Critical points for surfaces diffeomorphisms.

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Abstract

Using the definition of dominated splitting, we introduce the notion of critical set for any dissipative surface diffeomorphism as an intrinsically well defined object. We obtain a series of results related to this concept.

1 Introduction.

Uniform hyperbolicity has been a long standing paradigm of complete dynamical description: Systems such that the tangent bundle over the Limit set (the accumulation points of any orbit) split into two complementary subbundles that are uniformly forward (respectively backward) contracted by the tangent map, can be completely described from a geometrical and topological point of view.

Nevertheless, uniform hyperbolicity is a property less universal than it was initially thought: *there are open sets in the space of dynamics which are non-hyperbolic.* Actually, Newhouse showed that for smooth surface diffeomorphisms, the unfolding of a *homoclinic tangency* (a non transversal intersection of stable and unstable manifolds of a periodic point) generates open sets of diffeomorphisms such that their Limit set is non-hyperbolic (see [N1], [N2], [N3]).

A natural question arises; *which are the dynamical phenomenon that characterize non-hyperbolic systems?*

For one-dimensional endomorphism, the presence of *critical points* (points with zero derivative) in the Limit set is an obstruction to hyperbolicity. On the other hand, Mañé showed that smooth and generic (Kupka-Smale) one-dimensional endomorphisms without critical points are either hyperbolic or conjugate to an irrational rotation (see [M]). In other words, we could say that *for generic smooth one-dimensional endomorphisms, any compact invariant set is hyperbolic if, and only if, it does not contain critical points.*

So, taking in mind this scenario for one-dimensional dynamics, we could try to identify what are the two dimensional phenomenon whose *presence impede the hyperbolicity for a Kupka-Smale surfaces diffeomorphisms* and whose *absence guarantee it*? In other words, *what are two dimensional "critical points" for Kupka-Smale surfaces diffeomorphisms*?

The idea of trying to identify a critical set for two-dimensional maps, that it is to say, a set designated to play a role analogous to that played by critical points in one-dimensional dynamics, goes back to the seminal studies done for the Hénon attractor in [BC]. Here, we try to introduce the critical set for any dissipative diffeomorphism, as an intrinsically defined object.

To identify the critical sets, we need to look for weaker forms of hyperbolicity. Basically, there are two ways to relax hyperbolicity. One, called non-uniform hyperbolicity (or Oseledet's theory), where the tangent bundle splits for points almost everywhere with respect to some invariant measure, and vectors are asymptotically contracted or expanded in a rate that may depend on the base point. Other is the notion of *dominated splitting* which was first introduced independently by Mañé, Liao and Pliss, as a first step in the attempt to prove that structurally stable systems satisfy a hyperbolic condition on the tangent map. An f-invariant set Λ is said to have dominated splitting if we can decompose its tangent bundle in two invariant subbundles $T_\Lambda M = E \oplus F$, such that:

$$\|Df^n_{/E(x)}\|\|Df^{-n}_{/F(f^n(x))}\| \leq C\lambda^n, \text{ for all } x \in \Lambda, n \geq 0. \tag{1}$$

with $C > 0$ and $0 < \lambda < 1$. In this case we say that Λ has a $(C, \lambda)-$*dominated splitting*.

Since it was proved in [PS1] that any compact invariant set exhibiting dominated splitting of a generic C^2 surface diffeomorphism, is a hyperbolic set, *to look for dynamical obstructions to hyperbolicity is equivalent to look for dynamical obstructions to domination.*

Following this idea, we propose to define the *critical set as the region where domination fails*. This will provide us with a unique and intrinsic characterization of critical sets. Whit this definition in mind, we can conclude (see theorem C for details):

for generic smooth surface diffeomorphisms,
any (dissipative) compact invariant set is hyperbolic if, and only if,
it does not contain critical points.

To pursue this program, first we recall the projective tangent bundle dynamics associated to the derivative of a diffeomorphism and we rewrite the notion of domination in terms of the projective tangent bundle dynamics. In particular, we introduce the notion of non-uniformly dominated sets (this is asserted in the next subsection). In few words, these sets are sets that exhibit some kind of asymptotic domination. From an easy observation that follows from Oseledet's theorem, it is shown that under the assumption of dissipation the non-uniformly dominated sets always exists (this is asserted in subsection 1.2). Using these sets, and general results about the dynamical consequences of dominated splitting, we introduce the notion of critical points and values and we show that they are the generic obstruction for hyperbolicity (this is done in subsection 1.3). Using these notions of critical points, we explore different types of non-hyperbolic dynamics that could appear (see subsection 1.4). At the end of the present section, we give some application of those results. The proof of the theorem are given in the last section.

1.1 Projective tangent bundle dynamic.

Let $f \in Diff^1(M^2)$ and let Λ be a compact invariant set of f. Let us take the derivative of f acting in the unitarian tangent bundle:

$$G_x : (T_xM)_1 \to (T_{f(x)}M)_1$$

$$G_x(v) = \frac{D_xf(v)}{|D_xf(v)|}.$$

We denote with $G_x^n = \bigcirc_{i=0}^{n-1} G_{f^i(x)}$ and let us take

$$g_x(v) = D_wG_x(v).$$

To simplify notation, we denote $D_wG_x^n$ with g_x^n and sometimes only with g^n. Observe that $g_x^n(v) = \Pi g_{f^i(x)}(G_x^i(v))$,

It is possible to formulate the dominated spitting condition in terms of the dynamic of G (see lemma 2.0.2 for precisse formulation).

Definition 1 Blocks of domination: *Let Λ be a compact invariant set. Given $\delta > 0$, we define the following sets:*

$$H^-(\delta) = \{x \in \Lambda : \exists\, F_x \in (T_xM)_1 \ g^{-n}(F_x) > (1+\delta)^n, \forall n \geq 0\};$$

$$H^+(\delta) = \{x \in \Lambda : \exists\, E_x \in (T_xM)_1 \ g^n(E_x) > (1+\delta)^n, \forall n \geq 0\}.$$

It follows immediately that these sets are compacts and the directions E_x and F_y are unique. Observe that if Λ has a dominated splitting $T_\Lambda M = E \oplus F$, then there exists $\delta > 0$ such that any point has an iterate in $H^\pm(\delta) = \Lambda$ (for details see lemma 2.0.2).

1.2 Existences of Blocks of domination.

The goal of the next theorem is to show that under the assumption of "dissipation" (see below for the definition), for any compact invariant set Λ contained in the Limit set (noited from now on as $L(f)$) that there exists $\delta_0 > 0$ such that the sets $H^\pm(\delta_0)$ has *total measure*, in the sense that *for any invariant measure μ with support contained in Λ holds that $\mu(H^\pm(\delta_0)) = 1$*. The condition of dissipation is essential; in fact, observe that for the identity map the sets $H^+(\delta)$ and $H^-(\delta)$ are empty fro any $\delta > 0$.

From now on, we assume that f restricted to Λ is dissipative, i.e.: there exists a positive constant $b < 1$ such that

$$|det(D_xf)| \leq b \text{ for any } x \in \Lambda.$$

In this case, we say that Λ is a *dissipative compact invariant set* of f. From now on, with the constant b we denote

$$b = \max\{|det(D_xf)|,\ x \in \Lambda\}.$$

We also take the following positive constant larger than one,

$$b_0 = \frac{1}{b} - 1.$$

Moreover, for any set Λ we remove the attracting periodic points. More precisely, given a set Λ we consider the set $\hat{\Lambda} = \text{Closure}[\Lambda \setminus \mathcal{P}_0]$ where $\mathcal{P}_0$ is the set formed by attracting periodic points contained in Λ. From now on, and taking in mind the previous consideration, we assume that Λ does not contain attracting periodic points.

The next theorem should be considered a folklore's one and follows from an easy application of Oseledt's theorem. For completeness the proof is given in section 3.

Theorem A: *Let $f \in Diff^{1+\beta}(M^2)$ and let Λ be a dissipative compact invariant set contained in $L(f)$.. Then, there exists δ_0 such that the Blocks of domination $H^+(\delta_0)$ and $H^-(\delta_0)$ are not empty. Moreover, the sets*

$$\Lambda_0^+ = \cup_{n\in\mathbb{Z}} f^n(H^+(\delta_0)), \quad \Lambda_0^- = \cup_{n\in\mathbb{Z}} f^n(H^-(\delta_0))$$

have total measure.

Observe that points in $H^+(\delta_0)$ has a direction of uniform contraction for forward iterates and points in $H^-(\delta_0)$ has a non-positive Lyapunov exponent for backward iterates. In particular, observe that for points in $H^+(\delta_0)$ it is possible to obtain certain nice dynamical properties: their local stable set are uniformly embedded submanifolds. This is a classical result that can be adapted from Pesin's theory and it is recalled in section 2.

These sets are not necessarily invariant, however, if $x \in H^+(\delta_0)$ ($x \in H^-(\delta_0)$) then the forward (backward, respectively) orbit of x intersects $H^+(\delta_0)$ ($H^-(\delta_0)$) infinitely many times (see lemma 2.4).

1.3 Critical points and values.

Now, we define the critical sets. As we said, the critical sets corresponds to points where the domination fails. Following this idea, roughly speaking, the critical points are points in $H^-(\delta_0)$ such that, under forward iteration, their "unstable" direction is in tangent position to the "stable" direction of a point in $H^+(\delta_0)$. The definition is introduced for C^2-dynamics

Definition 2 Critical points and critical values. *Let $f \in Diff^1(M^2)$ and let Λ be a dissipative compact invariant set contained in $L(f)$. Let $H^{\pm}(\delta_0)$ be the sets given by corollary 2.2. We say that x is a critical point and y is a critical value if:*

1. *$x \in H^-(\delta_0)$ and $y \in H^+(\delta_0)$ verifyng that*

   (a) *for any $n \geq 0$ $f^n(x) \notin H^-(\delta_0)$ and;*

(b) for any $n \geq 0$ $f^{-n}(y) \notin H^+(\delta_0)$.

2. *There exist a sequence of positive integer* k_n *such that:*

   (a) $f^{k_n}(x) \to y$,

   (b) $G^{k_n}(F_x) \to E_y$.

We denote with CV_Λ*, the set of critical value of* Λ *and with* CP_Λ*, the set of critical points of* Λ.

Let us explain briefly the previous definition: The first item assert that a point x is a critical point, if it belongs to $H^-(\delta_0)$ and any forward iterate of x, does not belong to $H^-(\delta_0)$; a point y is a critical value, if it belongs to $H^+(\delta_0)$ and any backward iterate of y, does not belong to $H^+(\delta_0)$ The last item assert, that the unstable direction of x accumulates, by iteration, on the stable direction of y.

Observe from the definition that critical points and critical values are one related to the other. On the other hand, we are not assuming that the critical point and the critical values are in the orbit of each other. However, in the definition, it could hold that $k_n = k$ for any n and therefore $f^k(x) = y$. In this case holds for any critical point, we do not know if $k = k(x)$ is uniformly bounded for any critical point.

Lemma 1.3.1 *The set of critical points and critical values are closed sets.*

The proof is given in section 3 after the proof of theorem B.

An easy example of critical points and values is given by a homoclinic tangency. To define that, we need to recall that for a hyperbolic periodic point p of f, the stable and unstable sets

$$W^s(p) = \{y \in M : \ dist(f^n(y), f^n(p)) \to 0 \text{ as } n \to \infty\},$$

$$W^u(p) = \{y \in M : \ dist(f^n(y), f^n(p)) \to 0 \text{ as } n \to -\infty\}$$

are C^r-injectively immersed submanifolds. A point of intersection of these manifolds is called a homoclinic point.

Definition 3 Homoclinic tangency. *We say that* f *has a homoclinic tangency if there is a periodic point* p *such that there is a point* $x \in W^s(p) \cap W^u(p)$ *with* $T_xW^s(p) = T_xW^u(p)$. *In this case we say that* x *is a point of tangency.*

In the next lemma we relate the tangencies with the critical points and values. Observe that if x is a point of tangency then any iterate of x is also a point of tangency. However, the lemma says that there is only one critical point and only one critical value in the orbit of x. The proof is left to the reader.

Lemma 1.3.2 *If x is a point of tangency then there are integers $k_1 < k_2$ such that $f^{k_1}(x)$ is critical point and $f^{k_2}(x)$ is a critical value.*

The converse of the previous remark is not true (see next subsection); however, we could say that the presences of critical points are "almost tangencies": on one hand, recall that if $y \in H^+(\delta_0)$ then the local stable set is an embedded submanifold, on the other hand if $x \in H^-(\delta_0)$ and the local unstable set is also an embedded manifold then F_x coincides with the tangent direction of the local unstable set; therefore, if $Df^{k_n}(F_x)$ accumulates on E_y it holds that the unstable manifold accumulate in the stable manifold of y in a tangent way.

The next two theorems explain the role of critical points and values: *the presences of critical points (values) is an obstruction for the existences of dominated splitting and hyperbolicity; the absences of them guarantee domination properties and hyperbolicity.*

Theorem B: *Let $f \in Diff^2(M^2)$ be a Kupka-Smale system, and let Λ be a dissipative compact invariant set of f contained in the Limit set. The set Λ has dominated splitting if and only if $CP_\Lambda = \emptyset$ (so $CV_\Lambda = \emptyset$).*

Observe that to guarantee that a set has dominated splitting, it is enough to guarantee that there are no critical point. The above theorem hinges on a theorem that allows to understand the dynamics of invariant set displaying dominated splitting, assuming that the systems is smooth and Kupka-Smale. This theorem is proved in [PS1] (see theorem B of the mentioned paper):

Theorem 1.1 *([PS1]) Let $f \in Diff^2(M^2)$ be a Kupka-Smale system and let Λ be a compact invariant set contained in the Limit set exhibiting dominated splitting. Then Λ can be decomposed in a hyperbolic set and a finite number of invariant closed curve normally hyperbolic with dynamic conjugated to an irrational rotation.*

Moreover, from the previous result and theorem B we get the following:

Theorem C: *Let $f \in Diff^2(M^2)$ and let Λ be a dissipative transitive compact invariant set of f contained in $L(f)$, and let us assume that f is Kupka-Smale. Then Λ is either a hyperbolic set or an invariant closed curve normally hyperbolic with dynamic conjugated to an irrational rotation if and only if $CP_\Lambda = \emptyset$ (so $CV_\Lambda = \emptyset$).*

Since it is possible to show that generically, any smooth surface diffeomorphisms has not an invariant closed curve normally hyperbolic with dynamic conjugated to an irrational rotation, from previous theorem we can conclude the next corollary:

Corollary 1.1 *Let $f \in Diff^2(M^2)$ and let Λ be a dissipative transitive compact invariant set of f. Generically it follows that Λ is a hyperbolic set if and only if $CP_\Lambda = \emptyset$ (so $CV_\Lambda = \emptyset$).*

The previous theorem has immediate consequences about the non-hyperbolic dissipative compact invariant set: this set contains either critical points and critical values or they are invariant curves with dynamic conjugated to an irrational rotation.

Corollary 1.2 *Let $f \in C^2(M^2)$ and let x be a recurrent point such that $\omega(x)$ is a Kupka-Smale dissipative set. Then one of the next option holds:*

1. *$\omega(x)$ is a hyperbolic set;*
2. *$\omega(x)$ is a normally hyperbolic periodic closed curve;*
3. *there exists $v \in T_xM$ such that*

$$\limsup_{n\to+\infty} \frac{1}{n}\log|g^n(v)| \geq 1+\delta, \quad \lim_{n\to+\infty} \frac{1}{n}\log|g^{-n}(v)| > 1+\delta.$$

In particular, this implies that

$$\liminf_{n\to+\infty} \frac{1}{n}\log|Df^n(v)| < 0, \quad \lim_{n\to+\infty} \frac{1}{n}\log|Df^{-n}(v)| \leq 0.$$

The definition of critical points and values is naturally motivated by the definition of critical points and values of one dimensional endomorphisms: Let $h : I \to I$ be a one-dimensional map from an interval I having a critical point c_0 in I; therefore, $h(c_0)$ is the critical value of h. It is natural to embed h as a two dimensional dynamic from $I \times I$ to itself in the following way: $H(x, y) = (h(x), x)$. This map has a stable foliation given by the vertical lines: since $DH(0, 1) = (0, 0)$ follows that any point $(x, y) \in H^+(\delta_0)$ and $E_{(x,y)} = (0, 1)$. Moreover, $(h'(x), 1)$ is an invariant direction; in fact $DH_{(h(x),x))}(h'(x), 1) = h'(x)(h'(h(x)), 1)$. Let us also assume that there is a sequences $\{c_{-n}\}$ such that $h(c_{-n}) = c_{-n+1}$, $c_0 = c$ and c_{-n} is not a critical point for any $n > 0$. Let us take the sequences (c_{-n}, c_{-n-1}) and observe that $H((c_{-n}, c_{-n-1})) = (c_{-n+1}, c_{-n})$. Therefore, $(c_0, c_{-1}) \in H^-(\delta_0)$ and $F_{(c_0,c_{-1})} = (h'(c_{-1}), 1)$. Since, $DH_{(c_0,c_{-1}))}(h'(c_{-1}), 1) = (0, 1)$ follows that, (c_0, c_{-1}) is the critical point and $(h(c_0), c_0)$ is the critical value.

We have to point out, that it is not clear that it is possible to extract a complete geometric structure associated to the critical sets. Taking in mind the picture for one-dimensional endomorphisms, there is not hope to characterize any dynamics showing critical points. However, this can be done assuming properties on the dynamic of the critical orbit. We hope that the same approach could be done for the case of surface maps. In particular, in the next subsections it is formulated a theorem that characterized different kinds of dynamics that could hold for the critical sets.

1.4 Different types of critical dynamics.

It is possible to get a better description of the dynamic of the critical points and values. This is the purpose of the following theorem which strongly relies on theorem 1.1. Before to formulate it, recall that the local stable (unstable) manifold of a hyperbolic set Λ_0 is the union of the local (unstable) stable manifold of the points in Λ_0. To avoid notation, let us denote the sets $H^-(\delta_0)$ and $H^+(\delta_0)$ given by theorem A, with H^- and H^+.

Theorem D: *Let $f \in Diff^2(M^2)$ and let $\Lambda \subset L(f)$ be a dissipative compact invariant set of f without dominated splitting. Let also assume that f is Kupka-Smale. Then, one of the following alternatives holds:*

- *For any point $z \in \Lambda$ holds that $\omega(z) \cap CP_\Lambda \neq \emptyset$ (the same for $\alpha(z)$).*
- *There exist a compact invariant hyperbolic set $\Lambda_0 \subset \Lambda$, and for any hyperbolic set $\Lambda_0 \subset \Lambda$ holds:*
  1. *$closure(W^u(\Lambda_0)) \cap CP_\Lambda \neq \emptyset$*
  2. *$closure(W^s(\Lambda_0)) \cap CV_\Lambda \neq \emptyset$*

We would like to say a few words to explain the previous theorem.

In the first option, all points accumulates on the critical points and values. This is the case of a dissipative *pseudo-circle* which is a circle-like minimal nowhere locally connected set (see [Ha]).

In the second option, there are hyperbolic sets contained in Λ and for any of them, some critical points are accumulated by the unstable manifold of those hyperbolic sets and some critical values are accumulated by the stable manifold of those hyperbolic sets.

1.5 Some applications.

In this subsection we conclude some results that follows from the previous theorems.

1.5.1 Existences of periodic points

Given a compact set, it is natural to ask if this set has periodic points. Under certain conditions, related to the critical values and critical points, we can give a positive answer to the previous question in the dissipative case.

Definition 4 *We say that a compact invariant set Λ is maximal invariant if there exists a neighborhood U of Λ such that*

$$\Lambda = \cap_n f^n(U).$$

From theorem 1.1 it follows that any maximal invariant set with dominated splitting contains periodic points or it is a periodic normally hyperbolic curve with dynamic conjugated to an irrational rotation. In the next corollary we give a result about the existence of periodic points for sets without dominated splitting.

Corollary 1.3 *Let $f \in Diff^2(M)$ be a Kupka-Smale systems and let Λ be a transitive compact maximal invariant dissipative set. Let us assume that Λ does not have dominated splitting. If there is a point in Λ such that its orbit does not accumulate on the critical points, then Λ contains periodic points.*

1.5.2 Lyapunov stability.

The sets introduced in theorem A are useful to understand the Lyapunov stable systems (system for which the states will remain bounded for all time, see [Ly]). In this case, we do not assume that the map is Kupka-Smale.

Definition 5 *Let Λ be a compact invariant set of a homeomorphisms f. We say that Λ is Lyapunov stable, if for any $\epsilon_1 > 0$ there exists $\epsilon_2 > 0$ such that for any $x \in \Lambda$ follows that*

$$f^n(B_{\epsilon_1}(x)) \subset B_{\epsilon_2}(f^n(x)).$$

To characterize the attracting Lyapunov stable sets, we need the following theorem that gives a complete description of systems having dominated splitting for smooth surface maps without assuming that the map is Kupka Smale. This threm is a generalization of theorem 1.1.

Theorem 1.2 *([PS2]) Let $f \in Diff^2(M^2)$ and let Λ be an invariant compact set contained in $L(f)$ exhibiting a dominated splitting. Then Λ can be decomposed into $L(f) = \mathcal{I} \cup \tilde{\Lambda} \cup \mathcal{R}$ such that*

1. *$\mathcal{I}$ is a set of periodic points with bounded periods and contained in a disjoint union of finitely many normally hyperbolic periodic arcs or simple closed curves.*

2. *$\mathcal{R}$ is a finite union of normally hyperbolic periodic simple closed curves supporting an irrational rotation.*

3. *$\tilde{\Lambda}$ can be decomposed into a disjoint union of finitely many compact invariant and transitive sets. The periodic points are dense in $\tilde{\Lambda}$ and at most finitely many of them are non-hyperbolic periodic points. The (basic) sets above are the union of finitely many (nontrivial) homoclinic classes. Furthermore $f/\tilde{\Lambda}(f)$ is expansive.*

Roughly speaking, the above theorem says that the dynamics of a C^2 diffeomorphism having dominated splitting can be decomposed into two parts: one where the dynamics consists on periodic and almost periodic motions ($\mathcal{I}$, $\mathcal{R}$) with the diffeomorphism acting equicontinuously; and another, where the dynamics are expansive and similar to the hyperbolic case. Recall that a set is named expansive if any two different trajectories are eventually separated.

Theorem E: *Let $f \in Diff^2(M^2)$ and let Λ be a transitive dissipative Lyapunov stable set. Then, Λ is either an attracting periodic point or a closed invariant curve normally hyperbolic with dynamic conjugated to an irrational rotation.*

Observe that the minimal pseudo-circle introduce by Handel is not Lyapunov stable. Moreover, this set does not contain any embedded arc. In this direction we wonder the following

Questions: *Let $f \in Diff^2(M^2)$ and Λ be a locally connected transitive dissipative attractor. Is it true that Λ either contains a periodic point or a Λ is a closed invariant curve normally hyperbolic with dynamic conjugated to an irrational rotation.*

1.5.3 Newhouse phenomena.

It was through the seminal works of Newhouse (see [N1], [N2], [N3]) that were shown the existence of residual subsets of diffeomorphisms displaying infinitely many periodic attractors (nowadays called "Newhouse phenomena"). We show in this subsection, that the set of accumulation points of the infinitely many periodic attractors contains critical points.

Corollary 1.4 *Let $f \in Diff^2(M^2)$ be a Kupka-Smale system having infinitely many sinks with unbounded period. Let also assume that the attracting periodic points are contained in a dissipative region of the space. Then, the set of accumulation points of the infinitely many periodic attractors contains critical points.*

1.5.4 Dynamics conjugated to a hyperbolic ones.

In the next theorem, we address the problem about the characterization of dynamics conjugated to a hyperbolic one: under which conditions hold that a dynamic conjugated to a hyperbolic one is also hyperbolic? Observe that the conjugacy is not enough to guarantee hyperbolicity.

Theorem F: *Let $f \in Diff^2(M^2)$ be a Kupka-Smale map and let Λ be a compact maximal invariant transitive set. Let us assume that there exists $g \in Diff^1(M^2)$ and a hyperbolic compact invariant set Λ_g of g such that the following is verified:*

1. *there exists a homeomorphisms $h : \Lambda_g \to \Lambda$ such that*

$$f \circ h = h \circ g,$$

2. *for any* $x \in \Lambda_g$, $h(W^s_\epsilon(x))$ *and* $h(W^u_\epsilon(x))$ *are* C^1-*embeddings moving continuously with* x.

Therefore, it follows that Λ *is hyperbolic if and only if for any* $x \in \Lambda_g$ *follows that* $h(W^s_\epsilon(x))$ *and* $h(W^u_\epsilon(x))$ *are transversal.*

Corollary 1.5 *In the hypothesis of the previous theorem, it follows that if there exists a positive constant* α_0 *such that for any periodic point* $q \in \Lambda$ *holds that* $\alpha(E^s_q, E^u_q) > \alpha_0$, *where* E^s_q *and* E^u_q *are the eigenspaces of* $D_q f^{n_q}$ *(*n_q *being the period of* q*), then* Λ *is hyperbolic.*

Questions: *The previous theorem is true if it is assumed uniform transversality between the stable and unstable direction of the periodic points and it is not assumed that* $h(W^s_\epsilon(x))$ *and* $h(W^u_\epsilon(x))$ *are* C^1-*embeddings?*

Questions: *It is true that uniform transversality follows from the fact that* $h(W^s_\epsilon(x))$ *and* $h(W^u_\epsilon(x))$ *are* C^1-*embeddings that vary continuously? Observe that if there is a homoclinic tangency then* $h(W^s_\epsilon(x))$ *and* $h(W^u_\epsilon(x))$ *are* C^1-*embeddings that do not vary continuously.*

2 Preliminaries.

First we formulate a couple of lemma that characterize the dominated splitting in terms of the tangent bundle dynamic. In fact it translates the domination property in terms of hyperbolicity of G: roughly speaking, we would say that Λ exhibits a dominated splitting, if and only if there is "'a hyperbolic attracting" subbundle for G. To do that, first we have to relate the dynamic of Df with the dynamic of G. We also formulate a series of lemmas and remarks that are used in the proofs of the theorems. We start introducing the notion of angle of two vectors:

Definition 6 *Let v and w be two vectors of $\mathbb{R}^d$. It is defined the angle $\alpha(v,w)$ as the unique positive number in $[0, \frac{\pi}{2}]$ such that*

$$\cos(\alpha(v,w)) = \frac{< v, w >}{|v||w|}$$

where $< .,. >$ is the internal product induced by the riemanniam metric. Given two one-dimensional subspaces, it is defined the angle between them as the angle between two generators.

Lemma 2.0.1 *Let $f \in Diff^1(M^2)$ and let $G_x : (T_xM)_1 \to (T_{f(x)}M)_1$ defined by $G_x(v) = \frac{D_xf(v)}{|D_xf(v)|}$. Then it follows that*

$$g(v) = D_wG_{x|w=v} = \frac{\det(D_xf)}{|D_xf(v)|^2}. \tag{2}$$

Moreover, given two unitary vectors v and w it follows:

1. $\det(Df^n) = |Df^n(v)||Df^n(w)| \frac{\sin(\alpha(Df^n(v),Df^n(w)))}{\sin(\alpha(v,w))}$;
2. $g^n(w)g^n(v) = [\frac{\sin(\alpha(Df^n(v),Df^n(w)))}{\sin(\alpha(v,w))}]^2$, *where* $g^n(u) = \Pi_i = 0^{n-1}g(G^i(u))$.

Proof: The map G can be considered as a map from S^1 to S^1 and any unitarian vector v can be written as $v_\theta = (\cos(\theta), \sin(\theta))$ for some $\theta \in [0, 2\pi)$. Then it follows that

$$g(v_\theta) = \partial_\theta(G(\theta)) = \frac{1}{|Df(v_\theta)|^2}[|Df(v^*_\theta)||Df(v_\theta)| - \frac{< Df(v_\theta), Df(v^*_\theta) >}{|Df(v_\theta)|}|Df(v_\theta)|],$$

where $v^*_\theta = (\sin(\theta), -\cos(\theta))$.

It is not Difficult to check that

$$\det(Df) = |Df(v^*_\theta)||Df(v_\theta)| - \frac{< Df(v_\theta), Df(v^*_\theta) >}{|Df(v_\theta)|}|Df(v_\theta)|$$

therefore it is concluded the equality (2). In particular, it follows that

$$|Df^n(v)|^2 = \frac{\det(Df^n)}{|g^n(v)|}. \quad (3)$$

The other three equalities formulas are straightforward. The first two items are classical ones and the last one follows from equality (3).

■

Corollary 2.1 *For any $\delta > 0$ and any n_0 there is a unique direction v such that $g^n(v) > (1+\delta)^n$ for all positive $n > n_0$.*

Proof: The corollary follows from the previous formulas: Let us assume that there are two different directions v, w verifying that $g^n(v) > (1+\delta)^n$ and $g^n(w) > (1+\delta)^n$ for all integer $n > n_0$. Therefore by the last formula of lemma 2.0.1 follows that for any n large

$$(1+\delta)^{2n} < g^n(w)g^n(v) = [\frac{\sin(\alpha(Df^n(v), Df^n(w)))}{\sin(\alpha(v,w))}]^2 < \frac{1}{\sin(\alpha(v,w))}.$$

A contradiction.

■

The next corollary is used in the proof of theorem B.

Corollary 2.2 *Let $x \in H^-(\delta_0)$ and let $l > 0$ be such that $g^l(F_x) > (1+\delta_0)^l$. Then $g^l(x) \notin H^-(\delta_0)$.*

Proof: If $g^l(x) \in H^-(\delta_0)$ then there exists a direction $v \in (T_{f^l(x)}M)_1$ such that $g^{-n}(v) > (1+\delta_0)^n$ for $n > 0$. Since $g^{-n}(G^l(F_x)) > (1+\delta_0)^n$ for $n > l$ it follows that $v = G^l(F_x)$ and therefore, $g^{-l}(G^l(F_x)) > (1+\delta_0)^l$. A contradiction because $g^{-l}(G^l(F_x)) = \frac{1}{g^l(F_x)}$ and from hypothesis it is smaller that $\frac{1}{(1+\delta_0)^l}$.

■

Lemma 2.0.2 *Let $f \in Diff^1(M^2)$ and let Λ be a compact invariant set of f, then Λ have a dominated splitting if and only if there exists $\gamma > 1$, $C > 0$ and at least one of the following equivalent conditions is satisfied*

1. *there exists a subbundle $x \to E_x$ over Λ, such that for any x follows that $g^n(E_x) > C\gamma^n$ ($g^{-n}(E_x) < C\gamma^{-n}$) for $n > 0$*

2. *there exists a subbundle $x \to F_x$ over Λ such that for any x follows that $g^{-n}(F_x) > C\gamma^n$ ($g^n(F_x) < C\gamma^{-n}$) for $n > 0$.*

Corollary 2.3 *Let $x \in M$ and $v \in (T_xM)_1$. Let us assume that there exist positive integers n_0, m_0 and a positive constant $\gamma > 1$ such that $|g^{n_0}(G^m(v))| > \gamma$ for any $m > m_0$, then $\omega(x)$ has dominated splitting.*

Let $x \in M$ and $v \in (T_xM)_1$. Let us assume that there exist positive integers n_0, m_0 and a positive constant $\gamma > 1$ such that $|g^{-n_0}(G^-m(v))| > \gamma$ for any $m > m_0$, then $\alpha(x)$ has dominated splitting.

The next lemma is due to Pliss (see [Pl], [M2]).

Lemma 2.0.3 *Given $0 < \gamma_0 < \gamma_1$ and $a > 0$, there exist $n_0 = n_0(\gamma_0, \gamma_1, a)$ and $l = l(\gamma_0, \gamma_1, a) > 0$ such that for any sequences of numbers $\{a_i\}_{0 \leq i \leq n}$ with $n > n_0$, $a^{-1} < a_i < a$ and $\Pi_{i=0}^n a_i < \gamma_0^n$ then there exist $1 \leq n_1 \leq n_2 \leq .... \leq n_r$ with $r > ln$ and such that*

$$\Pi_{i=n_j+1}^k a_i < \gamma_1^{k-n_j} \quad n_j < k < n.$$

Corollary 2.4 *Let $f \in Diff^1(M)$. Given any positive constant γ_0, there exist a positive constant $\gamma_1 = \gamma_1(\gamma_0) < \gamma_0$, a positive integer n_0 and a positive constant l such that:*

- *for any x, $v \in T_xM$ and $n \geq n_0$ verifying $g^n(v) \geq (1+\gamma_0)^n$ follows that there exists a positive integers $j < n$ such that $n - j > ln$ and*

$$g^i(G^j(v)) > (1+\gamma_1)^i, \;\; 0 < i < n-j.$$

- *for any x, $v \in T_xM$ and $n \geq n_0$ verifying $g^{-n}(v) \geq (1+\gamma_0)^n$ follows that there exist positive integers $j < n$ such that $n - j > ln$ and*

$$g^{-i}(G^{-j}(v)) > (1+\gamma_1)^i, \;\; 0 < i < n-j.$$

There is an easy remark that follows from corollary 2.4, concerning points in $H^{\pm}(\delta_0)$.

Remark 2.1 *If $x \in H^-(\delta_0)$ then there is a sequences of positive integers $\{k_n\}$ such that $f^{-k_n}(x) \in H^-(\delta_0)$. If $x \in H^+(\delta_0)$ there is a sequences of positive integers $\{j_n\}$ such that $f^{j_n}(x) \in H^+(\delta_0)$. Observe that in this sense, a critical point verifies that it is the "last" iterate that belongs to $H^-(\delta_0)$ and a critical value verifies that it is the "first" iterate that belongs to $H^+(\delta_0)$.*

2.1 Complementary considerations.

From equation (3) it is obtained the following remark:

Remark 2.2 *Let $f \in Diff^{1+\beta}(M^2)$ and let Λ be a dissipative compact invariant set contained in the Limit set. Then, it holds that*

1. *if* $x \in H^+$ *then* $|D_x f^n(E_x)| < \lambda_0^n$, *for* $n > 0$,

2. *if* $x \in H^-$ *then for any* $\delta > 0$ *there exists* $m_0 = m_0(\delta)$ *such that* $|D_x f^{-n}(F_x)| < (1+\delta)^n$ *for any* $n > m_0(\delta)$.

The next lemma is an immediate application of Pesin's theory.

Lemma 2.1.1 *Let* Λ *be a compact invariant dissipative set. There exist a continuous functions* $\phi^s : H^+(\delta_0) \to Emb^1(I_1, M)$, *such that if* $W^s_\epsilon(x) = \phi^s(x)I_\epsilon$, *the following properties hold:*

1. $T_y W^s_\epsilon(y) = E_y$,

2. $W^s_\epsilon(x) = \{y \in M : dist(f^n(x), f^n(y)) < \epsilon\}$,

3. *there exists* $\lambda < 1$ *and* $C > 0$ *such that for any* $x \in H^+_\Lambda(\delta_0)$

   (a) $|Df^n(E_x)| < C\lambda^n$.

   (b) $\ell(f^n(W^s_\epsilon(x))) < C\lambda^n$.

Remark 2.3 *If* $x \in H^-(\delta)$ *then the local unstable set is not necessary neither trivial not an embedded submanifold. However, if* $W^u_{loc}(x)$ *is a submanifold, then* $T_x W^u_{loc}(x) = F_x$.

3 Proof of the Theorems A, B, C, D, E and F.

The proofs of the Theorem A is an immediate application of Oseldet's theorem, lemma 3.0.2 and lemma 2.0.3. Let us start recalling the definition of regular points: a point $x \in M$ is a *regular point* if there exist number $\lambda_{E_x} \leq \lambda_{F_x}$ and a splitting of $T_xM = E_x \oplus F_x$ such that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log ||D_x f^n(v)|| = \lambda_{E_x}, \quad v \in E_x; \quad \lim_{n\to\pm\infty} \frac{1}{n} \log ||D_x f^n(v)|| = \lambda_{F_x}, \quad v \in F_x.$$

Let us denote with Λ_0 the set of regular points of Λ.

The Oseledet's theorem asserts that given an invariant measure μ the set of regular points has total measure.

Theorem 3.1 Oseledet's theorem: *Let $f \in Diff^{1+\beta}(M)$ and let μ be an invariant measure. Then, the regular points has total measure.*

If the support of μ is dissipative set and the measure is not supported on a periodic attractor the next lemma follows:

Lemma 3.0.2 *Let $f \in Diff^{1+\beta}(M)$ and let μ be an invariant measure. If there are not attracting periodic points in the support of the measure, then the largest Lyapunov exponent, given by $\lim_{n\to\infty} \frac{1}{n} \log(||Df^n||)$ is non negative.*

Recalling that, for any regular point follows that

$$\lim_{n\to\infty} \frac{1}{n} \log(|det(D_x f^n)|) = \lambda_{E_x} + \lambda_{F_x}, \tag{4}$$

and recalling that there exists $b < 1$ such that

$$\log(|det(D_x f^n)|) \leq \log(b)$$

then from lemma 3.0.2 we can conclude that

$$\lambda_{E_x} \leq \log(b) < 0 \leq \lambda_{F_x}. \tag{5}$$

More precisely:

Corollary 3.1 *Let $f \in Diff^{1+\beta}(M^2)$ and let μ be an invariant measure. If the support of the measure is a dissipative set and there are not attracting periodic points in the support of the measure, then there are two Lyapunov exponents: one negative bounded by $\log(b)$ and one non-negative.*

Proof of Theorem A: Oseledet's theorem implies that the set Λ_0 is not empty. In particular, if $x \in \Lambda_0$ then from corollary 3.1 follows that

$$\lim_{n\to+\infty} \frac{1}{n} \log(|Df^n(E_x)|) \leq \log(b).$$

From lemma 2.0.1, corollary 3.1 and (4) it is concluded that

$$\lim_{n\to+\infty} \frac{1}{n} \log|g^n(E_x)| = \lambda_{F_x} - \lambda_{E_x} \geq -\lambda_{E_x} \geq -\log(b). \tag{6}$$

Therefore, for n sufficiently large follows that

$$|g^n(E_x)| \geq (\frac{1}{b})^n.$$

Recall that $b_0 = \frac{1}{b} - 1$, therefore, taking

$$\delta_0 = \frac{1}{2} b_0$$

and using the Pliss's lemma and the corollary 2.4, it follows that if $x \in \Lambda_0$ then there exist iterates that belong to $H^+(\delta_0)$.

Arguing in the same way, if follows that if $x \in \Lambda_0$ then there exist backward iterates that belong to $H^-(\delta_0)$. In fact, again from lemma 2.0.1, corollary 3.1 and (4) it is concluded that

$$\lim_{n\to+\infty} \frac{1}{n} \log|g^{-n}(F_x)| = \lambda_{E_x} - \lambda_{F_x} \geq -\lambda_{E_x} \geq -\log(b). \tag{7}$$

Therefore, for n sufficiently large follows that

$$|g^{-n}(F_x)| \geq (\frac{1}{b})^n.$$

and so, using corollary 2.4 there exist iterates of x that belong to $H^-(\delta_0)$.

Therefore, $H^+(\delta_0)$ and $H^-(\delta_0)$ are not empty. To conclude that those sets intersect $\omega(z)$ for any $z \in \Lambda$, it is enough to pick up an invariant measure with support contained in $\omega(x)$.

$\blacksquare$

Remark 3.1 *Observe that the proof is based on the fact that regular points has iterates in $H^+(\delta_0)$ and $H^-(\delta_0)$. However, the theorems does not assert that points in either $H^+(\delta_0)$ or $H^-(\delta_0)$ are regular points. Moreover, recall that if x is a point of tangency then an iterate of it belongs to $H^-(\delta_0)$ and an iterate of it belongs to $H^+(\delta_0)$, however, x is not a regular point. Moreover, recalling the definition of critical points and values, follows that the critical points and values are not regular points.*

The next lemma is extremely useful in the proof of theorem B.

Lemma 3.0.3 *Let $f \in Diff^{1+\beta}(M^2)$ and let Λ be a dissipative compact invariant set contained in the Limit set. Let Λ_0 be a hyperbolic compact set contained in Λ. Then, it holds that there exists $\delta^u > \delta_0$ and n_1 such that*

1. *if $x \in \Lambda_0$ then $|g^n(E^s_x)| > (1+\delta^u)^n$, for $n > n_1$;*
2. *if $x \in \Lambda_0$ then $|g^{-n}(E^u_x)| > (1+\delta^u)^n$, for $n > n_1$;*

where E^s_x and E^u_x are the contractive and expansive subbundle of the the hyperbolic splitting on Λ_0.

The proof is immediate and follows from inequalities (5) and (6), and using the hypothesis that Λ_0 is hyperbolic to use that there is a positive Lyapunov exponent. In fact, $\lim_{n\to+\infty} \frac{1}{n}\log|g^n(E_x)| = \lambda_{F_x} - \lambda_{E_x} > -\lambda_{E_x} \geq -\log(b) > \log(1+\delta_0)$.

The proof of theorem B follows from the following Main Propositions. Before to prove it (the proofs are given is subsection 3.1), we show how the theorem B is concluded.

Main Proposition. *Let $f \in Diff^{1+\beta}(M)$ and let $x \in M$ be such that $\omega(x)$ ($\alpha(x)$ respectively) is a dissipative compact invariant set. Then, there is a positive constant $\frac{2}{3}b_0 < \delta_1 < \frac{4}{5}b_0$ such that if for some $v \in (T_xM)_1$ and for some positive integers k_0, m_0 follows*

$$|g^k(G^m(v))| \leq (1+\delta_1)^k, \quad \textit{for any}\, m > m_0,\ k > k_0$$

then $\omega(x)$ has dominated splitting. Moreover, there exist $\lambda = \lambda(k_0, b)$ and $C = C(\lambda, k_0)$ such that $\omega(x)$ has a $(C,\lambda)-$dominated splitting.

A similar results holds for $\alpha(x)$; i.e.: if for some $v \in (T_xM)_1$ and for some positive integers k_0, m_0 follows that $|g^{-n}(G^{-m}(v))| \geq (1-\delta_1)^n$ then $\alpha(x)$ respectively has dominated splitting.

Remark 3.2 *Observe that rate of domination depends on b and k_0 and it does not depend on x.*

Proof of Theorem B: If Λ has dominated splitting, it follows immediately that there are not critical points (or values).
To show the converse, first we start proving the following lemma:

Lemma 3.0.4 *There exist $x \in H^-(\delta_0)$, $y_0 \in H^+(\delta_0)$ and a sequence of points $x_i \in H^-(\delta_0)$ such that:*

1. *$F_{x_i} \to F_x$;*
2. *there is a sequences of positive integer $\{k_i\}$ such that*
   (a) *$f^{k_i}(x_i) \to y_0$*

(b) $G^{k_i}(F_{x_i}) \to E_{y_0}$;

3. *for any* $n \geq 0$ *follows that* $f^n(x) \notin H^-(\delta_0)$;

Proof of lemma 3.0.4: The proof goes through a series of claims.

Claim 1 *Let* δ_1 *be the constant given by the main proposition. If* Λ *does not have dominated splitting then it follows that*

1. *there is a sequence of points* $\{x_i\}$ *such that* $x_i \in H^-(\delta_0)$,
2. *there is an increasing sequence* $\{n_i\}$ *verifying* $n_i \to +\infty$,
3. *there is a sequence* $\{k_i\}$ *not necessary increasing, such that for any* $i > 0$ *follows that*
$$g^{n_i}(G^{k_i}(F_{x_i})) > (1+\delta_1)^{n_i}.$$

Proof of claim 1: If the claim is false, follows that there exists k_0 such that for any $x \in H^-(\delta_0)$ and any $m > 0$ follows that

$$g^{k_0}(G^m(F_x)) \leq (1+\delta_1)^{k_0}.$$

Therefore from main proposition, follows that $\omega(x)$ has $(C,\lambda)-$dominated splitting. Using that the constant of domination are uniform for any $\omega(x)$, follows that

$$L^- = L(f_{|H^-(\delta_0)}) = closure(\cup_{x\in H^-(\delta_0)}\omega(x))$$

has a $(C,\lambda)-$dominated splitting. Let us take a neighborhood U of L^- and observe that there is m_0 such that for any $x \in H^-(\delta_0)$ and $m > m_0$ follows that $f^m(x) \in U$. In particular, since for any $z \in \Lambda$ holds that $\omega(z) \cap H^-(\delta_0) \neq \emptyset$, then for any $z \in \Lambda$ follows that

$$\omega(z) \subset \cap_{j\in\mathbb{Z}} f^j(U).$$

Since L^- has dominated splitting it is concluded that $\cap_{j\in\mathbb{Z}} f^j(U)$ also has dominated splitting and therefore the same holds for $\omega(z)$. Taking z such that $\omega(z) = \Lambda$ it follows that Λ has dominated splitting. Which is a contradiction with the assumption that Λ does not have dominated splitting. This finish the proof of claim 1.

Taking the sequence $\{x_i\}$ and the sequences $\{k_i\}$ and $\{n_i\}$ given by claim 1, follows from corollary 2.4 that there exists a sequence $\{r_i\}$ with $n_i - r_i \to \infty$ such that

$$g^l(G^{r_i}(F_x)) > (1+\delta_0)^l, \;\; 0 < l < n_i - r_i. \tag{8}$$

From the fact that for any $0 < l < n_i - r_i$ holds that $f^l(f_i^r(x)) \notin H^-(\delta_0)$ (recall corollary 2.2), we can assume that for any $0 < j \leq r_i$ follows that $f^j(x_i) \notin H^-(\delta_0)$; otherwise we changes x_i by $f^{j_i}(x_i)$ where j_i is the largest positive integer smaller or equal than r_i such that $f^{j_i}(x_i) \notin H^-(\delta_0)$.

Now we take an accumulation point x of the sequences $\{x_i\}$ selected before and F_x as an accumulation point of $\{F_{x_i}\}$. Observe that $x \in H^-(\delta_0)$ with

$$g^{-n}(F_x) > (1+\delta_0)^n.$$

Taking y_0 and E_{y_0} as an accumulation point of $\{f^{r_i}(x_i)\}$ and $\{G^{r_i}(F_{x_i})\}$ respectively (the sequences that verify (8)), it follows that $y_0 \in H^+(\delta_0)$ with

$$g^n(E_{y_0}) > (1+\delta_0)^n.$$

Claim 2 *For any $n > 0$ follows that $f^n(x) \notin H^-(\delta_0)$.*

Proof of claim 2: To check that we consider either the sequence $\{r_i\}$ is bounded or unbounded.

In the first case, it follows that $f^r(x) = y_0$ for some positive integer and $G^r(F_x) = E_{y_0}$ (where y_0 is the point selected before). Therefore, replacing x by an iterate of it up to $f^{r-1}(x)$ if necessary, observe that from corollary 2.2 it can not hold that a forward iterate of x belong to $H^-(\delta_0)$.

In the case that $\{r_i\}$ is unbounded, we can assume that $lim_{i\to+\infty} r_i = +\infty$. If the claim is false let n_0 be a positive integer arbitrarily large such that $f^{n_0}(x) \in H^-(\delta_0)$. Therefore, from corollary 2.1 follows that for any $0 \le j \le n_0$ $g^{-j}(G^{n_0}(F_x)) > (1+\delta_0)^j$. We can take a finite sequences of positive constants $s_1, s_2, ..., s_{n_0}$ such that each s_j is smaller than one and

$$g^{-(n_0-l)}(G^{n_0}(F_x)) > \Pi_{j=l}^{n_0}[s_j\, g^{-1}(G^j(F_x))] > (1+\delta_0)^{n_0-l} \quad 0 \le l \le n_0.$$

Now we take x_i close enough to x such that $f^j(x_i)$ is close enough to $f^j(x)$ and $G^j(F_{x_i})$ close to $G^j(F_x)$ for any $0 \le j \le n_0$ in such a way that $g^{-1}(G^j(F_{x_i})) > s_j\, g^{-1}(G^j(F_x))$. Therefore,

$$g^{-l}(G^{n_0}(F_{x_i})) > \Pi_{j=l}^{n_0}[s_j\, g^{-1}(G^j(F_x))] > (1+\delta_0)^{n_0-l} \quad 0 \le l \le n_0.$$

This implies that $f^{n_0}(x_i) \in H^-(\delta_0)$. Taking x_i such that $r_i > n_0$ it follows a contradiction with the election of the sequence $\{x_i\}$. This finish the proof of claim 2 and also the proof of lemma 3.0.4 is concluded.

Following with the proof of theorem B, we use the next claim:

Claim 3 *Let δ_1 be the constant given by the main proposition. Let x be the point in $H^-(\delta_0)$ obtained in lemma 3.0.4. There exists an increasing sequence $\{n_i\}$ verifying $n_i \to +\infty$, and a sequences $\{k_i\}$ not necessary increasing, such that for any $i > 0$ follows that*

$$g^{n_i}(G^{k_i}(F_x)) > (1+\delta_1)^{n_i}.$$

Proof of claim 3: If the sequence $\{k_i\}$ given by claim 1 verifies that is bounded, follows that for some $k > 0$ holds that $f^k(x) = y_0$ and $G^k(F_x) = E_{y_0}$ and therfore the lemma follows immediately.

Now we consider the case that the sequences $\{k_i\}$ is unbounded, and we can assume that $k_i \to +\infty$. If the thesis does not hold, then there exist positive integers n_0, m_0 such that $g^n(G^m(F_z)) \leq (1+\delta_1)^n$ for any $m > m_0, n > n_0$. From main proposition we conclude that $\omega(x)$ is a hyperbolic set. Let

$$\Lambda_0 = closure(\cap_{n\in\mathbb{Z}} f^n(V))$$

where V is a small neighborhood of $\omega(x)$. It follows that Λ_0 is a hyperbolic set and there is $k_0 > 0$ such that $f^{k_0}(x) \in W^s_{loc}(\Lambda_0)$. To avoid notation, we assume that $x \in W^s_{loc}(\Lambda_0)$ (where $W^s_{loc}(\Lambda_0)$ is the local stable manifold of Λ_0). Moreover, it holds that F_x is transversal to $T_xW_{loc}(\Lambda_0)$: otherwise, $g^n(F_x)$ would expand for positive iterates and by lemma 3.0.3 the expansion holds at rate $1+\delta_0$ and this is not the case because we are assuming that x does not verify the thesis of claim 3.

Now we have to choose a series of constants:

1. Let δ^u and n_1 be the constants given by lemma 3.0.3 for the set Λ_0.

2. Let δ_2 and δ_3 such that $\delta_0 < \delta_3 < \delta_2 < \delta^u$ and let $0 < s < 1$ such that $(1+\delta^u)s > 1+\delta_2 > (1+\delta_2)s > 1+\delta_3$.

3. Let s be the constant chosen in the previous item and let β_0 be a small positive constant such that if $z' \in W^u_{\beta_0}(z)$ for some $z \in \Lambda_0$ then $g^{-1}(E^u(z')) > s\, g^{-1}(E^u(z))$.

4. Let β_1 be a small positive constant such that if $dist(z, z_0) < \beta_1$ for some $z_0 \in \Lambda_0$ and $\alpha(F_z, E^u_{z_0}) < \beta_1$ then $g^{-1}(F_z) > s\, g^{-1}(E^u_{z_0})$.

Observe that from the first and second item, and the lemma 3.0.3 it follows that if $z' \in W^u_{\beta_0}(z)$ for some $z \in \Lambda_0$ then $g^{-n}(E^u(z')) > (1+\delta_2)^n$ if n is large enough.

Recall the sequences $\{x_i\}$ that accumulates on x (obtained in lemma 3.0.4). Let $\gamma > 0$ and let $V_\gamma = B_\gamma(\Lambda_0)$. From the fact that $x \in W^s_{loc}(\Lambda_0)$ follows that for any V_γ there exists $k_0 = k_0(\gamma)$ such that for any $n > 0$ if x_i is close enough to x then $f^j(x_i) \in V_\gamma$ for any $k_0 \leq j \leq n$. Using that Λ_0 is a hyperbolic set which is the closure of a maximal invariant set, follows that there exists a point $z_i \in W^u_{loc}(\Lambda_0)$ such that $f^j(z_i) \in V_\gamma$ and $dist(f^j(x_i), f^{j-n}(z_i)) < \gamma$ for any $k_0 \leq j \leq n$. Moreover, $f^{-k}(z_i) \to \Lambda_0$ as $k \to \infty$ and $z_i \to \Lambda_0$. To show that, let us take $f^n(x_i)$ and let us take $z_i \in W^u_\gamma(\Lambda_0) \cap W^s_{loc}(f^n(x_i))$ where $W^s_{loc}(f^n(x_i))$ is a C^1-curve close to the local stable manifold of Λ_0.

Since F_x is transversal to $T_xW_{loc}(\Lambda_0)$, follows that if n is large enough then $G^n(F_{x_i})$ is close to the expanding subbundle of the hyperbolic splitting of Λ_0. Therefore, we can take $\gamma < \min\{\beta_0, \beta_1\}$ such that if x_i is close enough to x and $f^j(x_i) \in V_\gamma$ for any $k_0 \leq j \leq n$

then it follows that $dist(f^j(x_i), f^{n-j}(z_i))) < \beta_1$ and $\alpha(G^j(F_{x_i}), E^u_{f^{n-j}(z_i)}) < \beta_1$. Then it follows that $g^{-1}(G^j(x_i)) > s\, g^{-1}(E^u_{f^{n-j}(z_i)})$.

Hence, taking n large and x_i close to x follows that

$$\begin{aligned} g^{-n}(G^n(x_i)) &= \Pi_{j=1}^{n-1} g^{-1}(G^j(x_i)) > C_0 \Pi_{j=k_0}^{n} g^{-1}(G^j(x_i)) > \\ &> C_0\, \Pi_{j=k_0}^{n} [g^{-1}(E^u_{f^{n-j}(z_i)}))\, s] = C_0 s^{n-k_0}\, \Pi_{j=k_0}^{n} g^{-1}(E^u_{f^{n-j}(z_i)})) > \\ &> C_0 s^{n-k_0}(1+\delta_2)^{n-k_0} \\ &> C_0(1+\delta_3)^{n-k_0} \end{aligned}$$

where $C_0 = C_0(\gamma) = \Pi_{j=1}^{k_0} g^{-1}(G^j(x_i))$. Since n can be taken arbitrarily large, C_0 is fixed (only depends on γ which is already chosen) and $\delta_3 > \delta_0$, follows that

$$C_0(1+\delta_3)^{n-k_0} > (1+\delta_0)^n$$

and so

$$g^{-n}(G^n(x_i)) > (1+\delta_0)^n.$$

Therefore, for n large, and x_i close enough to x we have that $f^n(x_i) \in H^-(\delta_0)$. On the other hand recall that for each x_i, there is a large k_i such that for any $0 < j < k_i$ follows that $f^j(x_i) \notin H^-(\delta_0)$. So, taking x_i such that $k_i > n$ we get a contradiction. This finished the proof of claim 3.

Taking the point x and the sequences $\{k_i\}$ and $\{n_i\}$ given by claim 3, follows from corollary 2.4 that there exists a sequence $\{r_i\}$ with $n_i - r_i \to \infty$ such that

$$g^l(G^{r_i}(F_x)) > (1+\delta_0)^l \;\; 0 < l < n_i - r_i. \tag{9}$$

We can assume that r_i is the first positive integer smaller than n_i such that the inequality (9) holds. Taking y and E_y as an accumulation point of $\{f^{r_i}(x_i)\}$ and $\{G^{r_i}(F_{x_i})\}$ respectively, it follows that $y \in H^+(\delta_0)$ with

$$g^n(E_y) > (1+\delta_0)^n.$$

Claim 4 *For any n large holds that $f^{-n}(y) \notin H^+(\delta_0)$.*

Proof of claim 4: To check that we consider either the sequence $\{r_i\}$ is bounded or unbounded.

In the first case, it follows that $f^r(x) = y$ for some positive integer and $G^r(F_x) = E_y$. Therefore, it can not hold that a backward iterate of y belong to $H^+(\delta_0)$. In the second case, we can assume that $r_i \to +\infty$ and we argue as in claim 2. More precisely, if there is $n > 0$ such that $y \in H^+(\delta_0)$ it follows that taking r_i sufficiently large it holds that $f^{r_i-n}(x_i)$ and $G^{r_i-n}(F_{x_i})$ are close enough to y and $G^{-n}(E_y)$ respectively. Therefore, for $j < n$ it follows that $g^j(G^{r_i-n}(F_{x_i})) > (1+\delta_0)^j$ and so r_i is not the first positive integer verifying inequality (9). This finish the proof claim of 4.

The proof of theorem B is therefore concluded.

■

Now we proceed to prove lemma 1.3.1. The proof is similar to claim 3.

Proof of lemma 1.3.1: Let $\{x_n\}$ be a sequences of critical points and let x be an accumulation point. It follows immediately that $x \in H^-$. To show that for any positive integer n holds that $f^n(x) \notin H^-$ we argue as in claim 2. Therefore, to conclude the lemma it is enough to show that

1. there is an increasing sequence $\{n_i\}$ verifying $n_i \to +\infty$,
2. there is a sequence $\{k_i\}$ not necessary increasing, such that for any $i > 0$ follows that
$$g^{n_i}(G^{k_i}(F_x)) > (1 + \delta_1)^{n_i}.$$

If these sequences do not exist, follows that there exists k_0 such that for any $m > 0$ follows that
$$g^{k_0}(G^m(F_x)) \leq (1 + \delta_1)^{k_0}.$$
Therefore, arguing as in claim 1 follows that $\omega(x)$ is a hyperbolic set and arguing as in claim 3 it is conclude that for x_n close enough to x follows that there exists a large positive integer k such that $f^k(x_n) \in H^-(\delta_0)$. A contradiction.

■

The proof of theorem C follows from theorem B and theorem 1.1. So it remains to prove theorem D.

Proof of Theorem D: Let us assume that the first option of theorem D does not hold. Therefore, there exists a point x such that $\omega(x)$ does not contains critical points and therefore is a hyperbolic set. Let $\Lambda_0 = closure(\cap_{n\in\mathbb{Z}} f^n(V))$ where V is an small neighborhood of $\omega(x)$. It follows that Λ_0 is a hyperbolic set.

Let us consider the fundamental domain
$$D^u = closure[f(W^u_{loc}(\Lambda_0)) \setminus W^u_{loc}(\Lambda_0)].$$
Observe that D^u is nonempty: otherwise Λ_0 would be an attractor and in this case it would coincide with Λ, which is a contradiction since we assume that Λ is non-hyperbolic and Λ_0 is a hyperbolic set.

Claim 5 *It follows that $D^u \cap \Lambda \neq \emptyset$.*

Proof of claim 5: Let $z \in \Lambda$ such that $\omega(z) = \Lambda$. It follows that there is a subsequences of iterates of z that accumulates on Λ_0. Since Λ_0 is a hyperbolic proper subset of Λ, it is concluded that there is a subsequences of iterates of z that accumulates on the unstable manifold of Λ_0 and does not accumulate on Λ_0. This conclude the proof of claim 5.

To conclude the theorem it is enough to shows the following:

Claim 6 *It follows that $closure(\cup_{z\in D^u} \cup_{n>0} \{f^n(z)\}) \cap CP_\Lambda \neq \emptyset$.*

Proof of claim 6: Let us take $\Lambda_1 = Closure(\cup_{z\in D^u}\omega(z))$. If the claim is false, follows that Λ_1 is a hyperbolic set. Moreover, there is $k_0 > 0$ such that for any $z \in D^u$ it follows that $f^{k_0}(z) \in W^s_{loc}(\Lambda_1)$ and $T_{f^{k_0}(z)}W^s_{loc}(\Lambda_1)$ is uniformly transversal to $G^{k_0}(F_z)$ for any $z \in D^u$: otherwise, it would hold that there is a critical point in D^u. Therefore, it follows that $\cup_{0\le j\le k_0} f^{-j}(W^s_{loc}(\Lambda_1))$ is a uniform neighborhood U of D^u. It follows that if z is such that $\omega(z) = \Lambda$ there is an iterate of it contained in U, which is a contradiction because in this case, $\omega(z) = \Lambda_0$ would coincide with Λ and so Λ would be hyperbolic. This conclude the proof of claim 6.

To prove the second item of the second option of theorem D it is argued as before, replacing f by f^{-1}.

■

Proof of Theorem E: It is enough to show that either Λ has dominated splitting or it is an attracting periodic point (maybe with complex eigenvalues). In fact, if Λ has dominated splitting, from theorem 1.2 then either it contains a normally hyperbolic invariant closed curve with dynamic conjugated to an irrational rotation, or contains a periodic closed arc, or it contains an expansive set. Since we are assuming that Λ is Lyapunov stable, the third option does not hold. Therefore, if Λ has dominated splitting and since we are assuming that Λ is transitive (so it can not contain a closed periodic arc), we conclude that Λ coincides with the invariant closed curve with dynamic conjugated to an irrational rotation.

First, we prove the following proposition:

Proposition 3.1 *Let Λ be a dissipative minimal topologically (any orbit is dense) and Lyapunov stable invariant set. Then, either Λ has dominated splitting or it is an attracting periodic point.*

Proof: Take a point $x \in H^+(\delta_0)$ and $\gamma > 0$ and $\epsilon > 0$ sufficient small such that $f^n(B_\epsilon(x)) \subset B_\gamma(f^n(x))$ for any positive integer n. So, there is $\delta_2 < \delta_0$ such that for any $y \in B_\epsilon(x)$ there is a unique direction E_y such that $g^n(E_y) > (1+\delta_2)^n$ for any positive integer n. Since Λ is minimal, there is $k_0 > 0$ such that for any $z \in \Lambda$ there is $n < k_0$ such that $f^n(z) \in B_\epsilon(x)$. On the other hand, if $y \in B_\epsilon(x)$ and $f^n(y) \in B_\epsilon(x)$ we have either that $Df^n(E_y)$ is collinear with $E_{f^n(y)}$ or is not the case. In the last case, we obtain an attracting periodic point, and so the proposition follows. In the former, we prove that for any $y \in \Lambda$ we obtain a direction E_y which is expanded by g for any positive iterate and therefore from remark 2.0.2 we obtain that Λ has a dominated splitting.

■

Coming back to the proof of the theorem E, let us assume that Λ has not dominated splitting. So, there is $\Lambda_0 \subset \Lambda$ a minimal set, in terms of Zorn's lemma, without dominated splitting. This set is topologically minimal: otherwise, it would contain a proper sets with dominated splitting and in this case we, it follows that this proper set is a normally hyperbolic invariant closed curve with dynamic conjugated to an irrational rotation; in particular, this curve is an attractor and therefore it follows that Λ is not transitive. After it was proved that Λ_0 is topologically minimal, we can apply the previous proposition and we conclude that Λ is an attracting periodic point.

■

Proof of corollary 1.4: Let $\{p_n\}$ be a sequence of periodic attracting points whose periods are unbounded and that there exists $b < 1$ such that for any p_n follows that $|det(D_{p_n})| < b$. . Let Λ_0 be the set of limits points of the orbits of the points p_n, i.e.:

$$\Lambda_0 = \cap_{m\geq 0}\overline{\cup_{n\geq m}\mathcal{O}(p_n)}.$$

This set is a compact invariant set. If there is not critical points in Λ_0 then it follows that it has dominated splitting. Then, by Theorem 1.1, we conclude that Λ_0 is a union of a hyperbolic set and a finite union of periodic simple closed curves normally hyperbolic. Since given a neighborhood of Λ_0 there exists n_0 such that, for any $n \geq n_0$, the orbit of p_n it is contained in this neighborhood, we get a contradiction. In fact, the orbits of p_n can not accumulate on the periodic simple closed curves since they are normally hyperbolic (attracting or repelling curves). Thus, Λ_0 is a hyperbolic set and so the maximal invariant set in an admissible compact neighborhood of Λ_0 is hyperbolic as well. In particular, for sufficient large n, p_n lies on this maximal invariant set and so it must be a hyperbolic periodic point of saddle type, a contradiction and so our assumption is false.

■

Proof of Theorem F: Observe that it is enough to show that Λ does not have critical points: Observe that for any $x \in H^-(\delta_0)$ and $y \in H^-(\delta_0)$ holds that

$$F_{f^k(x)} = T_{f^k(x)}h(W^s_\epsilon(h^{-1}(f^k(x)))), \quad E_y = T_y h(W^s_\epsilon(h^{-1}(y)))$$

therefore from the assumption of transversality can not exist points that satisfy the definition of critical points and values.

■

3.1 Proof of Main Proposition.

The Maim proposition is based in the following lemma and corollary 2.4.
Main Lemma. *Let $x \in M$ be such that $\omega(x)$ is dissipative. There exists $\frac{4}{5}b_0 < \delta_2 < \frac{5}{6}b_0$ such that if for some n_0 and some $v \in (T_xM)_1$ it is verified that $(1-\delta_2)^n < g^n(v) < (1+\delta_2)^n$ for any $n > n_0$, then $\omega(x)$ is a sink.*
Proof of Main Proposition: By lemma 2.0.2 it is enough to prove that there exists some positive integers n_1, m_1 such that for any $n > n_1$ and $m > m_1$ follows

$$|g^n(G^m(v))| < (1-\delta_1)^n.$$

If it is not the case, it follows that there exist increasing sequences $\{n_i\}, \{k_i\}$ such that $g^{n_i}(G^{k_i}(v)) > (1-\delta_1)^{n_i}$. Then, by corollary 2.4 follows that there exists a sequence $\{r_i\}$ with $n_i - r_i \to \infty$ and $l_0 > 0$ such that

$$g^l(G^{r_i}(v)) > (1-\delta_2)^l, \ \ \forall \, l_0 < l < n_i - r_i.$$

Taking z and w as an accumulation points of $\{f^{r_i}(x)\}$ and $\{G^{r_i}(v)\}$ respectively, follows that

$$(1-\delta_2)^l < |g^l(w)| < (1+\delta_2)^l, \ \ \forall \, l_0 < l.$$

Therefore, by the Maim Lemma we conclude that there is a sink in Λ which is a contradiction.

■

Proof of Main Lemma: Let us take $(TM)_1 = \{(x,v) \in TM : v \in T_xM, |v| = 1\}$ and $F : (TM)_1 \to (TM)_1$ defined as $F(x,v) = (f(x), G_x(v))$. Let us consider the points (x,v) in the hypothesis of the lemma. Let us define the measures $\mu_N = (\mu_N^1, \mu_N^2)$ where

$$\mu_N^1 = \frac{1}{N}\sum_{i=0}^{N} \delta_{f^i(x)},$$

$$\mu_N^2 = \frac{1}{N}\sum_{i=0}^{N} \delta_{G^i(v)}.$$

These measures has a convergent subsequences to an invariant measures μ of F. If the measure is supported on the orbit of a sink, then the lemma follows. So, let us assume that it is not the case. Let us take $g : (TM)_1 \to \mathbb{R}$ defined by $g(x,v) = g_x(v)$, the derivative of G_x. It follows that

$$\int \log(g_z(w))d\mu(z,w) = \lim_{N\to\infty} \int \log(g_z(w))d\mu_N(z,w), \tag{10}$$

and

$$\int \log(g_z(w))d\mu_N(z,w) = \frac{1}{N}\sum_{i=0}^{N} \log(g_{f^i(x)}(G^i(v))) = \frac{1}{N}\log(g_x^N(v)). \tag{11}$$

From the fact that

$$(1-\delta_0)^n < g^n(v) < (1+\delta_0)^n$$

for n large enough it follows from (10) and (11) that

$$\log(1-\delta_0) < \int \log(g_z(w))d\mu(z,w) < \log(1+\delta_0). \tag{12}$$

From lemma 3.1, Oseledet's theorem and equation (2) follows that either

1. $\int \log(g_z(w))d\mu(z,w) = \lambda_F - \lambda_E$ or

2. $\int \log(g_z(w))d\mu(z,w) = \lambda_E - \lambda_F$,

where λ_E is the negative Lyapunov exponent and λ_F is the non-negative Lyapunov exponent. From lemma 3.1 follows that either

1. $\int \log(g_z(w))d\mu(z,w) > -\lambda_E > \log(\frac{1}{b})$ or

2. $\int \log(g_z(w))d\mu(z,w) = \lambda_E < \log(b)$.

From the election of b_0 and b it holds that the last two inequalities contradict inequality (12).

■

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