1. Consider the fibered product $X^{(2)} := X \times_S X$ of two copies of f, and denote by p_1 and p_2 the projection maps. Denote by Δ the diagonal subscheme of $X^{(2)}$. Let ω_f denote the relative canonical bundle of f. Then $\omega := p_2^*\omega_f$ is the relative canonical bundle of p_1. Let

$$\mathcal{V} := p_{1*}(\omega((i+1)\Delta)).$$

A fiberwise analysis shows that $\mathcal{V}$ is a bundle of rank $g+i$ with formation commuting with base change. For each integer $\ell \geq 0$, denote by $\mathcal{J}^\ell$ the bundle of rank $\ell+1$ of p_1-relative jets of $\omega((i+1)\Delta)$ truncated in order ℓ, and denote by $\psi_\ell\colon p_1^*\mathcal{V} \to \mathcal{J}^\ell$ the map locally obtained by differentiating the sections of $\omega((i+1)\Delta)$ up to order ℓ along the fibers of p_1. Let $W_{i,1}$ (resp. $W_{i,2}$) be the closed subset of $X^{(2)}$ where ψ_{g+i-1} (resp. ψ_{g+i+1}) has rank at most $g+i-2$ (resp. $g+i-1$). Also, let W_i be the closed subset of $X^{(2)}$ where ψ_{g+i-1} has rank at most $g+i-1$. By Proposition 2.2, W_i contains Δ with multiplicity g. Let $W_i' := W_i - g\Delta$ and $Z_i := \Delta \cap W_i'$. Let $W_i'' \subset W_i'$ be the ramification scheme of the map $p_1|_{W_i'}$. Let $U_i \subseteq S$ be the intersection of $S - f(p_1(W_{i,1} \cup W_{i,2}))$ with $f(X - p_1(W_i'' \cup Z_i))$. Since p_1 is proper, and f is both proper and open, U_i is an open subscheme of S. Let $U := U_0 \cap \cdots \cap U_{i_0}$. The formation of U commutes with base change. Thus a fiberwise analysis reveals that U consists of the set of points $s \in S$ such that the proposition holds for $C := X(s)$.

Now, keeping in mind the existence of a versal family of smooth curves, it is enough to exhibit a single curve C for which the statement holds. We will actually show a somewhat stronger existence result:

3.2 Lemma. *Fix nonnegative integers i_0 and j_0. Let g be a positive integer. Then there is a smooth pointed curve (C, Q) of genus g for which the following three statements hold for each nonnegative integers $i \leq i_0$ and $j \leq j_0$:*

1. *The linear system $V_C(j, Q)$ ramifies at Q with weight g, and has otherwise at most simple ramification points.*

2. *For each $P \in C$ distinct from Q, either Q is an ordinary point or a simple ramification point of $V_C(i, P)$.*

3. *For each $P \in C$ distinct from Q, the linear system V of sections of*

$\omega_C((i+1)P+(j+1)Q)$ given by

$$V := H^0(\omega_C((i+1)P)) + H^0(\omega_C((j+1)Q))$$

satisfies

$$\dim V(-(g+i+j)R) \leq 1 \quad \text{and} \quad V(-(g+i+j+2)R) = 0$$

for each $R \in C$ distinct from P and Q.

We will first see how the lemma implies the proposition. Set $j_0 = i_0$, and consider the pointed curve (C, Q) given by the lemma. Then the two statements of Proposition 3.1 hold for C. Indeed, the first statement holds for $P = Q$, whence for P in a neighborhood of Q, that is, for a general P.

As for the second statement, first notice that (3) and (4) hold for $P = Q$ and every $R \in C$, a consequence of the first statement of the lemma for $j := i$. They hold as well for $R = Q$ and every $P \in C$ distinct from Q, a consequence of the second statement of the lemma. Furthermore, they hold for $R = P$ and any $P \in C$. Indeed, the first statement of the lemma for $j := 0$ implies that the canonical linear system has at most simple ramification points. Thus $h^0(\omega_C((1-g)P)) \leq 1$ and $h^0(\omega_C(-(g+1)P)) = 0$.

Finally, fix a point $P \in C$ distinct from Q and a point $R \in C$ distinct from P and Q. For $j := 0$, the linear system V defined in the lemma is the system of sections of $\omega_C((i+1)P+Q)$ that are zero on Q. Since $R \neq Q$, the third statement of the lemma yields (3) and (4).

It is thus enough to prove the lemma, what we do below. ∎

Proof. (Lemma 3.2) We will do induction on g. The initial step is taken care of below.

Let C be any elliptic curve and $Q \in C$ any point. Then the ramification points of the complete linear system of sections of $\omega_C((j+1)Q)$ are simple. (These are the $(j+1)^2$ points R for which $Q - R$ is $(j+1)$-torsion, what includes Q.) In fact, it follows from the Riemann–Roch theorem that every complete linear system has only simple ramification points. Thus Statements 1 and 2 of the lemma hold. Now, given $P \in C$ distinct from Q, since the vector subspace V of $H^0(\omega_C((i+1)P+(j+1)Q))$ defined in Statement 3 has codimension 1, the order sequence of V at a point R is obtained either from

$$0,\, 1,\, \ldots,\, g+i+j-1,\, g+i+j \quad \text{or} \quad 0,\, 1,\, \ldots,\, g+i+j-1,\, g+i+j+1$$

by removing an order. In any case, there is at most one order of V at R above $g+i+j-1$, that is $\dim V(-(g+i+j)R) \leq 1$, and all orders are at most $g+i+j+1$, that is $V(-(g+i+j+2)R)=0$.

Assume from now on that $g>1$, and that the claim holds for smaller genera and any integers i_0 and j_0. We will employ a degeneration technique in order to apply the induction hypothesis.

Let (Y,A) and (Z,B) be nonrational smooth pointed curves of genera g_Y and g_Z, with $g_Y+g_Z=g$. From the induction hypothesis, we may assume that the statements of the lemma hold for (C,Q) replaced by (Y,A) and all nonnegative integers $i \leq i_0$ and $j \leq g_Z+i_0+j_0+1$, and for (C,Q) replaced by (Z,B) and all nonnegative integers $i \leq i_0$ and $j \leq g_Y+i_0+j_0+1$.

Let C_0 be the curve of compact type that is the union of Y, of Z, and of a chain of rational curves $E_1,\dots,E_{n-1}$ connecting A to B, where $n \geq 2$. Our convention is that E_1 contains A and E_{n-1} contains B. Let v be any integer such that $0<v<n$, and let Q_0 be any point of E_v that is not a node of C_0.

Let $S := \mathrm{Spec}(\mathbb{C}[[t]])$, and denote its special point by 0 and generic point by η. Since there are no obstructions to deforming pointed nodal curves, there are a projective, flat map $f\colon X \to S$ and a section $\lambda\colon S \to X$ of f such that $(X(0),\lambda(0))=(C_0,Q_0)$ and $(X(\eta),\lambda(\eta))$ is a smooth pointed curve over the field of formal Laurent series $\mathbb{C}[[t]][1/t]$.

Let C be the base extension of $X(\eta)$ to the algebraic closure of $\mathbb{C}[[t]][1/t]$. Set $Q:=\lambda(\eta)$. It is enough to see that the statements of the lemma hold for (C,Q). Indeed, the argument is quite standard, and is summarized below. Though the pointed curve (C,Q) is not defined over $\mathbb{C}$, it is defined over a finitely generated extension L of $\mathbb{Q}$. If the statements of the lemma hold for (C,Q), they also hold for the base extension of (C,Q) over any algebraically closed field containing L. But, since $\mathbb{C}$ has many transcendentals over $\mathbb{Q}$, there is an algebraically closed field containing L which is isomorphic to $\mathbb{C}$. So, if the statements of the lemma hold for (C,Q), they hold as well for some pointed curve over $\mathbb{C}$.

Now, any finite set of points of C is defined over a finite field extension of $\mathbb{C}[[t]][1/t]$. Replacing S by its normalization in this field extension, we may assume that these are rational points of $X(\eta)$, and thus that there are sections of f intersecting $X(\eta)$ at them. By making a further base extension, if necessary, and a sequence of blowups at the singular points of the special fiber, we may assume that the total space X is regular, and that these sections factor through the smooth locus of f. The compensation for this is a change of the special fiber. However, the special fiber will have the same specification as the C_0 we described above. Thus, no confusion will ensue if we keep calling by C_0 this new fiber. Also, the section λ can be extended to a section of this new family.

Now, let P and R be points of C with P distinct from Q and R distinct from P and Q. As we mentioned above, we may assume there are sections $\gamma\colon S \to X$ and $\rho\colon S \to X$ through the smooth locus of f such that $\gamma(\eta) = P$ and $\rho(\eta) = R$. Set $P_0 := \gamma(0)$ and $R_0 := \rho(0)$. Let Γ and Λ be the images of γ and λ, respectively.

Fix nonnegative integers $i \leq i_0$ and $j \leq j_0$. Let ω be the relative dualizing bundle of $f\colon X \to S$. Let V_η be the linear system of sections of the line bundle $\omega(\eta)((i+1)P + (j+1)Q)$ given by

$$V_\eta := H^0(\omega(\eta)((i+1)P)) + H^0(\omega(\eta)((j+1)Q)).$$

Assume that R is a ramification point of V_η. To prove the statements of the lemma hold for (C, Q), it is enough to prove the following three statements:

1. For $i = 0$, the system V_η ramifies at Q with weight g, and R is a simple ramification point of V_η.

2. For $j = 0$, the point Q is a ramification point of V_η of weight $g + i$ or $g + i + 1$.

3. $\dim V_\eta(-(g + i + j)R) \leq 1$ and $V_\eta(-(g + i + j + 2)R) = 0$.

We will employ techniques of limit linear series, from [7], to show the above three statements. There are two cases to consider:

Case 1: Assume that $P_0 \in E_u$ for some u.

Since C_0 is of compact type, there is an effective divisor D of X supported on C_0 such that, letting

$$\mathcal{L} := \omega((i+1)\Gamma + (j+1)\Lambda + D),$$

we have $\mathcal{L}|_{E_m} \cong \mathcal{O}_{E_m}$ for each $m = 1, \dots, n-1$,

$$\mathcal{L}|_Z \cong \omega_Z((g_Y + i + j + 3)B) \quad \text{and} \quad \mathcal{L}|_Y \cong \omega_Y((1 - g_Y)A).$$

Since, from the induction hypothesis, A is a ramification point of weight g_Y of the complete linear system of sections of $\omega_Y(A)$, the point A is not a Weierstrass point of Y. Then $V := H^0(X, \mathcal{L}) \cap V_\eta$ restricts to a linear system

V_Z of dimension $g+i+j$ of sections of $\omega_Z((g_Y+i+j+3)B)$. Also from the induction hypothesis, B is not a Weierstrass point of Z. So the order sequence of B in the complete linear system of sections of $\omega_Z((g_Y+i+j+3)B)$ is

$$0,\,1,\,\ldots,\,g_Y+i+j+1,\,g_Y+i+j+3,\,\ldots,\,g+i+j+2.$$

As a consequence, the weight w_B of B as a ramification point of the linear system V_Z satisfies

$$w_B \leq 2(g_Y+i+j)+3g_Z, \tag{5}$$

with equality if and only if $V_Z = H^0(\omega_Z((g_Y+i+j+1)B))$.

Analogously, choosing an appropriate D, we obtain a linear system V_Y of dimension $g+i+j$ of sections of $\omega_Y((g_Z+i+j+3)A)$, and the weight w_A of A as a ramification point of V_Y satisfies

$$w_A \leq 2(g_Z+i+j)+3g_Y, \tag{6}$$

with equality if and only if $V_Y = H^0(\omega_Y((g_Z+i+j+1)A))$.

Let $r := g+i+j-1$. Using the Plücker formula, the number N of ramification points of V_Y and V_Z on $(Y-A)\cup(Z-B)$, counted with their respective weights, satisfies

$$\begin{aligned} N &= (r+1)\Big((2g_Z+g_Y+i+j+1)+r(g_Z-1)\Big)-w_B \\ &+ (r+1)\Big((2g_Y+g_Z+i+j+1)+r(g_Y-1)\Big)-w_A \\ &= N'+5g+4(i+j)-w_A-w_B, \end{aligned}$$

where

$$N' := (r+1)\Big((2g+i+j)+r(g-1)\Big)-2g-i-j.$$

Now, from the theory of limit linear series, each one of the ramification points of V_Y or V_Z on $(Y-A)\cup(Z-B)$ is a limit of ramification points of V_η, and its weight as a ramification point is the sum of the weights of the ramification points of V_η converging to it. Besides those, since P and Q are ramification points of V_η with weights at least $g+j$ and $g+i$, respectively, the points P_0 and Q_0 appear as limits of ramification points of V_η with weights summing up to at least $2g+i+j$. Thus, from the Plücker formula, at most N' ramification points of V_η, counted with their weights, converge to $(Y-A)\cup(Z-B)$. So

$$5g+4(i+j)-w_A-w_B \leq 0.$$

However, Inequalities (5) and (6) for w_B and w_A yield the opposite inequality:

$$5g + 4(i+j) - w_A - w_B \geq 0.$$

Thus, equalities hold, and hence

$$V_Y = H^0(\omega_Y((g_Z+i+j+1)A)) \quad \text{and} \quad V_Z = H^0(\omega_Z((g_Y+i+j+1)B)).$$

In addition, P and Q are ramification points of V_η of weights $g+j$ and $g+i$, respectively, and all the other ramification points of V_η converge to $(Y-A)\cup(Z-B)$. In particular, Statement 2 and the first part of Statement 1 are shown.

Now, since R is a ramification point of V_η, and R is distinct from P and Q, we have $R_0 \in (Y-A)\cup(Z-B)$. So R_0 is a ramification point of either V_Y or V_Z. From the induction hypothesis, the complete linear systems of sections of $\omega_Y((g_Z+i+j+1)A)$ and $\omega_Z((g_Y+i+j+1)B)$ have at most simple ramification points, other than A or B. Thus R is the unique ramification point of V_η converging to R_0 and its weight is 1. So the remainder of Statement 1 is shown.

As for Statement 3, assume, without loss of generality, that $R_0 \in Z$. Set $n := \dim V_\eta(-(g+i+j)R)$, and let $\sigma_1, \dots, \sigma_n$ form a $\mathbb{C}[[t]]$-basis of $V \cap V_\eta(-(g+i+j)R)$. Their restrictions to Z are sections of V_Z vanishing with multiplicity at least $g+i+j$ on R_0. Assume, by contradiction, that $n \geq 2$. Since R_0 is a simple ramification point of V_Z, the sections $\sigma_1|_Z, \dots, \sigma_n|_Z$ are linearly dependent. Thus, there is a nonzero n-tuple $(c_1, \dots, c_n) \in \mathbb{C}^n$ such that $c_1\sigma_1 + \cdots + c_n\sigma_n$ vanishes on Z, and hence on the whole C_0. Thus

$$c_1\sigma_1 + \cdots + c_n\sigma_n = t\sigma \tag{7}$$

for some $\sigma \in H^0(X, \mathcal{L})$. Also $\sigma \in V_\eta(-(g+i+j)R)$, and hence σ is a $\mathbb{C}[[t]]$-linear combination of $\sigma_1, \dots, \sigma_n$. Plugging this linear combination in (7) we obtain a nontrivial $\mathbb{C}[[t]]$-linear relation among the sections σ_i, a contradiction. Thus $n \leq 1$. A similar analysis, using that $V_Z(-(g+i+j+2)R_0) = 0$, shows that $V_\eta(-(g+i+2)R) = 0$, finishing the proof of Statement 3.

Case 2: Assume P_0 belongs to either Y or Z.

Without loss of generality, we may assume that $P_0 \in Z$. Again, since C_0 is of compact type, there is an effective divisor D of X supported on C_0 such that, letting

$$\mathcal{L} := \omega((i+1)\Gamma + (j+1)\Lambda + D),$$

we have $\mathcal{L}|_{E_m} \cong \mathcal{O}_{E_m}$ for each $m = 1, \dots, n-1$,

$$\mathcal{L}|_Z \cong \omega_Z((i+1)P_0 + (g_Y + j + 2)B) \quad \text{and} \quad \mathcal{L}|_Y \cong \omega_Y((1 - g_Y)A).$$

As before, $V := H^0(X, \mathcal{L}) \cap V_\eta$ restricts to a linear system V_Z of dimension $g + i + j$ of sections of $\omega_Z((i+1)P_0 + (g_Y + j + 2)B)$.

Now,

$$V \supseteq H^0(X, \omega((i+1)\Gamma)) + H^0(X, \omega((j+1)\Lambda + D)).$$

Reasoning as in Case 1, we can show that $H^0(X, \omega((j+1)\Lambda + D))$ restricts to $H^0(\omega_Z((g_Y + j + 1)B))$. On the other hand, the exact sequence

$$0 \to H^0(\omega_Z((i+1)P_0)) \to H^0(\omega((i+1)\Gamma)|_{C_0}) \to H^0(\omega_Y(A))$$

shows that $h^0(\omega((i+1)\Gamma)|_{C_0}) = g + i$, and hence that $H^0(X, \omega((i+1)\Gamma))$ restricts to a vector subspace of $H^0(\omega_Z((i+1)P_0 + B))$ containing the subspace $H^0(\omega_Z((i+1)P_0))$. Thus

$$V_Z \supseteq H^0(\omega_Z((i+1)P_0)) + H^0(\omega_Z((g_Y + j + 1)B)),$$

and a dimension count shows that equality holds.

The weight w_B of B as a ramification point of V_Z depends on its weight as a ramification point of $V_Z(i, P_0)$. Now, from the induction hypothesis, B is either an ordinary point or a simple ramification point of $V_Z(i, P_0)$. Hence, the order sequence at B of the linear system V_Z is either

$$1,\ 2,\ \dots,\ g_Y + j,\ g_Y + j + 2,\ g_Y + j + 3,\ \dots,\ g + i + j,\ g + i + j + 1$$

or

$$1,\ 2,\ \dots,\ g_Y + j,\ g_Y + j + 2,\ g_Y + j + 3,\ \dots,\ g + i + j,\ g + i + j + 2.$$

At any rate,

$$w_B \leq g_Y + j + 2(g_Z + i) + 1. \tag{8}$$

Notice that, if $i = 0$, then B is an ordinary point of $V_Z(0, P_0)$, as it is an ordinary point of Z, and thus Inequality (8) is strict.

On the other hand, let D' be an effective divisor of X supported on C_0 such that, letting

$$\mathcal{M} := \omega((i+1)\Gamma + (j+1)\Lambda + D'),$$

we have $\mathcal{M}|_{E_m} \cong \mathcal{O}_{E_m}$ for each $m = 1, \dots, n-1$,

$$\mathcal{M}|_Y \cong \omega_Y((g_Z + i + j + 3)A) \quad \text{and} \quad \mathcal{M}|_Z \cong \omega_Z((i+1)P_0 - (g_Z + i)B).$$

Since, as mentioned above, B is either an ordinary point or a simple ramification point of $V_Z(i, P_0)$, we have that $H^0(X, \mathcal{M}) \cap V_\eta$ restricts to a linear system V_Y of dimension $g + i + j$ of sections of $\omega_Y((g_Z + i + j + 3)A)$.

Since A is not a Weierstrass point of Y, the sequence of orders at A of the complete linear system of sections of $\omega_Y((g_Z + i + j + 3)A)$ is

$$0,\ 1,\ \dots,\ g_Z + i + j + 1,\ g_Z + i + j + 3,\ g_Z + i + j + 4,\ \dots,\ g + i + j + 2.$$

Since V_Y has codimension 2 in $H^0(\omega_Y((g_Z + i + j + 3)A))$, the weight w_A of V_Y at A satisfies

$$w_A \leq 2(g_Z + i + j) + 3g_Y, \tag{9}$$

with equality if and only if $V_Y = H^0(\omega_Y((g_Z + i + j + 1)A))$.

As in Case 1, using the Plücker formula, the number N of ramification points of V_Y and V_Z on $(Y - A) \cup (Z - B)$, counted with their respective weights, satisfies

$$N = N' + 4g + 4i + 3j - w_A - w_B,$$

where

$$N' := (g + i + j)(2g + i + j) + (g + i + j)(g + i + j - 1)(g - 1) - g - i.$$

As in Case 1, since Q is a ramification point of V_η with weight at least $g + i$, there are at most N' ramification points of V_η, counted with their respective weights, converging to $(Y - A) \cup (Z - B)$. So

$$4g + 4i + 3j - w_A - w_B \leq 0.$$

On the other hand, Inequalities (8) and (9) yield

$$w_A + w_B \leq 4g + 4i + 3j + 1.$$

In particular, $w_A \geq 2(g_Z + i + j) + 3g_Y - 1$, whence

$$V_Y \subset H^0(\omega_Y((g_Z + i + j + 2)A)).$$

Also, Q has weight $g+i$ or $g+i+1$ in V_η. Thus Statement 2 is shown. Furthermore, if $i=0$ we have $w_A+w_B=4g+4i+3j$. In this case, $V_Y=H^0(\omega_Y((g_Z+i+j+1)A))$ and Q has weight $g+i$ in V_η, showing the first part of Statement 1.

If Q has weight $g+i+1$ in V_η, all other ramification points converge to $(Y-A)\cup(Z-B)$. If Q has weight $g+i$, there is at most one ramification point of V_η, other than Q, converging outside $(Y-A)\cup(Z-B)$, and that point is simple. If R is that point, then $\dim V_\eta(-(g+i+j)R)\leq 1$ and $V_\eta(-(g+i+j+2)R)=0$ because of the simplicity of R.

Assume now that $R_0\in(Y-A)\cup(Z-B)$. Let us first consider the case $R_0\in Y-A$. In this case, since, from the induction hypothesis, the complete linear system of sections of $\omega_Y((g_Z+i+j+2)A)$ has at most simple ramification points, other than A, we have

$$\begin{aligned} h^0(\omega_Y((g_Z+i+j+2)A-(g+i+j)R_0) &\leq 1, \\ h^0(\omega_Y((g_Z+i+j+2)A-(g+i+j+2)R_0) &= 0. \end{aligned}$$

Thus $\dim V_Y(-(g+i+j)R_0)\leq 1$ and $V_Y(-(g+i+j+2)R_0)=0$ as well. It follows, as in Case 1, that

$$\dim V_\eta(-(g+i+j)R)\leq 1 \quad\text{and}\quad V_\eta(-(g+i+j+2)R)=0.$$

Furthermore, if $i=0$, since in this case $V_Y=H^0(\omega_Y((g_Z+i+j+1)A))$, all the ramification points of V_Y distinct from A are simple. Thus R_0 is simple in V_Y, and hence R is simple in V_η.

Assume now that $R_0\in Z-B$. There are two cases to consider. First, assume $R_0=P_0$. Since, by induction hypothesis, the complete linear system of sections of $\omega_Z((g_Y+j+1)B)$ has at most simple ramification points other than B, the weight of P_0 as a ramification point of V_Z is either $g+j$ or $g+j+1$. Since P has at least weight $g+j$ in V_η, and $R\neq P$, the latter must hold, and R must be a simple ramification point of V_η. In particular, $\dim V_\eta(-(g+i+j)R)\leq 1$ and $V_\eta(-(g+i+j+2)R)=0$.

Finally, assume $R_0\neq P_0$. Then

$$\dim V_Z(-(g+i+j)R_0)\leq 1 \quad\text{and}\quad V_Z(-(g+i+j+2)R_0)=0$$

from the induction hypothesis, and hence $\dim V_\eta(-(g+i+j)R)\leq 1$ and $V_\eta(-(g+i+j+2)R)=0$. Thus Statement 3 is shown. Also, if $i=0$, then $V_Z=H^0(\omega_Z((g_Y+j+1)B))$, and, since $R_0\neq P_0$, the weight of R_0

in V_Z is equal to its weight in the complete linear system of sections of $\omega_Z((g_Y + j + 1)B)$. By induction hypothesis, this weight is one, and thus R is a simple ramification point of V_η. So Statement 1 is shown. ∎

3.3 Corollary. *If C is a general smooth curve of genus $g \geq 1$, then all its Weierstrass points are simple.*

Proof. Apply Statement 1 of Proposition 3.1 for $i_0 := 0$ and $i := 0$. ∎

3.4 Proposition. *Fix an integer $i_0 \geq 0$. Let C be a general smooth curve of genus $g \geq 1$. Then for any two distinct points P and R of C, and any nonnegative integer $i \leq i_0$,*

$$h^0(C, \omega_C((i+1)P - (g+i-2)R)) = 2.$$

Proof. A line bundle of degree 2 on an elliptic curve has (at most) 2 linearly independent sections. Thus we may assume $g \geq 2$. Also, for $i = 0$,

$$h^0(C, \omega_C((i+1)P - (g+i-2)R) = h^0(C, \omega_C(-(g-2)R)) = 2,$$

since R is at most a simple Weierstrass point of C, a consequence of Corollary 3.3. So we need only show the stated equality for integers $i > 0$.

For each integer $j \geq 2$ (resp. $j \geq 1$), let M_j be the moduli space of smooth curves (resp. let $M_{j,1}$ be the moduli space of smooth pointed curves) of genus j. Let $\overline{M}_j$ and $\overline{M}_{j,1}$ denote their respective compactifications by stable (resp. stable, pointed) curves. For each positive integer $i \leq i_0$, let $D^{(i)} \subseteq M_{g+i}$ be the subset parameterizing curves admitting a covering of degree at most $g+i-2$ of the projective line totally ramified at a point. By [1], Thm. 3.11, p. 333, the subvariety $D^{(i)}$ is irreducible of codimension 2. Let $\overline{D}^{(i)} \subset \overline{M}_{g+i}$ be the closure of $D^{(i)}$.

Let μ_i: $M_{g,1} \times M_{i,1} \to \overline{M}_{g+i}$ be the natural map, associating to a pair of smooth pointed curves the stable uninodal curve which is the union of these curves identified at the marked points. Let $E^{(i)} := \mu_i^{-1}(\overline{D}^{(i)})$. Let ρ_i: $E^{(i)} \to M_g$ be the natural map, forgetting the second pointed curve and the marked point on the first curve. Since C is general, we may assume that, for each $i = 1, \ldots, i_0$, the curve C is parameterized by a point of M_g over which the fiber of ρ_i has minimum dimension. We claim this dimension is at most $3i - 3$, whence less than $\dim M_{i,1}$. Indeed, if the dimension were larger, then $E^{(i)}$ would have codimension at most 1 in $M_{g,1} \times M_{i,1}$, and hence would dominate $\overline{D}^{(i)}$ under μ_i. So $\overline{D}^{(i)}$ would be contained in the boundary $\overline{M}_{g+i} - M_{g+i}$, an absurd. From the claim, for each $i = 1, \dots, i_0$, the general smooth pointed curve (Y_i, B_i) of genus i is such that, for any $P \in C$, the pair of pointed curves $((C, P), (Y_i, B_i))$ is not parameterized by $E^{(i)}$. Consequently, the stable uninodal curve X_i, union of C and Y_i with P and B_i identified, is parameterized by a point of $\overline{M}_{g+i}$ off $\overline{D}^{(i)}$, for each $i = 1, \dots, i_0$.

Suppose, by contradiction, that for certain distinct points P and Q of C, and a certain positive integer $i \leq i_0$, we have

$$h^0(C, \omega_C((i+1)P - (g+i-2)Q) \geq 3.$$

Put $g' := g + i$. Since, by Riemann–Roch, $h^0(C, \omega_C((i+1)P - iQ)) = g$, there is an integer j with $2 \leq j < g$ such that

$$h^0(C, \omega_C((i+1)P - (g'-j)Q) = h^0(C, \omega_C((i+1)P - (g'-j-1)Q) = j+1.$$

Again by Riemann–Roch,

$$h^0(C, \mathcal{O}_C((g'-j)Q - (i+1)P)) > h^0(C, \mathcal{O}_C((g'-j-1)Q - (i+1)P)). \quad (10)$$

Thus, there is a map $\phi\colon C \longrightarrow \mathbb{P}^1$ of degree $g' - j$ such that $\phi^*(0) = (g'-j)Q$ and $\phi^*(\infty) \geq (i+1)P$. Let i' be the integer such that $i'+1$ is the multiplicity of P in $\phi^*(\infty)$. Then $i' \geq i$.

Set $Y := Y_i$ and $B := B_i$. Since B is general, B is not a Weierstrass point of Y. Thus, since $i' \geq i$, we have $h^0(Y, \mathcal{O}_Y(i'B)) < h^0(Y, \mathcal{O}_Y((i'+1)B))$. So, there is a map $\psi\colon Y \longrightarrow \mathbb{P}^1$ of degree $i'+1$ such that $\psi^*(\infty) = (i'+1)B$.

Putting together the maps ϕ and ψ, we may construct the covering with source X_i depicted in Figure 1 below,

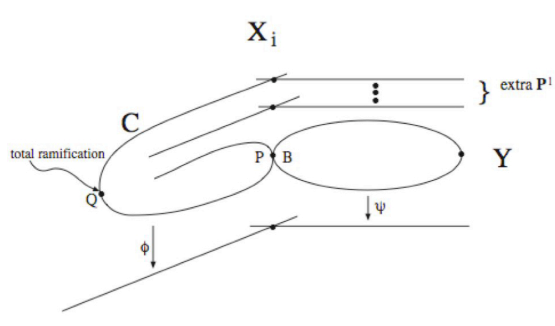

Figure 1: The covering.

which can be represented by a point $[X_i]$ of the (compactification of the) Hurwitz scheme parameterizing (pseudo)admissible coverings of the projective line of degree $(g'-j)$ totally ramified at a point; see Remark 3.5. Since coverings of $\mathbb{P}^1$ form a dense open subscheme of this compactification, the curve X_i is limit of smooth curves equipped with a degree-$(g'-j)$ map to the projective line totally ramified at a point. Since $j \geq 2$, it follows that $[X_i]$ lies on the boundary of $D^{(i)}$, a contradiction. ∎

3.5 Remark. The Hurwitz scheme we used in the proof of Proposition 3.4 is mentioned in [6], Section 5. It can be constructed following the same reasoning used in the construction of the Hurwitz scheme of (simple) admissible coverings, given in the proof of [10], Thm. 4, p. 58. Also, the local descriptions of both schemes are the same, given on [10], p. 62. From this description we see that the Hurwitz scheme is equidimensional. Now, there is a natural forgetful map from the Hurwitz scheme to a corresponding moduli space of pointed genus-0 curves, taking a covering to its target. This map is finite and surjective, also by [10], Thm. 4, p. 58. Since the moduli spaces of pointed genus-0 curves are irreducible (see [12] or [11]), it follows that each irreducible component of the Hurwitz scheme covers the target. So coverings of $\mathbb{P}^1$ form a dense open subscheme of the Hurwitz scheme, a fact used in the proof of Proposition 3.4.

3.6 Remark. We tried to prove Proposition 3.4 using the same induction argument used in the proof of Lemma 3.2. However, we could not prove the initial step, that is, the following statement: *Let C be a general elliptic curve, $Q \in C$ a general point, and $P \in C - \{Q\}$ any point. Let i and j be nonnegative integers. Then the linear system V of sections of the line bundle $\omega_C((i+1)P+(j+1)Q)$ generated by $H^0(\omega_C((i+1)P))$ and $H^0(\omega_C((j+1)Q))$ satisfies* $\dim V(-(i+j-1)R) = 2$ *for each $R \in C - \{P, Q\}$.*

4 Weierstrass divisors

4.1 (*Wronski maps*) Let C be a smooth curve of genus g. For each integer $j \geq 0$, consider the family of linear systems $V_C(j, P)$ for P varying on C. More precisely, let p_1 and p_2 denote the projections of $C \times C$ onto the first and second factors, and $\Delta \subset C \times C$ the diagonal. The relative canonical bundle of p_1 is simply the pullback $p_2^*\omega_C$ of the canonical bundle ω_C of C.

For each integer $j \geq 0$, let

$$\mathcal{L}_j := p_2^*\omega_C((j+1)\Delta), \quad \mathcal{E}_j := p_{1*}\mathcal{L}_j.$$

Notice that, for each point P of C, identifying $\{P\} \times C$ with C in the natural way, $\mathcal{L}_j|_{\{P\}\times C} = \omega_C((j+1)P)$. Also, as $h^0(\omega_C((j+1)P)) = g + j$ for every $P \in C$, the sheaf $\mathcal{E}_j$ is a bundle of rank $g + j$ and $\mathcal{E}_j|_P = H^0(\omega_C((j+1)P))$.

For each integer $\ell \geq 0$ and each line bundle $\mathcal{M}$ on $C \times C$, let $\mathcal{J}^\ell_{p_1}(\mathcal{M})$ be the bundle of rank $\ell + 1$ of p_1-relative jets of $\mathcal{M}$ truncated in order ℓ. Let

$$\rho_{j,\ell} \colon p_1^*\mathcal{E}_j \to \mathcal{J}^\ell_{p_1}(\mathcal{L}_j)$$

be the map of bundles locally obtained by differentiating up to order ℓ along the fibers of p_1 the sections of $\mathcal{L}_j$. We call $\rho_{j,\ell}$ a *Wronski map.*

The map $\rho_{j,g+j-1}$ is a map of bundles of the same rank. Taking determinants, we get a section z_j of the line bundle

$$\bigwedge^{g+j} \mathcal{J}^{g+j-1}_{p_1}(\mathcal{L}_j) \otimes \bigwedge^{g+j} p_1^*\mathcal{E}_j^\vee,$$

which is naturally isomorphic, using the truncation sequence of the bundles of jets, to

$$p_2^*\omega_C((j+1)\Delta)^{\otimes g+j} \otimes p_2^*\omega_C^{\otimes(g+j)(g+j-1)/2} \otimes \bigwedge^{g+j} p_1^*\mathcal{E}_j^\vee,$$

or more simply to

$$p_2^*\omega_C^{\otimes(g+j)(g+j+1)/2}\Big((g+j)(j+1)\Delta\Big) \otimes \bigwedge^{g+j} p_1^*\mathcal{E}_j^\vee.$$

4.2 (*Weierstrass divisors.*) Keep the notation used in Subsection 4.1. Let $Z_j \subseteq C \times C$ denote the zero scheme of z_j. The section z_j is a relative wronskian. More precisely, for each $P \in C$, on $\{P\} \times C$, identified with C in the natural way, the section z_j restricts to the wronskian of the linear system $V_C(j, P)$. Hence, Z_j consists of the pairs $(P, Q) \in C \times C$ such that $V_C(j, P)$ ramifies at Q. Now, since z_j is nonzero, being so on each fiber, Z_j is a Cartier divisor. By Proposition 2.2, the divisor Z_j intersects each fiber $\{P\} \times C$ at (P, P) with multiplicity $g + \mathrm{wt}(P)$, where $\mathrm{wt}(P)$ is the weight of P in the canonical system of C. Thus Z_j contains Δ with multiplicity exactly g. Let

$$W_j := Z_j - g\Delta.$$

Then W_j is, set-theoretically, the locus of pairs $(P,Q) \in C \times C$ such that either $P = Q$ and P is a Weierstrass point of C, or $P \neq Q$ and Q is a ramification point of $V_C(j,P)$. We call W_j the *j-th Weierstrass divisor of C.*

4.3 Proposition. *Let C be a smooth curve of genus $g \geq 1$ and j a nonnegative integer. Let Δ be the diagonal of $C \times C$, and p_1 and p_2 the projections of $C \times C$ onto the indicated factors. Let ω_C be the canonical bundle of C, and set $K_\ell := c_1(p_\ell^*\omega_C)$ for $\ell = 1, 2$. Let $W_j \subseteq C \times C$ be the j-th Weierstrass divisor of C. Then its class $[W_j]$ in the Chow group of $C \times C$ satisfies*

$$[W_j] = \frac{1}{2}(g+j)(g+j+1)K_2 + j(g+j+1)[\Delta] + \frac{1}{2}j(j+1)K_1. \tag{11}$$

Proof. Use the notation in Subsections 4.1 and 4.2. Since $W_j = Z_j - g\Delta$, and Z_j is the zero scheme of a section of the line bundle

$$p_2^*\omega_C^{\otimes(g+j)(g+j+1)/2}\Big((g+j)(j+1)\Delta\Big) \otimes \bigwedge^{g+j} p_1^*\mathcal{E}_j^\vee,$$

we get

$$[W_j] = \frac{1}{2}(g+j)(g+j+1)K_2 + j(g+j+1)[\Delta] - p_1^*c_1(\mathcal{E}_j). \tag{12}$$

To finish, we need only show that

$$c_1(\mathcal{E}_j) = -\frac{1}{2}j(j+1)c_1(\omega_C). \tag{13}$$

We show (13) by induction on j. First of all,

$$\mathcal{E}_0 = p_{1*}p_2^*\omega_C = H^0(\omega_C) \otimes \mathcal{O}_C.$$

Since $\mathcal{E}_0$ is free, $c_1(\mathcal{E}_0) = 0$.

Assume now that $j > 0$ and $c_1(\mathcal{E}_{j-1}) = -(j(j-1)/2)c_1(\omega_C)$. Consider the natural short exact sequence

$$0 \to p_2^*\omega_C(j\Delta) \to p_2^*\omega_C((j+1)\Delta) \to p_2^*\omega_C((j+1)\Delta)|_\Delta \to 0.$$

Since $H^1(\omega_C(jP)) = 0$ for each $P \in C$, applying p_{1*} to the sequence above, we get the exact sequence

$$0 \to \mathcal{E}_{j-1} \to \mathcal{E}_j \to p_{1*}p_2^*\omega_C((j+1)\Delta)|_\Delta \to 0.$$

Now, $p_\ell|_\Delta$ is an isomorphism for $\ell = 1, 2$. So $p_{1*}p_2^*\omega_C|_\Delta = \omega_C$. In addition, $p_{1*}\mathcal{O}_{C\times C}(-\Delta)|_\Delta = \omega_C$. Thus

$$\begin{aligned} c_1(\mathcal{E}_j) &= c_1(\mathcal{E}_{j-1}) + c_1(p_{1*}p_2^*\omega_C((j+1)\Delta)|_\Delta) \\ &= -(j(j-1)/2)c_1(\omega_C) + (1-(j+1))c_1(\omega_C) \\ &= -(j(j+1)/2)c_1(\omega_C), \end{aligned}$$

as claimed. ∎

4.4 Proposition. *Let C be a general smooth curve of genus $g \geq 1$ and j a nonnegative integer. Let $W_j \subseteq C \times C$ be the j-th Weierstrass divisor of C. Then W_j is nonsingular and intersects the diagonal Δ transversally, at the pairs (P, P) such that P is a Weierstrass point of C.*

Proof. Let us show first that W_j intersects Δ transversally. As pointed out in Subsection 4.2, the intersection $W_j \cap \Delta$ is, set-theoretically, the set of pairs (P, P) such that P is a Weierstrass point of C. As C is general, by Corollary 3.3, all its Weierstrass points are simple, and number $g^3 - g$ by Plücker Formula. Now, since the intersection $W_j \cap \Delta$ is finite, the number of points of intersection, weighted by their intersection multiplicities, is equal to the degree of the product $[W_j][\Delta]$. Using the notation and Formula (11) of Proposition 4.3, and using the Formulas

$$\int_{C\times C} K_2[\Delta] = \int_{C\times C} K_1[\Delta] = -\int_{C\times C} [\Delta]^2 = 2g - 2 \tag{14}$$

and

$$\int_{C\times C} K_1K_2 = 4(g-1)^2, \tag{15}$$

we get

$$\int_{C\times C} [W_j][\Delta] = (g+j)(g+j+1)(g-1) - 2j(g+j+1)(g-1) + j(j+1)(g-1),$$

which is exactly $g^3 - g$. Thus the intersection multiplicities are all one.

As a corollary of the transversal intersection, W_j is nonsingular at its points on Δ. So, let now $(P, Q) \in W_j$ for P and Q distinct, and let us show that W_j is nonsingular at (P, Q) as well.

Let $J := \mathrm{Pic}^{g-1}(C)$, the component of the Picard scheme of C parameterizing line bundles of degree $g - 1$. Let $\Theta \subset J$ be the theta divisor, parameterizing line bundles with nontrivial global sections. Let

$$\mu\colon C \times C \to J$$

be the map taking a pair (R, S) to the point of J representing the bundle $\omega_C((j+1)R - (g+j)S)$.

We claim that $\mu(W_j) \subseteq \Theta$. Indeed, let $C^{(3)} := C \times C \times C$, and denote by $p_{1,2}$ and p_3 the projection maps of $C^{(3)}$ onto the indicated factors. Let $\Delta_{1,3}$ and $\Delta_{2,3}$ be the indicated diagonals of $C^{(3)}$. Set

$$\mathcal{F} := p_3^*\omega_C((j+1)\Delta_{1,3} - (g+j)\Delta_{2,3}).$$

Recall the notation of Subsection 4.1. From the construction of Θ, to show that $\mu(W_j) \subseteq \Theta$, it is enough to show that the Wronski map $\rho_{j,g+j-1}$ represents universally the cohomology of $\mathcal{F}$ or, put more simply, that $\rho_{j,g+j-1}$ can be viewed as a presentation of the right derived image $R^1(p_{1,2})_*\mathcal{F}$.

Let $\mathcal{G} := p_3^*\omega_C((j+1)\Delta_{1,3})$. Then $\mathcal{F} \subseteq \mathcal{G}$. From the definition of the Wronski map $\rho_{j,g+j-1}$, we get that $\rho_{j,g+j-1}$ is the image under $(p_{1,2})_*$ of the quotient map $\mathcal{G} \to \mathcal{G}/\mathcal{F}$. Thus, the map $\rho_{j,g+j-1}$ is the first map in the following piece of the long derived sequence of $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{G}/\mathcal{F} \to 0$ under $(p_{1,2})_*$:

$$(p_{1,2})_*\mathcal{G} \to (p_{1,2})_*(\mathcal{G}/\mathcal{F}) \to R^1(p_{1,2})_*\mathcal{F} \to R^1(p_{1,2})_*\mathcal{G}.$$

Now, a fiberwise analysis shows that $R^1(p_{1,2})_*\mathcal{G} = 0$. Thus $\rho_{j,g+j-1}$ is a presentation for $R^1(p_{1,2})_*\mathcal{F}$, finishing the proof that $\mu(W_j) \subseteq \Theta$.

Let $\mathcal{L} := \omega_C((j+1)P - (g+j)Q)$, and denote by $[\mathcal{L}]$ the point of J representing $\mathcal{L}$. Since $(P, Q) \in W_j$, we have $[\mathcal{L}] \in \Theta$. By Proposition 3.1, $h^0(C, \mathcal{L}) = 1$. Thus, it follows from [2], Prop. (4.2), p. 189, that $[\mathcal{L}]$ is a nonsingular point of Θ. Furthermore, identifying the cotangent space of J at $[\mathcal{L}]$ with $H^0(C, \omega_C)$, the cotangent space of Θ at $[\mathcal{L}]$ is the quotient by the subspace $H^0(C, \omega_C(-F))$, where F is the unique effective divisor of C such that $\mathcal{L} = \mathcal{O}_C(F)$.

Identifying the cotangent space of $C \times C$ at (P, Q) with $\omega_C|_P \oplus \omega_C|_Q$, the induced map of cotangent spaces $d\mu^*\colon T^*_{J,[\mathcal{L}]} \to T^*_{C\times C,(P,Q)}$ is equivalent to the evaluation map,

$$\epsilon\colon H^0(C, \omega_C) \to \omega_C|_P \oplus \omega_C|_Q.$$

We claim that $\epsilon(H^0(C, \omega_C(-F))) \neq 0$. Indeed, if that were not the case, we would have $H^0(C, \omega_C(-F)) = H^0(C, \omega_C(-F - P - Q))$, that is,

$$h^0(C, \mathcal{O}_C((g+j-1)Q - (j+2)P)) = h^0(C, \mathcal{O}_C((g+j)Q - (j+1)P)).$$

By the Riemann–Roch theorem,

$$h^0(C, \mathcal{O}_C((g+j)Q - (j+1)P)) = h^0(C, \mathcal{L}) = 1,$$

and thus, also by the Riemann–Roch theorem,

$$h^0(C, \omega_C((j+2)P - (g+j-1)Q)) = 3.$$

However, this contradicts Proposition 3.4.

Since $\mu(W_j) \subseteq \Theta$, the image of $\epsilon(H^0(C, \omega_C(-F)))$ in the cotangent space of W_j at (P, Q) is zero. Since $\epsilon(H^0(C, \omega_C(-F))) \neq 0$, that cotangent space is a proper quotient of the cotangent space of $C \times C$ at (P, Q), and thus has dimension at most 1. Since W_j is a divisor, it follows that W_j is nonsingular at (P, Q). ∎

5 Special ramification classes

5.1 (*Special ramification loci.*) Let C be a smooth curve of genus $g \geq 1$. For each nonnegative integer i, consider the following loci in $C \times C$:

1. The locus D_i^+ of pairs $(P, Q) \in C \times C$ such that

$$(g+i-1)Q - (i+1)P$$

   is linearly equivalent to an effective divisor.

2. The locus E_i^+ of pairs $(P, Q) \in C \times C$ such that

$$(g+i+1)Q - (i+1)P$$

   is linearly equivalent to a moving effective divisor.

3. The locus SW_i^+ of pairs $(P, Q) \in C \times C$ such that Q is a special ramification point of $V_C(i, P)$.

We claim that, set-theoretically,

$$SW_i^+ = D_i^+ \cup E_i^+. \tag{16}$$

Indeed, by the Riemann–Roch Theorem, for a pair $(P, Q) \in C \times C$, the divisor $(g+i-1)Q - (i+1)P$ is linearly equivalent to an effective one if and only if

$$h^0(\omega_C((i+1)P - (g+i-1)Q)) \geq 2, \tag{17}$$

while $(g+i+1)Q-(i+1)P$ is linearly equivalent to a moving effective divisor if and only if

$$h^0(\omega_C((i+1)P-(g+i+1)Q))\geq 1. \tag{18}$$

At any rate, if $(P,Q)\in D_i^+\cup E_i^+$, then Q is a special ramification point of $V_C(i,P)$, that is, $(P,Q)\in SW_i^+$

On the other hand, let $(P,Q)\in C\times C-(D_i^+\cup E_i^+)$. Then

$$h^0(\omega_C((i+1)P-(g+i-1)Q))=1,$$
$$h^0(\omega_C((i+1)P-(g+i+1)Q))=0.$$

So, either Q is an ordinary or a simple ramification point of $V_C(i,P)$, that is, $(P,Q)\notin SW_i^+$.

Let Δ be the diagonal subscheme of $C\times C$. Notice that $E_i^+\cap\Delta$ consists of the pairs (P,P) such that P is a Weierstrass point of C. However, if $g>1$, both D_i^+ and SW_i^+ contain Δ. (If $g=1$, then $D_i^+=E_i^+=SW_i^+=\emptyset$.)

Let D_i, E_i and SW_i be the loci of points in D_i^+, E_i^+ and SW_i^+ that lie off Δ. Of course, Expression (16) implies $SW_i=D_i\cup E_i$. Our Proposition 5.4 claims that, if C is general, then $SW_i=D_i\cup E_i$ holds in a more refined way, in the cycle group of $C\times C$. Before stating it, we need to endow D_i, E_i and SW_i with natural subscheme structures.

5.2 (*Special ramification schemes*) Keep the notation of Subsection 5.1, and recall that of Subsections 4.1 and 4.2. Notice that the subsets D_i^+ and E_i^+ are the supports of the degeneracy schemes of $\rho_{i,g+i-2}$ and $\rho_{i,g+i}$, respectively. So we may give D_i^+ and E_i^+ the corresponding scheme structures. Give D_i and E_i the corresponding open subscheme structures. We say that D_i and E_i are the *i-th special ramification schemes of type Diaz and Cukierman*, respectively. Call E_i^+ the *i-th expanded special ramification scheme of type Cukierman*.

In addition, differentiating along the fibers of p_1 a section of $\mathcal{O}_{C\times C}(Z_i)$ defining Z_i, we obtain a section of $\mathcal{J}^1_{p_1}(\mathcal{O}_{C\times C}(Z_i))$, well-defined modulo $\mathbb{C}^*$. By functoriality, its zero scheme contains a pair (P,Q) if and only if Q is a special Weierstrass point of $V_C(i,P)$. Thus the zero scheme gives a scheme structure for SW_i^+. Give SW_i the induced open subscheme structure. We say that SW_i is the *i-th special ramification scheme of C*.

Now, $Z_i=W_i+g\Delta$. As done for Z_i, we can differentiate along the fibers of p_1 a section of $\mathcal{O}_{C\times C}(W_i)$ defining W_i to obtain a section of $\mathcal{J}^1_{p_1}(\mathcal{O}_{C\times C}(W_i))$. Its zero scheme S coincides with the scheme SW_i off Δ, because Z_i coincides with W_i there. Moreover, if C is general, then S does not intersect Δ, and hence $S = SW_i$ scheme-theoretically. Indeed, let P be a point of C. If $(P, P) \in W_i$, then P is a Weierstrass point of C. Moreover, as C is general, by Corollary 3.3, the point P is a simple Weierstrass point. So, it follows from Proposition 2.2 that W_i intersects the fiber $\{P\} \times C$ transversally at (P, P). Thus the derivative along $\{P\} \times C$ of a section defining W_i does not vanish at (P, P). So $S \cap \Delta = \emptyset$.

5.3 Lemma. *Let $\mathcal{O}$ be a local ring, and r a nonnegative integer. Let M be a matrix with $r+2$ rows and $r+1$ columns and entries in $\mathcal{O}$. Let M_1 and M_2 be the submatrices obtained from M by removing the last row, and the last two rows, respectively. Assume that the matrix obtained from M_1 by taking residues has rank at least r. Let z denote the determinant of M_1. Then there are $u, v \in \mathcal{O}$ such that*

1. *(z, u) is the ideal of all maximal minors of M_2,*
2. *(z, v) is the ideal of all maximal minors of M,*
3. *(z, uv) is the ideal generated by the two maximal minors of M obtained by removing each of the last two rows.*

Proof. We may write M in the form

$$M = \begin{bmatrix} A & a & b \\ c & f_1 & f_2 \\ d & g_1 & g_2 \\ e & h_1 & h_2 \end{bmatrix},$$

where A is a square matrix of size $r-1$, where a and b are column vectors of size $r-1$, where c, d and e are row vectors of size $r-1$, and where f_1, f_2, g_1, g_2, h_1 and h_2 are elements of $\mathcal{O}$.

Let I and J be the ideals of $\mathcal{O}$ generated, respectively, by all maximal minors of the submatrices

$$M_2 = \begin{bmatrix} A & a & b \\ c & f_1 & f_2 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} A & a & b \\ c & f_1 & f_2 \\ d & g_1 & g_2 \\ e & h_1 & h_2 \end{bmatrix}.$$

Also, let $K \subseteq \mathcal{O}$ be the ideal generated by the determinants of the square submatrices

$$M_1 = \begin{bmatrix} A & a & b \\ c & f_1 & f_2 \\ d & g_1 & g_2 \end{bmatrix} \quad \text{and} \quad M_1' := \begin{bmatrix} A & a & b \\ c & f_1 & f_2 \\ e & h_1 & h_2 \end{bmatrix}.$$

Notice that the determinant of the first matrix is z.

From the hypothesis, the matrix obtained from M_2 by taking residues has rank at least $r-1$. Thus, performing row and column operations on M, including column and row exchanges, we may, without changing the ideals I, J and K, assume that A is the identity matrix, $a = b = 0$ and $c = d = e = 0$. Then $z = f_1g_2 - f_2g_1$ and

$$\begin{aligned} I &= (f_1,\, f_2), \\ J &= (f_1g_2 - f_2g_1,\, f_1h_2 - f_2h_1,\, g_1h_2 - g_2h_1), \\ K &= (f_1g_2 - f_2g_1,\, f_1h_2 - f_2h_1). \end{aligned}$$

Now, since the matrix obtained from M_1 by taking residues has rank at least r, at least one among f_1, f_2, g_1, g_2 is invertible.

If f_1 is invertible, then

$$g_1h_2 - g_2h_1 = (g_1/f_1)(f_1h_2 - f_2h_1) - (h_1/f_1)(f_1g_2 - f_2g_1).$$

Thus, the lemma holds for $u = 1$ and $v = f_1h_2 - f_2h_1$. The case where f_2 is invertible is similar.

If g_1 is invertible, then

$$\begin{aligned} (f_1h_2 - f_2h_1) &= (f_1/g_1)(g_1h_2 - g_2h_1) + (h_1/g_1)(f_1g_2 - f_2g_1), \\ f_2 &= (g_2/g_1)f_1 - (1/g_1)(f_1g_2 - f_2g_1). \end{aligned}$$

Thus the lemma holds for $u = f_1$ and $v = g_1h_2 - g_2h_1$. A similar analysis holds if g_2 is invertible. ∎

5.4 Proposition. *Let C be a general smooth curve of genus $g \geq 1$ and i a nonnegative integer. Let Δ be the diagonal of $C \times C$ and W_i the i-th Weierstrass divisor. Let SW_i be the i-th special ramification scheme, and D_i and E_i the i-th special ramification schemes of type Diaz and Cukierman, respectively. Let E_i^+ be the i-th expanded special ramification scheme of* *type Cukierman. Then these ramification schemes are finite and satisfy, in the cycle group of $C \times C$:*

$$[SW_i] = [D_i] + [E_i] \quad \text{and} \quad [E_i^+] = [E_i] + (g+1)[W_i \cap \Delta].$$

Proof. Since C is general, by Statement 1 of Proposition 3.1, the set SW_i is finite for each $i \geq 0$. Thus, so are D_i and E_i by Expression (16). It follows that E_i^+ is finite, because $E_i^+ \cap \Delta$ is the set of points (P, P) such that P is Weierstrass, whence is finite.

Recall the notation of Subsections 4.1, 4.2, 5.1 and 5.2. Set $r := g+i-1$. Both equalities can be proved locally. Thus, let $(P, Q) \in C \times C$ and $\mathcal{O}$ be the local ring of $C \times C$ at (P, Q). As a map of $\mathcal{O}$-modules, $\rho_{i,r+1}$ is given by a matrix M of the form described in the proof of Lemma 5.3. Let us use the notation described in the statement of that lemma.

Let $K \subseteq \mathcal{O}$ define SW_i^+. Then $K = (z, z')$, where z (resp. z') is the maximal minor obtained from M by removing the last (resp. last but one) row. Notice that, from the nature of M as a "wronskian matrix", z' is also the derivative of z along p_1. Let I and J be the ideals of $\mathcal{O}$ defining D_i^+ and E_i^+, respectively. Then I and J are the ideals of all the maximal minors of M_2 and M, respectively.

Now, since C is a general curve, by Statement 2 of Proposition 3.1,

$$h^0(\omega_C((i+1)P - (g+i)Q)) \leq 1.$$

This translates in the matrix obtained from M_1 by evaluating at (P, Q) having rank at least r. Applying Lemma 5.3, there are $u, v \in \mathcal{O}$ such

$$I = (z, u), \quad J = (z, v), \quad K = (z, uv).$$

Now, since E_i^+ is finite-dimensional and $C \times C$ is smooth, the sequence z, v is regular. The same holds for the sequence z, u if $P \neq Q$. It follows that $[SW_i] = [D_i] + [E_i]$.

The second equality in the statement of the proposition is obvious off Δ. Thus, assume $Q = P$. Since $E_i^+ \cap \Delta = W_i \cap \Delta$, we may also assume that P is a Weierstrass point of C.

Let s be a local parameter of C at P, and denote by $t_1, t_2 \in \mathcal{O}$ its pullbacks with respect to the projections p_1 and p_2. Then $t := t_2 - t_1$ is a local equation for Δ. As we saw in Subsection 4.2, we have $z = t^g w$, where $w \in \mathcal{O}$ defines

W_i, and is not divisible by t. Letting ∂ denote the derivative with respect to t_2, we have

$$z' = \partial z = \partial(t^g w) = gt^{g-1}w + t^g \partial w.$$

Thus t^{g-1} divides z and z', and hence each element of K, in particular uv. Since E_i^+ is finite, t does not divide v, and hence $t^{g-1}|u$. Let $L := t^{1-g}K$. Then there are two expressions for L:

$$L = (tw, uv/t^{g-1}) \quad \text{and} \quad L = (tw, gw + t\partial w). \tag{19}$$

Since $W_i \cap \Delta$ is finite, the sequences $gw + t\partial w, t$ and w, t are regular. Thus, from the second expression for L above, we get

$$\ell(\mathcal{O}/L) = 2\ell(\mathcal{O}/(t, w)) + \ell(\mathcal{O}/(w, \partial w)).$$

Now, $\ell(\mathcal{O}/(w, \partial w)) = 0$ because w and ∂w cut out SW_i, and SW_i does not meet Δ. Also, by Lemma 4.4, W_i intersects Δ transversally. Thus $\ell(\mathcal{O}/(t, w)) = 1$, and hence $\ell(\mathcal{O}/L) = 2$.

Now, since the sequence z, v is regular, and $z = t^g w$, also the sequence tw, v is regular. Thus, from the first expression for L in (19), we get

$$\ell(\mathcal{O}/L) = \ell(\mathcal{O}/(tw, u/t^{g-1})) + \ell(\mathcal{O}/(tw, v)),$$

and whence $\ell(\mathcal{O}/(tw, v)) \leq 2$. Since $\mathcal{O}$ is regular, and the sequence tw, v is regular, so is the sequence v, w. Thus

$$\ell(\mathcal{O}/(tw, v)) = \ell(\mathcal{O}/(t, v)) + \ell(\mathcal{O}/(w, v)).$$

Since E_i^+ contains (P, P), the function v is zero on (P, P). Thus, since also t and w vanish on (P, P), we get $\ell(\mathcal{O}/(t, v)) = \ell(\mathcal{O}/(w, v)) = 1$. So, the multiplicity of E_i^+ at (P, P) is

$$\ell(\mathcal{O}/(z, v)) = g\ell(\mathcal{O}/(t, v)) + \ell(\mathcal{O}/(w, v)) = (g + 1).$$

Since, by Lemma 4.4, the multiplicity of $W_i \cap \Delta$ at (P, P) is 1, we are done.∎

5.5 Proposition. *Let C be a general smooth curve of genus $g \geq 1$ and i a nonnegative integer. Let SW_i be the i-th special ramification scheme of C. Then*

$$\int_{C\times C} [SW_i] = 2ig(g - 1)\Big((i + 2)(g + i)^2 + 2(g + i) + 2\Big). \tag{20}$$

Proof. Recall the notation of Subsections 4.1, 4.2, 5.1 and 5.2. Since C is general, SW_i is finite. Also, SW_i is the zero scheme of a section of the rank-2 bundle $\mathcal{J}^1_{p_1}(\mathcal{O}_{C\times C}(W_i))$. Thus its class in the Chow group of $C \times C$ satisfies

$$[SW_i] = c_2(\mathcal{J}^1_{p_1}(\mathcal{O}_{C\times C}(W_i))).$$

Using the truncation sequence for bundles of jets, we get

$$[SW_i] = [W_i](c_1(p_2^*\omega_C) + [W_i]).$$

Now, $c_1(p_2^*\omega_C) = K_2$. Using Expression (11) for $j = i$, and taking into account that $K_\ell^2 = 0$ for $\ell = 1, 2$, we get

$$\begin{aligned}[SW_i] &= i(g+i+1)\Big((g+i)^2+g+i+1\Big)K_2[\Delta] \\ &+ \frac{1}{2}i(i+1)\Big((g+i)^2+g+i+1\Big)K_1K_2 \\ &+ i^2(g+i+1)^2[\Delta]^2 + i^2(g+i+1)(i+1)K_1[\Delta].\end{aligned}$$

Using Formulas (14) and (15), we get

$$\begin{aligned}\int_{C\times C}[SW_i] &= i(g+i+1)\Big((g+i)^2+g+i+1\Big)(2g-2) \\ &+ \frac{1}{2}i(i+1)\Big((g+i)^2+g+i+1\Big)4(g-1)^2 \\ &- i^2(g+i+1)^2(2g-2) + i^2(g+i+1)(i+1)(2g-2).\end{aligned}$$

Simplifying, we get the claimed formula. ∎

5.6 Theorem. *Let C be a general smooth curve of genus $g \geq 1$, and i a nonnegative integer. Let D_i and E_i be the i-th special ramification schemes of type Diaz and Cukierman, respectively. Then D_i and E_i are reduced, and*

$$\int_{C\times C}[D_i] = g(g-1)\Big((g+i-1)^2(i+1)^2 - (g-1)^2\Big) \tag{21}$$

and

$$\int_{C\times C}[E_i] = g(g-1)\Big((g+i+1)^2(i+1)^2 - (g+1)^2\Big). \tag{22}$$

Proof. Recall the notation of Subsections 4.1, 4.2, 5.1 and 5.2. We will first compute the degrees of D_i and E_i. First of all, since E_i^+ is finite, and is the degeneracy scheme of $\rho_{i,g+i}$, applying Porteous formula ([8], Thm. 14.4,

p. 254), we get the following expression for the class $[E_i^+]$ in the Chow group of $C \times C$:

$$[E_i^+] = c_2(\mathcal{J}_{p_1}^{g+i}(\mathcal{L}_i) - p_1^*\mathcal{E}_i).$$

Now, $c_2(\mathcal{E}_i) = c_1(\mathcal{E}_i)^2 = 0$, since C is one-dimensional. Thus

$$[E_i^+] = c_2(\mathcal{J}_{p_1}^{g+i}(\mathcal{L}_i)) - c_1(\mathcal{J}_{p_1}^{g+i}(\mathcal{L}_i))c_1(p_1^*\mathcal{E}_i).$$

Using the truncation sequence of the bundles of jets, we get

$$\begin{aligned} c_1(\mathcal{J}_{p_1}^{g+i}(\mathcal{L}_i)) &= \sum_{\ell=1}^{g+i+1} (\ell K_2 + (i+1)[\Delta]); \\ c_2(\mathcal{J}_{p_1}^{g+i}(\mathcal{L}_i)) &= \sum_{m=2}^{g+i+1} \sum_{\ell=1}^{m-1} (\ell K_2 + (i+1)[\Delta])(mK_2 + (i+1)[\Delta]). \end{aligned}$$

Expanding, and using that $K_2^2 = 0$, we get

$$\begin{aligned} c_1(\mathcal{J}_{p_1}^{g+i}(\mathcal{L}_i)) &= \frac{1}{2}(g+i+1)(g+i+2)K_2 + (i+1)(g+i+1)[\Delta]; \\ c_2(\mathcal{J}_{p_1}^{g+i}(\mathcal{L}_i)) &= \frac{1}{2}(i+1)(g+i)(g+i+1)(g+i+2)K_2[\Delta] \\ &\quad + \frac{1}{2}(i+1)^2(g+i)(g+i+1)[\Delta]^2. \end{aligned}$$

Finally, using Formula (13) for $j = i$, and Formulas (14) and (15), we get

$$\int_{C\times C} [E_i^+] = (i+1)^2 g(g-1)(g+i+1)^2.$$

Now, it follows from Proposition 4.4 that W_i meets Δ transversally at $g^3 - g$ points. Thus, using Proposition 5.4, we get

$$\begin{aligned} \int_{C\times C} [E_i] &= \int_{C\times C} [E_i^+] - (g+1)(g^3 - g) \\ &= g(g-1)\Big((g+i+1)^2(i+1)^2 - (g+1)^2\Big), \end{aligned}$$

the stated formula for the degree of $[E_i]$.

Now, the expression for the degree of $[D_i]$ follows now from the equality $[SW_i] = [D_i] + [E_i]$ proved in Proposition 5.4 and Formula (20) for the degree of $[SW_i]$ proved in Proposition 5.5.

Let us now show that D_i and E_i are reduced. Let $(P, Q) \in SW_i$. Let $\hat{\mathcal{O}}$ be the completion of the local ring of $C \times C$ at (P, Q). Let t_1 and t_2 be local equations in $\hat{\mathcal{O}}$ for $\{P\} \times C$ and $C \times \{Q\}$, respectively. Then $\hat{\mathcal{O}} = \mathbb{C}[[t_1, t_2]]$. Let $w \in \mathcal{O}$ be a local equation for W_i. Since $(P, Q) \in SW_i$, and since W_i is nonsingular by Proposition 4.4, we may assume that $w = t_1 + u$, where $u \in \mathbb{C}[[t_2]]$. Now, let w' and u' be the derivatives of w and u with respect to t_2. Then the ideal defining SW_i at (P, Q) is (w, w'), and the multiplicity of the cycle $[SW_i]$ at (P, Q) is $\ell(\hat{\mathcal{O}}/(w, w'))$. Notice that $w' = u'$, and

$$\frac{\hat{\mathcal{O}}}{(w, w')} \cong \frac{\mathbb{C}[[t_2]]}{(u')} = \frac{\mathbb{C}[[t_2]]}{(u, u')} \cong \frac{\mathbb{C}[[t_1, t_2]]}{(t_1, w, w')}.$$

Thus the multiplicity of the cycle $[SW_i]$ at (P, Q) is the multiplicity m of $SW_i \cap (\{P\} \times C)$ at (P, Q).

Since the formation of SW_i commutes with base change, this multiplicity m satisfies

$$m = \mathrm{wt}_V(Q) - 1,$$

where V is the complete linear system of sections of $\omega_C((i+1)P)$. Now, by Propositions 3.1 and 3.4, the order sequence of V at Q satisfies

$$\begin{aligned} \epsilon_j(V, Q) &= j \quad (j = 0, 1, \ldots, g+i-3), \\ \epsilon_{g+i-2}(V, Q) &\leq g+i-1, \\ \epsilon_{g+i-1}(V, Q) &\leq g+i+1. \end{aligned}$$

Thus $m \leq 2$, with equality if and only if

$$h^0(\omega_C((i+1)P-(g+i-1)Q)) = 2 \quad \text{and} \quad h^0(\omega_C((i+1)P-(g+i+1)Q)) = 1,$$

that is, if and only if $(P, Q) \in D_i \cap E_i$. Since $[SW_i] = [D_i] + [E_i]$ by Proposition 5.4, it follows that D_i and E_i are reduced at (P, Q). ∎

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