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A CONTINUOUS BOWEN-MAÑÉ TYPE PHENOMENON

Esteban Muñoz-Young

Departamento de Matemática
Universidad Católica del Norte
Av. Angamos 0610, Antofagasta, Chile.

Andrés Navas

Departamento de Matemática
Facultad de Ciencias, Universidad de Chile
Las Palmeras 3425, Ñuñoa, Santiago, Chile.

Enrique Pujals

Instituto Nacional de Matemática Pura e Aplicada, IMPA
Estrada Dona Castorina 110, 22460-320, Rio de Janeiro, Brazil.

Carlos H. Vásquez

Departamento de Matemática
Universidad Católica del Norte
Av. Angamos 0610, Antofagasta, Chile.

Abstract. In this work we exhibit a one-parameter family of C^1-diffeomorphisms F_α of the 2-sphere, where $\alpha > 1$, such that the equator $\mathbb{S}^1$ is an attracting set for every F_α and $F_\alpha|_{\mathbb{S}^1}$ is the identity. For $\alpha > 2$ the Lebesgue measure on the equator is a non ergodic physical measure having uncountable many ergodic components. On the other hand, for $1 < \alpha \leq 2$ there is no physical measure for F_α. If $\alpha < 2$ this follows directly from the fact that the ω-limit of almost every point is a single point on the equator (and the basin of each of these points has zero Lebesgue measure). This is no longer true for $\alpha = 2$, and the non existence of physical measure in this critical case is a more subtle issue.

1. **Introduction.** Much of the recent progress in Dynamics arose from a probabilistic approach to the understanding of complicated dynamical systems. In this approach, one of the main topics is the study of the statistical properties of typical orbits, where *typical* means a positive volume (*i.e.* non zero Lebesgue measure) in the ambient space.

In this work we deal with discrete time dynamical systems, more precisely with diffeomorphisms $f : M \to M$ of compact boundaryless Riemannian manifolds. Given any invariant measure for such a diffeomorphism, the ergodic theorem asserts that time averages converge at almost every point. In addition, if the measure is ergodic, then these limit time averages coincide a.e. with the space average with respect to the measure. Nevertheless, invariant measures are in general singular with respect to the volume [1], and so these observations say nothing about the behavior for the orbits of any set of positive measure.

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A *physical* measure is an invariant probability measure such that the set of initial conditions for which the time averages converge and the limit coincides with the space average has positive Lebesgue measure. This set is called the *basin* of the measure. To be more precise, let us denote by Leb the normalized Riemannian volume on M (which we will also call the *Lebesgue measure* on M). If μ is an invariant probability measure for $f \in \mathrm{Diff}^r(M)$ (with $r \geq 1$), the *basin* $\mathcal{B}(\mu)$ of μ is the set

$$\mathcal{B}(\mu) = \left\{ x \in M \ : \ \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} = \mu \right\},$$

where δ_z denotes the Dirac measure on z and the convergence is in the weak* topology. An invariant measure μ is *physical* if $\mathcal{B}(\mu)$ has positive Lebesgue measure.

Sinai, Ruelle, and Bowen [4, 5, 9, 10], proved that for uniformly hyperbolic (Axiom A) diffeomorphisms and flows, time averages converge for Lebesgue almost every point and the limit coincides with one of finitely many physical measures. The problem of existence and finiteness of physical measures, beyond the Axiom A setting, has remained at the center of Dynamics ever since. We refer the reader to [3, 11, 13] for surveys of much of the progress obtained in this direction.

There exists a classical example of a family of systems exhibiting a physical measure with two ergodic components for certain parameters and such that there is no physical measure for the other parameters. This example is attributed by some authors to Bowen [12] and by others to Mañé [8], and so we will refer to it as the Bowen-Mañé's example. It goes roughly as follows. Consider a vector field in the plane with two saddle points s and s' which are joined by two trajectories, so that these trajectories bound a region U which contains one repelling fixed point r (see Figure 1). Suppose that all the orbits in $U \setminus \{r\}$ spiral outwards so as to accumulate on the boundary ∂U.

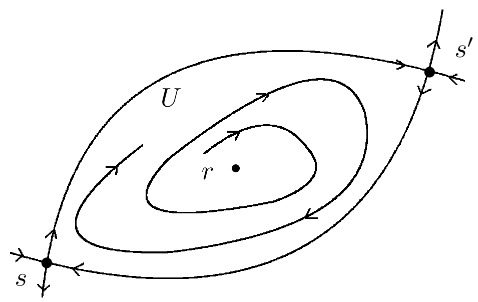
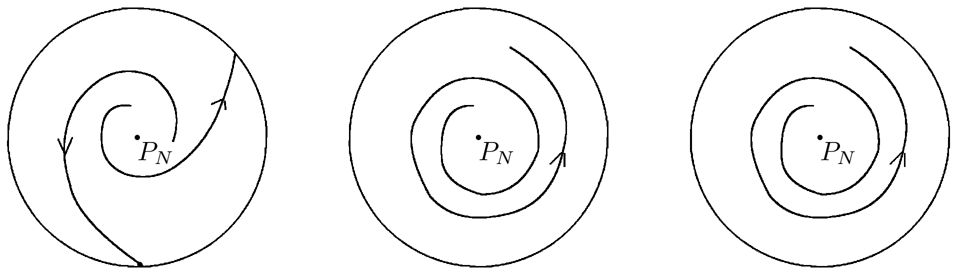


FIGURE 1. Bowen-Mañé's Example

Let $-\alpha, \beta$ be the eigenvalues associated to s, and $-\alpha', \beta'$ the eigenvalues associated to s', where $\alpha, \alpha', \beta, \beta'$ are all positive real numbers. (See [7] for the very interesting case where $\beta = 0$ and s is a saddle-node.) Of course, the hypothesis that the orbits spiral out to the boundary of U imposes the requirement that $\alpha\alpha' \geq \beta\beta'$. If $\alpha\alpha' > \beta\beta'$ there is no physical measure. This is due to the fact that sojourn times on small neighborhoods of s and s' are comparable with all the previous time. If $\alpha\alpha' = \beta\beta'$ (and the return maps to neighborhoods of s and s' have "nice" Taylor expansions), then there is a non ergodic physical measure which assigns positive weight to each of the two saddle fixed points (see [8] for details).

In this work we exhibit a one-parameter family of C^1-diffeomorphisms F_α : $\mathbb{S}^2 \to \mathbb{S}^2$, $\alpha > 1$, such that the equator $\mathbb{S}^1$ is the attracting set for each F_α and the restriction $F_\alpha|_{\mathbb{S}^1}$ is the identity. For $\alpha > 2$, the Lebesgue measure $\mathrm{Leb}_{\mathbb{S}^1}$ on the equator is a non ergodic physical measure having uncountably many ergodic components. If $1 < \alpha \le 2$ then $\mathrm{Leb}_{\mathbb{S}^1}$ is not a physical measure (and in fact, there is no physical measure at all). For $1 < \alpha < 2$ the orbit of almost every point converges to a single point on $\mathbb{S}^1$, and the basin of each of these points has zero Lebesgue measure. The case $\alpha = 2$ is special: almost every point has the whole equator $\mathbb{S}^1$ as ω-limit, but the speed in which the orbits turn around the sphere is too slow so that sojourn times on small regions near the equator are comparable with all the previous time. We summarize all these facts in the following Theorem.



$1 < \alpha < 2$ $\qquad$ $\alpha = 2$ $\qquad$ $\alpha > 2$

FIGURE 2. Dynamics of F_α on the north hemisphere

Theorem 1.1. *There exists a family of C^1-diffeomorphisms $F_\alpha : \mathbb{S}^2 \to \mathbb{S}^2, \alpha > 1$, such that:*

(a) *If $\alpha > 2$ the Lebesgue measure* $\mathrm{Leb}_{\mathbb{S}^1}$ *supported on the equator $\mathbb{S}^1$ is the unique physical measure for F_α. The basin of* $\mathrm{Leb}_{\mathbb{S}^1}$ *is equal to the sphere $\mathbb{S}^2$ minus the equator and the north and south poles. Furthermore, the restriction $F_\alpha|_{\mathbb{S}^1}$ equals the identity; in particular,* $\mathrm{Leb}_{\mathbb{S}^1}$ *has infinitely many ergodic components.*

(b) *If $1 < \alpha \le 2$ there is no physical measure for F_α. Moreover, for $1 < \alpha < 2$ the orbits of all the points in the open hemispheres and different from the poles converge to a single point on $\mathbb{S}^1$, and for $\alpha = 2$ the ω-limit of all such points coincides with the whole equator.*

It should be emphasized that all the Lyapunov exponents for the maps in our family are zero, and so the phenomenon described above is essentially "non hyperbolic", in contrast to the Bowen-Mañé's example which deeply relies on the hyperbolicity of the (saddle) fixed points. The following question remains completely open.

Question. What are the mild hyperbolic type conditions which ensure that physical measures have necessarily finitely many ergodic components ?

This work is organized as follows. In section 2 we define explicitly the family of diffeomorphisms F_α and we show a crucial estimate for the proof of our Theorem; we also discuss the (simplest) case $1 < \alpha < 2$. In Section 3 we deal with the particular case $\alpha = 2$, and we prove the non existence of physical measure. Finally, in Section 4 we prove that $\mathrm{Leb}_{\mathbb{S}^1}$ is a physical measure for F_α when $\alpha > 2$.

2. **The example.** We think of the 2-sphere $\mathbb{S}^2$ as being the surface obtained from the cylinder $\mathbb{T}^1 \times [-1,1]$ by identification of the upper boundary $\mathbb{T}^1 \times \{1\}$ and the lower boundary $\mathbb{T}^1 \times \{-1\}$ to the north pole P_N and the south pole P_S respectively. Having these identifications in mind, we define the family of C^1-diffeomorphisms $F_\alpha : \mathbb{S}^2 \to \mathbb{S}^2$, $\alpha > 1$, as follows:

(a) $F_\alpha(P_N) = P_N$ and $F_\alpha(P_S) = P_S$.

(b) If $(x,y) \in [0,1] \times [0,1)$, then

$$F_\alpha(x,y) = (x + y \ (\mathrm{mod}\ 1), f_\alpha(y)), \tag{2.1}$$

where $f_\alpha : [0,1] \to [0,1]$ is a C^1-diffeomorphism satisfying $f_\alpha(y) = y - y^\alpha$ for all $y \in [0, 1/e]$.

(c) If $(x,y) \in [0,1] \times [0,-1)$ then $F_\alpha(x,y) = F_\alpha(x,-y)$.

For any $\alpha > 1$ the poles P_N and P_S are repelling fixed points of F_α. From property (c) above, we can restrict our analysis of the dynamics of F_α to the north hemisphere. Since the restriction of F_α to the equator $\mathbb{S}^1$ is the identity, the Dirac measure on each point $(x,0) \in \mathbb{S}^1$ is an ergodic invariant measure for F_α. The Lebesgue measure $\mathrm{Leb}_{\mathbb{S}^1}$ on $\mathbb{S}^1$ is also invariant, and it has all these Dirac measures as ergodic components. Of course, none of the points in $\mathbb{S}^1$ belongs to the basin of $\mathrm{Leb}_{\mathbb{S}^1}$.

For all $n \geq 1$ and all $(x,y) \in [0,1] \times [0,1)$ one has

$$F_\alpha^n(x,y) = \left(x + \sum_{j=0}^{n-1} f_\alpha^j(y) \ (\mathrm{mod}\ 1), f_\alpha^n(y) \right). \tag{2.2}$$

Note that $f_\alpha(y) < y$ for every $y \in [0,1)$; thus $f_\alpha^{n+1}(y) < f_\alpha^n(y)$ for all $n \geq 1$. Furthermore, $f_\alpha^n(y) \to 0$ as $n \to \infty$. We then conclude that $\mathbb{S}^1$ contains the ω-limit of each point (x,y) in $\mathbb{S}^2 \setminus \{P_N, P_S\}$. Moreover, $\mathbb{S}^1$ is an attracting set for every F_α.

The following lemma concerns the dynamics of the maps f_α, and it is related to classical estimates for the iteration near parabolic fixed points in Complex Dynamics (see [6], Chapter II. 5). It is well known to the specialists, and we include a proof just for the convenience of the reader.

Lemma 2.1. *Let $g(y) = y - y^{1+\beta}$, where $\beta > 0$. Then for every $y_0 \in (0,1)$ one has*

$$\lim_{n\to\infty} \left(n^{1/\beta} g^n(y_0) \right) = \frac{1}{\beta^{1/\beta}}. \tag{2.3}$$

Proof. Letting $y_n = g^n(y_0)$ and $x_n = 1/y_n$ we have $\dfrac{1}{x_{n+1}} = \dfrac{1}{x_n} - \dfrac{1}{x_n^{1+\beta}} = \dfrac{x_n^\beta - 1}{x_n^{1+\beta}}$, and so $x_{n+1} = \dfrac{x_n^{1+\beta}}{x_n^\beta - 1}$. Thus $x_{n+1}^\beta = \dfrac{x_n^{\beta+\beta^2}}{(x_n^\beta - 1)^\beta}$, and the expansion of $\dfrac{1}{(x_n^\beta - 1)^\beta}$ in series shows that

$$x_{n+1}^\beta = x_n^{\beta+\beta^2} \left(x_n^{-\beta^2} + \beta x_n^{-\beta(\beta+1)} \right) + O(x_n^{-\beta}) \quad \text{and} \quad x_{n+1}^\beta \geq x_n^{\beta+\beta^2} \left(x_n^{-\beta^2} + \beta x_n^{-\beta(\beta+1)} \right).$$

In other words,

$$x_{n+1}^\beta = x_n^\beta + \beta + O(x_n^{-\beta}) \tag{2.4}$$

and

$$x_{n+1}^\beta \geq x_n^\beta + \beta. \tag{2.5}$$

From (2.5) one concludes that $x_n^\beta \geq x_0^\beta + n\beta \geq n\beta$, and from this and (2.4) one deduces

$$x_n^\beta \leq x_0^\beta + n\beta + \sum O(1/n\beta) = x_0^\beta + n\beta + O(\log n).$$

Therefore,

$$n\beta \leq x_n^\beta \leq x_0^\beta + n\beta + O(\log n).$$

Dividing by n and passing to the limit we obtain $x_n^\beta/n \to \beta$ as $n \to \infty$, which is equivalent to (2.3). □

Using the previous lemma we can come back to the study of our original maps F_α.

Proposition 2.2. *Let $(x,y) \in [0,1] \times (0,1)$. If $\alpha \geq 2$, then the ω-limit of (x,y) is $\mathbb{S}^1$. If $1 < \alpha < 2$, then the ω-limit of $(x,y) \in [0,1) \times (0,1)$ is a single point on the equator.*

Proof. To prove this Proposition first note that, by Lemma 2.1, the series $\sum_{j=0}^{n-1} f_\alpha^j(y)$ diverges if and only if $\alpha \geq 2$. Putting $z_n = x + \sum_{j=0}^{n-1} f_\alpha^j(y)$ we have $F_\alpha^n((x,y)) = (z_n \,(\mathrm{mod}\,1), f_\alpha^n(y))$. Since $f_\alpha^n(y) \to 0$ as $n \to \infty$, if z_n converges (*i.e.* if $1 < \alpha < 2$) then $F_\alpha^n((x,y))$ tends to $(\lim_{n\to\infty} z_n \,(\mathrm{mod}\,1), 0)$. If z_n diverges (*i.e.* if $\alpha \geq 2$) then, since $z_{n+1} - z_n = f_\alpha^n(y)$ goes to zero, the sequence $z_n \,(\mathrm{mod}\,1)$ is dense in $\mathbb{T}^1$. This easily implies that, in the latter case, the ω-limit of (x,y) is the whole equator. □

If $1 < \alpha < 2$ then $\mathbb{S}^2 \setminus \{P_N, P_S\}$ is foliated by the basins of the measures $\delta_{(x,0)}$, where $x \in \mathbb{T}^1$. The leaves of this foliation contain exactly one point on each level $\{(x,y) : y = c\}$, where $c \in (-1,1)$. Therefore, by Fubini's theorem, every basin has zero Lebesgue measure. This shows that, in this case, there is no physical measure for F_α.

3. Case $\alpha = 2$: There is no physical measure.

In this Section we deal only with the quite special diffeomorphism $F = F_2$, and to simplify notations we put $f = f_2$. Fix once and for all a point $(x_0, y_0) \in [0,1] \times (0,1)$. For $t \geq 0$ let us define

$$n(t) = \min\{n \geq 0 \,:\, x_0 + y_0 + \cdots + f^{n-1}(y_0) \geq t\}.$$

Given $k \in \mathbb{N}$ denote $n_k = n(k+1/2)$ and $n_k^* = n(k+1)$, so that $n_k \leq n_k^*$. Denoting by (μ_n) the sequence of probabilities

$$\mu_n = \frac{1}{n}\left[\delta_{(x_0,y_0)} + \delta_{F(x_0,y_0)} + \cdots + \delta_{F^{n-1}(x_0,y_0)}\right],$$

we will show that the sets of accumulation points of the sequences $(\mu_{n_k})_k$ and $(\mu_{n_k^*})_k$ are different. To do this, it suffices to exhibit a continuous function $\varphi : \mathbb{S}^2 \to \mathbb{R}$ such that

$$\liminf_{k\to\infty} \int \varphi \, d\mu_{n_k} > \limsup_{k\to\infty} \int \varphi \, d\mu_{n_k^*}. \tag{3.6}$$

Now, since $(\sqrt{e}-1)/(e-1) < 1/2$ and $(\sqrt{e}-1)/(\sqrt{e}-1/\sqrt{e}) > 1/2$, we can fix $\delta > 0$ and $\varepsilon > 0$ very small so that

$$\frac{\exp\left(\frac{1}{2(1-\varepsilon)}\right) - 1}{\exp\left(\frac{1}{1+\varepsilon}\right) - 1} < \frac{1}{2} \qquad \text{and} \qquad \frac{\exp\left(\frac{\frac{1}{2}-\delta}{1+\varepsilon}\right) - \exp\left(\frac{\delta}{1-\varepsilon}\right)}{\exp\left(\frac{1}{2(1-\varepsilon)}\right) - \exp\left(-\frac{1}{2(1-\varepsilon)}\right)} > \frac{1}{2}. \tag{3.7}$$

Let $g_1, g_2 : [0,1] \to [0,1]$ be the continuous functions illustrated below and which satisfy:

$$g_1(x) = \begin{cases} 1 & \text{for } x \in [\delta, 1/2-\delta], \\ 0 & \text{for } x \in [1/2, 1], \\ 0 & \text{for } x = 0; \end{cases} \qquad g_2(y) = \begin{cases} 1 & \text{for } y \in [0, y_0], \\ 0 & \text{for } y = 1. \end{cases}$$

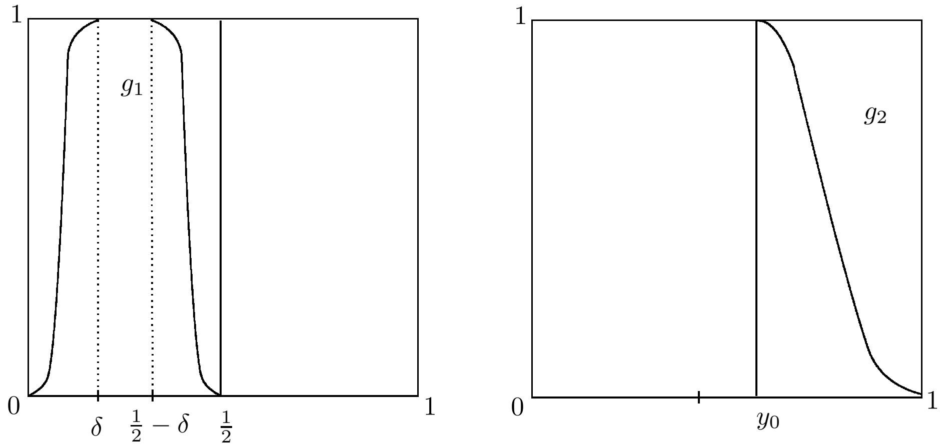


FIGURE 3. g_1, g_2 test functions

Put $\varphi(x,y) = g_1(x)g_2(y)$. Note that φ induces a continuous function on $\mathbb{S}^2$, which will be still denoted by φ. In order to verify (3.6) we will need the following estimates for the rates of growth of sojourn times.

Lemma 3.1. *There exists $T > 0$ such that for every $t \geq T$ and every $\Delta t \in (0,1]$ one has*

$$n(t+\Delta t) \leq n(t) \exp\left(\frac{\Delta t}{1-\varepsilon}\right) + 2, \tag{3.8}$$

$$n(t+\Delta t) \geq n(t) \exp\left(\frac{\Delta t}{1+\varepsilon}\right) - 4. \tag{3.9}$$

Proof. By Lemma 2.1 we have $nf^n(y_0) \to 1$ as $n \to \infty$; thus, we can fix $n_0 \in \mathbb{N}$ so that if $n \geq n_0$ then $1-\varepsilon \leq nf^n(y_0) \leq 1+\varepsilon$. We claim that the lemma holds for $T = x_0 + y_0 + \cdots + f^{n_0-1}(y_0)$. Let us prove (3.8) (the proof of (3.9) is analogous). Fix $t \geq T$ and $\Delta t > 0$. By definition, $x_0 + y_0 + \cdots + f^{n(t)-1}(y_0) \geq t$, and $n(t) \geq n_0$. If

$$(1-\varepsilon)\left[\frac{1}{n(t)} + \cdots + \frac{1}{m-1}\right] \geq \Delta t,$$

then

$$x_0 + y_0 + \cdots + f^{m-1}(y_0) \geq t + \frac{1-\varepsilon}{n(t)} + \cdots + \frac{1-\varepsilon}{m-1} \geq t + \Delta t,$$

and so $m \geq n(t+\Delta t)$. Since

$$(1-\varepsilon)\left[\frac{1}{n(t)} + \cdots + \frac{1}{m-1}\right] \geq (1-\varepsilon)\int_{n(t)}^{m-1} \frac{ds}{s} = (1-\varepsilon)\log\left(\frac{m-1}{n(t)}\right),$$

this shows that if $(1-\varepsilon)\log(\frac{m-1}{n(t)}) \geq \Delta t$ then $m \geq n(t+\Delta t)$, which proves (3.8). $\square$

We are now able to compare (asymptotically) the values of $\int \varphi d\mu_{n_k^*}$ and $\int \varphi d\mu_{n_k}$. First note that

$$\int \varphi \, d\mu_{n_k^*} \leq \frac{\sum_{i=0}^{k}[n(i+\frac{1}{2}) - n(i)]}{1 + \sum_{i=0}^{k}[n(i+1) - n(i)]},$$

and so

$$\limsup_{k\to\infty} \int \varphi \, d\mu_{n_k^*} \leq \limsup_{k\to\infty} \frac{n(k+1/2) - n(k)}{n(k+1) - n(k)}. \tag{3.10}$$

Now from (3.8) one gets (for k big enough)

$$n(k+1/2) \leq n(k) \exp\left(\frac{1}{2(1+\varepsilon)}\right) + 2,$$

and thus

$$n(k+1/2) - n(k) \leq n(k) \left[\exp\left(\frac{1}{2(1+\varepsilon)}\right) - 1\right] + 2. \tag{3.11}$$

On the other hand, (3.9) gives

$$n(k+1) \geq n(k) \exp\left(\frac{1}{1+\varepsilon}\right) - 4,$$

and therefore

$$n(k+1) - n(k) \geq n(k) \left[\exp\left(\frac{1}{1+\varepsilon}\right) - 1\right] - 4. \tag{3.12}$$

By combining (3.11) and (3.12) one concludes

$$\limsup_{k\to\infty} \frac{n(k+1/2) - n(k)}{n(k+1) - n(k)} \leq \frac{\exp\left(\frac{1}{2(1-\varepsilon)}\right) - 1}{\exp\left(\frac{1}{1+\varepsilon}\right) - 1},$$

and by (3.7) and (3.10) this gives

$$\limsup_{k\to\infty} \int \varphi \, d\mu_{n_k^*} < \frac{1}{2}.$$

Now let us deal with the sequence $(\mu_{n_k})_k$. Remark that, for k big enough,

$$\int \varphi \, d\mu_{n_k} \geq \frac{\sum_{i=0}^{k}[n(i+\frac{1}{2}-\delta) - n(i+\delta) - 1]}{n(1/2) + \sum_{i=1}^{k}[n(i+1/2) - n(i-1/2)]},$$

and since both $\big(n(k+1/2-\delta) - n(k+\delta)\big)$ and $\big(n(k+1/2) - n(k-1/2)\big)$ go to infinite as $k \to \infty$, this gives

$$\liminf_{k\to\infty} \int \varphi \, d\mu_{n_k} \geq \liminf_{k\to\infty} \frac{n(k+1/2-\delta) - n(k+\delta)}{n(k+1/2) - n(k-1/2)}. \tag{3.13}$$

By (3.8) and (3.9) we have (for k big enough)

$$n(k+1/2-\delta) \geq n(k) \exp\left(\frac{1/2-\delta}{1+\varepsilon}\right) - 4, \qquad n(k+\delta) \leq n(k) \exp\left(\frac{\delta}{1-\varepsilon}\right) + 2,$$

and so

$$n(k+1/2-\delta) - n(k+\delta) \geq n(k) \left[\exp\left(\frac{1/2-\delta}{1+\varepsilon}\right) - \exp\left(\frac{\delta}{1-\varepsilon}\right)\right] - 6. \tag{3.14}$$

Analogously,

$$n(k+1/2) \leq n(k) \exp\left(\frac{1}{2(1-\varepsilon)}\right) + 2, \quad n(k-1/2) \geq n(k) \exp\left(-\frac{1}{2(1-\varepsilon)}\right) - 2\exp\left(-\frac{1}{2(1-\varepsilon)}\right),$$

and so

$$n(k+1/2)-n(k-1/2) \leq n(k)\left[\exp\left(\frac{1}{2(1-\varepsilon)}\right)-\exp\left(-\frac{1}{2(1-\varepsilon)}\right)\right]+2+2\exp\left(-\frac{1}{2(1-\varepsilon)}\right). \tag{3.15}$$

By combining (3.14) and (3.15) this shows that

$$\liminf_{k\to\infty}\frac{n(k+1/2-\delta)-n(k+\delta)}{n(k+1/2)-n(k-1/2)} \geq \frac{\exp\left(\frac{1/2-\delta}{1+\varepsilon}\right)-\exp\left(\frac{\delta}{1-\varepsilon}\right)}{\exp\left(\frac{1}{2(1-\varepsilon)}\right)-\exp\left(-\frac{1}{2(1-\varepsilon)}\right)},$$

and by (3.7) and (3.13) this allows to conclude that

$$\liminf_{k\to\infty}\int \varphi\, d\mu_{n_k} > \frac{1}{2}.$$

3.1. **A general remark.** Let $F : \mathbb{S}^2 \to \mathbb{S}^2$ be any continuous map of the form $F(x,y) = (x+y \,(\mathrm{mod}\, 1), f(y))$, where $f : [-1,1] \to [-1,1]$ is a strictly increasing homeomorphism satisfying $f(-y) = -f(y)$ for all y. For each $\theta \in \mathbb{R}$ denote by R_θ the rotation by angle θ in the first coordinate, that is $R_\theta(x,y) = (x+\theta \,(\mathrm{mod}\, 1), y)$. Note that R_θ (preserves the Lebesgue measure and) centralizes F, that is $R_\theta \circ F = F \circ R_\theta$. If μ is a F-invariant measure, then for every $n \geq 1$ the measure $R_\theta^n(\mu)$ is F-invariant as well. (Here $R_\theta^n(\mu)$ is the probability measure defined by $R_\theta^n(\mu)(A) = \mu(R_\theta^{-n}(A))$). One easily checks that

$$R_\theta^n(\mathcal{B}(\mu)) = \mathcal{B}(R_\theta^n(\mu)). \tag{3.16}$$

Since R_θ preserves the Lebesgue measure,

$$\mathrm{Leb}\,(\mathcal{B}(\mu)) = \mathrm{Leb}\,(R_\theta^n(\mathcal{B}(\mu))) = \mathrm{Leb}\,(\mathcal{B}(R_\theta^n(\mu))). \tag{3.17}$$

Therefore, if μ is a physical measure for F, then $R_\theta^n(\mu)$ is also a physical measure for F. A simple argument shows that in fact a little bit more is true.

Proposition 3.2. *If μ is a physical measure then $R_\theta(\mu) = \mu$ for all θ.*

Proof. Fix $\theta \in \mathbb{R}\setminus\mathbb{Q}$. If we assume that $R_\theta(\mu) \neq \mu$, then the measures $R_\theta^n(\mu)$ are two by two distinct physical measures for F, and by (3.17) their basin have the same positive Lebesgue measure for all $n \in \mathbb{N}$, which is absurd. Hence, $R_\theta(\mu) = \mu$ for all $\theta \in \mathbb{R}\setminus\mathbb{Q}$. Finally, by continuity, or just because every rational number is the sum of two irrationals, μ is invariant by the rational rotations as well. □

Now assume that $0 < f(y) < y$ for all $y \in (0,1)$. In this case, the ω-limit of the orbit of every point in $\mathbb{S}^2 \setminus \{P_N, P_S\}$ is contained in the equator $\mathbb{S}^1$. Hence, the support of every physical measure is contained in $\mathbb{S}^1$, and by the Proposition 3.2 the only possible physical measure is the Lebesgue measure on $\mathbb{S}^1$. This can be applied to the map $f(y) = y - y^2$ to give another proof of the non-existence of physical measure for F_2.

Proposition 3.3. *If $\alpha = 2$ then* $\mathrm{Leb}_{\mathbb{S}^1}$ *is not a physical measure for F_α.*

The proof of this Proposition can be made by a direct computation. Indeed, one can show that $\frac{d}{dx}\left(\frac{e^x-1}{e-1}\right) = \frac{e^x}{e-1}$ is the density of the limit measure as $k \to \infty$ along the sequence

$$\frac{1}{n(k)}\left[\delta_{(x_0,y_0)} + \delta_{F(x_0,y_0)} + \cdots + \delta_{F^{n(k)-1}(x_0,y_0)}\right].$$

4. **Case $\alpha > 2$: $\mathrm{Leb}_{\mathbb{S}^1}$ is a physical measure.** Let us fix $\alpha > 2$ and let us denote $F = F_\alpha$ and $f = f_\alpha$. According to Theorem 2.2 of [2], in order to prove that a sequence of probability measures μ_n on $\mathbb{S}^2$ converges to some probability measure μ in the weak* topology, it is enough to verify that, for (the projection on $\mathbb{S}^2$ of) each square $A = [a,b] \times [c,d] \subset [0,1] \times [-1,1]$, one has $\mu_n(A) \to \mu(A)$. In our case we have to deal with the measures $\mu = \mathrm{Leb}_{\mathbb{S}^1}$ and

$$\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{F^j(z)},$$

where z belongs to $\mathbb{S}^2 \setminus \mathbb{S}^1$ and is different from the poles. Since F is symmetric with respect to the equator and $\mathbb{S}^1$ separates the dynamics between the two hemispheres, we can restrict our analysis to squares of the form $A = [a,b] \times [d',d]$, where $d' \geq 0$, and to points $z = (x_0, y_0)$ in $[0,1] \times (0,1)$. Therefore, denoting $z_j = F^j(z)$, we are reduced to show that, for every $d \geq d' \geq 0$ and all $a \leq b$ in $[0,1]$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{F^j(z)}([a,b] \times [d',d]) = \mathrm{Leb}_{\mathbb{S}^1}([a,b] \times [d',d]). \tag{4.18}$$

Since $f^n(y) \to 0$ as $n \to \infty$, if $d > 0$ then $f^n(y_0) \leq d$ for every $n \in \mathbb{N}$ sufficiently large. In particular, if $d' > 0$ (and also if $a = b$) then (4.18) is trivially satisfied: both sides are equal to zero.

To summarize, if we denote $y_n = f^n(y_0)$, then we need to verify that, for all $a < b$ in $[0,1]$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{y_j}([a,b]) = b - a. \tag{4.19}$$

To do this let us define again, for $t \geq 0$,

$$n(t) = \min\{n \geq 0 \,:\, x_0 + y_0 + \cdots + f^{n-1}(y_0) \geq t\}.$$

The *a priori* bounds for the rate of growth of $n(t)$ now take the following form.

Lemma 4.1. *Given $\varepsilon > 0$ there exists $T > 0$ such that for every $t \geq T$ and every $\Delta t > 0$ one has, for $c = 1/(\alpha-1)^{1/(\alpha-1)}$,*

$$n(t+\Delta t) \leq \left[\frac{\Delta t(\alpha-2)}{(1-\varepsilon)c(\alpha-1)} + n(t)^{\frac{\alpha-2}{\alpha-1}} \right]^{\frac{\alpha-1}{\alpha-2}} + 2, \tag{4.20}$$

$$n(t+\Delta t) \geq \left[\frac{\Delta t(\alpha-2)}{(1+\varepsilon)c(\alpha-1)} + (n(t)-2)^{\frac{\alpha-2}{\alpha-1}} \right]^{\frac{\alpha-1}{\alpha-2}} + 2. \tag{4.21}$$

Proof. By Lemma 2.1 we have $n^{1/(\alpha-1)} f^n(y_0) \to c$ as $n \to \infty$. Thus, we can find $n_0 \in \mathbb{N}$ so that if $n \geq n_0$ then

$$\frac{(1-\varepsilon)c}{n^{1/(\alpha-1)}} \leq f^n(y_0) \leq \frac{(1+\varepsilon)c}{n^{1/(\alpha-1)}}.$$

We claim that the lemma holds for $T = x_0 + y_0 + \cdots + f^{n_0-1}(y_0)$. Let us prove (4.20) (the proof of (4.21) is analogous). Fix $t \geq T$ and $\Delta t > 0$. By definition, $x_0 + y_0 + \cdots + f^{n(t)-1}(y_0) \geq t$, and $n(t) \geq n_0$. If

$$(1-\varepsilon)c \left[\frac{1}{n(t)^{1/(\alpha-1)}} + \cdots + \frac{1}{(m-1)^{1/(\alpha-1)}} \right] \geq \Delta t,$$

then

$$x_0 + y_0 + \cdots + f^{m-1}(y_0) \geq t + \frac{(1-\varepsilon)c}{n(t)^{1/(\alpha-1)}} + \cdots + \frac{(1-\varepsilon)c}{(m-1)^{1/(\alpha-1)}} \geq t + \Delta t,$$

and so $m \geq n(t+\Delta t)$. Since

$$\begin{aligned} \frac{1}{n(t)^{1/(\alpha-1)}} + \cdots + \frac{1}{(m-1)^{1/(\alpha-1)}} &\geq \int_{n(t)}^{m-1} \frac{ds}{s^{1/(\alpha-1)}} \\ &= \frac{\alpha-1}{\alpha-2}\left[(m-1)^{\frac{\alpha-2}{\alpha-1}} - (n(t))^{\frac{\alpha-2}{\alpha-1}}\right], \end{aligned}$$

this shows that if

$$\frac{(1-\varepsilon)c(\alpha-1)}{\alpha-2}\left[(m-1)^{\frac{\alpha-2}{\alpha-1}} - (n(t))^{\frac{\alpha-2}{\alpha-1}}\right] \geq \Delta t$$

then $m \geq n(t+\Delta t)$, which proves (4.20). □

Now denote

$$\begin{aligned} m_k &= \#\{n \in \mathbb{N} :\ x_0 + y_0 + \cdots + f^{n-1}(y_0) \in [a+k, b+k]\}, \\ m_k^* &= \#\{n \in \mathbb{N} :\ x_0 + y_0 + \cdots + f^{n-1}(y_0) \in [a+k-1, a+k]\}, \\ m_k^{**} &= \#\{n \in \mathbb{N} :\ x_0 + y_0 + \cdots + f^{n-1}(y_0) \in [a+k, a+k+1]\}. \end{aligned}$$

Since $f^n(y_0) \to 0$ as $n \to \infty$, the sequences (m_k), (m_k^*) and (m_k^{**}) go to infinity with k. Thus, in order to show (4.19), it is enough to verify that

$$\lim_{n\to\infty} \frac{m_k}{m_k^*} = \lim_{n\to\infty} \frac{m_k}{m_k^{**}} = b - a.$$

To do this first note that

$$\begin{aligned} m_k &\geq n(b+k) - n(a+k) = n(a+k+(b-a)) - n(a+k), \\ m_k^* &\leq n(a+k) - n(a+k-1) + 1. \end{aligned}$$

By Lemma 4.1, given $\varepsilon > 0$ one has, for k sufficiently large,

$$\frac{m_k}{m_k^*} \geq \frac{\left[\frac{(b-a)(\alpha-2)}{(1+\varepsilon)c(\alpha-1)} + (n(a+k)-2)^{\frac{\alpha-2}{\alpha-1}}\right]^{\frac{\alpha-1}{\alpha-2}} - (n(a+k)-2)}{n(a+k) - \left[(n(a+k)-2)^{\frac{\alpha-2}{\alpha-1}} - \frac{\alpha-2}{(1-\varepsilon)c(\alpha-1)}\right]^{\frac{\alpha-1}{\alpha-2}} + 1}.$$

Since $n(a+k) \to \infty$ as $k \to \infty$, this gives

$$\liminf_{k\to\infty} \frac{m_k}{m_k^*} \geq \liminf_{k\to\infty} \frac{\left[\frac{(b-a)(\alpha-2)}{(1+\varepsilon)c(\alpha-1)} + (n(a+k)-2)^{\frac{\alpha-2}{\alpha-1}}\right]^{\frac{\alpha-1}{\alpha-2}} - (n(a+k)-2)}{(n(a+k)-2) - \left[(n(a+k)-2)^{\frac{\alpha-2}{\alpha-1}} - \frac{\alpha-2}{(1-\varepsilon)c(\alpha-1)}\right]^{\frac{\alpha-1}{\alpha-2}}}. \tag{4.22}$$

Now recalling that, if u, v are real numbers and $\gamma > 0$ then

$$\lim_{x\to\infty} \frac{(u + x^\gamma)^{1/\gamma} - x}{x - (x^\gamma - v)^{1/\gamma}} = \frac{u}{v},$$

and applying this fact to

$$u = \frac{(b-a)(\alpha-2)}{(1+\varepsilon)c(\alpha-1)}, \qquad v = \frac{\alpha-2}{(1-\varepsilon)c(\alpha-1)}, \qquad \gamma = \frac{\alpha-2}{\alpha-1},$$

one concludes that the right hand side expression in (4.22) is equal to $(b-a)(1-\varepsilon)/(1+\varepsilon)$. Thus,

$$\liminf_{k\to\infty} \frac{m_k}{m_k^*} \geq (b-a)\Big(\frac{1-\varepsilon}{1+\varepsilon}\Big),$$

and since this inequality is true for all $\varepsilon > 0$, one deduces that

$$\liminf_{k\to\infty} \frac{m_k}{m_k^*} \geq b - a. \tag{4.23}$$

On the other hand, note that

$$m_k \leq n(b+k) - n(a+k) + 1 = n(a+k+(b-a)) - n(a+k) + 1,$$

$$m_k^* \geq n(a+k) - n(a+k-1).$$

Thus, given $\varepsilon > 0$ one has, for k large enough,

$$\frac{m_k}{m_k^*} \leq \frac{\left[\frac{(b-a)(\alpha-2)}{(1-\varepsilon)c(\alpha-1)} + n(a+k)^{\frac{\alpha-2}{\alpha-1}}\right]^{\frac{\alpha-1}{\alpha-2}} - n(a+k) + 3}{(n(a+k)-2) - \left[(n(a+k)-2)^{\frac{\alpha-2}{\alpha-1}} - \frac{\alpha-2}{(1+\varepsilon)c(\alpha-1)}\right]^{\frac{\alpha-1}{\alpha-2}}}.$$

As in the previous case, by passing to the limit in this inequality one deduces that

$$\limsup_{k\to\infty} \frac{m_k}{m_k^*} \leq (b-a)\Big(\frac{1+\varepsilon}{1-\varepsilon}\Big),$$

and since $\varepsilon > 0$ is arbitrary this shows that

$$\limsup_{k\to\infty} \frac{m_k}{m_k^*} \leq b - a. \tag{4.24}$$

By combining (4.23) and (4.24) one finally obtains

$$\lim_{k\to\infty} \frac{m_k}{m_k^*} = b - a.$$

We leave to the reader the proof of the (analogous) equality

$$\lim_{k\to\infty} \frac{m_k}{m_k^{**}} = b - a,$$

which together with the previous one allows to finish the proof of our Theorem.

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E-mail address: jmunoz@ucn.cl
E-mail address: andnavas@uchile.cl
E-mail address: enrique@impa.br
E-mail address: cavasquez@ucn.cl