Antipodal pairs, critical pairs, and Nash angular equilibria in convex cones

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Abstract. In this paper we discuss three related geometric concepts that arise in connection with the angular analysis of a convex cone $K \subset \mathbb{R}^d$. Antipodal pairs and critical pairs in convex cones have been studied in earlier works of ours, but several additional results are presented here for the first time. The concept of Nash angular equilibrium is new and we set up the basic ingredients of the theory.

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1 Introduction

The present paper is the last component of a triptych initiated in [3] and continued in [5]. Everything started with the variational problem which consist in finding a pair of unit vectors achieving the maximal angle

$$\theta_{\max}(K) = \sup_{u,v\ \in K\cap S_d} \arccos\langle u, v\rangle \tag{1}$$

of a closed convex cone $K \subset \mathbb{R}^d$. This theme was addressed in detail in our work [3] and was motivated by the fact that the term $\theta_{\max}(K)$ not only has an intrinsic geometric interest, but also serves to measure the degree of pointedness of the cone K. For a more specialized use of the expression (1), see for instance the paper by Peña and Renegar [8].

The notation employed in (1) is standard: the underlying space $\mathbb{R}^d$ is equipped with the usual inner product $\langle u, v\rangle = u^T v$ and the associated norm $\|\cdot\|$. The symbol S_d refers to the unit sphere in $\mathbb{R}^d$. The dimension d is assumed to be greater than or equal to 2. We introduce also the notation

$$K \in \Xi(\mathbb{R}^d) \Longleftrightarrow \begin{cases} K \subset \mathbb{R}^d \text{ is a closed convex cone} \\ \text{different from } \{0\} \text{ and different from } \mathbb{R}^d. \end{cases}$$

The angle maximization problem (1) gives rise to a concept of antipodality that one may qualify as strong or absolute:

Definition 1. *Let $K \in \Xi(\mathbb{R}^d)$. One says that $(\bar{u}, \bar{v}) \in \mathbb{R}^d \times \mathbb{R}^d$ is a (strong or absolute) antipodal pair of K if $\bar{u}, \bar{v} \in K \cap S_d$ and $\|\bar{u} - \bar{v}\| = \mathrm{diam}(K \cap S_d)$.*

As usual, $\mathrm{diam}(C)$ stands for the diameter of a nonempty bounded set $C \subset \mathbb{R}^d$. In view of the general relation

$$\|u - v\|^2 = 2(1 - \langle u, v\rangle) \qquad \forall u, v \in K \cap S_d, \tag{2}$$

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one sees that $(\bar{u}, \bar{v})$ is an antipodal pair of K if and only if $\bar{u}, \bar{v} \in K \cap S_d$ and $\arccos\langle \bar{u}, \bar{v}\rangle = \theta_{\max}(K)$. Antipodal pairs always exist because the diameter of a compact set is necessarily attained. By writing out the first-order optimality conditions for the corresponding diameter maximization problem one arrives at the concept of critical pair as formulated in the next definition (cf. [3, Theorem 4.1]). The notation $K^+ = \{y \in \mathbb{R}^d : \langle y, x\rangle \geq 0 \ \forall x \in K\}$ stands for the dual cone of K.

Definition 2. *By a critical pair of $K \in \Xi(\mathbb{R}^d)$ one understands any pair $(\bar{u}, \bar{v})$ of vectors satisfying*

$$\bar{u}, \bar{v} \in K \cap S_d, \tag{3}$$

$$\bar{v} - \langle \bar{u}, \bar{v}\rangle \bar{u} \in K^+, \tag{4}$$

$$\bar{u} - \langle \bar{u}, \bar{v}\rangle \bar{v} \in K^+. \tag{5}$$

The angle $\theta(\bar{u}, \bar{v}) = \arccos\langle \bar{u}, \bar{v}\rangle$ formed by a critical pair $(\bar{u}, \bar{v})$ is called a critical angle. The adjective proper is added when $\bar{u}$ and $\bar{v}$ are not collinear, that is to say, $|\langle \bar{u}, \bar{v}\rangle| \neq 1$. The set of all proper critical angles of K, denoted by $\Omega(K)$, is called the angular spectrum of K.

Angular spectra of convex cones is the second component of our triptych. In reference [5] we established a number of results dealing with the concept of critical angle, but some aspects of this theory were not treated in an exhaustive manner. In this paper we will address a few questions that were left unattended.

There is yet another reason which motivates us to write this paper. It has to do with the necessity of examining more carefully the very definition of antipodality. The concept introduced in Definition 1 is not the first idea that comes to mind when one refers to antipodality in a classical sense. According to our definition, an antipodal pair is formed by unit vectors in K that are as far away from each other as possible. Now, imagine that we are looking for antipodal points on the surface of the earth. The north pole and the south pole do form a pair of antipodal points in the common parlance, but they are not located at maximal distance because the earth is not perfectly spherical. This short and intuitive discussion leads us to introduce an alternative concept of antipodality which is less restrictive:

Definition 3. *A Nash angular equilibrium of $K \in \Xi(\mathbb{R}^d)$ is a pair $(\bar{u}, \bar{v}) \in \mathbb{R}^d \times \mathbb{R}^d$ satisfying*

$$\begin{aligned} \bar{u}, \bar{v} &\in K \cap S_d, \\ \theta(\bar{u}, \bar{v}) \geq \theta(\bar{u}, v) &\quad \forall v \in K \cap S_d, \\ \theta(\bar{u}, \bar{v}) \geq \theta(u, \bar{v}) &\quad \forall u \in K \cap S_d. \end{aligned}$$

If $(\bar{u}, \bar{v})$ is as above, then one says that $\theta(\bar{u}, \bar{v})$ is a Nash angle of K.

The terminology that we are using is motivated by the general equilibrium theory developed by the economist John Nash [7] in the context of non-cooperative games. Observe that the concept of Nash angular equilibrium for K can be formulated also as a Nash problem involving the diameter maximization of $K \cap S_d$. Indeed, relation (2) yields straightforwardly

$$(\bar{u}, \bar{v}) \text{ is a Nash angular equilibrium of } K \Longleftrightarrow \left\{ \begin{array}{ll} \bar{u}, \bar{v} \in K \cap S_d, & \\ \|\bar{u} - \bar{v}\| \geq \|\bar{u} - v\| & \forall v \in K \cap S_d, \\ \|\bar{u} - \bar{v}\| \geq \|u - \bar{v}\| & \forall u \in K \cap S_d. \end{array} \right.$$

In this paper we will explore the concept of Nash angle and we shall try to better understand what makes these angles so special. As preamble to our work, nothing is better than starting with the following elementary but important observation.

Proposition 1. *For any $K \in \Xi(\mathbb{R}^d)$, the following two conditions hold:*

(a) every antipodal pair is a Nash angular equilibrium,

(b) every Nash angular equilibrium is a critical pair.

Proof. Part (a) is obvious. In order to prove (b), suppose that $(\bar{u}, \bar{v})$ is a Nash angular equilibrium. In particular, $\bar{u}$ is a solution to the problem

$$\begin{aligned} &\text{minimize } \langle u, \bar{v} \rangle \\ &u \in K \cap S_d. \end{aligned} \tag{6}$$

Pick an arbitrary vector $h \in K$ and consider the curve $\varphi : [0, \varepsilon] \to \mathbb{R}^d$ defined by

$$\varphi(t) = \frac{(1-t)\bar{u} + th}{\|(1-t)\bar{u} + th\|}\,. \tag{7}$$

To make sure that the denominator in (7) doesn't vanish we ask $\varepsilon \in]0, 1]$ to be small enough. We call such φ an "admissible curve" emanating from $\bar{u}$ because

$$\varphi(0) = \bar{u} \quad \text{and} \quad \varphi(t) \in K \cap S_d \quad \forall t \in [0, \varepsilon].$$

Since $\bar{u}$ solves the problem (6), the choice $t = 0$ yields a minimum for the univariate function

$$t \in [0, \varepsilon] \mapsto f(t) = \langle \varphi(t), \bar{v} \rangle, \tag{8}$$

and therefore the derivative

$$f'(0) = \langle \varphi'(0), \bar{v} \rangle = \Big\langle h - \langle \bar{u}, h \rangle \bar{u}, \bar{v} \Big\rangle$$

is nonnegative. One arrives in this way to the variational inequality

$$\Big\langle \bar{v} - \langle \bar{u}, \bar{v} \rangle \bar{u}, h \Big\rangle \geq 0 \quad \forall h \in K$$

which expresses that $\bar{v} - \langle \bar{u}, \bar{v} \rangle \bar{u}$ belongs to the dual cone of K. The criticality condition $\bar{u} - \langle \bar{u}, \bar{v} \rangle \bar{v} \in K^+$ is obtained in a similar way. □

In view of Proposition 1, one can split the angular spectrum of K in two disjoint pieces:

$$\Omega(K) = \Omega_{\text{nash}}(K) \cup \Omega_{\text{ord}}(K).$$

The first piece, denoted $\Omega_{\text{nash}}(K)$, collects the proper critical angles that are formed by a Nash angular equilibrium. As a general rule, this portion is usually very small compared with the full collection of proper critical angles. In fact, most proper critical pairs are "ordinary" in the sense that they don't enjoy any type of antipodality property. The angles formed by these ordinary proper critical pairs are thrown in the set $\Omega_{\text{ord}}(K)$.

2 Preliminary Results

2.1 Are Nash Angular Equilibria Formed with Extreme Rays?

Extreme rays (or extreme directions) play a very important role in the description of a convex cone. The definition of an extreme ray varies slightly from one author to another, so it is not a bad idea to fix clearly the meaning of this concept. We adhere to the following definition:

Definition 4. *Let $K \in \Xi(\mathbb{R}^d)$. An extreme ray of K is a nonzero vector $z \in K$ such that $z = z_1 + z_2$ with $z_1, z_2 \in K\backslash\{0\}$ implies $z = \beta_1 z_1 = \beta_2 z_2$ for some $\beta_1 > 0$ and $\beta_2 > 0$. The symbol*

$$\mathrm{ext}(K) = \{z/\|z\| : \; z \textit{ is an extreme ray of } K\}$$

refers to the set of normalized extreme rays of K.

Lemma 1. *Let $K \in \Xi(\mathbb{R}^d)$. Let $\bar{v} \in K \cap S_d$ be such that $\langle g, \bar{v}\rangle > 0$ for all $g \in \mathrm{ext}(K)$. If $\bar{u}$ minimizes the linear form $\langle \cdot\,, \bar{v}\rangle$ over $K \cap S_d$, then $\bar{u} \in \mathrm{ext}(K)$.*

Proof. Since every nonzero vector in K is representable as a positive linear combination of finitely many vectors taken from $\mathrm{ext}(K)$, we write $\bar{u} = \big\| \sum_{i=1}^p \alpha_i g^i \big\|^{-1} \sum_{i=1}^p \alpha_i g^i$ for some integer $p \in \mathbb{N}$, vectors $g^1, \ldots, g^p$ in $\mathrm{ext}(K)$, and positive scalars $\alpha_1, \ldots, \alpha_p$. Given that

$$\big\| \sum_{i=1}^p \alpha_i g^i \big\| \leq \sum_{i=1}^p \alpha_i, \tag{9}$$

we distinguish between two cases:
Case I: equality occurs in (9). By strict convexity of $\|\cdot\|^2$, this situation can happen only if $p = 1$, in which case we are done.
Case II: the inequality (9) is strict. Relabeling the vectors $g^1, \ldots, g^p$ if necessary and taking into account that $\langle \cdot\,, \bar{v}\rangle$ is positive over $\mathrm{ext}(K)$, one can write $0 < \langle g^1, \bar{v}\rangle \leq \cdots \leq \langle g^p, \bar{v}\rangle$. Hence,

$$\begin{aligned} \langle \bar{u}, \bar{v}\rangle &= \big\| \sum_{i=1}^p \alpha_i g^i \big\|^{-1} \sum_{i=1}^p \alpha_i \langle g^i, \bar{v}\rangle = \frac{\sum_{i=1}^p \alpha_i}{\| \sum_{i=1}^p \alpha_i g^i \|} \, \frac{\sum_{i=1}^p \alpha_i \langle g^i, \bar{v}\rangle}{\sum_{i=1}^p \alpha_i} \\ &> \frac{\sum_{i=1}^p \alpha_i \langle g^i, \bar{v}\rangle}{\sum_{i=1}^p \alpha_i} \geq \langle g^1, \bar{v}\rangle \end{aligned}$$

Since we are contradicting the optimality of $\bar{u}$, the case II must be ruled out. □

Proposition 2. *Let $K \in \Xi(\mathbb{R}^d)$ be strictly acute in the sense that $\langle x, y\rangle > 0$ for all $x, y \in K\backslash\{0\}$. Then, every Nash angular equilibrium of K is a pair of normalized extreme rays of the cone.*

Proof. It follows directly from Lemma 1. □

Below we specialize Proposition 2 to the particular case of a polyhedral cone, that is to say, a convex cone representable in the form

$$K = \mathrm{cone}\{g^1, \ldots, g^p\} = \Big\{ \sum_{i=1}^p x_i g^i : \; x \in \mathbb{R}^p_+ \Big\}. \tag{10}$$

As customary, we say that $\{g^1, \ldots, g^p\} \subset \mathbb{R}^d$ is a set of generators for K. There is no loss of generality in assuming that all generators have unit length and that no generator is a positive linear combination of the remaining ones.

Corollary 1. *Consider a polyhedral cone $K \subset \mathbb{R}^d$ with generators $\{g^1, \ldots, g^p\}$ satisfying $\langle g^i, g^j \rangle > 0$ whenever $i \neq j$. If $(\bar{u}, \bar{v})$ is a Nash angular equilibrium of K, then $\bar{u}$ and $\bar{v}$ are in $\{g^1, \ldots, g^p\}$.*

Proof. It suffices to apply Proposition 2 and observe that $\mathrm{ext}(K) = \{g^1, \ldots, g^p\}$. □

Remark 1. Corollary 1 applies only to Nash angular equilibria and not to ordinary critical pairs. It is possible to construct a strictly acute polyhedral cone having a critical pair which is not formed with a couple of generators.

We state next a sufficient condition for a pair to be a Nash angular equilibrium of an acute polyhedral cone. This result will be used later in Section 4.

Proposition 3. *Let $K \subset \mathbb{R}^d$ be polyhedral cone with generators $\{g^1, \ldots, g^p\}$ satisfying $\langle g^i, g^j \rangle \geq 0$ whenever $i \neq j$. For a pair $(\bar{u}, \bar{v})$ of unit vectors in K to be a Nash angular equilibrium of K it suffices that*

$$\|\bar{u} - \bar{v}\| \geq \|\bar{u} - g^i\| \tag{11}$$

$$\|\bar{u} - \bar{v}\| \geq \|\bar{v} - g^i\| \tag{12}$$

for every $i \in \{1, \ldots, p\}$.

Proof. Suppose that (11)-(12) is in force. Take an arbitrary vector $z \in K \cap S_d$ and express it in the form $z = \big\| \sum_{i=1}^p \alpha_i g^i \big\|^{-1} \sum_{i=1}^p \alpha_i g^i$. Let us prove that $\langle \bar{u}, \bar{v} \rangle \leq \langle \bar{u}, z \rangle$. As in the proof of Lemma 1, one has

$$\langle \bar{u}, z \rangle = \Big\| \sum_{i=1}^p \alpha_i g^i \Big\|^{-1} \sum_{i=1}^p \alpha_i \langle \bar{u}, g^i \rangle = \frac{\sum_{i=1}^p \alpha_i}{\| \sum_{i=1}^p \alpha_i g^i \|} \frac{\sum_{i=1}^p \alpha_i \langle \bar{u}, g^i \rangle}{\sum_{i=1}^p \alpha_i}.$$

In view of the convexity inequality (9) and the acuteness hypothesis made on the generators of K, one can write

$$\langle \bar{u}, z \rangle \geq \frac{\sum_{i=1}^p \alpha_i \langle \bar{u}, g^i \rangle}{\sum_{i=1}^p \alpha_i} \geq \min_{1 \leq i \leq p} \langle \bar{u}, g^i \rangle \ \geq \langle \bar{u}, \bar{v} \rangle,$$

the last inequality being due to Assumption (11). The proof of $\langle \bar{u}, \bar{v} \rangle \leq \langle \bar{v}, z \rangle$ is similar but relies on (12). □

2.2 Stability of Critical Pairs and Nash Angular Equilibria

We endow the set $\Xi(\mathbb{R}^d)$ with the usual bounded Pompeiu-Hausdorff metric

$$\delta(K_1, K_2) = \sup_{\|z\| \leq 1} \big| \mathrm{dist}[z, K_1] - \mathrm{dist}[z, K_2] \big|.$$

Since we are working in a finite dimensional setting, convergence in the metric space $(\Xi(\mathbb{R}^d), \delta)$ is equivalent to convergence in the sense of Painlevé-Kuratowski (cf. [9]).

Endowing $\Xi(\mathbb{R}^d)$ with a metric enables one to address all kind of relevant topological questions. As shown in the next proposition, the set-valued maps

$$\begin{aligned} K \mapsto \mathcal{C}(K) &= \text{critical pairs of } K, \\ K \mapsto \mathcal{N}(K) &= \text{Nash angular equilibria of } K \end{aligned}$$

both enjoy a certain upper stability property called graph-closedness.

Proposition 4. *The set-valued maps $\mathcal{C} : \Xi(\mathbb{R}^d) \to \mathbb{R}^d \times \mathbb{R}^d$ and $\mathcal{N} : \Xi(\mathbb{R}^d) \to \mathbb{R}^d \times \mathbb{R}^d$ are graph-closed, that is to say, the sets*

$$\begin{aligned} \operatorname{gr}(\mathcal{C}) &= \{(K,u,v) :\ (u,v) \text{ is a critical pair of } K\}, \\ \operatorname{gr}(\mathcal{N}) &= \{(K,u,v) :\ (u,v) \text{ is Nash angular equilibrium of } K\} \end{aligned}$$

are closed in the product space $\Xi(\mathbb{R}^d) \times (\mathbb{R}^d \times \mathbb{R}^d)$.

Proof. The Walkup-Wets isometry theorem [10] asserts that $\delta(K_1^+, K_2^+) = \delta(K_1, K_2)$ for all $K_1, K_2 \in \Xi(\mathbb{R}^d)$. Closedness of $\operatorname{gr}(\mathcal{C})$ is then a direct consequence of (3)-(5) and the fact that $K \mapsto K^+$ is a continuous map over the metric space $(\Xi(\mathbb{R}^d), \delta)$. For proving that $\operatorname{gr}(\mathcal{N})$ is a closed set, take converging sequences $\{K_n\}_{n\in\mathbb{N}} \to K$ and $\{(\bar{u}_n, \bar{v}_n)\}_{n\in\mathbb{N}} \to (\bar{u}, \bar{v})$ such that

$$\bar{u}_n, \bar{v}_n \in K_n \cap S_d, \tag{13}$$

$$\|\bar{u}_n - \bar{v}_n\| \geq \|\bar{u}_n - v\| \quad \forall v \in K_n \cap S_d, \tag{14}$$

$$\|\bar{u}_n - \bar{v}_n\| \geq \|u - \bar{v}_n\| \quad \forall u \in K_n \cap S_d. \tag{15}$$

By passing to the limit in (13) one easily gets $\bar{u}, \bar{v} \in K \cap S_d$. Take now an arbitrary $v' \in K$ and express it in the form $v' = \lim_{n\to\infty} v_n$ with $v_n \in K_n$. By choosing $v = v_n$ in (14) and then letting $n \to \infty$, one arrives at the inequality $\|\bar{u} - \bar{v}\| \geq \|\bar{u} - v'\|$. The proof of the condition

$$\|\bar{u} - \bar{v}\| \geq \|u' - \bar{v}\| \quad \forall u' \in K \cap S_d$$

is analogous but relies on (15). $\square$

Remark 2. Since $\mathcal{C}$ and $\mathcal{N}$ are graph-closed and take values in the compact set $S_d \times S_d$, these set-valued maps turn out to be upper-semicontinuous in Berge's sense [1]. One can find examples showing that these set-valued maps are not lower-semicontinuous.

3 Higher-Order Analysis

First-order information was obtained by working with the admissible curve (7), which can be written in the equivalent form

$$\varphi(t) = \frac{\bar{u} + tz}{\|\bar{u} + tz\|}$$

with $z = h - \bar{u}$. We now take the analysis one step further. Readers who are familiar with the theory of second-order optimality conditions for constrained optimization problems know that the Hessian matrices of the cost and constraint functions play a prominent role in the discussion. In our case, second-order information is captured by a matrix having the the form of a shifted tensor product $a \star b = ab^T - \langle a, b\rangle$. In the sequel we use the symbol $c^\perp$ for indicating the hyperplane that is orthogonal to c.

Proposition 5. *For $(\bar{u}, \bar{v})$ to be a Nash angular equilibrium of $K \in \Xi(\mathbb{R}^d)$, the following second-order criticality conditions are necessary:*

(a) $\left\langle z, [\bar{u} \star \bar{v}]z \right\rangle \geq 0$ *for all* $z \in [\bar{v} - \langle \bar{u}, \bar{v}\rangle \bar{u}]^\perp \cap \mathbb{R}_+(K - \bar{u})$,

(b) $\left\langle z, [\bar{u} \star \bar{v}]z \right\rangle \geq 0$ *for all* $z \in [\bar{u} - \langle \bar{u}, \bar{v}\rangle \bar{v}]^\perp \cap \mathbb{R}_+(K - \bar{v})$.

Proof. Let $(\bar{u}, \bar{v})$ be a Nash angular equilibrium of K. Let us prove that $\bar{u} \star \bar{v}$ is positive semidefinite over the convex cone

$$\mathcal{T}_K(\bar{u}, \bar{v}) = [\bar{v} - \langle \bar{u}, \bar{v}\rangle \bar{u}]^\perp \cap \mathbb{R}_+(K - \bar{u}).$$

By positive homogeneity, there is no loss of generality in taking z in $[\bar{v} - \langle \bar{u}, \bar{v}\rangle \bar{u}]^\perp \cap (K - \bar{u})$. We write then $z = h - \bar{u}$ with $h \in K$. Since $\bar{u}$ solves the problem (6), the choice $t = 0$ yields a minimum for the function f introduced in (8). By writing the second-order Taylor expansion

$$f(t) = f(0) + tf'(0) + \frac{1}{2}t^2 f''(0) + t^2\delta(t),$$

with $\delta(t) \to 0$ as $t \to 0^+$, one sees that $f''(0) \geq 0$ if $f'(0) = 0$. A matter of computation shows that

$$f'(0) = \left\langle \bar{v} - \langle \bar{u}, \bar{v}\rangle \bar{u}, z \right\rangle = 0,$$

the last equality being due to the fact that z has been taken in the orthogonal of $\bar{v} - \langle \bar{u}, \bar{v}\rangle \bar{u}$. Hence

$$f''(0) = \langle \bar{u}, z\rangle\langle z, \bar{v}\rangle - \|z\|^2\langle \bar{u}, \bar{v}\rangle = \left\langle z, [\bar{u} \star \bar{v}]z \right\rangle$$

must be nonnegative. Condition (b) is derived in a similar way. $\square$

Proposition 5 seems a promising result, but a more careful examination of the second-order criticality conditions forces us to be less optimistic. Two cases must be distinguished:

- if the pair $(\bar{u}, \bar{v})$ forms an angle greater than or equal to $\pi/2$, then the second-order criticality conditions are automatically satisfied, so they don't provide any new information.
- if the pair $(\bar{u}, \bar{v})$ forms an angle less than $\pi/2$, then second-order criticality conditions are transformed into a tangency test as explained below.

Theorem 1. *Let $(\bar{u}, \bar{v})$ be a Nash angular equilibrium of $K \in \Xi(\mathbb{R}^d)$ with angle $\theta(\bar{u}, \bar{v}) < \pi/2$. Then, $(\bar{u}, \bar{v})$ satisfies the tangency test*

$$\left\{ \begin{array}{l} [\bar{v} - \langle \bar{u}, \bar{v}\rangle \bar{u}]^\perp \cap \mathbb{R}_+(K - \bar{u}) \subset \mathbb{R}\bar{u}, \\ [\bar{u} - \langle \bar{u}, \bar{v}\rangle \bar{v}]^\perp \cap \mathbb{R}_+(K - \bar{v}) \subset \mathbb{R}\bar{v}. \end{array} \right. \tag{16}$$

Proof. Take $z \in \mathbb{R}_+(K - \bar{u})$ such that $\langle \bar{v}, z\rangle - \langle \bar{u}, \bar{v}\rangle\ \langle \bar{u}, z\rangle = 0$. By combining the latter equality and Proposition 5(a) one gets

$$\langle \bar{u}, \bar{v}\rangle \left[\langle \bar{u}, z\rangle^2 - \|z\|^2\right] \geq 0. \tag{17}$$

But, $\langle \bar{u}, \bar{v}\rangle > 0$ because $\theta(\bar{u}, \bar{v})$ is smaller than $\pi/2$. On the other hand, the term between brackets is nonpositive by the Cauchy-Schwarz inequality. The conclusion is that $\langle \bar{u}, z\rangle^2 - \|z\|^2 = 0$, and therefore z is a multiple of $\bar{u}$. This proves the first inclusion in (16). The second inclusion is proved in a similar way. $\square$

Remark 3. The inequality (17) holds automatically if $\langle \bar{u}, \bar{v} \rangle \leq 0$. We insist on the fact that the tangency test applies only to pairs with an angle less than $\pi/2$. Consider, for instance, the positive orthant $K = \mathbb{R}^3_+$ in the space $\mathbb{R}^3$. The vectors $\bar{u} = (1, 0, 0)$ and $\bar{v} = (0, 1, 0)$ form not just a Nash angular equilibrium, but also an antipodal pair. Notice that $[\bar{v} - \langle \bar{u}, \bar{v} \rangle \bar{u}]^\perp \cap \mathbb{R}_+(K - \bar{u}) = \mathbb{R} \times \{0\} \times \mathbb{R}_+$ is a two-dimensional convex cone, and therefore it is not contained in the line $\mathbb{R}\bar{u}$.

Since K is a closed convex cone, one has $\mathbb{R}_+(K - \bar{u}) = K + \mathbb{R}\bar{u}$, showing that both inclusions in (16) always hold in the opposite sense. Thus, the tangency test can be written in the equivalent form

$$\begin{cases} [\bar{v} - \langle \bar{u}, \bar{v} \rangle \bar{u}]^\perp \cap \mathbb{R}_+(K - \bar{u}) = \mathbb{R}\bar{u}, \\ [\bar{u} - \langle \bar{u}, \bar{v} \rangle \bar{v}]^\perp \cap \mathbb{R}_+(K - \bar{v}) = \mathbb{R}\bar{v}. \end{cases}$$

Except for a closure operation, $\mathbb{R}_+(K - \bar{u})$ corresponds to the tangent cone to K at $\bar{u}$. In general, one cannot reinforce the tangency test by writing $\mathrm{cl}[\mathbb{R}_+(K - \bar{u})]$ instead of $\mathbb{R}_+(K - \bar{u})$.

4 Threshold Condition for Nash Angular Equilibria

It is reasonable to expect a Nash angular equilibrium to form an angle which is large, or at least not too small while compared with the maximal angle of the cone. This idea will be expressed in a more formal way in Theorem 2, the main result of this section. Before stating this result we need however to prepare the ground by introducing some notation and terminology. Recall that $\mathcal{N}(K)$ stands for the set of all Nash angular equilibria of K.

Definition 5. *The Nash threshold coefficient of $K \in \Xi(\mathbb{R}^d)$, denoted by β_K, is the largest constant $\beta \in [0, 1]$ such that*

$$\|\bar{u} - \bar{v}\| \geq \beta \operatorname{diam}(K \cap S_d) \qquad \forall (\bar{u}, \bar{v}) \in \mathcal{N}(K). \tag{18}$$

If K is a ray, then one has $\beta_K = 1$. Otherwise, the Nash threshold coefficient of K admits the obvious characterization

$$\beta_K = \frac{1}{\operatorname{diam}(K \cap S_d)} \inf_{(u,v) \in \mathcal{N}(K)} \|u - v\|.$$

Even when K has a simple structure, evaluating the Nash threshold coefficient β_K may be a quite cumbersome task. In [6] we work out in detail the case of an elliptic cone. What is clear, however, is that

$$\beta_K = 1 \iff \begin{cases} \text{every Nash angular equilibrium} \\ \text{of } K \text{ is an antipodal pair,} \end{cases}$$

or equivalently,

$$\beta_K < 1 \iff \begin{cases} K \text{ admits a Nash angular equilibrium} \\ \text{which is not an antipodal pair.} \end{cases}$$

The question that we address now is that of finding the largest constant β for which (18) is uniform with respect to all elements in $\Xi(\mathbb{R}^d)$. In other words, we want to compute the infimal value

$$\beta^* = \inf_{K \in \Xi(\mathbb{R}^d)} \beta_K. \tag{19}$$

To start with, observe that $\beta^* \geq 1/2$. The proof of this inequality is straightforward and runs as follows: pick an arbitrary antipodal pair (a,b) of K. Then, for any $(\bar{u},\bar{v}) \in \mathcal{N}(K)$, one has

$$\operatorname{diam}(K \cap S_d) = \|a-b\| \leq \|a-\bar{u}\| + \|\bar{u}-b\| \leq 2\|\bar{u}-\bar{v}\|.$$

Obtaining a better lower estimate for β^* requires, however, a more elaborate line of thought.

4.1 An Auxiliary Problem

We open a parenthesis and study the auxiliary minimization problem

$$\mu(a,b) = \inf_{(u,v)\in F(a,b)} \|u-v\|, \tag{20}$$

where the feasible set $F(a,b)$ is given by

$$(u,v) \in F(a,b) \Longleftrightarrow \left\{ \begin{array}{l} \|u-a\| \leq \|u-v\|, \\ \|u-b\| \leq \|u-v\|, \\ \|v-a\| \leq \|u-v\|, \\ \|v-b\| \leq \|u-v\|. \end{array} \right. \tag{21}$$

At a later stage we will choose (a,b) as an antipodal pair of K, but for the time being a and b are arbitrary vectors taken from $\mathbb{R}^d$. In the next lemma we fully characterize the solution set

$$S(a,b) = \{(u,v) \in F(a,b) : \|u-v\| = \mu(a,b)\}$$

and the infimal value $\mu(a,b)$ of problem (20). Lemma 2 has nothing to do with angular analysis of convex cones. In fact, it is a geometric result concerning the intrinsic structure of the Euclidean space $(\mathbb{R}^d, \langle\cdot,\cdot\rangle)$.

Lemma 2. *Let a,b be two different vectors taken from $\mathbb{R}^d$. Then,*

(a) The solution set of (20) is nonempty and bounded. In fact, it admits the characterization

$$S(a,b) = \left\{ \Big(\frac{a+b}{2} + \frac{\sqrt{3}}{6}\|a-b\|w, \frac{a+b}{2} - \frac{\sqrt{3}}{6}\|a-b\|w\Big) : w \in S_d \cap [a-b]^\perp \right\}.$$

(b) The infimal value of (20) is given by $\mu(a,b) = (\sqrt{3}/3)\|a-b\|$.

Proof. The feasible set $F(a,b)$ is nonempty because $(a,b) \in F(a,b)$. The solution set of (20) does not change if one adds the extra constraint $\|u-v\| \leq \|a-b\|$, in which case the new feasible set becomes bounded. This simple argument shows that $S(a,b)$ is nonempty and bounded. Consider now a pair (u^*,v^*) of the form

$$u^* = \frac{a+b}{2} + \frac{\sqrt{3}}{6}\|a-b\|w, \quad v^* = \frac{a+b}{2} - \frac{\sqrt{3}}{6}\|a-b\|w,$$

with $w \in S_d \cap [a-b]^\perp$. One can easily check that (u^*,v^*) belongs to the set $G(a,b)$ defined by

$$(u,v) \in G(a,b) \quad \Longleftrightarrow \quad \|u-v\| = \|u-a\| = \|u-b\| = \|v-a\| = \|v-b\|.$$

In particular, (u^*, v^*) is feasible for (20) and

$$\mu(a,b) \le \|u^* - v^*\| = (\sqrt{3}/3)\|a-b\|.$$

Notice that (u^*, v^*) is strictly better than (a,b) in the sense that $\|u^* - v^*\| < \|a-b\|$. We will prove that (u^*, v^*) is in fact optimal. Consider an arbitrary solution $(\tilde{u}, \tilde{v})$ of problem (20). Observe that the vectors $\{a, b, \tilde{u}, \tilde{v}\}$ cannot be on the same line, because otherwise one would contradict the optimality of $(\tilde{u}, \tilde{v})$. We claim that $(\tilde{u}, \tilde{v}) \in G(a,b)$, that is to say, all the inequality constraints in (21) become active at $(\tilde{u}, \tilde{v})$. Suppose on the contrary that at least one inequality constraint is inactive, say $\|\tilde{u} - a\| < \|\tilde{u} - \tilde{v}\|$. We distinguish between two cases.
Case I: $\|\tilde{v} - a\| < \|\tilde{u} - \tilde{v}\|$. Take $t \in]0,1[$ and form the pair

$$(u_t, v_t) = \big(tb + (1-t)\tilde{u}, tb + (1-t)\tilde{v}\big).$$

Clearly $\|u_t - v_t\| = (1-t)\|\tilde{u} - \tilde{v}\| < \|\tilde{u} - \tilde{v}\|$ and

$$\begin{aligned} \|u_t - b\| = (1-t)\|\tilde{u} - b\| &\le (1-t)\|\tilde{u} - \tilde{v}\| = \|u_t - v_t\|, \\ \|v_t - b\| = (1-t)\|\tilde{v} - b\| &\le (1-t)\|\tilde{u} - \tilde{v}\| = \|u_t - v_t\|. \end{aligned}$$

On the other hand, if t is sufficiently small, then (u_t, v_t) is near $(\tilde{u}, \tilde{v})$ and one can write

$$\|u_t - a\| < \|u_t - v_t\|, \quad \|v_t - a\| < \|u_t - v_t\|.$$

In short, $(u_t, v_t) \in F(a,b)$ and $\|u_t - v_t\| < \|\tilde{u} - \tilde{v}\|$, contradicting the optimality of $(\tilde{u}, \tilde{v})$.
Case II: $\|\tilde{v} - a\| = \|\tilde{u} - \tilde{v}\|$. Take an orthonormal matrix Q of size $d \times d$ and form the pair

$$(\hat{u}, \hat{v}) = \big(b + Q(\tilde{u} - b), b + Q(\tilde{v} - b)\big).$$

By orthonormality, one has

$$\|\hat{u} - \hat{v}\| = \|\tilde{u} - \tilde{v}\|, \quad \|\hat{u} - b\| = \|\tilde{u} - b\|, \quad \|\hat{v} - b\| = \|\tilde{v} - b\|.$$

Moreover, Q can be chosen in such a way as to get $\|\hat{u} - a\| < \|\hat{u} - \hat{v}\|$ and $\|\hat{v} - a\| < \|\hat{u} - \hat{v}\|$. This can be done by taking as Q a suitable rotation matrix that slightly moves the vector $\tilde{v} - b$ so that it gets closer to $a - b$. Notice that with the new pair $(\hat{u}, \hat{v})$ we are in the situation discussed in Case I, so again we arrive at a contradiction. We have established that all inequalities are tight at an optimal solution. It is then easy to check that all solutions must be of the form stated in (a). □

4.2 Exact Estimate for β^*

With the help of Lemma 2 we are now ready to establish:

Proposition 6. *Consider an arbitrary $K \in \Xi(\mathbb{R}^d)$. Then,*

$$\|\bar{u} - \bar{v}\| \ge (\sqrt{3}/3)\,\mathrm{diam}(K \cap S_d) \qquad \forall (\bar{u}, \bar{v}) \in \mathcal{N}(K). \tag{22}$$

Proof. Choose (a,b) as an antipodal pair of K. Since a, b are unit vectors in K, one has $\mathcal{N}(K) \subset F(a,b)$. In view of Lemma 2, for an arbitrary Nash angular equilibrium $(\bar{u}, \bar{v})$, one gets

$$\begin{aligned} \|\bar{u} - \bar{v}\| &\ge \inf_{(u,v)\in\mathcal{N}(K)} \|u - v\| \ge \inf_{(u,v)\in F(a,b)} \|u - v\| \\ &= (\sqrt{3}/3)\,\|a - b\| = (\sqrt{3}/3)\,\mathrm{diam}(K \cap S_d). \end{aligned}$$

□

What Proposition 6 says is that $\beta^* \geq \sqrt{3}/3 \approx 0.577$. We have thus obtained an improvement with respect to the previous lower estimate $\beta^* \geq 1/2$. As shown next, a further improvement is simply not possible.

Proposition 7. *Let $d \geq 3$. For any $\beta > \sqrt{3}/3$, one can find a (strictly acute, solid, and polyhedral) $K \in \Xi(\mathbb{R}^d)$ and a pair $(\bar{u}, \bar{v}) \in \mathcal{N}(K)$ such that $\|\bar{u} - \bar{v}\| < \beta \operatorname{diam}(K \cap S_d)$.*

Proof. For simplicity we work in $\mathbb{R}^3$, but a similar argument applies in a higher dimensional space. Pick any $\beta > \sqrt{3}/3$. The idea behind the construction of K is getting a convex cone whose Nash angular equilibria are easy to identify. We rely on Proposition 3 to see what happens with the polyhedral cone K_η generated by the following unit vectors:

$$\begin{aligned} g^1 &= [\eta^2 + 3)]^{-1/2} \, (\sqrt{3}, 0, \eta) \, , \\ g^2 &= [\eta^2 + 3]^{-1/2} \, (-\sqrt{3}, 0, \eta) \, , \\ g^3 &= [\eta^2 + 1]^{-1/2} \, (0, 1, \eta) \, , \\ g^4 &= [\eta^2 + 1)]^{-1/2} \, (0, -1, \eta) \, . \end{aligned}$$

We choose $\eta > \sqrt{3}$, so that the inner products

$$\langle g_1, g_2 \rangle = \frac{\eta^2 - 3}{\eta^2 + 3}, \quad \langle g_3, g_4 \rangle = \frac{\eta^2 - 1}{\eta^2 + 1},$$

$$\langle g_1, g_3 \rangle = \langle g_1, g_4 \rangle = \langle g_2, g_3 \rangle = \langle g_2, g_4 \rangle = \frac{\eta^2}{\sqrt{\eta^2 + 3}\sqrt{\eta^2 + 1}}$$

are all strictly positive. We work then in a context in which K_η is strictly acute. By Corollary 1, the Nash angular equilibria (and, in particular, the antipodal pairs) of K_η are formed exclusively with generators of the cone. Notice that

$$\frac{\eta^2 - 3}{\eta^2 + 3} \leq \frac{\eta^2 - 1}{\eta^2 + 1} \leq \frac{\eta^2}{\sqrt{\eta^2 + 3}\sqrt{\eta^2 + 1}} \, . \tag{23}$$

This means that $\langle g_1, g_2 \rangle$ is smaller than all the other inner products, and therefore (g_1, g_2) is an antipodal pair of K_η. On the other hand, by using the sufficiency test of Proposition 3, one can prove that (g_3, g_4) is a Nash angular equilibrium of K_η. According to such test, we must check that

$$\|g_3 - g_4\| \leq \min\{\|g_3 - g_1\|, \|g_3 - g_2\|, \|g_4 - g_1\|, \|g_4 - g_2\|\},$$

but, in the present context, everything reduces to checking $\langle g_3, g_4 \rangle \leq \langle g_1, g_3 \rangle$, which is precisely the second inequality in (23). Observe finally that the ratio

$$\beta_{K_\eta} = \frac{\|g_3 - g_4\|}{\|g_1 - g_2\|} = \frac{\sqrt{3}}{3} \sqrt{\frac{\eta^2 + 3}{\eta^2 + 1}}$$

goes to $\sqrt{3}/3$ as $\eta \to \infty$. Hence, for η sufficiently large, one gets $\beta_{K_\eta} < \beta$ and the desired conclusion. $\square$

Corollary 2. *If the dimension d of the underlying Euclidean space is at least three, then the infimal value of problem (19) is $\beta^* = \sqrt{3}/3$.*

Some further comments on the proof of Proposition 7 are in order. Notice that:

- If the parameter η is an integer, then $\{K_\eta\}_{\eta\in\mathbb{N}}$ is a minimizing sequence for (19) in the sense that $\lim_{\eta\to\infty}\beta_{K_\eta}=\beta^*$.
- When η goes to ∞, the four generators of K_η approach $e=(0,0,1)$, and therefore K_η gets closer to the ray $\mathbb{R}_+e=\{(0,0,t):t\in\mathbb{R}_+\}$.

These two items seem to be contradicting each other because a ray has absolutely no chance of being a minimizer of the function $K\mapsto\beta_K$ over the compact metric space $(\Xi(\mathbb{R}^d),\delta)$. This paradox is explained with the help of the next proposition.

Proposition 8. *The function $K\mapsto\beta_K$, defined over the metric space $(\Xi(\mathbb{R}^d),\delta)$, is lower-semicontinuous at $K^*\in\Xi(\mathbb{R}^d)$ if and only if K^* is not a ray.*

Proof. Suppose that K^* is not a ray. In view of Proposition 4 and Berge's minimum theorem [1], the function

$$K\mapsto\inf_{(u,v)\in\mathcal{N}(K)}\|u-v\|$$

is lower-semicontinuous over $(\Xi(\mathbb{R}^d),\delta)$. On the other hand, the function $K\mapsto[\mathrm{diam}(K\cap S_d)]^{-1}$ is well defined and continuous on a neighbourhood of K^*. Hence, $K\mapsto\beta_K$ is lower-semicontinuous at K^*. Suppose now that K^* is a ray. Then, $\beta_{K^*}=1$. But

$$\liminf_{K\to K^*}\beta_K<1$$

because one can construct a sequence $\{K_\eta\}_{\eta\in\mathbb{N}}$ converging to K^* and such that $\lim_{\eta\to\infty}\beta_{K_\eta}=\beta^*<1$. □

4.3 A Localization Result for Nash Angles

We now establish a relationship existing between an arbitrary Nash angle and the maximal angle of the cone.

Theorem 2. *If θ is a Nash angle of $K\in\Xi(\mathbb{R}^d)$, then*

$$\arccos\left[\frac{2+\cos\theta_{\max}(K)}{3}\right]\le\theta\le\theta_{\max}(K). \tag{24}$$

Proof. The second inequality in (24) is obvious and has been written here only for the sake of completeness. The first inequality is obtained from Proposition 6 and the relation (2). We also use the general formula

$$[\mathrm{diam}(K\cap S_d)]^2=2[1-\cos\theta_{\max}(K)]$$

linking the terms $\mathrm{diam}(K\cap S_d)$ and $\theta_{\max}(K)$. □

To fix the ideas, if the maximal angle of K has, for instance, 120 degrees, then every Nash angle of K has at least 51.3 degrees. By contrast, the ordinary critical angles could be as closed to 0 as one wishes!

On the other hand, if a critical angle θ is above the threshold value (24), it does not follow necessarily that θ is a Nash angle. In fact, the situation can be even worse: we exhibit next an example of a critical pair whose angle is arbitrarily close to the maximal angle of the cone, and which is not a Nash angular equilibrium. In other words, proximity with respect to the maximal angle is not a guarantee for a critical angle to be of the Nash type.

Example 1. Consider the polyhedral cone $K \subset \mathbb{R}^3$ generated by the unit vectors

$$g^1 = \frac{1}{4}(-1, \sqrt{3}, 2\sqrt{3}), \quad g^2 = \frac{1}{4}(-1, -\sqrt{3}, 2\sqrt{3}), \quad g^3 = \frac{1}{4}(2-\varepsilon, \sqrt{\varepsilon(4-\varepsilon)}, 2\sqrt{3}),$$

with $\varepsilon \in [0, 1]$. It is easy to compute

$$\langle g^1, g^2 \rangle = \frac{5}{8}, \quad \langle g^1, g^3 \rangle = \frac{5}{8} + \frac{\varepsilon + \sqrt{3\varepsilon(4-\varepsilon)}}{16} > \frac{5}{8}, \quad \langle g^2, g^3 \rangle = \frac{5}{8} + \frac{\varepsilon - \sqrt{3\varepsilon(4-\varepsilon)}}{16} < \frac{5}{8}\,.$$

We take ε close enough to 0 so that the condition $\gamma \geq \Gamma^2$ of Proposition 9 holds. According to this proposition (which is stated a few lines below), all pairs of generators are critical. Since the cone K is strictly acute, all antipodal pairs are formed with a couple of generators. We conclude that $\theta_{\max}(K) = \arccos \langle g^2, g^3 \rangle$. The critical pair (g^1, g^2) makes an angle arbitrarily close to $\theta_{\max}(K)$ for ε close enough to 0, but is not a Nash angular equilibrium because $\langle g^2, g^3 \rangle < \langle g^2, g^1 \rangle$.

5 Angular Spectra of Polyhedral Cones

In this section we address several questions concerning the angular spectrum of a polyhedral cone K, which we represent always in the form (10) with the usual assumptions on the set of generators $\{g^1, \ldots, g^p\}$.

When is a pair of generators critical? This issue has been already addressed in [3, Section 7]. We just state here a minor technical result which will be used in the sequel.

Proposition 9. *Suppose that $K \subset \mathbb{R}^d$ is a polyhedral cone with generators $\{g^1, \ldots, g^p\}$. Let*

$$\gamma = \min_{\substack{1 \leq k, \ell \leq p \\ k \neq \ell}} \langle g^k, g^\ell \rangle \quad \text{and} \quad \Gamma = \max_{\substack{1 \leq k, \ell \leq p \\ k \neq \ell}} \langle g^k, g^\ell \rangle.$$

If $\gamma \geq \Gamma^2$, then K is acute and all pairs (g^i, g^j) $(1 \leq i, j \leq p)$ are critical.

Proof. Checking (4)-(5) for $(\bar{u}, \bar{v}) = (g^i, g^j)$ is the same as verifying

$$\langle g^i, g^j \rangle \langle g^j, g^q \rangle \leq \langle g^i, g^q \rangle, \tag{25}$$

$$\langle g^i, g^j \rangle \langle g^i, g^q \rangle \leq \langle g^j, g^q \rangle, \tag{26}$$

for all triplets (i, j, q). Let us check, for instance, the inequality (25). To avoid trivialities, one can assume that the indices i, j, q are different. Under the hypothesis $\gamma \geq \Gamma^2$, all the inner products $\langle g^k, g^\ell \rangle$ are nonnegative and

$$\langle g^i, g^j \rangle \langle g^j, g^q \rangle \leq \Gamma^2 \leq \gamma \leq \langle g^i, g^q \rangle.$$

A similar argument shows that (26) holds. □

We now explain how to compute the critical angles of a polyhedral cone by solving a family of Perron-type eigenvalue problems. We need first to introduce some additional notation and terminology. The symmetric positive semidefinite matrix

$$M = \left[\langle g^i, g^j \rangle\right]_{i,j=1,\ldots,p}$$

of size $p \times p$ is referred to as the Gramian matrix associated to the set $\{g^1, \ldots, g^p\}$. For nonempty subsets $I, J \subset \{1, \ldots, p\}$, denote by $M_{I,J}$ the principal matrix of M obtained by deleting the i-th row and the

j-th column of M, whenever $i \notin I$ and $j \notin J$. If the symbol $|I|$ stands for the cardinality of I, then $M_{I,J}$ is a rectangular matrix of size $|I| \times |J|$. Because the generators $\{g^1, \ldots, g^p\}$ are not necessarily linearly independent, it is helpful to write

$$I \in \mathcal{M}(g^1, \ldots, g^p) \iff \begin{cases} I \subset \{1, \ldots, p\} \text{ is nonempty and the set} \\ \{g^i : i \in I\} \text{ is linearly independent.} \end{cases}$$

Notice that $M_{I,I}$ is the Gramian matrix of the sub-collection $\{g^i : i \in I\}$. So, for $I \in \mathcal{M}(g^1, \ldots, g^p)$, the matrix $M_{I,I}$ is nonsingular.

Theorem 3. *Let $K \in \Xi(\mathbb{R}^d)$ be a polyhedral cone with generators $\{g^1, \ldots g^p\}$ and let M be the corresponding Gramian matrix. For $\theta \in]0, \pi[$ the following two statements are equivalent:*

(a) θ is a proper critical angle of K,

(b) there are sets $I, J \in \mathcal{M}(g^1, \ldots, g^p)$, with $I \neq J$, and vectors $\xi \in \operatorname{int}(\mathbb{R}_+^{|I|})$, $\eta \in \operatorname{int}(\mathbb{R}_+^{|J|})$ such that

$$\begin{bmatrix} 0 & M_{I,J} \\ M_{J,I} & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \cos\theta \begin{bmatrix} M_{I,I} & 0 \\ 0 & M_{J,J} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \tag{27}$$

$$\sum_{j \in J} M_{kj}\eta_j - \cos\theta \sum_{i \in I} M_{ki}\xi_i \geq 0 \quad \forall k \notin I,$$

$$\sum_{i \in I} M_{li}\xi_i - \cos\theta \sum_{j \in J} M_{lj}\eta_j \geq 0 \quad \forall l \notin J,$$

$$\langle \xi, M_{I,I}\xi \rangle = 1,$$

$$\langle \eta, M_{J,J}\eta \rangle = 1.$$

Furthermore, when these equivalent statements hold, the proper critical angle θ is formed with the proper critical pair $(\bar{u}, \bar{v}) = \left(\sum_{i \in I} \xi_i g^i, \sum_{j \in J} \eta_j g^j\right)$.

Proof. The implication (a) $\Rightarrow$ (b) has been shown in our previous work [3]. A careful examination of that proof shows that the reverse implication is also true. □

The next corollary is a localization result for angular spectra of polyhedral cones in terms of Perron-type spectra of square matrices. One says that $\lambda \in \mathbb{R}$ is a Perron-type eigenvalue of the square matrix E if the system $Ez = \lambda z$ has a solution z in the interior of the positive orthant. For convenience we denote by $\sigma_{\text{int}}(E)$ the set of Perron-type eigenvalues of E.

Corollary 3. *Let $K \in \Xi(\mathbb{R}^d)$ be a polyhedral cone with generators $\{g^1, \ldots, g^p\}$ and let M be the corresponding Gramian matrix. Then,*

$$\Omega(K) \subset \bigcup_{I,J} \left\{ \arccos\, \lambda : \ \lambda \in \sigma_{\text{int}}(B^{I,J}) \right\}, \tag{28}$$

where the union is taken with respect to $I, J \in \mathcal{M}(g^1, \ldots, g^p)$ with $I \neq J$, and

$$B^{I,J} = \begin{bmatrix} M_{I,I} & 0 \\ 0 & M_{J,J} \end{bmatrix}^{-1} \begin{bmatrix} 0 & M_{I,J} \\ M_{J,I} & 0 \end{bmatrix} = \begin{bmatrix} 0 & M_{I,I}^{-1} M_{I,J} \\ M_{J,J}^{-1} M_{J,I} & 0 \end{bmatrix}. \tag{29}$$

Proof. This follows directly from Theorem 3 by identifying λ with $\cos\theta$. We are keeping the eigenvalue equation (27) and dropping the remaining conditions. $\square$

Remark 4. In fact, the index sets $I, J \in \mathcal{M}(g^1, \ldots, g^p)$ can be chosen in a more restrictive manner. For instance, there is no loss of generality in assuming that

$$\max\{|I|, |J|\} \leq \dim[\mathrm{span}(K)] - 1, \tag{30}$$

$$I \not\subset J, \quad J \not\subset I \tag{31}$$

If K happens to be acute, then the requirement (31) can be reinforced by writing instead $I \cap J = \emptyset$. The reason behind (30) is that the components of a critical pair are not to be sought in the relative interior of the cone. The notation $\mathrm{span}(K)$ refers, of course, to the vector space spanned by K. The justification of (31), which is a bit more technical, can be found implicitly in our work [5].

Needless to say, evaluating the right-hand side of (28) involves a huge amount of numerical work. To save time and computational effort, it is convenient to keep in mind that the matrices $B^{I,J}$ and $B^{J,I}$ may well be different, but they yield exactly the same eigenvalues. So, if the system (27) has been worked out for a given choice (I, J), then it is superfluous to do similar computations with the pair (J, I). Geometrically speaking, exchanging the order of I and J corresponds to exchanging the order of u and v.

5.1 Equi-Angular Polyhedral Cones

In this section we consider a polyhedral cone K generated by an equi-angular collection $\{g^1, \ldots, g^p\}$ of unit vectors in $\mathbb{R}^d$. Equi-angularity of the set of generators simply means that

$$\text{there is a constant } \psi \in]0, \pi[\text{ such that } \langle g^i, g^j \rangle = \cos\psi \quad \forall i \neq j. \tag{32}$$

By abuse of language, one applies the adjective "equi-angular" to the cone K itself. Of course, one recognizes the term ψ as being the common angle formed by the generators.

The equi-angularity assumption (32) imposes a restriction on the number of generators. Indeed, the integer p is bounded from above by a constant depending on the angle ψ and the dimension d of the underlying Euclidean space. This point is clarified in the next proposition. We start by writing:

Lemma 3. *Consider an equi-angular collection $\{g^1, \ldots, g^p\}$ of unit vectors with $\psi \in]0, \pi[$ as common angle. Let M be the Gramian matrix associated to this collection of generators. Then,*

$$\det(M) = (1 - \cos\psi)^{p-1}\big\{1 + (p-1)\cos\psi\big\}. \tag{33}$$

Furthermore, p and ψ are bound by the relation

$$1 + (p-1)\cos\psi \geq 0. \tag{34}$$

Proof. Note that $M = (1 - \cos\psi)\big\{I + [\cos\psi/(1 - \cos\psi)]\,(ee^T)\big\}$, with $e = (1, 1, \ldots, 1)$. Then the formula in (33) follows from the well known fact that $\det(A + uv^T) = (1 + v^T A^{-1} u)\,\det(A)$ for all invertible $A \in \mathbb{R}^{p\times p}$ and all $u, v \in \mathbb{R}^p$ (see, e.g., [2]). The binding constraint (34) is a consequence of the positive semidefiniteness of the Gramian matrix M. $\square$

An angle $\psi \in]0, \pi[$ is declared *unstable* if $\cos\psi \in \{-1/2, -1/3, -1/4, \ldots\}$, otherwise it is said to be *stable*. The concept of instability is motivated by the possibility of the binding constraint (34) being active. Notice that only obtuse angles can be unstable.

Proposition 10. *Let $\{g^1, \ldots, g^p\} \subset \mathbb{R}^d$ be an equi-angular collection of unit vectors with $\psi \in]0, \pi[$ as common angle. One has:*

(a) if ψ is stable, then $p \leq d$.

(b) if ψ is unstable, then $p \leq d + 1$.

Proof. If ψ is stable, then the Gramian matrix M has a nonzero determinant. So, the vectors $\{g^1, \ldots, g^p\}$ are linearly independent and necessarily $p \leq d$. Suppose now that ψ is unstable. Then $1 + (q-1)\cos\psi = 0$ for some $q \in \{2, 3, \ldots\}$. If $q \neq p$, then we are still under the condition $\det(M) \neq 0$, so one gets again $p \leq d$. If $q = p$, then by dropping one generator from the collection $\{g^1, \ldots, g^p\}$ one gets a subcollection having a Gramian matrix with nonzero determinant. This subcollection of $p - 1$ vectors is then linearly independent and therefore $p - 1 \leq d$. $\square$

We now come back to the main stream of our discussion. A striking feature of acute equi-angular cones is that any pair (g^i, g^j) of generators happens to be critical. This can be checked with the help of Proposition 9. In the next theorem we shall see how to construct other pairs whose criticality is less obvious to prove. The basic idea is to pick suitable index sets I, J and form

$$(u_I, u_J) = \left(\frac{1}{\| \sum_{i \in I} g^i \|} \sum_{i \in I} g^i, \frac{1}{\| \sum_{j \in J} g^j \|} \sum_{j \in J} g^j \right).$$

Notice that (u_I, u_J) is well defined whenever $\sum_{i \in I} g^i \neq 0$ and $\sum_{j \in J} g^j \neq 0$. Geometrically speaking, the unit vectors u_I and u_J correspond to the barycenters of the polyhedral subcones $K_I = \text{cone}\{g^i : i \in I\}$ and $K_J = \text{cone}\{g^j : j \in J\}$, respectively.

Theorem 4. *Let K be an acute cone generated by an equi-angular collection $\{g^1, \ldots, g^p\}$ of unit vectors in $\mathbb{R}^d$. Pick disjoint index sets $I, J \subset \{1, \ldots, p\}$ such that $\sum_{i \in I} g^i \neq 0$ and $\sum_{j \in J} g^j \neq 0$. Then, (u_I, u_J) is a critical pair of K.*

Proof. Let ψ be the common angle formed by the generators, and let $b = \cos\psi$. For notational convenience, we set

$$c_{I,J} = \Big\langle \sum_{i \in I} g^i, \sum_{j \in J} g^j \Big\rangle, \quad c_{I,I} = \| \sum_{i \in I} g^i \|^2, \quad c_{J,J} = \| \sum_{j \in J} g^j \|^2.$$

For arbitrary disjoint index sets $I, J \subset \{1, \ldots, p\}$, one gets

$$\begin{aligned} c_{I,J} &= b\,|I|\,|J|, \\ c_{I,I} &= |I|\,\big\{1 + (|I| - 1)b\big\}, \\ c_{J,J} &= |J|\,\big\{1 + (|J| - 1)b\big\}. \end{aligned} \tag{35}$$

Since $\sum_{i \in I} g^i \neq 0$ and $\sum_{j \in J} g^j \neq 0$, the pair (u_I, u_J) is well defined and

$$\langle u_I, u_J \rangle = \frac{c_{I,J}}{\sqrt{c_{I,I}}\sqrt{c_{J,J}}} = \frac{b\,|I|\,|J|}{\sqrt{|I|}\sqrt{1 + (|I| - 1)b}\,\sqrt{|J|}\sqrt{1 + (|J| - 1)b}}.$$

A matter of simplification leads to the expression

$$\langle u_I, u_J \rangle = \frac{b\sqrt{|I|}\sqrt{|J|}}{\sqrt{1 + (|I| - 1)b}\,\sqrt{1 + (|J| - 1)b}}. \tag{36}$$

Let us prove that $u_I - \langle u_I, u_J\rangle u_J \in K^+$, i.e., that $\Delta_k(u_I,u_J) := \langle u_I, g^k\rangle - \langle u_I,u_J\rangle\langle u_J, g^k\rangle$ is nonnegative for each $k \in \{1,\ldots,p\}$. Consider first the case $k \in J$. A short computation shows that

$$\Delta_k(u_I,u_J) = \frac{b|I|}{\sqrt{c_{I,I}}} - \frac{b\sqrt{|I|}\sqrt{|J|}}{\sqrt{1+(|I|-1)b}\,\sqrt{1+(|J|-1)b}}\,\frac{1+(|J|-1)b}{\sqrt{c_{J,J}}}$$

After plugging (35) into the above line and simplifying, one gets $\Delta_k(u_I,u_J) = 0$. Consider now the case $k \notin J$. This time one has

$$\begin{aligned}
\Delta_k(u_I,u_J) &= \frac{1+(|I|-1)b}{\sqrt{c_{I,I}}} - \frac{b\sqrt{|I|}\sqrt{|J|}}{\sqrt{1+(|I|-1)b}\,\sqrt{1+(|J|-1)b}}\,\frac{b|J|}{\sqrt{c_{J,J}}} \qquad \text{if } k\in I,\\
\Delta_k(u_I,u_J) &= \frac{b|I|}{\sqrt{c_{I,I}}} - \frac{b\sqrt{|I|}\sqrt{|J|}}{\sqrt{1+(|I|-1)b}\,\sqrt{1+(|J|-1)b}}\,\frac{b|J|}{\sqrt{c_{J,J}}} \qquad \text{if } k\notin I.
\end{aligned}$$

We recall again (35) and simplify. Regardless of whether I contains k or not, one ends up with $\Delta_k(u_I,u_J) > 0$. The details are omitted. The proof of the symmetric condition $u_J - \langle u_I,u_J\rangle u_I \in K^+$ is similar. The conclusion is that (u_I,u_J) is a critical pair of K. □

The main merit of Theorem 4 is providing a simple way of constructing critical pairs in a polyhedral cone with special structure. As shown in the next result, the critical angle formed by (u_I,u_J) depends on the choice of I and J only in terms of the cardinality of these sets. The next theorem says however more than that: it provides a full characterization of the angular spectrum of an acute equi-angular cone.

Theorem 5. *Let K be generated by an equi-angular collection $\{g^1,\ldots,g^p\}$ of unit vectors in $\mathbb{R}^d$. Suppose that the common angle ψ formed by the generators is acute, more precisely, $\psi \in]0,\pi/2[$. Then, the angular spectrum of K is given by*

$$\Omega(K) = \{\theta_{\ell,k} :\ 1\le \ell\le k\le p-1,\ \ell+k\le p\}, \tag{37}$$

where

$$\theta_{\ell,k} = \arccos\left[\frac{\sqrt{\ell k}\cos\psi}{\sqrt{1+(\ell-1)\cos\psi}\,\sqrt{1+(k-1)\cos\psi}}\right]. \tag{38}$$

Furthermore, $\theta_{\max}(K) = \psi$ is the only Nash angle of K, i.e. all other critical angles are ordinary.

Proof. The acuteness assumption rules out the possibility of instability in ψ. Hence, the determinant of the Gramian matrix M is nonzero. The generators are then linearly independent and, in particular, $\sum_{i\in I} g^i \neq 0$ and $\sum_{j\in J} g^j \neq 0$ for any choice of I and J. Pick disjoint index sets I, J, and denote their cardinality by ℓ and k, respectively. If one plugs this information in (36), then one gets the critical angle $\theta_{\ell,k}$ given in (38). We write the condition $\ell+k\le p$ because the index sets I and J are required to be disjoint. The condition $\ell\le k$ takes care of the symmetry relation $\theta_{\ell,k} = \theta_{k,\ell}$.

We have proven so far that $\theta_{\ell,k}\in\Omega(K)$ for any pair (ℓ,k) as in (37). We now must check that Ω doesn't contain other critical angles. This fact can be proven directly by invoking [5, Theorem 6], a quite technical and difficult result concerning the uniqueness of critical angles for a given pair (I,J). Another possibility is exploiting Theorem 3 and the acuteness assumptions made on ψ. According to Theorem 3 and Remark 4, if θ is a proper critical angle of K, then for a suitable pair (I,J) of disjoint index sets, the term $\cos\theta$ will be a Perron-type eigenvalue of the matrix $B^{I,J}$ given by (29). The acuteness assumption and a little bit of standard linear algebra show that the matrix $B^{I,J}$ admits at most one Perron-type eigenvalue. In short, a given choice (I,J), with $|I|=\ell, |J|=k$, produces the critical pair $\theta_{\ell,k}$ and no other.

In relation to the last part of the theorem, two comments will do. Firstly, the proper critical angle ψ is obtained by setting $\ell=1$ and $k=1$, that is to say, ψ is formed with a pair of generators of K. In view of the expression (38), the angle ψ is clearly the largest element in $\Omega(K)$. And, secondly, Corollary 1 shows that the critical angle $\theta_{\ell,k}$ is of the ordinary type if $(\ell,k)\neq(1,1)$. □

Notice that $\theta_{\ell,k}\neq\theta_{\ell',k'}$ whenever $(\ell,k)\neq(\ell',k')$. This simple observation leads to the following cardinality result.

Corollary 4. *Let K be generated by an equi-angular collection of p unit vectors in $\mathbb{R}^d$. If the common angle formed by the generators belongs to $]0,\pi/2[$, then the angular spectrum of K has exactly*

$$m_p = \left\lfloor \frac{p}{2} \right\rfloor \left(p - \left\lfloor \frac{p}{2} \right\rfloor\right)$$

elements, with $\lfloor\cdot\rfloor$ denoting the lower integer part function.

Proof. It suffices to apply Theorem 5 and count the number of pairs (ℓ,k) as in (37). To get the number m_p one must work out the expression

$$\sum_{\substack{1\leq \ell\leq k\leq p-1\\ \ell+k\leq p}} 1 \;=\; \sum_{\ell=1}^{\lfloor p/2\rfloor} \{p-1-2(\ell-1)\}.$$

□

Remark 5. Notice that m_p increases with respect to p. For constructing a polyhedral cone with a large number of critical angles, we take p as big as possible. According to Proposition 10(a) we can go up to $p=d$ but not beyond this threshold. To see that it is always possible to construct an acute equi-angular polyhedral cone in $\mathbb{R}^d$ with exactly d generators, consider for instance the vectors

$$\begin{aligned}
g^1 &= (\sqrt{1-(d-1)a^2}, a, a, \ldots, a),\\
g^2 &= (a, \sqrt{1-(d-1)a^2}, a, \ldots, a),\\
&\;\;\vdots\\
g^d &= (a, a, \ldots, a, \sqrt{1-(d-1)a^2}),
\end{aligned}$$

with $0<a<(d-1)^{-1/2}$. Notice that m_d corresponds roughly to $d^2/4$ when d is large.

Remark 6. It is possible to exhibit a strictly acute polyhedral cone in $\mathbb{R}^d$ with more than m_d critical angles, but this can happen only outside the class of equi-angular cones. We suggest slightly perturbing the generators $\{g^1,\ldots,g^d\}$ of Remark 5, namely, in the i-th generator g^i we change a by $a(1+\varepsilon_i)$. The perturbation parameters $\varepsilon_1>0,\ldots,\varepsilon_d>0$ are taken small enough and such that $\langle g^i,g^j\rangle \neq \langle g^{i'},g^{j'}\rangle$ whenever $(i,j)\neq(i',j')$. According to Proposition 9, every pair of generators is critical. So, we have constructed in this way a strictly acute polyhedral cone having at least $d(d-1)/2$ proper critical angles.

6 Final Remarks

While dealing with acute equi-angular polyhedral cones we know how to compute exactly all the critical angles and how to recognize those that are of the Nash type. As we have seen in Section 5, the task of identifying critical angles in a general polyhedral cone can be done with the help of Theorem 3. Identifying Nash angles is in general quite problematic and one must proceed with extreme care.

It is very tempting to conjecture that the "big" critical angles correspond to Nash angles and the "small" critical angles are of the ordinary type. It would make life much easier if for each convex cone K there is a number κ (depending on K) such that $\Omega_{\rm ord}(K) \subset]0,\kappa[$ and $\Omega_{\rm nash}(K) \subset [\kappa,\pi[$. As illustrated in the next example, it is not always possible to separate the sets $\Omega_{\rm ord}(K)$ and $\Omega_{\rm nash}(K)$ by means of two disjoint intervals. More precisely, in any space $\mathbb{R}^d$ (with $d \geq 3$) one can construct a polyhedral cone having an ordinary critical angle located between two Nash angles.

Example 2. For simplicity we take $d = 3$ but this choice is not essential. We will construct a polyhedral cone $K \subset \mathbb{R}^3$ generated by 5 unit vectors. The first 4 vectors are:

$$g^1 = (-\sqrt{3}/3, 0, \sqrt{6}/3), \qquad g^2 = (0, 1/2, \sqrt{3}/2),$$
$$g^3 = (\sqrt{3}/3, 0, \sqrt{6}/3), \qquad g^4 = (0, -1/2, \sqrt{3}/2).$$

A simple computation shows that these vectors have unit length and

$$\langle g^1, g^3\rangle = 1/3 \approx 0.333, \quad \langle g^2, g^4\rangle = 0.5,$$
$$\langle g^2, g^3\rangle = \langle g^1, g^4\rangle = \langle g^3, g^4\rangle = \langle g^1, g^2\rangle = \sqrt{2}/2 \approx 0.707.$$

So, the pair (g^1, g^3) is a good candidate for antipodality and the pair (g^2, g^4) has some chances of being a Nash angular equilibrium. The last vector g^5 will be constructed so as to create an ordinary critical angle between $\arccos\langle g^2, g^4\rangle$ and $\arccos\langle g^1, g^3\rangle$. We consider a unit vector

$$g^5 = \left(\frac{\sqrt{3}}{3} - \varepsilon, \sqrt{\frac{7\sqrt{3}}{6}\varepsilon - \frac{3}{2}\varepsilon^2}, \sqrt{\frac{2}{3} - \frac{\sqrt{3}}{2}\varepsilon + \frac{1}{2}\varepsilon^2}\right)$$

depending on a parameter $\varepsilon > 0$. This choice of g^5 has been obtained after a long and tedious series of technical evaluations. It it not worthwhile entering into the details on how we arrived at such an expression. Anyway, observe that if ε gets close to 0, then g^5 approaches g^3. A matter of computation shows that

$$\langle g^1, g^5\rangle = \frac{-1}{3} + \frac{\sqrt{3}}{3}\varepsilon + \sqrt{\frac{4}{9} - \frac{\sqrt{3}}{3}\varepsilon + \frac{1}{3}\varepsilon^2} \tag{39}$$

$$\langle g^3, g^5\rangle = \frac{1}{3} - \frac{\sqrt{3}}{3}\varepsilon + \sqrt{\frac{4}{9} - \frac{\sqrt{3}}{3}\varepsilon + \frac{1}{3}\varepsilon^2} \tag{40}$$

For a small ε one ends up with

$$\begin{aligned} 0 < \langle g^1, g^3\rangle < \langle g^1, g^5\rangle < \langle g^2, g^4\rangle &< \langle g^2, g^3\rangle = \langle g^1, g^4\rangle = \langle g^3, g^4\rangle = \langle g^1, g^2\rangle, \\ \langle g^1, g^3\rangle &\leq \min\{\langle g^2, g^5\rangle, \langle g^3, g^5\rangle, \langle g^4, g^5\rangle\}, \\ \langle g^2, g^4\rangle &\leq \min\{\langle g^2, g^5\rangle, \langle g^5, g^4\rangle\}. \end{aligned}$$

Given that K is strictly acute, all antipodal pairs are composed of generators (cf. Corollary 1). We conclude that the only antipodal pair of K is (g^1, g^3). Concerning (g^2, g^4), observe that

$$\langle g^2, g^4 \rangle \leq \langle g^2, g^i \rangle \quad \text{and} \quad \langle g^2, g^4 \rangle \leq \langle g^i, g^4 \rangle,$$

for every $i \in \{1, \ldots, 5\}$. By using Proposition 3, one deduces that (g^2, g^4) is a Nash angular equilibrium. Summarizing, we have a sandwich situation

$$\arccos\langle g^2, g^4 \rangle < \arccos\langle g^1, g^5 \rangle < \arccos\langle g^1, g^3 \rangle$$

in which both extremes are Nash angles. Concerning the angle in the middle of the sandwich, notice that (g^1, g^5) is not a Nash angular equilibrium because $\langle g^1, g^3 \rangle < \langle g^1, g^5 \rangle$. So, if $\arccos\langle g^1, g^5 \rangle$ happens to be a critical angle, then it must be of the ordinary type. In view of [3, Proposition 7.1], for (g^1, g^5) to be a critical pair it is necessary and sufficient that

$$\langle g^1, g^i \rangle \geq \langle g^1, g^5 \rangle \langle g^5, g^i \rangle, \tag{41}$$

$$\langle g^5, g^i \rangle \geq \langle g^1, g^5 \rangle \langle g^1, g^i \rangle, \tag{42}$$

for every $i \in \{1, \ldots, 5\}$. Checking this system is painful and time consuming. However, if ε is small enough, the inequalities in (42) are immediate. The inequalities in (41) are also easy to check except for $i = 3$, i.e.

$$\langle g^1, g^3 \rangle \geq \langle g^1, g^5 \rangle \langle g^5, g^3 \rangle.$$

By using (39)-(40), one gets after some simplification

$$\langle g^1, g^5 \rangle \langle g^5, g^3 \rangle = \frac{1}{3} - \frac{\sqrt{3}}{9}\,\varepsilon < \frac{1}{3} = \langle g^1, g^3 \rangle.$$

This confirms that (g^1, g^5) is indeed a critical angle.

We hope we have not frightened the reader with the technicalities of Example 2. Some aspects of the theory of critical angles are unfortunately highly technical and sometimes we must use brute force to arrive at the desired conclusion. However, most of the theory is built up by using powerful and elegant tools of geometry and analysis.

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