Global Well-Posedness and Non-linear Stability of Periodic Travelling Waves Solutions for a Schrödinger-Benjamin-Ono System

Jaime Angulo P. [1], Carlos Matheus [2], and Didier Pilod [2]

January 26, 2007

[1]Department of Mathematics, IMECC-UNICAMP
C.P. 6065, CEP 13083-970, Campinas, SP, Brazil.
[2] IMPA, Estrada Dona Castorina 110,
CEP 22460-320 Rio de Janeiro, RJ, Brazil.

Abstract

The objective of this paper is two-fold: firstly, we develop a local and global (in time) well-posedness theory of a system describing the motion of two fluids with different densities under capillary-gravity waves in a deep water flow (namely, a Schrödinger-Benjamin-Ono system) for *low-regularity* initial data in both periodic and continuous cases; secondly, a family of new periodic travelling waves for the Schrödinger-Benjamin-Ono system is given: by fixing a minimal period we obtain, via the implicit function theorem, a smooth branch of periodic solutions bifurcating of the Jacobian elliptic function called *dnoidal*, and, moreover, we prove that all these periodic travelling waves are nonlinearly stable by perturbations with the same wavelength.

1 Introduction

In this paper we are interested in the study of the following Schrödinger-Benjamin-Ono (SBO) system

$$\begin{cases} iu_t + u_{xx} = \alpha v u, \\ v_t + \gamma D v_x = \beta(|u|^2)_x, \end{cases} \tag{1.1}$$

where u is a complex-valued function, v is a real-valued function, $t \in \mathbb{R}$, $x \in \mathbb{R}$ or $\mathbb{T}$, α, β and γ are real constants, and $D\partial_x$ is a linear differential operator representing the dispersive term. Here $D = \mathcal{H}\partial_x$ where $\mathcal{H}$ denotes the *Hilbert transform* defined as

$$\widehat{\mathcal{H}f}(k) = -i\text{sgn}(k)\widehat{f}(k),$$

where

$$\text{sgn}(k) = \begin{cases} -1, & k < 0, \\ 1, & k > 0. \end{cases}$$

[1]Partially supported by CNPq/Brazil under grant No. 300654/96-0, email: angulo@ime.unicamp.br.
AMS Subject Classifications: 76B25, 35Q51, 35Q53.
[2]Partially supported by CNPq/Brazil; emails: matheus@impa.br and pilod@impa.br

Note that from these definitions we have that D is a linear positive Fourier operator with symbol $|k|$. The system (1.1) was deduced by Funakoshi and Oikawa ([22]). It describes the motion of two fluids with different densities under capillary-gravity waves in a deep water flow. The short surface wave is usually described by a Schrödinger type equation and the long internal wave is described by some sort of wave equation accompanied with a dispersive term which in this case is given by a Benjamin-Ono type equation. This system is also of interest in the sonic-Langmuir wave interaction in plasma physics [28], in the capillary-gravity interaction wave [21], [26], and in the general theory of water wave interaction in a nonlinear medium [14], [15]. We note that the Hilbert transform considered in [22] for describing system (1.1) is given as $-\mathcal{H}$.

When studying an initial value problem, the first step is usually to investigate in which function space well-posedness occurs. In our case, smooth solutions of the SBO system (1.1) enjoy the following conserved quantities

$$\begin{cases} G(u,v) \equiv Im \int_0^L u(x)\overline{u_x(x)}\,dx + \frac{\alpha}{2\beta}\int_0^L |v(x)|^2\,dx, \\ \\ E(u,v) \equiv \int_0^L |u_x(x)|^2\,dx + \alpha\int_0^L v(x)|u(x)|^2\,dx - \frac{\alpha\gamma}{2\beta}\int_0^L |D^{1/2}v(x)|^2\,dx, \\ \\ H(u,v) \equiv \int_0^L |u(x)|^2\,dx, \end{cases} \tag{1.2}$$

where $D^{1/2}$ is the multiplier of Fourier defined as $\widehat{D^{1/2}v}(k) = |k|^{\frac{1}{2}}\widehat{v}(k)$ for $k \in \mathbb{Z}$. Therefore, the natural spaces to study the well-posedness are the Sobolev H^s-type spaces. Moreover, due to the scaling property of the SBO system (1.1) (see [10] Remark 2), we are leading to investigate the well-posedness in the spaces $H^s \times H^{s-1/2}$, $s \in \mathbb{R}$.

In the continuous case Bekiranov, Ogawa and Ponce [11] proved local well-posedness for initial data in $H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R})$ when $|\gamma| \neq 1$ and $s \geqq 0$. Thus, because of the conservation laws in (1.2), the solutions extend globally in time when $s \geqq 1$, in the case $\frac{\alpha\gamma}{\beta} < 0$. Recently, Pecher [34] has shown local well-posedness in $H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R})$ when $|\gamma| = 1$ and $s > 0$. He also used the Fourier restriction norm method to extend the global well-posedness result when $1/3 < s < 1$, always in the case $\frac{\alpha\gamma}{\beta} < 0$. Here, we improve the global well-posedness result in the case $|\gamma| \neq 1$ via a trick based on the decoupled formulation of the SBO system and the conservation of the L^2 mass of the Schrödinger part. It is worth to point out that this scheme applies for other systems. In fact, we realized this trick a little bit after the ICM 2006, but when the argument started to be written up, we noticed that Colliander, Holmer and Tzirakis [19] (March 2006) already known this trick and they successfully applied it to the Zakharov and Klein-Gordon-Schrödinger systems; moreover, they surely know that the same argument should apply to other systems and, in fact, Colliander, Holmer and Tzirakis remarked that the global well-posedness theorem above would be established in a forthcoming article (see Remark 1.5 in [19]). Naturally, we decided to exchange some emails with Colliander *et al.* and they told us that their current projects are taking a lot of time and consequently they are not planning to write up this argument; in particular, Colliander *et al.* allowed us to include the result of Theorem 3.1 in this paper. We take the opportunity to express again our gratitude to Colliander, Holmer and Tzirakis for the fruitful interaction about the Schrödinger-Benjamin-Ono system.

In the periodic setting, there does not exist, until we know, any result about the well-posedness of the SBO system (1.1). Nevertheless, Bourgain [17] proved well-posedness for the cubic nonlinear Schrödinger equation (NLS) (see (1.3)) in $H^s(\mathbb{T})$ for $s \geq 0$ using the Fourier transform restriction method. Unfortunatly, this method does not appply directly for the Benjamin-Ono equation. Nevertheless, using an appropriate Gauge transform introduced by Tao [37], Molinet and Ribaud [32] and Molinet [31] proved well-posednees in $H^{1/2}(\mathbb{T})$ and $L^2(\mathbb{T})$ respectively. Here we apply Bourgain's method for the SBO system and prove well-posedness in $H^s(\mathbb{T}) \times H^{s-\frac{1}{2}}(\mathbb{T})$ when $s \geq 1/2$ in the case $\gamma \neq 0$, $|\gamma| \neq 1$, which lead as in the continuous case to global well-posedness in the energy space $H^1(\mathbb{T}) \times H^{1/2}(\mathbb{T})$ in the case $\frac{\alpha\gamma}{\beta} < 0$. We also show that our results are sharp in the sense that the bilinear estimates in the Bourgain spaces fail whenever $s < 1/2$ and $|\gamma| \neq 1$ or $s \in \mathbb{R}$ and $|\gamma| = 1$.

In a second time, we turn our attention to another important aspect of dispersive nonlinear evolution equations: the translating waves or so-called travelling-waves. These solutions imply a balance between the effects of nonlinearity and dispersion. By depending of specific boundary conditions on the wave's shape, for instance in the case of water waves, these special states of motion can arise either solitary or periodic waves. The study of these special steady waveform is essential to the explanation of many wave phenomena observed in the practice, for instance, in surface water waves propagating in a canal, or in propagation of internal waves, or in the interaction between long waves and short waves. Then questions about the stability of travelling waves and their existence as exact solutions of the dynamical equations are very important.

The solitary waves are in general a single crested, symmetric, localized travelling waves, whose hyperbolic sech-profiles are well known (see Ono [33] and Benjamin [12] for the existence of solitary waves of algebraic type or with a finite number of oscillations). The study of the nonlinear stability or unstability in form of solitary waves has had a big development and refinement in recent years. The proofs have been simplified and sufficient conditions have been obtained to insure the stability to small localized perturbations in the waveform. Those conditions have showed to be effective in a variety of circumstances, see for example [1], [2], [3], [13], [16], [25], [36].

The situation regarding to periodic travelling waves is very different. The stability and the existence of explicit formulas of these progressive wavetrains have received comparatively little attention. Recently many research papers about this issue have appeared for specify dispersive equations, such as the existence and stability of *cnoidal waves* for the Korteweg-de Vries equation ([6]) and the Hirota-Satsuma system ([4]), as well as, the stability of *dnoidal waves* for the one-dimensional cubic nonlinear Schrödinger equation

$$iu_t + u_{xx} + |u|^2 u = 0, \tag{1.3}$$

where $u = u(t,x) \in \mathbb{C}$ and $x, t \in \mathbb{R}$ (Angulo [5]. See also Gallay&Hărăgus [23], [24]).

In this paper we are also interested in giving a stability theory of periodic travelling waves solutions for the nonlinear dispersive system SBO (1.1). The periodic travelling waves solutions for (1.1) considered here will be of the general form

$$\begin{cases} u(x,t) = e^{i\omega t} e^{ic(x-ct)/2} \phi(x-ct), \\ v(x,t) = \psi(x-ct), \end{cases} \tag{1.4}$$

where $\phi, \psi : \mathbb{R} \to \mathbb{R}$ are smooth, L-periodic functions (with a prescribed period L), $c > 0$, $\omega \in \mathbb{R}$ and we will suppose that there is a $q \in \mathbb{N}$ such that

$$4q\pi/c = L.$$

So, by replacing these permanent waves form into (1.1) we obtain the pseudo-differential system

$$\begin{cases} \phi'' - \sigma\phi = \alpha\, \psi\phi \\ \gamma \mathcal{H}\psi' - c\psi = \beta\phi^2 + A_{\phi,\psi} \end{cases} \tag{1.5}$$

where $\sigma = \omega - \frac{c^2}{4}$ and $A_{\phi,\psi}$ is an integration constant which we will set equal zero in our theory. By finding analytic solutions of system (1.5) for $\gamma \neq 0$ is very difficult. In the framework of travelling waves of type solitary waves, namely, the profiles ϕ, ψ satisfy $\phi(\xi), \psi(\xi) \to 0$ as $|\xi| \to \infty$, it is well known the existence of solutions for (1.5) in the form

$$\phi_{0,s}(\xi) = \sqrt{\frac{2c\sigma}{\alpha\beta}}\, sech(\sqrt{\sigma}\xi), \qquad \psi_{0,s}(\xi) = -\frac{\beta}{c}\phi_{0,s}^2(\xi) \tag{1.6}$$

when $\gamma = 0$, $\sigma > 0$, and $\alpha\beta > 0$. For $\gamma \neq 0$ a theory of even solutions of these permanent waves solutions has been established in [7] (see also [8]) by using the concentration-compactness method.

For $\gamma = 0$ and $\sigma > 2\pi^2/L^2$ we will prove, by following the ideas in Angulo ([5]) with regard to (1.3), the existence of a smooth curve of even periodic travelling wave solutions of (1.5) (with $\alpha = 1, \beta = 1/2$, naturally this restriction does not imply loss of generality) which will depend of the Jacobian elliptic function *dnoidal*, namely,

$$\begin{cases} \phi_0(\xi) = \eta_1\, dn\Big(\frac{\eta_1}{2\sqrt{c}}\, \xi; k\Big) \\ \psi_0(\xi) = -\frac{\eta_1^2}{2c}\, dn^2\Big(\frac{\eta_1}{2\sqrt{c}}\, \xi; k\Big), \end{cases} \tag{1.7}$$

where η_1 and k are positive smooth functions depending of the parameter σ. We note that the solution in (1.7) gives us in " the limit " the solitary waves solutions (1.6) when

$\eta_1 \to \sqrt{4c\sigma}$ and $k \to 1^-$, because in this case the elliptic function dn converges, uniformly on compacts sets, to the hyperbolic function *sech*.

In the case of our main interest, $\gamma \neq 0$, for obtaining periodic solutions is a delicate issue. Our approach for obtaining these solutions will use the implicit function theorem together with the explicit formulas in (1.7) and a detailed study of the periodic eigenvalue problem associated to the Jacobian form of Lame's equation

$$\begin{cases} \frac{d^2}{dx^2}\Psi + [\rho - 6k^2 sn^2(x;k)]\Psi = 0 \\ \Psi(0) = \Psi(2K(k)), \quad \Psi'(0) = \Psi'(2K(k)), \end{cases} \tag{1.8}$$

where $sn(\cdot;k)$ is the Jacobi elliptic function of type snoidal and $K = K(k)$ represents the complete elliptic integral of the first kind and defined for $k \in (0,1)$ as

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

So by fixing a period L, and choosing c and ω such that $\sigma \equiv \omega - \frac{c^2}{4}$ satisfies $\sigma > 2\pi^2/L^2$, we obtain a smooth branch $\gamma \in (-\delta, \delta) \to (\phi_\gamma, \psi_\gamma)$ of periodic travelling wave solutions of (1.5) with a fundamental period L bifurcating of (ϕ_0, ψ_0) in (1.7). Moreover, we obtain that for γ near zero $\phi_\gamma(x) > 0$ for all $x \in \mathbb{R}$ and $\psi_\gamma(x) < 0$ for $\gamma < 0$ and $x \in \mathbb{R}$.

Now, with regarding to the non-linear stability of this branch of periodic solutions we extend the classical approach developed by Benjamin [13], Bona [16] and Weinstein [36] to the periodic case. So, by making use of the conservation laws (1.2), we prove that the solutions $(\phi_\gamma, \psi_\gamma)$ are stable in $H^1_{per}([0,L]) \times H^{\frac{1}{2}}_{per}([0,L])$ at least when γ is negative near zero. We use essentially the Benjamin&Bona&Weinstein's stability ideas because it gives us an easy form of manipulating with the required spectral conditions and the positivity property of the quantity $\frac{d}{d\sigma}\int \phi_\gamma^2(x)dx$, which are basic information in our stability analysis. We do not use the abstract stability theory of Grillakis *et al.* basically because of these circumstance. We recall that Grillakis *et al.* theory in general requires a study of the Hessian for the function

$$d(c,\omega) = L(e^{ic\xi/2}\phi_\gamma, \psi_\gamma) \equiv E(e^{ic\xi/2}\phi_\gamma, \psi_\gamma) + cG(e^{ic\xi/2}\phi_\gamma, \psi_\gamma) + \omega H(e^{ic\xi/2}\phi_\gamma, \psi_\gamma)$$

with $\gamma = \gamma(c,\omega)$, and a specific spectrum information of the matrix linear operator $H_{c,\omega} = L''(e^{ic\xi/2}\phi_\gamma, \psi_\gamma)$. In our case these facts do not seem clear to be obtained. So, for $\gamma < 0$ we reduce our spectral analysis (see formula (5.6)) to study the self-adjoint operator $\mathcal{L}_\gamma$,

$$\mathcal{L}_\gamma = -\frac{d^2}{d\xi^2} + \sigma + \alpha\psi_\gamma - 2\alpha\beta\phi_\gamma \circ \mathcal{K}_\gamma^{-1} \circ \phi_\gamma,$$

where $\mathcal{K}_\gamma^{-1}$ is the inverse operator of $\mathcal{K}_\gamma = -\gamma D + c$, and we obtain via the min-max principle that $\mathcal{L}_\gamma$ has a simple negative eigenvalue and zero is simple with eigenfunction $\frac{d}{dx}\phi_\gamma$ provide that γ is small enough.

Finally, we close this introduction with the organization of this paper: in Section 2, we introduce some notations to be used throughout the whole article; in Section 3, we prove the global well-posedness results in the periodic and continuous settings via some appropriate bilinear estimates; in Section 4, we show the existence of periodic travelling waves by the implicit function theorem; then, in Section 5, we derive the stability of these waves based on the ideas of Benjamin and Weinstein, that is, to manipulate the information from the spectral theory of certain self-adjoint operators and the positivity of some relevant quantities.

2 Notation

For any positive numbers a and b, the notation $a \lesssim b$ means that there exists a positive constant θ such that $a \leq \theta b$. Here, θ may depend only on certain parameters related to the equation (1.1) such as γ, α, β. Also, we denote $a \sim b$ when, $a \lesssim b$ and $b \lesssim a$.

For $a \in \mathbb{R}$, we denote by $a+$ and $a-$ a number slightly larger and smaller than a, respectively.

In the sequel, we fix ψ a smooth function supported on the interval $[-2, 2]$ such that $\psi(x) \equiv 1$ for all $|x| \leq 1$ and, for each $T > 0$, $\psi_T(t) := \psi(t/T)$.

The norm of a function $f \in L^p(\Omega)$ (equivalence class), for Ω an open subset of $\mathbb{R}$, is written $|f|^p_{L^p(\Omega)} = \int_\Omega |f|^p dx$, $p \geqq 1$. The inner product of two functions $f, g \in L^2(\Omega)$ is written as $< f, g >= \int_\Omega f\bar{g}dx$. To define the Sobolev spaces (of L^2 type) of periodic functions, we need the following notions (for further information see Iorio&Iorio [27]). Let $\mathcal{P} = C^\infty_{per}$ denote the collection of all the functions $f : \mathbb{R} \to \mathbb{C}$ which are $C^\infty(\mathbb{R})$ and periodic with period $2\ell > 0$. The topological dual of $\mathcal{P}$ will be denoted by $\mathcal{P}'$, i.e., $\mathcal{P}'$ is the collection of all continuous linear functionals from $\mathcal{P}$ into $\mathbb{C}$. $\mathcal{P}'$ is called the set of all *periodic distributions*. If $\Psi \in \mathcal{P}'$ we denote the value of Ψ in φ by $\Psi(\varphi) = (\Psi, \varphi)$. Define the functions $\Theta_k(x) = \exp(ik\pi x/\ell)$, $k \in \mathbb{Z}$, $x \in \mathbb{R}$. The Fourier transform of $\Psi \in \mathcal{P}'$ is the function $\widehat{\Psi} : \mathbb{Z} \to \mathbb{C}$ defined by the formula $\widehat{\Psi}(k) = \frac{1}{2\ell}(\Psi, \Theta_{-k})$, $k \in \mathbb{Z}$.

Let $s \in \mathbb{R}$. The Sobolev space $H^s_{2\ell}(\mathbb{R}) = H^s_{per}([-\ell, \ell])$ is the set of all $f \in \mathcal{P}'$ such that $\|f\|^2_{H^s_{2\ell}} \equiv 2\ell \sum_{k=-\infty}^{\infty}(1 + |k|^2)^s |\widehat{f}(k)|^2 < \infty$. $H^s_{2\ell}(\mathbb{R})$ is a Hilbert space with respect to the inner product $(f|g)_s = 2\ell \sum_{k=-\infty}^{\infty}(1 + |k|^2)^s \widehat{f}(k)\overline{\widehat{g}(k)}$. For $s = 0$, $H^0_{2\ell} \equiv L^2_{2\ell}$. Moreover, Sobolev's Lemma states that, if $s > \frac{1}{2}$ then $H^s_{2\ell}(\mathbb{R}) \hookrightarrow C_{per}$ where $C_{per} = \{f : \mathbb{R} \to \mathbb{C} :$ f is continuous and periodic with period $2\ell\}$. Sometimes we also write $H^s(\mathbb{T})$ to denote the space $H^s_{per}([-\ell, \ell])$ when the period 2ℓ does not play a fundamental role.

When the function u is of the two time-space variables $(t, x) \in \mathbb{R} \times [-\ell, \ell]$, periodic of period 2ℓ, we define its Fourier transform by

$$\widehat{u}(\tau, n) = \frac{1}{(2\pi)^{1/2} 2\ell} \int_{\mathbb{R}\times[-\ell,\ell]} u(t, x) e^{-i(n\pi x/\ell + t\tau)} dt dx,$$

and similarly, when $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, we define

$$\widehat{u}(\tau, \xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} u(t,x) e^{-i(x\xi + t\tau)} dt dx.$$

Next, we introduce the Bourgain spaces related to the Schrödinger-Benjamin-Ono system in the periodic case:

$$\|u\|_{X^{s,b}_{per}} := \left(\sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \langle \tau + n^2 \rangle^{2b} \langle n \rangle^{2s} |\widehat{u}(\tau, n)|^2 d\tau \right)^{1/2}, \tag{2.9}$$

$$\|u\|_{Y^{s,b}_{\gamma,per}} := \left(\sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \langle \tau + \gamma |n| n \rangle^{2b} \langle n \rangle^{2s} |\widehat{u}(\tau, n)|^2 d\tau \right)^{1/2}, \tag{2.10}$$

and the continuous case:

$$\|u\|_{X^{s,b}} := \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \langle \tau + \xi^2 \rangle^{2b} \langle \xi \rangle^{2s} |\widehat{u}(\tau, \xi)|^2 d\xi d\tau \right)^{1/2}, \tag{2.11}$$

$$\|u\|_{Y^{s,b}_{\gamma}} := \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \langle \tau + \gamma |\xi| \xi \rangle^{2b} \langle \xi \rangle^{2s} |\widehat{u}(\tau, \xi)|^2 d\xi d\tau \right)^{1/2}, \tag{2.12}$$

where $\langle x \rangle := 1 + |x|$. The relevance of these spaces are related to the fact that they are well-adapted to the linear part of the system and, after some time-localization, the coupling terms of (1.1) verifies particularly nice bilinear estimates. Consequently, it will be a standard matter to conclude our global well-posedness results (via Picard fixed point method).

3 Global Well-Posedness of the Schrödinger-Benjamin-Ono System

This section is devoted to the proof of our well-posedness results for (1.1) in both continuous and periodic settings.

3.1 *Global well-posedness on* $\mathbb{R}$

The bulk of this subsection is to show the following theorem:

Theorem 3.1 (Global well-posedness in $\mathbb{R}$)**.** *Let* $|\gamma| \neq 1$*, then the SBO system (1.1) is globally well-posed for initial data* $(u_0, v_0) \in H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R})$*, when* $s \geq 0$*.*

Recall that the local well-posedness of the SBO system for $(u_0, v_0) \in H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R})$, $|\gamma| \neq 1$ and $s \geq 0$ was showed by Bekiranov, Ogawa and Ponce [10]. Subsequently, Pecher [34] proved local well-posedness also in the case $|\gamma| = 1$, $s > 0$ and global well-posedness if $\gamma < 0$, $\alpha/\beta > 0$, $1 \geq s > 1/3$ via the I-method of Colliander, Keel, Staffilani, Takaoka and Tao.

Hence, Theorem 3.1 above improves Pecher's global well-posedness results in the case $|\gamma| \neq 1$; furthermore, as we are going to see, our proof of this theorem does not relies on sophisticated techniques such as the I-method but instead it uses the local well-posedness for the *decoupled system* and the L^2 conservation law (for the Schrödinger part of the system). More precisely, the integral equation associated to the SBO equation is

$$\begin{cases} u(t) = U(t)u_0 - i\alpha \int_0^t U(t-t')u(t')v(t')dt', \\ \\ v(t) = V_\gamma(t)v_0 + \beta \int_0^t V_\gamma(t-t')\partial_x |u(t')|^2 dt' \end{cases}$$

where $U(t) := e^{it\Delta}$ and $V_\gamma(t) := e^{-\gamma t \partial_x |\partial_x|}$ are the linear Schrödinger and Benjamin-Ono propagators, respectively. Note that this integral equation is equivalent to

$$\begin{aligned} u(t) :=& U(t)u_0 - i\alpha \int_0^t U(t-t')u(t')V_\gamma(t')v_0 dt' \\ &- i\alpha\beta \int_0^t U(t-t')u(t') \left(\int_0^{t'} V_\gamma(t'-t'')\partial_x |u(t'')|^2 dt'' \right) dt'. \end{aligned} \tag{3.1}$$

The advantage of this "decoupled" formulation of the SBO system now is clear: the (local and global) well-posedness of the SBO equation reduces to obtain some good control of u, because the function v_0 is fixed. On the other hand, the evolution of u has fine properties (for instance, the L^2 norm of u is conserved), so that the desired control of u comes for free.

Coming back to the proof of Theorem 3.1, the following preliminary linear estimates for the Schrödinger and Benjamin-Ono propagators will be useful (see [11] for details):

Lemma 3.1. *For any $s, k, \gamma \in \mathbb{R}$, $-1/2 < b < b' < 1/2$,*

$$\|\psi_T(t)F\|_{X^{s,b}} \lesssim T^{b'-b} \|F\|_{X^{s,b'}}$$

and

$$\|\psi_T(t)G\|_{Y_\gamma^{k,b}} \lesssim T^{b'-b} \|G\|_{Y_\gamma^{k,b'}}.$$

Lemma 3.2. *For any $s, k, \gamma \in \mathbb{R}$, $0 < b < 1$, it holds*

$$\|\psi(t)U(t)u_0\|_{X^{s,b}} \lesssim \|u_0\|_{H^s}$$

and

$$\|\psi(t)V_\gamma(t)v_0\|_{Y_\gamma^{k,b}} \lesssim \|v_0\|_{H^k}.$$

Lemma 3.3. *For any $s, k, \gamma \in \mathbb{R}$, $0 < b < 1$, we have*

$$\left\| \int_0^t U(t-t')F(t')dt' \right\|_{X^{s,b}} \lesssim \|F\|_{X^{s,b-1}},$$

and

$$\left\| \int_0^t V_\gamma(t-t')G(t')dt' \right\|_{Y_\gamma^{k,b}} \lesssim \|G\|_{Y_\gamma^{k,b-1}}.$$

Now we slightly modificate the bilinear estimates obtained in [11] for the coupling terms uv and $\partial_x(|u|^2)$ of the SBO system:

Lemma 3.4. *For any $1/2 < b < 3/4$, $a \leq -1/4$, $s \geq 0$ and $|\gamma| \neq 1$,*

$$\|uv\|_{X^{s,a}} \lesssim \|u\|_{X^{0,b}} \|v\|_{Y_\gamma^{s-\frac{1}{2},b}} + \|u\|_{X^{s,b}} \|v\|_{Y_\gamma^{-1/2,b}}.$$

Also, for any $b > 1/2$, $a \leq 0$ and $s \geq 0$,

$$\|\partial_x(|u|^2)\|_{Y_\gamma^{s-\frac{1}{2},a}} \lesssim \|u\|_{X^{0,b}} \|u\|_{X^{s,b}}.$$

Proof. Recall that the Lemma 3.4 of [11] states that, under our hypothesis, it holds

$$\|uv\|_{X^{s,a}} \lesssim \|u\|_{X^{s,b}} \|v\|_{Y_\gamma^{s-\frac{1}{2},b}}.$$

On the other hand, taking two indices $s \geq s'$, the triangular inequality says that $\langle\xi\rangle^s \lesssim \langle\xi\rangle^{s'}\langle\xi_1\rangle^{s-s'} + \langle\xi\rangle^{s'}\langle\xi_2\rangle^{s-s'}$, where $\xi = \xi_1 + \xi_2$. Denoting by $\widehat{J^s u}(\xi) := \langle\xi\rangle^s \widehat{u}(\xi)$, we obtain that, for any $s \geq 0$, the previous estimate implies

$$\begin{aligned}
\|uv\|_{X^{s,a}} &\lesssim \|J^s u \cdot v\|_{X^{0,a}} + \|u \cdot J^s v\|_{X^{0,a}} \\
&\lesssim \|J^s u\|_{X^{0,b}} \|v\|_{Y_\gamma^{-\frac{1}{2},b}} + \|u\|_{X^{0,b}} \|J^s v\|_{Y_\gamma^{-\frac{1}{2},b}} \\
&= \|u\|_{X^{s,b}} \|v\|_{Y_\gamma^{-1/2,b}} + \|u\|_{X^{0,b}} \|v\|_{Y_\gamma^{s-\frac{1}{2},b}}.
\end{aligned}$$

This proves the first part of the lemma.

On the other hand, similarly to the previous case, since $\langle\xi\rangle^s \lesssim \langle\xi\rangle^{s'}\langle\xi_1\rangle^{s-s'} + \langle\xi\rangle^{s'}\langle\xi_2\rangle^{s-s'}$, we get for all $s \geq 0$ that the corollary 3.3 of [10] forces

$$\begin{aligned}
\|\partial_x(|u|^2)\|_{Y_\gamma^{s-\frac{1}{2},a}} &\lesssim \|\partial_x(J^s u \cdot \overline{u})\|_{Y_\gamma^{-\frac{1}{2},a}} + \|\partial_x(u \cdot \overline{J^s u})\|_{Y_\gamma^{-\frac{1}{2},a}} \\
&\lesssim \|J^s u\|_{X^{0,b}} \|u\|_{X^{0,b}} + \|u\|_{X^{0,b}} \|J^s v\|_{X^{0,b}} \\
&= 2\|u\|_{X^{0,b}} \|u\|_{X^{s,b}}.
\end{aligned}$$

□

Finally, we are ready to prove Theorem 3.1: consider the integral operator Φ_T defined by a time-localization of the right-hand side of (3.1):

$$\begin{aligned}\Phi_T(u)(t) :=&\psi(t)U(t)u_0 - i\alpha\psi_T(t)\int_0^t U(t-t')u(t')\psi(t')V_\gamma(t')v_0 dt' \\ &- i\alpha\beta\psi_T(t)\int_0^t U(t-t')u(t')\left(\psi_T(t')\int_0^{t'} V_\gamma(t'-t'')\partial_x|u(t'')|^2 dt''\right)dt'.\end{aligned}$$

The linear and bilinear estimates above clearly implies that, given $1/2 < b < 3/4$ and $s \geq 0$,

$$\|\Phi_T u\|_{X^{s,b}} \leq C_0\|u_0\|_{H^s} + T^\mu C_0\|v_0\|_{H^{s-\frac{1}{2}}}\|u\|_{X^{s,b}} + T^\mu C_0\|u\|^2_{X^{0,b}}\|u\|_{X^{s,b}}, \tag{3.2}$$

for some constants $\mu > 0$ and C_0 depending only on γ.

We start with the case of initial data $(u_0, v_0) \in L^2(\mathbb{R}) \times H^{-1/2}(\mathbb{R})$. Choosing $R = 2C_0\|u_0\|_{L^2}$ and $T > 0$ small enough such that

$$C_0 T^\mu(1 + \|v_0\|_{H^{-1/2}} + 4C_0^2\|u_0\|^2_{L^2}) \leq 1/2,$$

it follows that Φ_T is a contraction of the ball $B_R(0)$ of $X^{0,b}$. Hence, it has a fixed point $u(t)$ solving the SBO equation on the time interval $[0, T]$. Furthermore, it holds

$$\|u\|_{X^{0,b}} \leq 2C_0\|u_0\|_{L^2}.$$

On the other hand, the theorem 1.2 of [11] asserts that the L^2-norm of u is preserved during the evolution of the solution $u(t)$ of the SBO equation, i.e.,

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}.$$

Therefore, once v_0 is *fixed*, we can iterate the argument above as follows: take the solution $u(t)$ of the SBO equation with respect to the initial data u_0 just constructed (by the fixed point method applied to Φ_T) on the time interval $[0, T_0]$; observe that $T_0 = \left[2C_0(1 + \|v_0\|_{H^{-1/2}} + 4C_0^2\|u_0\|^2_{L^2})\right]^{-1/\mu}$ depends only on $\|u_0\|_{L^2}$ and $\|v_0\|_{H^{-1/2}}$; next, we consider the SBO equation with initial data $u(T_0)$ and the corresponding solution on a time interval $[T_0, T_0 + T_1]$; by the discussion above, we already know that T_1 depends only on $\|u(T_0)\|_{L^2}$ and $\|v_0\|_{H^{-1/2}}$; by the conservation of the L^2 norm, this means that

$$\begin{aligned}T_1 &= \left[2C_0(1 + \|v_0\|_{H^{-1/2}} + 4C_0^2\|u(T_0)\|^2_{L^2})\right]^{-1/\mu} \\ &= \left[2C_0(1 + \|v_0\|_{H^{-1/2}} + 4C_0^2\|u_0\|^2_{L^2})\right]^{-1/\mu} \\ &= T_0.\end{aligned}$$

In particular, the solution of the SBO equation can be extended from the time interval $[0, T_0]$ to $[0, 2T_0]$; repeating indefinitely this procedure, we obtain a global solution $u(t)$ of the SBO equation with initial data $(u_0, v_0) \in L^2(\mathbb{R}) \times H^{-1/2}(\mathbb{R})$ such that

$$\|u\|_{X^{0,b}} \leq 2C_0\|u_0\|_{L^2}.$$

Next, we turn to the general case $(u_0, v_0) \in H^s(\mathbb{R}) \times H^{s-\frac{1}{2}}(\mathbb{R})$, $s \geq 0$. Taking $R := 2C_0\|u_0\|_{H^s}$ and $T > 0$ sufficiently small with

$$C_0 T^{\mu}(1 + \|v_0\|_{H^{s-\frac{1}{2}}} + \|u\|^2_{X^{0,b}}) \leq 1/2,$$

it follows from (3.2) that Φ_T is a contraction of the ball $B_R(0)$ of $X^{s,b}$. Thus, it has a fixed point $u(t)$ solving the SBO equation on the time interval $[0, T]$. Moreover, since $u(t)$ is also a solution of the SBO equation for initial data $(u_0, v_0) \in L^2(\mathbb{R}) \times H^{-1/2}(\mathbb{R})$ (because $s \geq 0$), we have an *a priori* bound

$$\|u\|_{X^{0,b}} \leq 2C_0\|u_0\|_{L^2}.$$

This allows us to conclude that the solution $u(t)$ is defined on a time interval $[0, T]$ with

$$T = \left[2C_0(1 + \|v_0\|_{H^{-1/2}} + 4C_0^2\|u_0\|^2_{L^2})\right]^{-1/\mu}.$$

Using the conservation of the L^2-norm exactly as before, we obtain the desired theorem. □

3.2 *Global well-posedness on* $\mathbb{T}$

This subsection contains sharp bilinear estimates for the coupling terms uv and $\partial_x(|u|^2)$ of the SBO system in the periodic setting and the global well-posedness result in the energy space $H^1(\mathbb{T}) \times H^{1/2}(\mathbb{T})$ (which is necessary for our subsequent stability theory).

Let us state our well-posedness result:

Theorem 3.2 (Local well-posedness in $\mathbb{T}$). *Let $\gamma \in \mathbb{R}$ such that $\gamma \neq 0$, $|\gamma| \neq 1$ and $s \geq 1/2$, then the SBO system (1.1) is locally well-posed in $H^s(\mathbb{T}) \times H^{s-1/2}(\mathbb{T})$, i.e. for all $(u_0, v_0) \in H^s(\mathbb{T}) \times H^{s-1/2}(\mathbb{T})$, there exists $T = T(\|u_0\|_{H^s}, \|v_0\|_{H^{s-1/2}})$ (with $T(\rho_1, \rho_2) \to \infty$ as $(\rho_1, \rho_2) \to 0$), and a unique solution of the Cauchy problem (1.1) of the form $(\psi_T u, \psi_T v)$ such that $(u, v) \in X^{s,1/2+}_{per} \times Y^{s-1/2,1/2+}_{\gamma,per}$. Moreover, (u, v) satisfies the additional regularity*

$$(u, v) \in C([0, T]; H^s(\mathbb{T})) \times C([0, T]; H^{s-1/2}(\mathbb{T})) \tag{3.1}$$

and the map solution $S : (u_0, v_0) \mapsto (u, v)$ is smooth.

Using the conservation laws (1.2) as in [34], our local existence result implies

Theorem 3.3 (Global well-posedness in $\mathbb{T}$). *Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\gamma \neq 0$, $|\gamma| \neq 1$ and $\frac{\alpha\gamma}{\beta} < 0$, then the SBO system (1.1) is globally well-posed in $H^s(\mathbb{T}) \times H^{s-1/2}(\mathbb{T})$, when $s \geq 1$.*

The fundamental technical points in the proof of Theorem 3.2 are the following bilinear estimates. The rest of the proof follows by standard arguments, as in [20] (see also [9]).

Proposition 3.1. *Let $\gamma \in \mathbb{R}$ such that $\gamma \neq 0$, $|\gamma| \neq 1$ and $s \geq 1/2$, then*

$$\|uv\|_{X^{s,-1/2+}_{per}} \lesssim \|u\|_{Y^{s-1/2,1/2}_{\gamma,per}} \|v\|_{X^{s,1/2}_{per}}, \tag{3.2}$$

$$\|\partial_x(u\bar{v})\|_{Y^{s-1/2,-1/2+}_{\gamma,per}} \lesssim \|u\|_{X^{s,1/2}_{per}} \|v\|_{X^{s,1/2}_{per}}, \tag{3.3}$$

where the implicit constants depend on γ.

These estimates are sharp in the following sense

Proposition 3.2. *Let $\gamma \neq 0$, $|\gamma| \neq 1$, then*

(i) The estimate (3.2) fails for any $s < 1/2$.

(ii) The estimate (3.3) fails for any $s < 1/2$.

Proposition 3.3. *Let $\gamma \in \mathbb{R}$ such that $|\gamma| = 1$, then*

(i) The estimate (3.2) fails for any $s \in \mathbb{R}$.

(ii) The estimate (3.3) fails for any $s \in \mathbb{R}$.

The following Bourgain-Strichartz estimates will be needed in the proof of Proposition 3.1

Proposition 3.4. *We have*

$$\|u\|_{L^4_{t,x}} \lesssim \|u\|_{X^{0,3/8}}, \tag{3.4}$$

and

$$\|u\|_{L^4_{t,x}} \lesssim \|u\|_{Y^{0,3/8}_{\gamma}}, \tag{3.5}$$

for $u : \mathbb{R} \times \mathbb{T} \to \mathbb{C}$ and $\gamma \in \mathbb{R}$, $\gamma \neq 0$.

Proof. The first estimate of (3.4) was proved by Bourgain in [17] and the second one is a simple consequence of the first one (see for example [31]). □

Proof of Proposition 3.1. Fix $s \geq 1/2$. Without loss of generality we can suppose that $|\gamma| < 1$ in the rest of the proof.

In order to prove estimate (3.2), it is sufficient to prove that

$$\|uv\|_{X^{s,-3/8}_{per}} = \|\frac{\langle n\rangle^s}{\langle \tau + n^2\rangle^{3/8}}(uv)^\wedge(\tau,n)\|_{l^2_n L^2_\tau} \lesssim \|u\|_{Y^{s-1/2,1/2}_{\gamma,per}} \|v\|_{X^{s,1/2}_{per}}. \tag{3.6}$$

Letting $f(\tau,n) = \langle n\rangle^{s-1/2}\langle \tau + \gamma n|n|\rangle^{1/2}\widehat{u}(\tau,n)$, $g(\tau,n) = \langle n\rangle^s \langle \tau + n^2\rangle^{1/2}\widehat{v}(\tau,n)$ and using duality, we deduce that the estimate (3.6) is equivalent to prove that

$$I \lesssim \|f\|_{L^2_\tau l^2_n} \|g\|_{L^2_\tau l^2_n} \|h\|_{L^2_\tau l^2_n}, \tag{3.7}$$

where

$$I := \sum_{n,n_1\in\mathbb{Z}} \int_{\mathbb{R}^2} \frac{\langle n\rangle^s h(\tau,n) f(\tau_1,n_1)}{\langle \tau+n^2\rangle^{3/8}\langle n_1\rangle^{s-1/2}\langle \tau_1+\gamma|n_1|n_1\rangle^{1/2}} \times \frac{g(\tau-\tau_1,n-n_1)}{\langle n-n_1\rangle^s\langle \tau-\tau_1+(n-n_1)^2\rangle^{1/2}} d\tau d\tau_1. \tag{3.8}$$

In order to estimate integral (3.8), we split the integration domain $\mathbb{R}^2\times\mathbb{Z}^2$ in the following regions,

$$\begin{aligned}\mathcal{M} &= \{(\tau,\tau_1,n,n_1)\in\mathbb{R}^2\times\mathbb{Z}^2 \ : \ n_1=0 \text{ or } |n|\le c(\gamma)^{-1}|n-n_1|\},\\ \mathcal{N} &= \{(\tau,\tau_1,n,n_1)\in\mathbb{R}^2\times\mathbb{Z}^2 \ : \ n_1\neq 0 \text{ and } |n-n_1|\le c(\gamma)|n|\},\end{aligned}$$

where $c(\gamma)$ is a positive constant depending on γ to be fixed later. We also denote by $I_{\mathcal{M}}$ and $I_{\mathcal{N}}$ the integral I restricted to the regions $\mathcal{M}$ and $\mathcal{N}$, respectively.
Estimate in the region $\mathcal{M}$. We observe that, since $s\ge 1/2$, in the region $\mathcal{M}$ holds $\frac{\langle n\rangle^s}{\langle n_1\rangle^{s-1/2}\langle n-n_1\rangle^s}\lesssim 1$ (where the implicit constant depends on γ). Thus, we deduce using the Plancherel identity and the $L^2_{t,x}L^4_{t,x}L^4_{t,x}$-Hölder inequality that

$$\begin{aligned} I_{\mathcal{M}} &\lesssim \sum_{n,n_1\in\mathbb{Z}} \int_{\mathbb{R}^2} \frac{h(\tau,n)f(\tau_1,n_1)g(\tau-\tau_1,n-n_1)d\tau d\tau_1}{\langle\tau+n^2\rangle^{3/8}\langle\tau_1+\gamma|n_1|n_1\rangle^{1/2}\langle\tau-\tau_1+(n-n_1)^2\rangle^{1/2}}\\ &\lesssim \int_{\mathbb{R}\times\mathbb{T}} \Big(\frac{h(\tau,n)}{\langle\tau+n^2\rangle^{3/8}}\Big)^{\vee}\Big(\frac{f(\tau,n)}{\langle\tau+\gamma|n|n\rangle^{1/2}}\Big)^{\vee}\Big(\frac{g(\tau,n)}{\langle\tau+n^2\rangle^{1/2}}\Big)^{\vee} dtdx\\ &\lesssim \|\Big(\frac{h(\tau,n)}{\langle\tau+n^2\rangle^{3/8}}\Big)^{\vee}\|_{L^2_{t,x}}\|\Big(\frac{f(\tau,n)}{\langle\tau+\gamma|n|n\rangle^{1/2}}\Big)^{\vee}\|_{L^4_{t,x}}\|\Big(\frac{g(\tau,n)}{\langle\tau+n^2\rangle^{1/2}}\Big)^{\vee}\|_{L^4_{t,x}}. \end{aligned} \tag{3.9}$$

This implies, together with the estimates (3.4) and (3.5), that

$$I_{\mathcal{M}} \lesssim \|f\|_{L^2_\tau l^2_n}\|g\|_{L^2_\tau l^2_n}\|h\|_{L^2_\tau l^2_n}. \tag{3.10}$$

Estimate in the region $\mathcal{N}$. The dispersive smoothing effect associated to the SBO system (1.1) can be translated by the following algebraic relation

$$-(\tau+n^2)+(\tau_1+\gamma n_1|n_1|)+(\tau-\tau_1+(n-n_1)^2) = Q_\gamma(n,n_1), \tag{3.11}$$

where

$$Q_\gamma(n,n_1) = (n-n_1)^2+\gamma|n_1|n_1-n^2. \tag{3.12}$$

We have in the region $\mathcal{N}$, $|n_1|\le(1+c(\gamma))|n|$, so that

$$|Q_\gamma(n,n_1)| \ge \big(1-|\gamma|(1+c(\gamma))^2-c(\gamma)^2\big)(1+c(\gamma))^{-2}|n_1|^2.$$

Now, we choose $c(\gamma)$ positive, small enough such that

$$\big(1-|\gamma|(1+c(\gamma))^2-c(\gamma)^2\big)=\frac{1-|\gamma|}{2},$$

which is positive remembering that $|\gamma|<1$. Therefore, we divide the region $\mathcal{N}$ in three parts according which term of the left-hand side of (3.11) is dominant,

$$\begin{aligned}
\mathcal{N}_1 &= \{(\tau,\tau_1,n,n_1)\in\mathcal{N} \ : \ |\tau+n^2|\geq|\tau_1+\gamma|n_1|n_1|,|\tau-\tau_1+(n-n_1)^2|\},\\
\mathcal{N}_2 &= \{(\tau,\tau_1,n,n_1)\in\mathcal{N} \ : \ |\tau_1+\gamma|n_1|n_1|\geq|\tau+n^2|,|\tau-\tau_1+(n-n_1)^2|\},\\
\mathcal{N}_3 &= \{(\tau,\tau_1,n,n_1)\in\mathcal{N} \ : \ |\tau-\tau_1+(n-n_1)^2|\geq|\tau+n^2|,|\tau_1+\gamma|n_1|n_1|\},
\end{aligned}$$

and denote by $I_{\mathcal{N}_1}$, $I_{\mathcal{N}_2}$ and $I_{\mathcal{N}_3}$ the restriction of the integral I to the regions $\mathcal{N}_1$, $\mathcal{N}_2$ and $\mathcal{N}_3$, respectively.

In the region $\mathcal{N}_1$, we have $\frac{\langle n\rangle^s}{\langle n_1\rangle^{s-1/2}\langle n-n_1\rangle^s}\times\frac{1}{\langle\tau+n^2\rangle^{3/8}}\lesssim 1$, so that we can conclude

$$I_{\mathcal{N}_1}\lesssim\|f\|_{L^2_\tau l^2_n}\|g\|_{L^2_\tau l^2_n}\|h\|_{L^2_\tau l^2_n},\tag{3.13}$$

exactly as in (3.9). We note that $\frac{\langle n\rangle^s}{\langle n_1\rangle^{s-1/2}\langle n-n_1\rangle^s}\times\frac{1}{\langle\tau_1+|n_1|n_1\rangle^{1/2}}\lesssim 1$ in the region $\mathcal{N}_2$. Then, using the $L^4_{t,x}L^2_{t,x}L^4_{t,x}$-Hölder inequality, that

$$I_{\mathcal{N}_2}\lesssim\|\left(\frac{h(\tau,n)}{\langle\tau+n^2\rangle^{3/8}}\right)^\vee\|_{L^4_{t,x}}\|f\|_{L^2_\tau l^2_n}\|\left(\frac{g(\tau,n)}{\langle\tau+n^2\rangle^{1/2}}\right)^\vee\|_{L^4_{t,x}}.$$

This implies, using the Bourgain-Strichartz (3.4), that

$$I_{\mathcal{N}_2}\lesssim\|f\|_{L^2_\tau l^2_n}\|g\|_{L^2_\tau l^2_n}\|h\|_{L^2_\tau l^2_n}.\tag{3.14}$$

Similarly, $\frac{\langle n\rangle^s}{\langle n_1\rangle^{s-1/2}\langle n-n_1\rangle^s}\times\frac{1}{\langle\tau-\tau_1+(n-n_1)^2\rangle^{1/2}}\lesssim 1$ in $\mathcal{N}_3$ so that

$$\begin{aligned}
I_{\mathcal{N}_3}&\lesssim\|\left(\frac{h(\tau,n)}{\langle\tau+n^2\rangle^{3/8}}\right)^\vee\|_{L^4_{t,x}}\|\left(\frac{f(\tau,n)}{\langle\tau+\gamma|n|n\rangle^{1/2}}\right)^\vee\|_{L^4_{t,x}}\|g\|_{L^2_\tau l^2_n}\\
&\lesssim\|f\|_{L^2_\tau l^2_n}\|g\|_{L^2_\tau l^2_n}\|h\|_{L^2_\tau l^2_n}.
\end{aligned}\tag{3.15}$$

Then, we gather (3.10), (3.13), (3.14) and (3.15) to deduce (3.7), which concludes the proof of the estimate (3.2).

Next in order to prove the estimate (3.3), we reason as previously to deduce that it is sufficient to prove that

$$\|\partial_x(u\bar{v})\|_{Y^{s-1/2,-3/8}_{\gamma,per}}\lesssim\|u\|_{X^{s,1/2}_{per}}\|v\|_{X^{s,1/2}_{per}},\tag{3.16}$$

which is equivalent by duality and after performing the change of variable $f(\tau,n)=\langle n\rangle^s\langle\tau+n^2\rangle^{1/2}\widehat{u}(\tau,n)$ and $g(\tau,n)=\langle n\rangle^s\langle\tau-n^2\rangle^{1/2}\widehat{\bar{v}}(\tau,n)$ to

$$J\lesssim\|f\|_{L^2_\tau l^2_n}\|g\|_{L^2_\tau l^2_n}\|h\|_{L^2_\tau l^2_n},\tag{3.17}$$

where

$$J := \sum_{n,n_1\in\mathbb{Z}} \int_{\mathbb{R}^2} \frac{|n|\langle n\rangle^{s-1/2} h(\tau,n) f(\tau_1,n_1)}{\langle \tau+\gamma|n|n\rangle^{3/8}\langle n_1\rangle^s \langle \tau_1+n_1^2\rangle^{1/2}}$$
$$\times \frac{g(\tau-\tau_1,n-n_1)}{\langle n-n_1\rangle^s\langle \tau-\tau_1-(n-n_1)^2\rangle^{1/2}} d\tau d\tau_1. \tag{3.18}$$

The algebraic relation associated to (3.18) is given by

$$-(\tau+\gamma|n|n)+(\tau_1+n_1^2)+(\tau-\tau_1-(n-n_1)^2)=\tilde{Q}_\gamma(n,n_1),$$

where

$$\tilde{Q}_\gamma(n,n_1) = -(n-n_1)^2-\gamma|n|n+n_1^2.$$

Therefore we can prove estimate (3.17) using exactly the same arguments as for estimate (3.7). □

Remark 3.1. Observe that we obtain our bilinear estimates in the spaces $X^{s,1/2+}_{per}$ and $Y^{s-1/2,1/2+}_{\gamma,per}$ which control the $L^\infty_t H^s_x$ and $L^\infty_t H^{s-1/2}_x$ norms respectively. Therefore, we do not need to use other norms as in the case of the periodic KdV equation [20].

Remark 3.2. Observe that the proof of Proposition 3.1 actually shows that the following bilinear estimates hold:

$$\|uv\|_{X^{s,-3/8}_{per}} \lesssim \|u\|_{X^{s,3/8}_{per}}\|v\|_{Y^{s-1/2,1/2}_{\gamma,per}} + \|u\|_{X^{s,1/2}_{per}}\|v\|_{Y^{s-1/2,3/8}_{\gamma,per}},$$

$$\|\partial_x(u\bar{w})\|_{Y^{s-1/2,-3/8}_{\gamma,per}} \lesssim \|u\|_{X^{s,3/8}_{per}}\|w\|_{X^{s,1/2}_{per}} + \|u\|_{X^{s,1/2}_{per}}\|w\|_{X^{s,3/8}_{per}},$$

While we are not attempting to use this refined version of proposition 3.1 in this paper, we plan to apply these estimates combined with the I-method of Colliander, Keel, Stafillani, Takaoka and Tao to get global well-posedness results for the periodic SBO system below the energy space. Indeed, this issue will be adressed in a forthcoming paper.

In the proof of Proposition 3.2, we will use the following lemma which is a direct consequence of the Dirichlet theorem.

Lemma 3.5. *Let $\gamma\in\mathbb{R}$ such that $|\gamma|<1$ and Q_γ defined as in (3.12), then there exists a sequence of positive integers $\{N_j\}_{j\in\mathbb{N}}$ such that*

$$N_j\to\infty \quad and \quad |Q_\gamma(N_j,N_j^0)|\le 1, \tag{3.19}$$

where $N_j^0=[\frac{2N_j}{1+\gamma}]$. Here $[x]$ denotes the closest integer to x.

Theorem 3.4 (Dirichlet)**.** *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then the inequality*

$$0 < \left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2} \tag{3.20}$$

has an infinity of rational solutions $\frac{p}{q}$.

Proof of Lemma 3.5. Fix $\gamma \in \mathbb{R}$ such that $|\gamma| < 1$. Let N a positive integer, $N \geq 2$, we define $\alpha = \frac{2}{1+\gamma}$ and $N^0 = [\alpha N]$, then we deduce from the definition in (3.12) that

$$|Q_\gamma(N, N^0)| \leq 1 \quad \Longleftrightarrow \quad \left| \alpha - \frac{[\alpha N]}{N} \right| \leq \frac{1}{N^2}. \tag{3.21}$$

When $\alpha \in \mathbb{Q}$, $\alpha = \frac{p}{q}$, it is clear that we can find an infinity of positive integer N satisfying the right-hand side of (3.21) choosing $N_j = jq$, $j \in \mathbb{N}$. When $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, this is guaranteed by the Dirichlet theorem. □

Proof of Proposition 3.2. We will only show that the estimate (3.2) fails, since a counterexample for the estimate (3.3) can be constructed by a similar way. First observe that, letting $f(\tau, n) = \langle n \rangle^{s-1/2} \langle \tau + \gamma n|n| \rangle^{1/2} \widehat{u}(\tau, n)$ and $g(\tau, n) = \langle n \rangle^s \langle \tau + n^2 \rangle^{1/2} \widehat{v}(\tau, n)$, the estimate (3.2) is equivalent to

$$\|B_\gamma(f, g; s)\|_{L^2_\tau l^2_n} \lesssim \|f\|_{L^2_\tau l^2_n} \|g\|_{L^2_\tau l^2_n}, \ \forall \ f, \ g \in L^2_\tau l^2_n, \tag{3.22}$$

where

$$\begin{aligned} B_\gamma(f, g; s)(\tau, n) := \frac{\langle n \rangle^s}{\langle \tau + n^2 \rangle^{1/2}} \sum_{n_1 \in \mathbb{Z}} \int_{\mathbb{R}} & \frac{f(\tau_1, n_1)}{\langle n_1 \rangle^{s-1/2} \langle \tau_1 + \gamma |n_1| n_1 \rangle^{1/2}} \\ & \times \frac{g(\tau - \tau_1, n - n_1)}{\langle n - n_1 \rangle^s \langle \tau - \tau_1 + (n - n_1)^2 \rangle^{1/2}} d\tau_1, \end{aligned} \tag{3.23}$$

for all s and $\gamma \in \mathbb{R}$.

Fix $s < 1/2$ and γ such that $|\gamma| \neq 1$, we suppose for example that $|\gamma| < 1$. Consider the sequence of integer $\{N_j\}$ obtained in Lemma 3.5, that we can always suppose to verify $N_j \gg 1$, then we define

$$f_j(\tau, n) = a_n \chi_{1/2}(\tau + \gamma |n| n) \quad \text{with} \quad a_n = \begin{cases} 1, \, n = N_j^0, \\ 0, \text{ elsewhere,} \end{cases} \tag{3.24}$$

and

$$g_j(\tau, n) = b_n \chi_{1/2}(\tau + n^2) \quad \text{with} \quad b_n = \begin{cases} 1, \, n = N_j - N_j^0, \\ 0, \text{ elsewhere,} \end{cases} \tag{3.25}$$

where χ_r is the characteristic function of the interval $[-r, r]$. Hence,

$$\|f_j\|_{L^2_\tau l^2_n} \sim \|g_j\|_{L^2_\tau l^2_n} \sim 1, \tag{3.26}$$

$$a_{n_1}b_{n-n_1} \neq 0 \quad \text{if and only if} \quad n_1 = N_j^0 \text{ and } n = N_j,$$

and using (3.11), we deduce that

$$\int_{\mathbb{R}} \chi_{1/2}(\tau_1 + \gamma|N_j^0|N_j^0)\chi_{1/2}(\tau - \tau_1 + (N_j - N_j^0)^2)d\tau_1 \sim \chi_1(\tau + N^2 + Q_\gamma(N_j, N_j^0)).$$

Therefore, we have from the definition in (3.23)

$$B_\gamma(f_j, g_j; s)(\tau, N_j) \gtrsim \frac{N_j^s \chi_1(\tau + N_j^2 + Q_\gamma(N_j, N_j^0))}{\langle \tau + N_j^2 \rangle^{1/2} N_j^{s-1/2} N_j^s}, \tag{3.27}$$

where the implicit constant depends on γ. Thus, we deduce using (3.19) that

$$\|B_\gamma(f_j, g_j; s)\|_{L^2_\tau l^2_n} \gtrsim N_j^{1/2-s}, \quad \forall\, j \in \mathbb{N} \tag{3.28}$$

which combined with (3.19) and (3.26) contradicts (3.22), since $s < 1/2$. □

Proof of Proposition 3.3. Let $s \in \mathbb{R}$, we fix $\gamma = 1$. As in the proof of Proposition 3.2, we will only show that the estimate (3.2) fails, since a counterexample for the estimate (3.3) can be constructed by a similar way. In this case, (3.2) is equivalent to

$$\|B_1(f, g; s)\|_{L^2_\tau l^2_n} \lesssim \|f\|_{L^2_\tau l^2_n} \|g\|_{L^2_\tau l^2_n}, \ \forall\ f,\ g \in L^2_\tau l^2_n, \tag{3.29}$$

where

$$B_1(f, g; s)(\tau, n) := \frac{\langle n \rangle^s}{\langle \tau + n^2 \rangle^{1/2}} \sum_{n_1 \in \mathbb{Z}} \int_{\mathbb{R}} \frac{f(\tau_1, n_1)}{\langle n_1 \rangle^{s-1/2} \langle \tau_1 + |n_1|n_1 \rangle^{1/2}} \times \frac{g(\tau - \tau_1, n - n_1)}{\langle n - n_1 \rangle^s \langle \tau - \tau_1 + (n - n_1)^2 \rangle^{1/2}} d\tau_1. \tag{3.30}$$

Fix a positive integer N, such that $N \gg 1$, and define

$$f_N(\tau, n) = a_n \chi_{1/2}(\tau + |n|n) \quad \text{with} \quad a_n = \begin{cases} 1, & n = N, \\ 0, & \text{elsewhere}, \end{cases} \tag{3.31}$$

and

$$g_N(\tau, n) = b_n \chi_{1/2}(\tau + n^2) \quad \text{with} \quad b_n = \begin{cases} 1, & n = 0, \\ 0, & \text{elsewhere}, \end{cases} \tag{3.32}$$

where χ_r is the characteristic function of the interval $[-r, r]$. Hence,

$$\|f_N\|_{L^2_\tau l^2_n} \sim \|g_N\|_{L^2_\tau l^2_n} \sim 1, \tag{3.33}$$

$$a_{n_1}b_{n-n_1} \neq 0 \quad \text{if and only if} \quad n_1 = N \text{ and } n = N,$$

and

$$\int_{\mathbb{R}} \chi_{1/2}(\tau_1 + N^2)\chi_{1/2}(\tau - \tau_1)d\tau_1 \sim \chi_1(\tau + N^2).$$

Therefore, we deduce from (3.30) that

$$\|B_1(f_N, g_N; s)\|_{L^2_\tau l^2_n} \gtrsim N^{1/2}, \quad \forall\, N \gg 1, \tag{3.34}$$

which combined with (3.33) contradicts (3.29). The case $\gamma = -1$ is similar. □

4 Existence of Periodic Travelling-Wave Solutions

The idea in this section is to show the existence of a smooth branch of periodic travelling-wave solutions for (1.5). Initially we show a novel smooth branch of dnoidal waves solutions for (1.5) in the case $\gamma = 0$. So, by using the implicit function theorem we show a smooth curve in the case $\gamma \neq 0$ of periodic solutions bifurcating of the dnoidal waves.

4.1 *Dnoidal Waves Solutions*

We start finding solutions for the case $\gamma = 0$ and $\sigma > 0$ in (1.5). Henceforth, without loss of generality, we will assume that $\alpha = 1$ and $\beta = \frac{1}{2}$. Hence, we need to solve the system

$$\begin{cases} \phi_0'' - \sigma\phi_0 = \psi_0\phi_0 \\ \psi_0 = -\frac{1}{2c}\phi_0^2. \end{cases} \tag{4.1}$$

Then by replacing the second equation of (4.1) into the first one we obtain that ϕ_0 satisfies

$$\phi_0'' - \sigma\phi_0 + \frac{1}{2c}\phi_0^3 = 0. \tag{4.2}$$

Equation (4.2) can be solved in a similar fashion to the method used by Angulo in [5] to find periodic travelling-wave solutions to the nonlinear Schrödinger equation (1.3) . For the sake of completeness we provide here a sketch of the proof in an adapted manner. Indeed, from (4.2) ϕ_0 must satisfy the first-order equation

$$[\phi_0']^2 = \frac{1}{4c}[-\phi_0^4 + 4c\sigma\phi_0^2 + 4cB_{\phi_0}] = \frac{1}{4c}(\eta_1^2 - \phi_0^2)(\phi_0^2 - \eta_2^2),$$

where B_{ϕ_0} is an integration constant and $-\eta_1, \eta_1, -\eta_2, \eta_2$ are the zeros of the polynomial $F(t) = -t^4 + 4c\sigma t^2 + 4cB_{\phi_0}$. Moreover,

$$\begin{cases} 4c\sigma = \eta_1^2 + \eta_2^2 \\ 4cB_{\phi_0} = -\eta_1^2\eta_2^2. \end{cases} \tag{4.3}$$

We suppose, without loss of generality, that $\eta_1 > \eta_2 > 0$. Then $\eta_2 \leqq \phi_0 \leqq \eta_1$ and so ϕ_0 will be a positive solution. Note that $-\phi_0$ is also a solution of (4.2). Next, define $\zeta = \phi_0/\eta_1$ and $k^2 = (\eta_1^2 - \eta_2^2)/\eta_1^2$. It follows from (4.3) that

$$[\zeta']^2 = \tfrac{\eta_1^2}{4c}(1-\zeta^2)(\zeta^2 + k^2 - 1).$$

Let us now define a new function χ through $\zeta^2 = 1 - k^2 \sin^2 \chi$. So we get that $4c(\chi')^2 = \eta_1^2(1 - k^2 \sin^2 \chi)$. Then for $l = \frac{\eta_1}{2\sqrt{c}}$, and assuming that $\zeta(0) = 1$, we have

$$\int_0^{\chi(\xi)} \frac{dt}{\sqrt{1 - k^2 sin^2 t}} = l\ \xi.$$

Then from the definition of the *Jacobian elliptic function* $sn(u;k)$, we have that $\sin\chi = sn(l\xi;k)$ and hence $\zeta(\xi) = \sqrt{1 - k^2 sn^2(l\xi;k)} \equiv dn(l\xi;k)$. Then by returning to the variable ϕ_0, we obtain the novel **dnoidal waves** solutions associated to equation (4.1),

$$\begin{cases} \phi_0(\xi) \equiv \phi_0(\xi;\eta_1,\eta_2) = \eta_1\ dn\Big(\frac{\eta_1}{2\sqrt{c}}\ \xi;k\Big) \\ \psi_0(\xi) \equiv \psi_0(\xi;\eta_1,\eta_2) = -\frac{\eta_1^2}{2c}\ dn^2\Big(\frac{\eta_1}{2\sqrt{c}}\ \xi;k\Big), \end{cases} \tag{4.4}$$

where

$$0 < \eta_2 < \eta_1, \qquad k^2 = \frac{\eta_1^2 - \eta_2^2}{\eta_1^2}, \qquad \eta_1^2 + \eta_2^2 = 4c\sigma. \tag{4.5}$$

Next, since dn has fundamental period $2K(k)$, it follows that ϕ_0 in (4.4) has fundamental period or wavelength T_{ϕ_0} given by

$$T_{\phi_0} \equiv \frac{4\sqrt{c}}{\eta_1}\ K(k).$$

Given $c > 0$ fixed such that $\sigma > 0$, it follows from (4.5) that $0 < \eta_2 < \sqrt{2c\sigma} < \eta_1 < 2\sqrt{c\sigma}$. Moreover we can write

$$T_{\phi_0}(\eta_2) = \frac{4\sqrt{c}}{\sqrt{4c\sigma - \eta_2^2}}\ K(k(\eta_2)) \qquad \text{with} \qquad k^2(\eta_2) = \frac{4c\sigma - 2\eta_2^2}{4c\sigma - \eta_2^2}. \tag{4.6}$$

Then by using these formulas and properties of the function K we see that $T_{\phi_0} \in (\sqrt{\frac{2}{\sigma}}\ \pi, +\infty)$ for $\eta_2 \in (0, \sqrt{2c\sigma})$. Moreover, we will see in Theorem 4.1 below that $\eta_2 \to T_{\phi_0}(\eta_2)$ is a strictly decreasing mapping and so we obtain the basic inequality

$$T_{\phi_0} > \sqrt{\frac{2}{\sigma}}\ \pi. \tag{4.7}$$

Two relevant solutions of (4.1) are hidden in (4.4). Namely, the constants and solitary waves solutions. Indeed, when $\eta_2 \to \sqrt{2c\sigma}$, i.e. $\eta_2 \to \eta_1$, it follows that $k \to 0^+$. Then since $d(u; 0^+) \to 1$ we obtain the constants solutions

$$\phi_0(\xi) = \sqrt{2c\sigma} \qquad \text{and} \qquad \psi_0(\xi) = -\sigma. \tag{4.8}$$

Next, for $\eta_2 \to 0$ we have $\eta_1 \to 4c\sigma^-$ and so $k \to 1^-$. Then since $dn(u; 1^-) \to sechu$ we obtain the classical solitary wave solutions

$$\phi_{0,s}(\xi) = 2\sqrt{c\sigma}sech(\sqrt{\sigma}\xi) \qquad \text{and} \qquad \psi_{0,s}(\xi) = -2\sigma sech^2(\sqrt{\sigma}\xi). \tag{4.9}$$

We recall that the stability of the solutions in (4.9) by the flow of equation (**??**) was shown in [**?**].

Next theorem is the main result of this subsection and proves that for a fixed period $L > 0$ there exists a smooth branch of dnoidal waves solutions with the same period L to the system (4.1) or equivalently to equation (4.2). The construction of a family of dnoidal waves with a fixed period L it is an immediate consequence of the analysis made above. Indeed, let $L > 0$ be a fixed number. Let $c > 0$ and $w \in \mathbb{R}$ be real fixed numbers such that $\sigma \equiv \omega - c^2/4 > 2\pi^2/L^2$. Since the function $\eta_2 \in (0, \sqrt{2c\sigma}) \to T_{\phi_0}(\eta_2)$ is strictly decreasing (see Theorem 4.1 below), there is a unique $\eta_2 = \eta_2(\sigma) \in (0, \sqrt{2c\sigma})$ such that $\phi_0(\cdot; \eta_1(\sigma), \eta_2(\sigma))$ has fundamental period $T_{\phi_0}(\eta_2(\sigma)) = L$. Next we show that the choice of $\eta_2(\sigma)$ depends smoothly of σ.

Theorem 4.1. *Let L and c be fixed but arbitrary positive numbers. Let $\sigma_0 > 2\pi^2/L^2$ and $\eta_{2,0} = \eta_2(\sigma_0)$ be the unique number in the interval $(0, \sqrt{2c\sigma})$ such that $T_{\phi_0}(\eta_{2,0}) = L$. Then,*

(1) *there are interval $I(\sigma_0)$ and $B(\eta_{2,0})$ around of σ_0 and $\eta_2(\sigma_0)$ respectively, and an unique smooth function $\Lambda : I(\sigma_0) \to B(\eta_{2,0})$, such that $\Lambda(\sigma_0) = \eta_{2,0}$ and*

$$\frac{4\sqrt{c}}{\sqrt{4c\sigma - \eta_2^2}} K(k(\sigma)) = L, \tag{4.10}$$

where $\sigma \in I(\sigma_0)$, $\eta_2 = \Lambda(\sigma)$, and

$$k^2 \equiv k^2(\sigma) = \frac{4c\sigma - 2\eta_2^2}{4c\sigma - \eta_2^2} \in (0, 1). \tag{4.11}$$

(2) *Solutions $(\phi_0(\cdot; \eta_1, \eta_2), \psi_0(\cdot; \eta_1, \eta_2))$ given by (4.4) and determined by $\eta_1 = \eta_1(\sigma)$, $\eta_2 = \eta_2(\sigma) = \Lambda(\sigma)$, with $\eta_1^2 + \eta_2^2 = 4c\sigma$, have fundamental period L and satisfy (4.1). Moreover, the mapping*

$$\sigma \in I(\sigma_0) \to \phi_0(\cdot; \eta_1(\sigma), \eta_2(\sigma)) \in H^n_{per}([0, L])$$

is a smooth function.

(3) *$I(\sigma_0)$ can be chosen as $(\frac{2\pi^2}{L^2}, +\infty)$.*

(4) *The mapping $\Lambda : I(\sigma_0) \to B(\eta_{2,0})$ is a strictly decreasing function. Therefore from (4.11), $\sigma \to k(\sigma)$ is a strictly increasing function.*

Proof. The key of the proof is to apply the implicit function theorem. In fact, it considers the open set $\Omega = \{(\eta,\sigma) :\ \sigma > \frac{2\pi^2}{L^2},\ \eta \in (0, \sqrt{2c\sigma})\ \} \subseteq \mathbb{R}^2$ and define $\Psi : \Omega \to \mathbb{R}$ by

$$\Psi(\eta,\sigma) = \frac{4\sqrt{c}}{\sqrt{4c\sigma - \eta^2}}\ K(k(\eta,\sigma)) \tag{4.12}$$

where $k^2(\eta,\sigma) = \frac{4c\sigma - 2\eta^2}{4c\sigma-\eta^2}$. By hypotheses $\Psi(\eta_{2,0}, \sigma_0) = L$. Next, we show $\partial_\eta \Psi(\eta,\sigma) < 0$. In fact, it is immediate that

$$\partial_\eta \Psi(\eta,\sigma) = \frac{4\sqrt{c}\,\eta}{(4c\sigma - \eta^2)^{3/2}}\ K(k) + \frac{4\sqrt{c}}{\sqrt{4c\sigma-\eta^2}} \frac{dK}{dk}\frac{dk}{d\eta}.$$

Next, from

$$\frac{dk}{d\eta} = -\frac{4c\sigma\eta}{k(4c\sigma-\eta^2)^2},$$

and from the relations (see [18])

$$\begin{cases} \frac{dE}{dk} = \frac{E-K}{k}, & \frac{d^2E}{dk^2} = -\frac{1}{k}\frac{dK}{dk}, \\ kk'^2\frac{d^2E}{dk^2} + k'^2\frac{dE}{dk} + kE = 0, \end{cases} \tag{4.13}$$

with $k'^2 = 1 - k^2$, and $E = E(k)$ being the complete elliptic integral of second kind defined as

$$E(k) = \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\ dt,$$

we have the following formal equivalences

$$\begin{aligned} \partial_\eta\Psi(\eta,\sigma) < 0 &\Leftrightarrow k(4c\sigma-\eta^2)\Big(E - k\tfrac{dE}{dk}\Big) < -4c\sigma k\tfrac{d^2E}{dk^2} \\ &\Leftrightarrow k(4c\sigma-\eta^2)\Big(E - k\tfrac{dE}{dk}\Big) < \Big(\tfrac{dE}{dk} + \tfrac{k}{k'^2}E\Big)(4c\sigma-\eta^2)(2-k^2) \\ &\Leftrightarrow 2k'^2\tfrac{dE}{dk} + kE > 0 \Leftrightarrow \tfrac{dE}{dk} - k\tfrac{d^2E}{dk^2} > 0 \Leftrightarrow \tfrac{dE}{dk} + \tfrac{dK}{dk} > 0. \end{aligned}$$

So, since $E + K$ is a strictly increasing function, we obtain our affirmation.

Therefore, there is a unique smooth function, Λ, defined in a neighborhood $I(\sigma_0)$ of σ_0, such that $\Psi(\Lambda(\sigma),\sigma) = L$ for every $\sigma \in I(\sigma_0)$. So, we obtain (4.10). Moreover, since σ_0 was chosen arbitrarily in $\mathcal{I} = (\frac{2\pi^2}{L^2}, +\infty)$, it follows from the uniqueness of the function Λ that it can be extended to $\mathcal{I}$.

Next we show that Λ is a strictly decreasing function. We know that $\Psi(\Lambda(\sigma),\sigma) = L$ for every $\sigma \in I(\sigma_0)$, then

$$\frac{d}{d\sigma}\Lambda(\sigma) = -\frac{\partial\Psi/\partial\sigma}{\partial\Psi/\partial\eta} < 0 \Leftrightarrow \partial\Psi/\partial\sigma < 0.$$

So, by using the relation $\eta^2 = (4c\sigma - \eta^2)(1-k^2) \equiv (4c\sigma - \eta^2)k'^2$, we obtain the following formal equivalences

$$\frac{\partial\Psi}{\partial\sigma} < 0 \Leftrightarrow (4c\sigma - \eta^2)K > \frac{\eta^2}{k}\frac{dK}{dk} \Leftrightarrow K > \frac{k'^2}{k}\frac{dK}{dk}.$$

Then, since $\frac{dK}{dk} = (E - k'^2K)/(kk'^2)$, it follows that

$$\frac{\partial\Psi}{\partial\sigma} < 0 \Leftrightarrow k^2K > E - k'^2K \Leftrightarrow (k^2 + k'^2)K > E \Leftrightarrow K > E.$$

This completes the Theorem. □

Remark 4.1. In the case that the polynomial $F(t) = -t^4 + 4c\sigma t^2 + 4cB_{\phi_0}$ has a pure imaginary root and the other two roots are real we can show the existence of two smooth curves of periodic solutions for (4.2) of cnoidal type, more precisely we have

$$1- \qquad \omega \in (0,+\infty) \to b\, cn\Big(\sqrt{b^2-\omega}\;\xi; k\Big) \in H^1_{per}([0,L])$$
$$2- \; \omega \in \Big(-\tfrac{4\pi^2}{L^2}, 0\Big) \to \sqrt{a^2+2\omega}\; cn\Big(\sqrt{a^2+\omega}\;\xi; k\Big) \in H^1_{per}([0,L]),$$

where a, b, k are smooth functions of ω.

The following result will be used in our stability theory.

Corollary 4.1. *Let L and c be fixed but arbitrary positive numbers. It considers the smooth curve of dnoidal waves $\sigma \in (\frac{2\pi^2}{L^2}, \infty) \to \phi_0(\cdot;\eta_1(\sigma),\eta_2(\sigma))$ determined by Theorem 4.1. Then*

$$\frac{d}{d\sigma}\int_0^L \phi_0^2(\xi)\, d\xi > 0.$$

Proof. By (4.4), (4.10), and the formula $\int_0^{K(k)} dn^2(x;k)\, dx = E(k)$ (see page 194 in [18]) it follows that

$$\begin{aligned}\int_0^L \phi_0^2(\xi)\, d\xi = 2\eta_1\sqrt{c}\int_0^{2K(k)} dn^2(x;k)\, dx &= \tfrac{16cK}{L}\int_0^{K(k)} dn^2(x;k)\, dx \\ &= \tfrac{16c}{L}E(k)K(k).\end{aligned}$$

So, since $k \to K(k)E(k)$ and $\sigma \to k(\sigma)$ are strictly increasing functions we have that

$$\frac{d}{d\sigma}\int_0^L \phi_0^2(\xi)\, d\xi = \frac{16c}{L}\frac{d}{dk}[K(k)E(k)]\frac{dk}{d\sigma} > 0.$$

This finishes the Corollary. $\square$

4.2 *Periodic Travelling Waves Solutions for Eq. (1.5)*

In this subsection we show the existence of a branch of periodic travelling waves solutions of (1.5) for γ close to zero, such that these solutions will bifurcate of the dnoidal waves solutions found in Theorem 4.1.

We start our analysis by studying the periodic eigenvalue problem considered on $[0, L]$,

$$\begin{cases} \mathcal{L}_0\chi \equiv (-\frac{d^2}{dx^2} + \sigma - \frac{3}{2c}\phi_0^2)\chi = \lambda\chi \\ \chi(0) = \chi(L), \quad \chi'(0) = \chi'(L), \end{cases} \tag{4.14}$$

where for $\sigma > 2\pi^2/L^2$, ϕ_0 is given by Theorem 4.1 and satisfies (2.2).

Theorem 4.2. *The linear operator $\mathcal{L}_0$ defined in (4.14) with domain $H^2_{per}([0, L]) \subseteq L^2_{per}([0, L])$, has its first three eigenvalues simple, being the eigenvalue zero the second one with eigenfunction $\frac{d}{dx}\phi_0$. Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues which are double and converging to infinity.*

Theorem 4.2 is a consequence of the Floquet theory (Magnus&Winkler [30]). By convenience of the readers we will give some basic results of this theory. From the classical theory of compact symmetric linear operator we have that problem (4.14) determines a countable infinity set of eigenvalues $\{\lambda_n | n = 0, 1, 2, ...\}$ with $\lambda_0 \leqq \lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \lambda_4 \leqq ...$, where double eigenvalue is counted twice and $\lambda_n \to \infty$ as $n \to \infty$. We shall denote by χ_n the eigenfunction associated to the eigenvalue λ_n. By the conditions $\chi_n(0) = \chi_n(L)$, $\chi_n'(0) = \chi_n'(L)$, χ_n can be extended to the whole of $(-\infty, \infty)$ as a continuously differentiable function with period L.

We know that with the *periodic eigenvalue problem* (4.14) there is an associated semi-periodic eigenvalue problem in $[0, L]$, namely,

$$\begin{cases} \mathcal{L}_0\xi = \mu\xi \\ \xi(0) = -\xi(L), \quad \xi'(0) = -\xi'(L). \end{cases} \tag{4.15}$$

As in the periodic case, there is a countable infinity set of eigenvalues $\{\mu_n | n = 0, 1, 2, 3, ...\}$, with $\mu_0 \leqq \mu_1 \leqq \mu_2 \leqq \mu_3 \leqq \mu_4 \leqq ...$, where double eigenvalue is counted twice and $\mu_n \to \infty$ as $n \to \infty$. We shall denote by ξ_n the eigenfunction associated to the eigenvalue μ_n. So, we have that the equation

$$\mathcal{L}_0 f = \gamma f \tag{4.16}$$

has a solution of period L if and only if $\gamma = \lambda_n$, $n = 0, 1, 2, \cdots$, whilst the only periodic solutions of period $2L$ are either those associated with $\gamma = \lambda_n$, but viewed on $[0, 2L]$, or those corresponding to $\gamma = \mu_n$, but extended in form $\xi_n(L+x) = \xi_n(L-x)$ for $0 \leqq x \leqq L$, $n = 0, 1, 2, \cdots$. If all solutions of (4.16) are bounded we say that they are *stable*; otherwise we say that they are *unstable*. From the Oscillation Theorem of the Floquet theory (see [30]) we have that

$$\lambda_0 < \mu_0 \leqq \mu_1 < \lambda_1 \leqq \lambda_2 < \mu_2 \leqq \mu_3 < \lambda_3 \leqq \lambda_4 \cdots. \tag{4.17}$$

The intervals $(\lambda_0, \mu_0), (\mu_1, \lambda_1), \cdots$, are called *intervals of stability*. At the endpoints of these intervals the solutions of (4.16) are in general unstable. This is always true for $\gamma = \lambda_0$ (λ_0 is always simple). The intervals, $(-\infty, \lambda_0), (\mu_0, \mu_1), (\lambda_1, \lambda_2), (\mu_2, \mu_3), \cdots$, are called *intervals of instability*, omitting however any interval which is absent as a result of having a double eigenvalue. The interval of instability $(-\infty, \lambda_0)$ will always be present. We note that *the absence of an instability interval means that there is a value of γ for which all solutions of (4.16) have either period L or semi-period L, in other words, coexistence of solutions of (4.16) with period L or period $2L$ occurs for that value of γ.*

Proof of Theorem 4.2. From (4.17) we have that $\lambda_0 < \lambda_1 \leqq \lambda_2$. Since $\mathcal{L}_0 \frac{d}{dx}\phi_0 = 0$ and $\frac{d}{dx}\phi_0$ has 2 zeros in $[0, L)$, it follows that 0 is either λ_1 or λ_2. We will show that $0 = \lambda_1 < \lambda_2$. We consider $\Psi(x) \equiv \chi(\gamma x)$ with $\gamma^2 = 4c/\eta_1^2$. Then from (4.14) and from the identity $k^2 sn^2 x + dn^2 x = 1$, we obtain

$$\begin{cases} \frac{d^2}{dx^2}\Psi + [\rho - 6k^2 sn^2(x;k)]\Psi = 0 \\ \Psi(0) = \Psi(2K(k)), \quad \Psi'(0) = \Psi'(2K(k)), \end{cases} \tag{4.18}$$

where

$$\rho = \frac{4c(\lambda - \sigma)}{\eta_1^2} + 6. \tag{4.19}$$

Now, from Floquet theory it follows that $(-\infty, \rho_0), (\mu_0', \mu_1')$ and (ρ_1, ρ_2) are the instability intervals associated to this Lamé's equation, where for $i \geqq 0$, μ_i' are the eigenvalues associated to the semi-periodic problem. Therefore, ρ_0, ρ_1, ρ_2 are simple eigenvalues for (4.18) and the rest of eigenvalues $\rho_3 \leqq \rho_4 < \rho_5 \leqq \rho_6 < \cdots$, satisfy that $\rho_3 = \rho_4, \rho_5 = \rho_6, \cdots$, in other words, they are double eigenvalues.

It is easy to verify that the first three eigenvalues ρ_0, ρ_1, ρ_2 and its corresponding eigenfunctions Ψ_0, Ψ_1, Ψ_2 are given by the formulas

$$\begin{cases} \rho_0 = 2[1 + k^2 - \sqrt{1 - k^2 + k^4}], \quad \Psi_0(x) = 1 - (1 + k^2 - \sqrt{1 - k^2 + k^4}\,)sn^2(x), \\ \rho_1 = 4 + k^2, \quad \Psi_1(x) = snx\, cnx, \\ \rho_2 = 2[1 + k^2 + \sqrt{1 - k^2 + k^4}], \quad \Psi_2(x) = 1 - (1 + k^2 + \sqrt{1 - k^2 + k^4}\,)sn^2(x). \end{cases} \tag{4.20}$$

Next, Ψ_0 has no zeros in $[0, 2K]$ and Ψ_2 has exactly 2 zeros in $[0, 2K)$, then ρ_0 is the first eigenvalue to (4.18). Since $\rho_0 < \rho_1$ for every $k^2 \in (0, 1)$, we obtain from (4.19) and (4.5) that

$$4c\lambda_0 = \eta_1^2(k^2 - 2 - 2\sqrt{1-k^2+k^4}) < 0 \Leftrightarrow \rho_0 < \rho_1.$$

Therefore λ_0 is the first negative eigenvalue to $\mathcal{L}_0$ with eigenfunction $\chi_0(x) = \Psi_0(x/\gamma)$. Similarly, since $\rho_1 < \rho_2$ for every $k^2 \in (0,1)$, we obtain from (4.19) that

$$4c\lambda_2 = \eta_1^2(k^2 - 2 + 2\sqrt{1-k^2+k^4}) > 0 \Leftrightarrow \rho_1 < \rho_2.$$

Hence λ_2 is the third eigenvalue to $\mathcal{L}_0$ with eigenfunction $\chi_2(x) = \Psi_2(x/\gamma)$. Finally, since $\chi_1(x) = \Psi_1(x/\gamma) = \beta\frac{d}{dx}\phi_0(x)$ we finish the proof. □

Next we have our theorem of existence of solutions for (1.5). For $s \geqq 0$, let $H^s_{per,e}([0,L])$ denote the closed subspace of all even functions in $H^s_{per}([0,L])$.

Theorem 4.3. *Let $L, \alpha, \beta, c > 0$ and $\sigma > 2\pi^2/L^2$ fixed numbers. Then there exist $\gamma_1 > 0$ and a smooth branch*

$$\gamma \in (-\gamma_1, \gamma_1) \to (\phi_\gamma, \psi_\gamma) \in H^2_{per,e}([0,L]) \times H^1_{per,e}([0,L])$$

of solutions for Eq. (1.5). In particular, for $\gamma \to 0$, $(\phi_\gamma, \psi_\gamma)$ converges to (ϕ_0, ψ_0) uniformly for $x \in [0,L]$, where (ϕ_0, ψ_0) is given by Theorem 4.1 and is defined as in (4.4). Moreover, the mapping

$$\gamma \in (-\gamma_1, \gamma_1) \to (\frac{d}{d\sigma}\phi_\gamma, \frac{d}{d\sigma}\psi_\gamma)$$

is continuous.

Proof. Without loss of generality we take $\alpha = 1$ and $\beta = 1/2$. Let $X_e = H^2_{per,e}([0,L]) \times H^1_{per,e}([0,L])$ and define the map

$$G : \mathbb{R} \times (0, +\infty) \times X_e \to L^2_{per,e}([0,L]) \times L^2_{per,e}([0,L])$$

by

$$G(\gamma, \lambda, \phi, \psi) = (-\phi'' + \lambda\phi + \phi\psi, -\gamma D\psi + c\psi + \phi^2).$$

A calculation shows that the Fréchet derivative $G_{(\phi,\psi)} = \partial G(\gamma,\lambda,\phi,\psi)/\partial(\phi,\psi)$ exists and is defined as a map from $\mathbb{R} \times (0,+\infty) \times X_e$ to $B(X_e; L^2_{per,e}([0,L]) \times L^2_{per,e}([0,L])$ by

$$G_{(\phi,\psi)}(\gamma,\lambda,\phi,\psi) = \begin{pmatrix} -\frac{d^2}{dx^2} + \lambda + \psi & \phi \\ \phi & -\gamma D + c \end{pmatrix}.$$

From Theorem 4.1 it follows that for $\Phi_0 = (\phi_0, \psi_0)$, $G(0, \sigma, \Phi_0) = \vec{0}^t$. Moreover, from Theorem 4.2 we have that $G_{(\phi,\psi)}(0,\sigma,\Phi_0)$ has a kernel generated by ${\Phi'_0}^t$. Next, since $\Phi'_0 \notin X_e$, it follows that $G_{(\phi,\psi)}(0,\sigma,\Phi_0)$ is invertible. Hence, since G and $G_{(\phi,\psi)}$ are smooth maps on their domains we have from the Implicit Function Theorem that there are $\gamma_1 > 0$, $\lambda_1 \in (0,\sigma)$, and a smooth curve

$$(\gamma, \lambda) \in (-\gamma_1, \gamma_1) \times (\sigma - \lambda_1, \sigma + \lambda_1) \to (\phi_{\gamma,\lambda}, \psi_{\gamma,\lambda}) \in X_e$$

such that $G(\gamma,\lambda,\phi_{\gamma,\lambda},\psi_{\gamma,\lambda}) = 0$. Then, for $\lambda = \sigma$ we obtain a smooth branch $\gamma \in (-\gamma_1,\gamma_1) \to (\phi_{\gamma,\sigma},\psi_{\gamma,\sigma}) \equiv (\phi_\gamma,\psi_\gamma)$ of solutions of Eq. (1.5) such that $\gamma \in (-\gamma_1,\gamma_1) \to (\frac{d}{d\sigma}\phi_\gamma, \frac{d}{d\sigma}\psi_\gamma)$ is continuous. This shows the Theorem. □

Remark 4.2. Since ϕ_0 is strictly positive on $[0,L]$ and $\phi_\gamma \to \phi_0$, as $\gamma \to 0$, uniformly in $[0,L]$, we have that for γ near zero $\phi_\gamma(x) > 0$ for $x \in \mathbb{R}$. Moreover, since the linear operator $-\gamma D + c$ is a strictly positive operator from $H^1_{per}([0,L])$ to $L^2_{per}([0,L])$ for γ negative, we have that $\psi_\gamma(x) < 0$ for all $x \in \mathbb{R}$.

5 Stability of Periodic Travelling-Wave Solutions

We start this section by defining the type of stability which we are interested. For any $c \in \mathbb{R}^+$ define the functions $\Phi(\xi) = e^{ic\xi/2}\phi(\xi)$ and $\Psi(\xi) = \psi(\xi)$, where (ϕ,ψ) is a solution of (1.5). Then we say that the orbit generated by (Φ,Ψ), namely,

$$\Omega_{(\Phi,\Psi)} = \{(e^{i\theta}\Phi(\cdot + x_0), \Psi(\cdot + x_0)) \ : (\theta, x_0) \in [0,2\pi) \times \mathbb{R}\},$$

is stable in $H^1_{per}([0,L]) \times H^{\frac{1}{2}}_{per}([0,L])$ by the flow generated by Eq. (1.1), if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for (u_0,v_0) satisfying $\|u_0 - \Phi\|_1 < \delta$ and $\|v_0 - \Psi\|_{\frac{1}{2}} < \delta$, we have that (u,v) solution of (1.1) with $(u(0),v(0)) = (u_0,v_0)$, satisfies that $(u,v) \in C(\mathbb{R}; H^1_{per}([0,L]) \times C(\mathbb{R}; H^{\frac{1}{2}}_{per}([0,L])$ and

$$\inf_{x_0\in\mathbb{R},\theta\in[0,2\pi)} \|e^{i\theta}u(\cdot + x_0,t) - \Phi\|_1 < \epsilon, \quad \inf_{x_0\in\mathbb{R}} \|v(\cdot + x_0,t) - \Psi\|_{\frac{1}{2}} < \epsilon, \tag{5.1}$$

for all $t \in \mathbb{R}$.

The main result to be proved in this section is that the periodic travelling waves solutions of (1.1) determined by Theorem 4.3 are stable for $\sigma > 2\pi^2/L^2$ and γ negative and near 0.

Theorem 5.1. *Let $L,\alpha,\beta,c > 0$ and $\sigma > 2\pi^2/L^2$ fixed numbers. We consider the smooth curve of periodic travelling waves solutions for (1.5), $\gamma \to (\phi_\gamma,\psi_\gamma)$, determined by Theorem 4.3. Then there exists $\gamma_0 > 0$ such that for each $\gamma \in (-\gamma_0, 0)$, the orbit generated by $(\Phi_\gamma(\xi),\Psi_\gamma(\xi))$ with*

$$\Phi_\gamma(\xi) = e^{ic\xi/2}\phi_\gamma(\xi) \ \ \text{and} \ \ \Psi_\gamma(\xi) = \psi_\gamma(\xi),$$

is orbitally stable in $H^1_{per}([0,L]) \times H^{\frac{1}{2}}_{per}([0,L])$.

The proof of Theorem 5.1 is based in the ideas developed by Benjamin ([13]) and Weinstein ([36]) which given us an easy form of manipulating with the required spectral information and the positivity property of the quantity $\frac{d}{d\sigma}\int \phi^2_\gamma(x)dx$, which are basic in our stability theory. We do not use the abstract stability theory of Grillakis *et al.* basically by these circumstance. So, it considers $(\phi_\gamma, \psi_\gamma)$ a solution of (1.5) obtained in Theorem 4.3. For $(u_0, v_0) \in H^1_{per}([0,L]) \times H^{\frac{1}{2}}_{per}([0,L])$ and (u,v) the global solution to (1.1) corresponding to these initial data given by theorem 3.3, we define for $t \geqq 0$ and $\sigma > 2\pi^2/L^2$

$$\Omega_t(x_0,\theta) = \|e^{i\theta}(T_c u)'(\cdot + x_0, t) - \phi'_\gamma\|^2 + \sigma\|e^{i\theta}(T_c)u(\cdot + x_0, t) - \phi_\gamma\|^2 \tag{5.2}$$

where we denote by T_c the bounded linear operator defined by

$$(T_c u)(x,t) = e^{-ic(x-ct)/2}u(x,t).$$

Then, the deviation of the solution $u(t)$ from the orbit generated by Φ is measured by

$$\rho_\sigma(u(\cdot,t),\phi_\gamma)^2 \equiv \inf_{x_0\in[0,L],\theta\in[0,2\pi]} \Omega_t(x_0,\theta). \tag{5.3}$$

Hence, from (5.3) we have that the $\inf \Omega_t(x_0,\theta)$ is attained in $(\theta, x_0) = (\theta(t), x_0(t))$.

Proof of Theorem 5.1. It considers the perturbation of the periodic travelling wave $(\phi_\gamma, \psi_\gamma)$

$$\begin{cases} \xi(x,t) = e^{i\theta}(T_c u)(x + x_0, t) - \phi_\gamma(x) \\ \eta(x,t) = v(x + x_0, t) - \psi_\gamma(x). \end{cases} \tag{5.4}$$

Hence by the property of minimum of $(\theta, x_0) = (\theta(t), x_0(t))$ we obtain from (5.4) that $p(x,t) = \mathrm{Re}(\xi(x,t))$ and $q(x,t) = \mathrm{Im}(\xi(x,t))$ satisfy the compatibility relations

$$\begin{cases} \int_0^L q(x,t)\phi_\gamma(x)\psi_\gamma(x)\, dx = 0 \\ \int_0^L p(x,t)(\phi_\gamma(x)\psi_\gamma(x))'\, dx = 0. \end{cases} \tag{5.5}$$

Now we consider the continuous functional L defined on $H^1_{per}([0,L]) \times H^{\frac{1}{2}}_{per}([0,L])$ by

$$L(u,v) = E(u,v) + c\, G(u,v) + \omega\, H(u,v),$$

where E, G, H are defined by (1.2). Then from (5.4) and (1.5) we have

$$\begin{aligned} \Delta L(t) &:= L(u(t),v(t)) - L(\Phi_\gamma,\Psi_\gamma) = L(\Phi_\gamma + e^{icx/2}\xi, \psi_\gamma + \eta) - L(\Phi_\gamma,\psi_\gamma) \\ &=< \mathcal{L}_\gamma p, p > + < \mathcal{L}^+_\gamma q, q > + \tfrac{\alpha}{2\beta}\int_0^L \left[\mathcal{K}_\gamma^{1/2}\eta + 2\beta\mathcal{K}_\gamma^{-1/2}(\phi_\gamma p) + \beta\mathcal{K}_\gamma^{-1/2}(p^2+q^2)\right]^2 dx \\ &- \tfrac{\alpha\beta}{2}\int_0^L \left[|\mathcal{K}_\gamma^{-1/2}(p^2+q^2)|^2 + 4\mathcal{K}_\gamma^{-1/2}(\phi_\gamma p)\mathcal{K}_\gamma^{-1/2}(p^2+q^2)\right] dx, \end{aligned} \tag{5.6}$$

where, for $\gamma < 0$ we define $\mathcal{K}_\gamma^{-1}$ as

$$\widehat{\mathcal{K}_\gamma^{-1} f}(k) = \frac{1}{-\gamma|k| + c}\widehat{f}(k) \qquad \text{for } k \in \mathbb{Z},$$

which is the inverse operator of $\mathcal{K}_\gamma : H^s_{per}([0, L]) \to H^{s-1}_{per}([0, L])$ defined by $\mathcal{K}_\gamma = -\gamma D + c$. The operator $\mathcal{L}_\gamma$ is

$$\mathcal{L}_\gamma = -\frac{d^2}{d\xi^2} + \sigma + \alpha\psi_\gamma - 2\alpha\beta\phi_\gamma \circ \mathcal{K}_\gamma^{-1} \circ \phi_\gamma, \tag{5.7}$$

with $\phi_\gamma \circ \mathcal{K}_\gamma^{-1} \circ \phi_\gamma$ given by $[\phi_\gamma \circ \mathcal{K}_\gamma^{-1} \circ \phi_\gamma](f) = \phi_\gamma \mathcal{K}_\gamma^{-1}(\phi_\gamma f)$. $\mathcal{L}_\gamma^+$ is given by

$$\mathcal{L}_\gamma^+ = -\frac{d^2}{d\xi^2} + \sigma + \alpha\psi_\gamma \tag{5.8}$$

and $\mathcal{K}_\gamma^{1/2}$, $\mathcal{K}_\gamma^{-1/2}$ are the positive roots of $\mathcal{K}_\gamma$ and $\mathcal{K}_\gamma^{-1}$ respectively.

Now, we need to find a lower bound for $\Delta L(t)$. The first step will be to obtain a suitable lower bound of the last term on the right-hand side of (5.6). In fact, since $\mathcal{K}_\gamma^{-1/2}$ is a bounded operator on $L^2_{per}([0, L])$, ϕ_γ is uniformly bounded, and from the continuous embedding of $H^1_{per}([0, L])$ in $L^4_{per}([0, L])$ and in $L^\infty([0, L])$, we have that

$$-\frac{\alpha\beta}{2}\int_0^L \Big[|\mathcal{K}_\gamma^{-1/2}(p^2 + q^2)|^2 + 4\mathcal{K}_\gamma^{-1/2}(\phi_\gamma p)\mathcal{K}_\gamma^{-1/2}(p^2 + q^2)\Big] dx \geq -C_1\|\xi\|_1^3 - C_2\|\xi\|_1^4 \tag{5.9}$$

where C_1 and C_2 are positive constants.

The estimates for $< \mathcal{L}_\gamma p, p >$ and $< \mathcal{L}_\gamma^+ q, q >$ will be obtained from the following theorem.

Theorem 5.2. *Let $L, \alpha, \beta, c > 0$ and $\sigma > 2\pi^2/L^2$ fixed numbers. Then there exists $\gamma_2 > 0$ such that if $\gamma \in (-\gamma_2, 0)$ then the self-adjoint operators $\mathcal{L}_\gamma$ and $\mathcal{L}_\gamma^+$ defined in (5.7) and (5.8), respectively, with domain $H^2_{per}([0, L])$ have the following properties:*

(1) *$\mathcal{L}_\gamma$ has a simple negative eigenvalue λ_γ with eigenfunction φ_γ and $\int_0^L \phi_\gamma\varphi_\gamma dx \neq 0$.*

(2) *$\mathcal{L}_\gamma$ has a simple eigenvalue at zero with eigenfunction $\frac{d}{dx}\phi_\gamma$.*

(3) *There is $\eta_\gamma > 0$ such that for $\beta_\gamma \in \Sigma(\mathcal{L}_\gamma) - \{\lambda_\gamma, 0\}$, we have that $\beta_\gamma > \eta_\gamma$.*

(4) *$\mathcal{L}_\gamma^+$ is a non-negative operator which has zero as its first eigenvalue with eigenfunction ϕ_γ. The remainder of the spectrum is constituted by a discrete set of eigenvalues.*

Proof. From (1.5) it follows that $\mathcal{L}_\gamma\phi_\gamma = 2\phi_\gamma\psi_\gamma$ and so from Remark 4.2 we have that for $\gamma < 0$, $< \mathcal{L}_\gamma\phi_\gamma, \phi_\gamma >= 2\int_{\mathbb{R}} \phi_\gamma^2\psi_\gamma dx < 0$. Therefore $\mathcal{L}_\gamma$ has a negative eigenvalue.

Moreover, we have that $\mathcal{L}_\gamma \frac{d}{dx}\phi_\gamma = 0$. Next for $f \in H^1_{per}([0,L])$ and $\|f\| = 1$ we have

$$
\begin{aligned}
&< \mathcal{L}_\gamma f, f >=< \mathcal{L}_0 f, f > -\tfrac{\gamma}{c}\alpha^2 < \phi_0 f, D\mathcal{K}_\gamma^{-1}(\phi_0 f) > +\alpha \textstyle\int_0^L (\psi_\gamma - \psi_0) f^2 dx \\
&+\alpha^2 \textstyle\int_0^L [\phi_0 f \mathcal{K}_\gamma^{-1}(\phi_0 f) - \phi_\gamma f \mathcal{K}_\gamma^{-1}(\phi_\gamma f)] dx \qquad (5.10)\\
&\geqq < \mathcal{L}_0 f, f > +\alpha \textstyle\int_0^L (\psi_\gamma - \psi_0) f^2 dx + \alpha^2 \int_0^L [\phi_0 f \mathcal{K}_\gamma^{-1}(\phi_0 f) - \phi_\gamma f \mathcal{K}_\gamma^{-1}(\phi_\gamma f)] dx,
\end{aligned}
$$

where the last inequality is due to that $\gamma < 0$ and $D\mathcal{K}_\gamma^{-1}$ is a positive operator. So, since

$$
\begin{aligned}
&\Big| \textstyle\int_0^L (\psi_\gamma - \psi_0) f^2 dx \Big| \leqq \|\psi_\gamma - \psi_0\|_\infty \\
&\Big| \textstyle\int_0^L [\phi_0 f \mathcal{K}_\gamma^{-1}(\phi_0 f) - \phi_\gamma f \mathcal{K}_\gamma^{-1}(\phi_\gamma f)] dx \Big| \leqq (\|\phi_\gamma\| + \|\phi_0\|)\|\phi_\gamma - \phi_0\|_\infty,
\end{aligned}
\qquad (5.11)
$$

we have from Theorem 4.3 that for γ near 0^- and ϵ small, $< \mathcal{L}_\gamma f, f >\geqq< \mathcal{L}_0 f, f > -\epsilon$. Hence, for $f \perp \chi_0$ and $f \perp \frac{d}{dx}\phi_0$, where $\mathcal{L}_0 \chi_0 = \lambda_0 \chi_0$ with $\lambda_0 < 0$, we have from the spectral structure of $\mathcal{L}_0$ (Theorem 4.2) that $< \mathcal{L}_\gamma f, f >\geqq \eta_\gamma > 0$. Therefore, from min-max principle ([35]) we obtain the desired spectral structure for $\mathcal{L}_\gamma$. Moreover, let φ_γ be such that $\mathcal{L}_\gamma \varphi_\gamma = \lambda_\gamma \varphi_\gamma$ with $\lambda_\gamma < 0$. Therefore, if $\phi_\gamma \perp \varphi_\gamma$ then from the spectral structure of $\mathcal{L}_\gamma$ we must have that $< \mathcal{L}_\gamma \phi_\gamma, \phi_\gamma >\geqq 0$. But we know that $< \mathcal{L}_\gamma \phi_\gamma, \phi_\gamma >< 0$. Hence, $< \phi_\gamma, \varphi_\gamma >\neq 0$. Finally, since $\mathcal{L}_\gamma^+ \phi_\gamma = 0$ with $\phi_\gamma > 0$, it follows that zero is simple and it is the first eigenvalue. The remainder of the spectrum is discrete. ☐

Theorem 5.3. *It considers $\gamma < 0$ and near zero such that Theorem 5.2 is true. Then*

(a) $\inf \{< \mathcal{L}_\gamma f, f > : \ \|f\| = 1, < f, \phi_\gamma >= 0, \ \} \equiv \beta_0 = 0$.

(b) $\inf \{< \mathcal{L}_\gamma f, f > : \ \|f\| = 1, < f, \phi_\gamma >= 0, < f, (\phi_\gamma \psi_\gamma)' >= 0 \ \} \equiv \beta > 0$.

Proof. Part (a). Since $\mathcal{L}_\gamma \frac{d}{dx}\phi_\gamma = 0$ and $< \frac{d}{dx}\phi_\gamma, \phi_\gamma >= 0$ then $\beta_0 \leqq 0$. Next we will show that $\beta_0 \geqq 0$ by using Lemma E.1 in Weinstein [38]. So, we shall show initially that the infimum is attained. Let $\{\psi_j\} \subseteq H^1_{per}([0,L])$ with $\|\psi_j\| = 1$, $< \psi_j, \phi_\gamma >= 0$ and $\lim_{j\to\infty} < \mathcal{L}_\gamma \psi_j, \psi_j >= \beta_0$. Then there is a subsequence of $\{\psi_j\}$, which we denote again by $\{\psi_j\}$, such that $\psi_j \rightharpoonup \psi$ weakly in $H^1_{per}([0,L])$, so $\psi_j \to \psi$ in $L^2_{per}([0,L])$. Hence $\|\psi\| = 1$ and $< \psi, \phi_\gamma >= 0$. Since $\|\psi'\|^2 \leqq \liminf \|\psi'\|^2$ and $\mathcal{K}_\gamma^{-1}(\phi_\gamma \psi_j) \to \mathcal{K}_\gamma^{-1}(\phi_\gamma \psi)$ in $L^2_{per}([0,L])$ we have, $\beta_0 \leqq< \mathcal{L}_\gamma \psi, \psi >\leqq \liminf < \mathcal{L}_\gamma \psi_j, \psi_j >= \beta_0$. Next we show that $< \mathcal{L}_\gamma^{-1} \phi_\gamma, \phi_\gamma >\leqq 0$. From (1.5) and Theorem 4.3 we obtain for $\chi_\gamma = -\frac{d}{d\sigma}\phi_\gamma$ that $\mathcal{L}_\gamma \chi_\gamma = \phi_\gamma$. Moreover, from Corollary 4.1 it follows that $< -\frac{d}{d\sigma}\phi_0, \phi_0 >< 0$ and so for γ small enough $< -\frac{d}{d\sigma}\phi_\gamma, \phi_\gamma >< 0$. Hence from [38] we obtain that $\beta \geqq 0$. This shows part (a).

Part (b). From (a) we have that $\beta \geqq 0$. Suppose $\beta = 0$. Then following a similar analysis to that used in part (a) above, we have that the infimum define in (b) is attained at an admissible function ζ. So from Lagrange's multiplier theory there are λ, θ, η such that

$$\mathcal{L}_\gamma \zeta = \lambda \zeta + \theta \phi_\gamma + \eta (\phi_\gamma \psi_\gamma)'. \tag{5.12}$$

Using (5.12) and $< \mathcal{L}_\gamma \zeta, \zeta >= 0$ we obtain that $\lambda = 0$. Taking the inner product of (5.12) with ϕ'_γ, we have from $\mathcal{L}_\gamma \phi'_\gamma = 0$ that

$$0 = \eta \int_0^L \phi'_\gamma (\phi_\gamma \psi_\gamma)' dx, \tag{5.13}$$

but the integral in (5.13) converges to

$$\int_0^L \phi'_0 (\phi_0 \psi_0)' dx = \frac{-3\beta}{c} \int_0^L \phi_0^2 (\phi'_0)^2 dx < 0$$

as $\gamma \to 0$. Then from (5.13) we obtain $\eta = 0$ and therefore $\mathcal{L}_\gamma \zeta = \theta \phi_\gamma$. So, since $\mathcal{L}_\gamma (-\frac{d}{d\sigma} \phi_\gamma) = \phi_\gamma$ we obtain $0 =< \zeta, \phi_\gamma >= \theta < \phi_\gamma, -\frac{d}{d\sigma} \phi_\gamma >$. Therefore $\theta = 0$ and $\mathcal{L}_\gamma \zeta = 0$. Then $\zeta = \nu \phi'_\gamma$ for some $\nu \neq 0$, which is a contradiction. Then $\beta > 0$ and the proof of the Theorem is completed. □

Theorem 5.4. *It considers $\gamma < 0$ and near zero such that Theorem 5.2 is true. If $\mathcal{L}_\gamma^+$ is defined as in (5.8) then*

$$\inf \{< \mathcal{L}_\gamma^+ f, f >: \ \|f\| = 1, < f, \phi_\gamma \psi_\gamma >= 0, \ \} \equiv \mu > 0.$$

Proof. From Theorem 5.2 we have that $\mathcal{L}_\gamma^+$ is a non-negative operator and so $\mu \geqq 0$. Suppose $\mu = 0$. Then following the ideas of the proof of Theorem 5.3 we have that the minimum is attained at an admissible function $g^* \neq 0$ and there is $(\lambda, \theta) \in \mathbb{R}^2$ such that

$$\mathcal{L}_\gamma^+ g^* = \lambda g^* + \theta \phi_\gamma \psi_\gamma. \tag{5.14}$$

Thus, it follows that $\lambda = 0$. Now, taking the inner product of (5.14) with ϕ_γ it is deduced that $0 =< \mathcal{L}_\gamma^+ \phi_\gamma, g^* >= \theta \int_0^L \phi_\gamma^2 \psi_\gamma dx$, and therefore $\theta = 0$. Then, since zero is a simple eigenvalue for $\mathcal{L}_\gamma^+$ it follows that $g^* = \nu \phi_\gamma$ for some $\nu \neq 0$, which is a contradiction. This completes the proof. □.

Next we complete the proof of Theorem 5.1 by returning to (5.6). So we estimate the terms $< \mathcal{L}_\gamma p, p >$ and $< \mathcal{L}_\gamma^+ q, q >$ where p and q satisfy (5.5). Thus, from Theorem 5.4 and from the particular form of $\mathcal{L}_\gamma^+$ we have that there is $C_1 > 0$ such that

$$< \mathcal{L}_\gamma^+ q, q >\geqq C_1 \|q\|_1^2. \tag{5.15}$$

Now we estimate $< \mathcal{L}_\gamma p, p >$. Suppose without loss of generality that $\|\phi_\gamma\| = 1$. We write $p_\perp = p - p_{||}$, where $p_{||} =< p, \phi_\gamma > \phi_\gamma$. Then, from (5.5) and the positivity of the operator $\mathcal{K}_\gamma^{-1}$ it follows that $< p_\perp, (\phi_\gamma \psi_\gamma)' >= 0$. Therefore from Theorem 5.3, it follows $< \mathcal{L}_\gamma p_\perp, p_\perp > \geqq D\|p_\perp\|^2$. Now we suppose that $\|u_0\| = \|\phi_\gamma\| = 1$. Since $\|u(t)\|^2 = 1$ for all t, we have that $< p, \phi_\gamma >= -\|\xi\|^2/2$. So, $< \mathcal{L}_\gamma p_\perp, p_\perp > \geqq \beta_0\|p\|^2 - \beta_1\|\xi\|_1^4$. Since $< \mathcal{L}_\gamma \phi_\gamma, \phi_\gamma > < 0$ it follows that $< \mathcal{L}_\gamma p_{||}, p_{||} > \geqq -\beta_3\|\xi\|_1^4$. Moreover, Cauchy-Schwarz inequality implies $< \mathcal{L}_\gamma p_{||}, p_\perp > \geqq -\beta_4\|\xi\|_1^3$. Therefore we conclude from the specific form of $\mathcal{L}_\gamma$ that

$$< \mathcal{L}_\gamma p, p > \geqq D_1\|p\|_{1,\sigma}^2 - D_2\|\xi\|_{1,\sigma}^3 - D_3\|\xi\|_{1,\sigma}^4, \tag{5.16}$$

with $D_i > 0$ and $\|f\|_{1,\sigma}^2 = \|f'\|^2 + \sigma\|f\|^2$.

Next by collecting the results in (5.9), (5.15) and (5.16) and substituting them in (5.6), we obtain

$$\Delta L(t) \geqq d_1\|\xi\|_{1,\sigma}^2 - d_2\|\xi\|_{1,\sigma}^3 - d_3\|\xi\|_{1,\sigma}^4, \tag{5.17}$$

where $d_i > 0$. Therefore, from standard arguments for $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that if $\|u_0 - \Phi_\gamma\|_{1,\sigma} < \delta(\epsilon)$ and $\|v_0 - \Psi_\gamma\|_{\frac{1}{2}} < \delta(\epsilon)$,

$$\rho_\sigma(u(t), \phi_\gamma)^2 = \|\xi(t)\|_{1,\sigma}^2 < \epsilon \tag{5.18}$$

for $t \in [0, \infty)$, and so we obtain the first inequality in (5.1).

Now, it follows from (5.6) and from the above study of ξ that

$$\epsilon \geqq \frac{\alpha}{2\beta} \int_{\mathbb{R}} \left[\mathcal{K}_\gamma^{1/2}\eta + 2\beta\mathcal{K}_\gamma^{-1/2}(\phi_\gamma p) + \beta\mathcal{K}_\gamma^{-1/2}(p^2 + q^2)\right]^2 dx.$$

Thus, from (5.18) and the equivalence of the norms $\|\mathcal{K}_\gamma^{1/2}\eta\|$ and $\|\eta\|_{\frac{1}{2}}$ we obtain (5.1). We have thus proved that $(\Phi_\gamma, \Psi_\gamma)$ is stable relative to small perturbation which preserves the $L^2_{per}([0, L])$ norm of Φ_γ. The general case follows from that $\gamma \in (-\gamma_1, \gamma_1) \to (\phi_\gamma, \psi_\gamma)$ is a smooth branch of solutions for Eq. (1.5). This finishes the Theorem. □

References

[1] J. P. Albert, *Positivity properties and stability of solitary-wave solutions of model eqautions for long waves*, Comm. PDE, 17 (1992) p. 1-22.

[2] J. P. Albert, and J. L. Bona, *Total positivity and the stability of internal waves in fluids of finite depth*, IMA J. Applied Math. 46 (1991), p. 1-19.

[3] J. P. Albert, J. L. Bona, J.L. and D. Henry, *Sufficient conditions for stability of solitary-wave equation of model equations for long waves*, Physica D 24 (1987), p. 343-366.

[4] J. Angulo, *Stability of cnoidal waves to Hirota-Satsuma systems*, Matemática Contemporânea 27 (2004), p. 189-223.

[5] J. Angulo, *Non-linear stability of periodic travelling-wave solutions to the Schrödinger and the modified Korteweg-de Vries* . To appear in J. Differential Equations (2007).

[6] J. Angulo, J. L. Bona, and M. Scialom, *Stability of cnoidal waves*, Advances in Differential Equations 11 (2006), p. 1321-1374.

[7] J. Angulo and F. Montenegro, *Existence and evenness of solitary waves solutions for an interaction equation of short and long dispersive waves*, Nonlinearity., 13 (2000), p. 1595-1611.

[8] J. Angulo and F. Montenegro, *Orbital stability of Solitary Wave Solutions for an Interaction equation of Short and Long Dispersive waves*, J. Differential equations., 174 (2001), p. 181-199.

[9] A. Arbieto, A. J. Corcho, C. Matheus, *Rough solutions for the periodic Schrödinger-Korteweg-de Vries system*, Journal of Differential Equations, 230 (2006), p. 295-336.

[10] D. Bekiranov, T. Ogawa and G. Ponce, *On the well-posedness of Benney's interaction equation of short and long waves*, Advances Diff. Equations 1, (1996), p.919–937.

[11] D. Bekiranov, T. Ogawa and G. Ponce, *Interaction equation for short and long dispersive waves*, J. Functional Analysis 158 (1998), p. 357-388

[12] T. B. Benjamin, *Internal waves of permanent form in fluids of great depht*, J. Fluid Mech. 29 (1967), p. 559-592.

[13] T. B. Benjamin, *The stability of solitary waves* , Proc. Roy. Soc. London A., 338 (1972), p. 153-183.

[14] D. J. Benney, *Significant interactions between small and large scale surface waves*, Stud. Appl. Math., 55 (1976), p. 93-106.

[15] D. J. Benney, *A general theory for interactions between short and long waves*, ibid., 56 (1977), p. 81-94.

[16] J. L. Bona, *On the stability theory of solitary waves*, Proc Roy. Soc. London Ser. A 344 (1975), p. 363-374.

[17] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and application to the nonlinear evolution equations I, II*, Geom. Funct. Anal. **3** (1993), 107-156, 209-262.

[18] P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists*, Sec. Ed. Springer-Verlag, New-York, 1971.

[19] J. Colliander, J. Holmer and N. Tzirakis, *Low regularity global well-posedness for the Zakharov and Klein-Gordon-Schrödinger systems*, Arxiv preprint math.AP/0603595 (2006).

[20] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Sharp global well-posedness for KdV and modified KdV on* $\mathbb{R}$ *and* $\mathbb{T}$, J. Amer. Math. Soc. **16** (2003), 705-749.

[21] V. D. Djordjevic and L. G. Redekopp, *On two-dimensional packet of capillary-gravity waves*, J. Fluid Mech., 79 (1977), p. 703-714.

[22] M. Funakoshi and M. Oikawa, *The resonant interaction between a long internal gravity wave and a surface gravity wave packet*, J. Phys. Soc. Japan., 52 (1983), p. 1982-1995.

[23] T. Gallay and M. Hărăguş, *Stability of small periodic waves for the nonlinear Schrödinger equation*, Preprint (2006).

[24] T. Gallay and M. Hărăguş, *Orbital Stability of periodic waves for the nonlinear Schrödinger equation*, Preprint (2006).

[25] M. Grillakis, J. Shatah, and W. Strauss, *Stability theory of solitary waves in the presence of symmetry II*, J. Funct. Anal. 94 (1990), p. 308-348.

[26] R. H. J. Grimshaw, *The modulation of an internal gravity-wave packet and the resonance with the mean motion*, Stud. Appl. Math., 56 (1977), p. 241-266.

[27] R. Iorio and V. de Magalhães Iorio, *Fourier Analysis and Partial Differential Equations*, Cambridge Stud. in Advan. Math., 70, 2001.

[28] V. I. Karpman *On the dynamics of sonic-Langmuir soliton*, Physica Scripta., 11 (1975), p. 263-265.

[29] C. E. Kenig, G. Ponce, L. Vega, *A bilinear estimate with applications to the KdV equation*, J. Amer. Math. Soc. **9** (1996), 573-603.

[30] W. Magnus and S. Winkler, *Hill's Equation*, Interscience, Tracts in Pure and Appl. Math. Wiley, NY., 20, 1976.

[31] L. Molinet, *Global well-posedness in* L^2 *for the periodic Benjamin-Ono equation*, preprint (2006).

[32] L. Molinet, F. Ribaud, *Global well-posedness in the energy space for the Benjamin-Ono equation in the circle*, preprint (2005).

[33] H. Ono, *Algebraic solitary waves in stratified fluids*, J. Phys. Soc. Japan 39 (1975), 1082-1091.

[34] H. Pecher, *Rough solutions of a Schrödinger-Benjamin-Ono system* , Diff. Integral Eq., 19 (2006), p. 517-535.

[35] S. Reed and B. Simon, *Methods of Modern Mathematical Physics: Analysis of Operator*, v. IV, Academic Press, 1975.

[36] M. I. Weinstein *Liapunov stability of ground states of nonlinear dispersive evolutions equations*, Comm. Pure Appl. Math., 39 (1986), p. 51-68.

[37] T. Tao *Global well-posedness of the Benjamin-Ono equation in $H^1(\mathbb{R})$*, J. Hyperbolic Diff. Eq. 1 (2004), p. 27-49.

[38] M. I. Weinstein *Modulational stability of ground states of nonlinear Schrödinger equations*, Siam J. Math. Anal., 16 (1985), p. 472-491.

[39] N. Yajima and J. Satsuma, *Soliton solutions in a diatomic lattice system*, Prog. Theor. Phys., 62 (1979), p. 370-378.