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STOCHASTIC GALERKIN METHOD FOR ELLIPTIC SPDES: A WHITE NOISE APPROACH

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Abstract. An equation that arises in mathematical studies of the transport of pollutants in groundwater and of oil recovery processes is of the form: $-\nabla_x \cdot (\kappa(x,\cdot)\nabla_x u(x,\omega)) = f(x)$, for $x \in D$, where $\kappa(x,\cdot)$, the permeability tensor, is random and models the properties of the rocks, which are not know with certainty. Further, geostatistical models assume $\kappa(x,\cdot)$ to be a log-normal random field. The use of Monte Carlo methods to approximate the expected value of $u(x,\cdot)$, higher moments, or other functionals of $u(x,\cdot)$, require solving similar system of equations many times as trajectories are considered, thus it becomes expensive and impractical. In this paper, we present and explain several advantages of using the *White Noise* probability space as a natural framework for this problem. Applying properly and timely the Wiener-Itô Chaos decomposition and an eigenspace decomposition, we obtain a symmetric positive definite linear system of equations whose solutions are the coefficients of a Galerkin-type approximation to the solution of the original equation. Moreover, this approach reduces the simulation of the approximation to $u(x,\omega)$ for a fixed ω, to the simulation of a finite number of independent normally distributed random variables.

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1. **Introduction.** In the mathematical studies of the transport of pollutants in groundwater and of oil recovery processes one faces a system of stochastic partial differential equations, which models the two-phase flow in a porous medium. The system is composed of two equations, a transport equation for the saturation (the relative volume of one of the two fluids) coupled with an equation for the velocity field, which is given by Darcy's Law and the incompressibility condition of the flow. The randomness enters the problem through the unknown properties of the rocks, especially the permeability tensor. With no sources of sink, and neglecting the effects of gravity and capillarity these equations are of the form(see [3, 4, 5]):

$$\mathbf{u} = \lambda(s)K\nabla p \qquad \nabla\cdot\mathbf{u} = 0$$
$$\frac{\partial s}{\partial t} + \nabla\cdot(F(s)\mathbf{u}) = 0$$

Here, $\mathbf{u}$ is the total seepage velocity, s is the water saturation, K is the permeability, and p is the pressure. The constitutive functions $\lambda(s)$ and $F(s)$ represent the total mobility and the fractional flow of water. In Furtado-Pereira ([3, 4, 5]), they take K to be a log-normal random field so that $\xi(x,\cdot) = \log K(x,\cdot)$ is Gaussian so its distribution is determined by its mean $E[\xi(x,\cdot)] = 0$, and its covariance function $C_\xi(x,y) = E[\xi(x,\cdot)\xi(y,\cdot)]$. They assumed also that $\{\xi(x,\cdot)\}_{x\in D}$ is stationary, isotropic and fractal. In this paper we deal with one of the equations derived from (1), specifically we consider an equation of the form:

$$\begin{cases} -\nabla_x\cdot(\kappa(x,\cdot)\nabla_x u(x,\cdot)) = f(x), \text{ for } x\in D \\ \\ u(x,\cdot) = 0, \text{ on } \partial D, \end{cases} \tag{1}$$

where $\kappa(x,\cdot) = \rho_0 + e^{W_\phi(x,\cdot)}$, for some $\rho_0 > 0$, and $W_\phi(x,\cdot)$ is the well-known 1-dimensional *White Noise*, which is a stationary Gaussian process. The permeability function $\kappa(x,\cdot)$ also might be the one characterized by its covariance function $C_\kappa(x,y) \doteq E[(\kappa(x,\cdot) - E[\kappa(x,\cdot)])(\kappa(y,\cdot) - E[\kappa(y,\cdot)])]$, and not the process $\xi(x,\cdot)$ in the exponent. Knowing the covariance function $C_\kappa(x,y)$ is the same assumption taken in [1, 2, 10, 14, 15]. Here in this paper we consider both approaches since all the calculations can be carried with very little changes if one considers the covariance of $\xi(x,\cdot)$ instead of the one for $\kappa(x,\cdot)$ or vice-versa. With these assumptions $\kappa(x,\cdot)$ satisfies:

$$0 < \rho_0 < \kappa(x,\omega) \tag{2}$$

for all $x\in D$, and ω in the probability space where $W_\phi(x,\cdot)$ is defined. The addition of the positive constant ρ_0 is to guarantee the existence of the solution and, in practice, this is what is done when numerical solutions of the coupled system (1) are to be found. Besides, notice that the positive constant ρ_0 added to the original permeability function does not change the form of the covariance function. We emphasize that the assumption of the stochastic structure of the permeability function $\kappa(x,\cdot)$ is due, more than anything else, to lack of accuracy in the measurements of the media. Further, the isotropic assumption and specific forms of the covariance function are features suggested by empirical data. On the other hand, observe that in principle, given a trajectory or path of the permeability function, i.e., $\kappa(x,\omega) : \mathbb{R}^d \to \mathbb{R}$, for fixed $\omega\in\Omega$, we can approximate the solution of (1) by a finite element technique. This procedure will produce a linear system whose dimensions depend on the discretization implemented. In any case, typically these are very large systems, then in order to approximate the expected value of $u(x,\cdot)$, or other moments, we could use Monte Carlo methods, in which case, we will have to solve a similar system of equations as many times as trajectories are going to be used in the Monte Carlo approximation. This whole work would be expensive and slow since the stiffness matrices associated to each trajectory are distinct and therefore need to be assembled and factorized each time. To find alternatives ways to solve (1) using less expensive and faster methods is still a very interesting research problem for many scientists and engineers (see [1, 2, 3, 4, 6, 7, 8, 10, 14, 15] among others). One attempt to approach this problem without having to solve very large linear systems many times, is considered in the work of Babuška in [1, 2], where they proposed an auxiliary deterministic partial differential equation in higher dimension, whose solution is an approximation to the solution of (1), but, under very restrictive assumptions on the function $\kappa(x,\cdot)$. In fact, they consider (1) on a bounded domain D in $\mathbb{R}^d$, and assumed that there exist positive constants α_1, α_2, such that $\alpha_1 \le \kappa(x,\cdot) \le \alpha_2$, $\mathbb{P}$-almost everywhere, for all $x \in D$. They further assumed that $\kappa(x,\cdot)$ has an expansion of the form

$$\kappa(x,\cdot) = E[\kappa(x,\cdot)] + \sum_{n=1}^{\infty} b_n(x) X_n(\cdot), \tag{3}$$

where the X_n are mutually independent random variables such that $E[X_n] = 0$, $E[X_n^2] = 1$, and the $b_n(x)$ are uniformly bounded functions. In addition, they require that the $X_{n's}$ have bounded images $\Gamma_n \doteq X_n(\Omega)$, with $\Omega_n = (-\gamma_n, \gamma_n)$, $\gamma_n > 0$. Furthermore, they impose yet another restriction on the coefficients of this expansion such that they can consider a finite truncation which satisfies also the ellipticity condition. These assumptions are not satisfied neither in model (1) nor in other permeability functions used in practice, indeed they are very restrictive. On the other hand, in [6, 10, 14] a broader and similar approach to ours is taken, however, they do not explore the specific form of our κ, and moreover, the equations they get after truncating terms in the expansion for κ is not guaranteed to satisfy a condition like (2), and thus the existence of their approximation is questionable. Here, we present and explain several features which show the advantages of using the *white noise* probability space as a natural framework for this problem. Applying the Wiener-Itô Chaos decomposition, we obtain a symmetric positive definite linear system of equations whose solutions are the coefficients of a Galerkin-type approximation to the solution of the original equation. We would like to emphasize that a remarkable difference between the works in [1, 2, 10, 14] and ours, is that we do not need to truncate the permeability function $\kappa(x,\cdot)$, therefore the ellipticity condition is maintained. In addition, using the white noise approach, we propose a modification on the Karnunen-Loeve expansion in order to have the process $\kappa(x,\cdot)$ statistically stationary.

This paper is structured as follows: in Section 2 we introduce the framework under which these equations can be studied, it is included to make this manuscript more self-contained and accessible to an audience not necessarily familiar with white noise theory. Next, Section 3 presents and discuss the problem with more details and gives results on the Wiener-Itô chaos expansion of κ. Section 4 develops the variational formulation associated to the original problem (1) and then we introduce the finite-dimensional stochastic Galerkin method approximation. Finally, in Section 5 we explain how to choose the function ϕ for the white noise $W_\phi(x,.)$ so that a given covariance function is matched, and last, details on how to use the Karhunen-Loeve expansion to choose the underlying functions $\eta_{j'_s}$ in the construction of the white noise are given.

2. **Framework: White Noise.** Let d be a positive integer and $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing functions. $\mathcal{S}(\mathbb{R}^d)$ is a Fréchet space under the family of semi-norms:

$$\|f\|_{k,\alpha} \doteq \sup_{x\in\mathbb{R}^d} \big\{(1+|x|^k)|\partial^\alpha f(x)|\big\},$$

where k is nonnegative integer, $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multi-index of nonnegative integers and

$$\partial^\alpha f \doteq \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$$

where $|\alpha| = \alpha_1 \dots \alpha_d$. Denote by $\mathcal{S}'$ the dual of $\mathcal{S}(\mathbb{R}^d)$, which equipped with the weak-star topology is known as the space of *tempered distributions*. This space is the one we will use as our basic probability space. As events we will use the family $\mathcal{B}(\mathcal{S}'(\mathbb{R}^d))$ of Borel subsets of $\mathcal{S}'(\mathbb{R}^d)$, and the probability measure μ is given by the Bochner-Minlos theorem (see [9], page 12). We will call the triplet $(\mathcal{S}'(\mathbb{R}^d), \mathcal{B}(\mathcal{S}'(\mathbb{R}^d)), \mu)$ the 1-*dimensional white noise probability space*, and μ is called the *white noise measure*. The measure μ is also often called the (normalized) *Gaussian measure* on $\mathcal{S}'(\mathbb{R}^d)$. The measure μ has the following property

$$E_\mu\left[e^{i<\cdot,\phi>}\right] \doteq \int_{\mathcal{S}'} e^{i<\omega,\phi>} d\mu(\omega) = e^{-\frac{1}{2}\|\phi\|^2}, \tag{4}$$

for all $\phi \in \mathcal{S}(\mathbb{R}^d)$, where $\|\phi\|^2 = \|\phi\|^2_{L^2(\mathbb{R}^d)}$, $<\omega, \phi>= \omega(\phi)$ is the action of $\omega \in \mathcal{S}'(\mathbb{R}^d)$ on $\phi \in \mathcal{S}(\mathbb{R}^d)$, and E_μ denotes the expectation with respect to μ.

Remark 1. *Equation (4) says that: for any $\phi \in \mathcal{S}(\mathbb{R}^d)$, the random variable $\phi \longrightarrow <\cdot, \phi>$, from $L^2(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$, is normal distributed with zero mean and variance $\|\phi\|^2$.*

Remark 2. *We will denote the expectation with respect to the normalized Gaussian measure μ by E_μ.*

Further, the lemma below will be used throughout this paper:

Lemma 3. *([9], Lemma 2.1.2) Let $\eta_1, \dots, \eta_n$ be functions in $\mathcal{S}(\mathbb{R}^d)$ that are orthonormal in $L^2(\mathbb{R}^d)$. Let λ_n be the normalized Gaussian measure in $\mathbb{R}^n$, i.e.,*

$$d\lambda_n(x) = (2\pi)^{-n/2} e^{-\frac{1}{2}|x|^2} dx_1 \dots dx_n;\ x = (x_1, \dots, x_n) \in \mathbb{R}^n$$

Then the random variable

$$\omega \longrightarrow (<\omega, \eta_1>, \dots, <\omega, \eta_n>)$$

has distribution λ_n. Equivalently,

$$E_\mu\left[f(<\omega, \eta_1>, \dots, <\omega, \eta_n>)\right] = \int_{\mathbb{R}^n} f(x) d\lambda_n(x),\ \textit{for all } f \in L^1(\lambda_n) \tag{5}$$

Remark 4. *Lemma (3) implies that: if $\eta_1, \dots, \eta_n$ are orthonormal in $L^2(\mathbb{R}^d)$ then the random variables $<\cdot, \eta_1>, \dots, <\cdot, \eta_n>$ defined on the probability space $(\mathcal{S}'(\mathbb{R}^d), \mathcal{B}(\mathcal{S}'(\mathbb{R}^d)), \mu)$, are independent and normally distributed with zero mean and variance equal to one.*

Definition 5. *The* 1*-dimensional (d-parameter) smoothed white noise is the map*

$$w : \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d) \longrightarrow \mathbb{R}$$

given by

$$w(\phi) = w(\phi, \omega) =< \omega, \phi >,$$

for $\omega \in \mathcal{S}'(\mathbb{R}^d)$, $\phi \in \mathcal{S}(\mathbb{R}^d)$.

Using Lemma 3 it is not difficult to prove that if $\phi \in L^2(\mathbb{R}^d)$ and we choose $\phi_m \in \mathcal{S}(\mathbb{R}^d)$ such that $\phi_m \to \phi$ in $L^2(\mathbb{R}^d)$, then

$$< \omega, \phi > \doteq \lim_{m \to +\infty} < \omega, \phi_m >$$

exists in $L^2(\mu)$, and is independent of the choice of ϕ_m. Thus, the definition of smoothed white noise can be extended to functions in $L^2(\mathbb{R}^d)$.

Definition 6. *Using* $w(\phi, \omega)$ *we can construct a stochastic process, called the smoothed white noise process* $W_\phi(x, \omega)$, *as follows:*

$$W_\phi(x, \omega) \doteq w(\phi_x, \omega), \ x \in \mathbb{R}^d, \ \omega \in \mathcal{S}'(\mathbb{R}^d),$$

where $\phi_x(y) = \phi(y - x)$, *is the* x*-shift of* ϕ; $x, y \in \mathbb{R}^d$.

Remark 7. *Note that the process* $\{W_\phi(x, \cdot)\}_{x \in \mathbb{R}^d}$ *has the following properties:*

i): *If supp* $\phi_{x_1} \cap$ *supp* $\phi_{x_2} = \emptyset$, *then* $W_\phi(x_1, \cdot)$ *and* $W_\phi(x_2, \cdot)$ *are independent,*
ii): $\{W_\phi(x, \cdot)\}_{x \in \mathbb{R}^d}$ *is a stationary process,*
iii): *For each* $x \in \mathbb{R}^d$, *the random variable* $W_\phi(x, \cdot)$ *is normally distributed with mean* 0 *and variance* $\|\phi\|^2$.

So $\{W_\phi(x, \cdot)\}_{x \in \mathbb{R}^d}$ is indeed a mathematical model for what one usually intuitively thinks of as white noise. In explicit applications the test function or "*window*" ϕ can be chosen such that the diameter supp ϕ is the maximal distance within which $W_\phi(x_1, \cdot)$ and $W_\phi(x_2, \cdot)$ might be correlated.

2.1. **The Wiener-Itô Chaos Expansion.**

The *Hermite polynomials* $h_k(x)$ are defined by

$$h_k(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k}(e^{-x^2/2}); \text{ for } k = 0, 1, 2, \ldots. \tag{6}$$

for $k = 0, 1, 2, \ldots,$. The *Hermite functions* $\xi_k(x)$ are defined by

$$\xi_k(x) = \pi^{-1/4}((k-1)!)^{-1/2} e^{-x^2/2} h_{k-1}(\sqrt{2}x); \text{ for } k = 1, 2, \ldots. \tag{7}$$

The collection $\{\xi_k\}_{k=1}^\infty$ constitutes an orthonormal basis for $L^2(\mathbb{R})$ and $\xi_k \in \mathcal{S}(\mathbb{R})$ for all k. In the following we show how to construct a set of orthonormal fuctions $\eta_{j'_s}$ on $L^2(\mathbb{R}^d)$. We note that on the last section we propose an alternative way to define the $\eta_{j'_s}$ taking into account that a covariance function associated to the permeability stochastic field is given.

Let $\delta = (\delta_1, \ldots, \delta_d)$ denote d-dimensional multi-indices with $\delta_1, \ldots, \delta_d \in \mathbb{N}$. It follows that the family of tensors products

$$\xi_\delta \doteq \xi_{(\delta_1, \ldots, \delta_d)} \doteq \xi_{\delta_1} \otimes \cdots \otimes \xi_{\delta_d}; \ \delta \in \mathbb{N}^d$$

forms an orthonormal basis for $L^2(\mathbb{R}^d)$. Let $\delta^{(j)} = (\delta_1^{(j)}, \ldots, \delta_d^{(j)})$ be the jth multi-index in some fixed ordering of all d-dimensional multi-indices $\delta = (\delta_1, \ldots, \delta_d) \in \mathbb{N}^d$. We can, and will assume that this ordering has the property that

$$i < j \Longrightarrow \delta_1^{(i)} + \cdots + \delta_d^{(i)} \leq \delta_1^{(j)} + \cdots + \delta_d^{(j)}$$

i.e., the $\left\{\delta^{(j)}\right\}_{j=1}^\infty$ occur in increasing order. Now define

$$\eta_j \doteq \xi_{\delta^{(j)}} = \xi_{\delta_1^{(j)}} \otimes \cdots \otimes \xi_{\delta_d^{(j)}}; \ j = 1, \ldots. \tag{8}$$

. We will need to consider multi-index of arbitrary length. To simplify the notation, we regard multi-indices as elements of the space $(\mathbb{N}_0^{\mathbb{N}})_c$ of all sequences $\alpha = (\alpha_1, \alpha_2, \dots)$ with elements $\alpha_j \in \mathbb{N}_0$ and with compact support, i.e., with only finitely many $\alpha_j \neq 0$. We write $\mathcal{J} = (\mathbb{N}_0^{\mathbb{N}})_c$. Given $\alpha \in \mathcal{J}$, define the order and length of α, denoted by $d(\alpha)$ and $|\alpha|$ respectively, by

$$d(\alpha) \doteq \max\{j : \alpha_j \neq 0\}$$

and

$$|\alpha| \doteq \alpha_1 + \alpha_2 + \cdots + \alpha_{d(\alpha)}.$$

Definition 8. *Let $\alpha = (\alpha_1, \alpha_2, \dots) \in \mathcal{J}$. Then we define*

$$H_\alpha(\omega) \doteq \prod_{j=1}^{\infty} h_{\alpha_j}(<\omega, \eta_j>);\ \omega \in \mathcal{S}'(\mathbb{R}^d)$$

The following are two important results about the $\{H_\alpha\}_{\alpha\in\mathcal{J}}$:

Theorem 9. *([9], Theorem 2.2.3) The family $\{H_\alpha\}_{\alpha\in\mathcal{J}}$ constitutes an orthogonal basis for $L^2_\mu(\mathcal{S}')$. Moreover, if $\alpha \in \mathcal{J}$, we have the norm expression*

$$\|H_\alpha\|^2_{L^2_\mu(\mathcal{S}')} = \alpha! \doteq \alpha_1!\alpha_2!\dots.$$

and

Theorem 10. *(Wiener-Itô chaos expansion theorem, [9], Theorem 2.2.4) Every $f \in L^2_\mu(\mathcal{S}')$ has a unique representation*

$$f(\omega) = \sum_{\alpha\in\mathcal{J}} c_\alpha H_\alpha(\omega) \tag{9}$$

where $c_\alpha \in \mathbb{R}$ for all α. Moreover, we have the isometry

$$\|f\|^2_{L^2_\mu(\mathcal{S}')} = \sum_{\alpha\in\mathcal{J}} \alpha! c_\alpha^2 \tag{10}$$

3. **The Problem.** Given a function $\phi \in L^2(\mathbb{R}^d)$ (fixed), the problem that we want to consider is the following:

$$\begin{cases} -\nabla_x.\left(\kappa(x,w;\phi)\nabla_x u(x,w;\phi)\right) = f(x),\ \text{for } x \in D \\ \\ u(x,\cdot,\phi) = 0,\ \text{on } \partial D \end{cases} \tag{11}$$

for all $w \in \mathcal{S}'(\mathbb{R}^d)$, where $\kappa(x,\omega;\phi) \doteq \rho_0 + e^{W_\phi(x,\omega)}$, and $W_\phi(x,\omega)$ is the 1-dimensional smoothed white noise process defined on the 1-dimensional white noise probability space $\left(\mathcal{S}'(\mathbb{R}^d), \mathcal{B}(\mathcal{S}'(\mathbb{R}^d)), \mu\right)$ constructed in the previous section, and $\rho_0 > 0$. Thus, κ is the sum of a positive constant and a log-normal random process, i.e., $\kappa - \rho_0$ is the exponential of a Gaussian (i.e. normal) stochastic process. Observe that for different $\phi \in L^2(\mathbb{R}^d)$ there is a different permeability function $\kappa(\cdot,\cdot,\phi)$ associated to it. In what follows, we will omit, whenever there is no danger of confusion, the dependence of κ on the function ϕ just to make the notation less cumbersome. The log-normal assumption on $\kappa - \rho_0$ is the same as, up to normalization, the one used in [3, 4, 5], where in practice they take the absolute permeability K as $K(x,w) = \rho_0 + e^{\xi(x,w)}$, where $\{\xi(x,\cdot)\}_{x\in\mathbb{R}^d}$ is assumed to be a stationary, isotropic, and fractal Gaussian field characterized by its mean $E\xi(x,\cdot) = 0$, and its covariance function $C_\xi(x,y) = (a + b|x-y|)^\beta$, for $x,y \in \mathbb{R}^d$, $a,b > 0$, and, $\beta < 0$. In the context of equation (11), $\kappa - \rho_0$, defined for all $x \in D$, and $\omega \in \mathcal{S}'(\mathbb{R}^d)$, is a stationary and log-normal distributed in the 1-dimensional white noise probability space. It is important to mention that in [9] they consider a similar problem to this one, but $\kappa(x,\cdot) = e^{\diamond W_\phi(x,\cdot)}$, where $\diamond$ stands for the *Wick product*. However, it is well-known that $e^{\diamond W_\phi(x,\cdot)} = e^{W_\phi(x,\cdot) - \frac{\|\phi\|^2}{2}}$, and therefore both situations are the same up to a multiplicative constant, and all the calculations presented here can be carried over.

3.1. **The Wiener-Chaos Decomposition of $\kappa(x,\cdot)$.** In this section we develop on the computation of the coefficients $\kappa_\gamma(x)$, which can be used later in equation (31). First of all, let us define

$$L^\infty(D, L^2_\mu(\mathcal{S}')) \doteq \left\{ \zeta : D \times \mathcal{S}' \longrightarrow \mathbb{R} : \sup_{x \in D} E_\mu[\zeta^2(x,\cdot)] < +\infty \right\} \tag{12}$$

Since $W_\phi(x,\cdot)$ is normally distributed with mean 0 and variance $\|\phi\|^2$ it follows that

$$\begin{aligned} E_\mu[\kappa(x,\cdot,\phi)] &= \rho_0 + e^{\|\phi\|^2/2} \\ E_\mu[\kappa^2(x,\cdot,\phi)] &= \rho_0^2 + 2\rho_0 e^{\|\phi\|^2/2} + e^{2\|\phi\|^2} \end{aligned} \tag{13}$$

therefore, we have that $\kappa \in L^\infty(D, L^2_\mu(\mathcal{S}'))$. Then by the Wiener-Itô chaos expansion Theorem 10, (or see [9], Theorem 2.2.4), it follows that $\kappa(x,\omega)$ has a unique representation of the form:

$$\kappa(x,\omega) = \rho_0 + \sum_{\gamma \in \mathcal{J}} \kappa_\gamma(x) H_\gamma(\omega) \tag{14}$$

Remark 11. *Notice that from (14) and properties of the basis $\{H_\gamma\}$, it yields*

$$E_\mu[\kappa^2(x,\cdot,\phi)] = \rho_0^2 + 2\rho_0\kappa_0(x) + \sum_{\gamma \in \mathcal{J}} \gamma! \kappa_\gamma^2(x) \tag{15}$$

Thus, if $\kappa \in L^\infty(D, L^2_\mu(\mathcal{S}'))$, it follows easily that $\kappa_\gamma \in L^\infty(D)$ uniformly for all $\gamma \in \mathcal{J}$.

Next, we compute explicitly the coefficients $\kappa_\gamma(x)$:

Lemma 12. *Let $\gamma \in \mathcal{J}$ such that $d(\gamma) = n \le N$, then*

$$\kappa_\gamma(x) = e^{\|\phi\|^2/2} \prod_{j=1}^{n} \frac{a_j(x)^{\gamma_j}}{\gamma_j!},$$

where $a_j(x) = (\phi_x, \eta_j)$.

Proof: The 1-dimensional smoothed white noise $w(\phi,\omega)$ has the expansion

$$\begin{aligned} w(\phi,\omega) &= <\omega,\phi> = <\omega, \sum_{j=1}^{\infty} (\phi,\eta_j)\eta_j> \\ &= \sum_{j=1}^{\infty} (\phi,\eta_j) <\omega,\eta_j> \end{aligned}$$

Therefore,

$$W_\phi(x,\omega) = w(\phi_x,\omega) = \sum_{j=1}^{\infty} (\phi_x,\eta_j) <\omega,\eta_j> = \sum_{j=1}^{\infty} a_j(x) <\omega,\eta_j>$$

where $a_j(x) = (\phi_x, \eta_j)$. Now, let $\gamma \in \mathcal{J}$ such that $d(\gamma) = n \leq N$, then

$$H_\gamma(\omega) = \prod_{j=1}^{n} h_{\gamma_j}(< \omega, \eta_j >)$$

and

$$\kappa_\gamma(x)\gamma! = E_\mu\left[(\kappa(x,\cdot) - \rho_0)H_\gamma(\cdot)\right]$$

Thus,

$$\begin{aligned}
\kappa_\gamma(x)\gamma! &= E_\mu\left[e^{\sum_{j=1}^{\infty} a_j(x)<\cdot,\eta_j>} \prod_{j=1}^{n} h_{\gamma_j}(<\cdot,\eta_j>)\right] \\
&= E_\mu\left[\prod_{j=1}^{n} e^{a_j(x)<\cdot,\eta_j>} h_{\gamma_j}(<\cdot,\eta_j>) \prod_{j=n+1}^{\infty} e^{a_j(x)<\cdot,\eta_j>}\right] \quad (16) \\
&= \prod_{j=1}^{n} E_\mu\left[e^{a_j(x)<\cdot,\eta_j>} h_{\gamma_j}(<\cdot,\eta_j>)\right] \prod_{j=n+1}^{\infty} E_\mu[e^{a_j(x)<\cdot,\eta_j>}] \quad (17) \\
&= e^{\frac{1}{2}\sum_{j=n+1}^{\infty} a_j^2(x)} \prod_{j=1}^{n} \int_{\mathbb{R}} e^{a_j(x)y_j} h_{\gamma_j}(y_j) \frac{e^{-y_j^2/2}}{\sqrt{2\pi}} dy_j \quad (18)
\end{aligned}$$

where from equation (16) to (17) we have used the independence of the $<\cdot,\eta_j>'$ s, and from (17) to (18) Lemma 3. On the other hand, the generating function of Hermite polynomials is $e^{-\frac{t^2}{2}+tx}$, i.e., we have that

$$e^{-\frac{t^2}{2}+tx} = \sum_{k=0}^{+\infty} \frac{t^k}{k!} h_k(x) \quad (19)$$

therefore substituting $t = a$ in (19) we get that

$$e^{ax} = \sum_{k=0}^{+\infty} c_k h_k(x) \quad (20)$$

where $c_k = \frac{a^k}{k!} e^{a^2/2}$. Using this expansion for $e^{a_j(x)y_j}$ and substituting in equation (18) we obtain that

$$\begin{aligned}
\kappa_\gamma(x)\gamma! &= e^{\frac{1}{2}\sum_{j=n+1}^{\infty} a_j^2(x)} \prod_{j=1}^{n} \frac{(a_j(x))^{\gamma_j}}{\gamma_j!} e^{a_j^2(x)/2} E_\mu[h_{\gamma_j}^2(<\cdot,\eta_j>)] \\
&= e^{\frac{1}{2}\sum_{j=n+1}^{\infty} a_j^2(x)} \prod_{j=1}^{n} (a_j(x))^{\gamma_j} e^{a_j^2(x)/2} \\
&= e^{\|\phi\|^2/2} \prod_{j=1}^{n} (a_j(x))^{\gamma_j} \quad (21)
\end{aligned}$$

where we have used the fact that $\|H_\gamma\|_{L^2(\mu)}^2 = \gamma!$ from Theorem 9. Hence it yields that

$$\kappa_\gamma(x) = e^{\|\phi\|^2/2} \prod_{j=1}^{n} \frac{(a_j(x))^{\gamma_j}}{\gamma_j!} \quad (22)$$

for $\gamma = (\gamma_1, \dots, \gamma_n, 0, \dots, 0, \dots)$.

□

4. **Variational and Galerkin Formulations.** We next introduce the variational formulation associated to the problem (11) to define the exact solution $u(x,\cdot)$ and its Galerkin approximation. Given a permeability function $\kappa(x,\cdot) \in L^\infty(D, L^2_\mu(\mathcal{S}'))$ such that there exists a positive constant ρ_0 for which $\rho_0 < \kappa(x,\omega)$ for all $x \in D$, and all $\omega \in \mathcal{S}'$, define the following spaces:

$$\mathcal{H}^1(\kappa, D) \doteq \left\{ v : D \times \mathcal{S}' \longrightarrow \mathbb{R} : E_\mu \int_D \kappa(x,\cdot)|\nabla v(x,\cdot)|^2 dx < +\infty \right\} \tag{23}$$

and

$$\mathcal{H}^1_0(\kappa, D) \doteq \left\{ v \in \mathcal{H}^1(\kappa, D) : v(x,\cdot) = 0 \text{ on } \partial D \right\}. \tag{24}$$

Remark 13. *Notice that if* $v \in \mathcal{H}^1_0(\kappa, D)$ *then* $v \in L^2_\mu(\mathcal{S}', H^1_0(D))$ *since* κ *is assumed to be bounded from below away from zero, and hence it has a Wiener-Itô chaos expansion of the form*

$$v(x,\omega) = \sum_{\alpha \in \mathcal{J}} v_\alpha(x) H_\alpha(\omega) \tag{25}$$

Next, let $v_\beta(x)H_\beta(\omega) \in \mathcal{H}^1_0(\kappa, D)$, for $\beta \in \mathcal{J}$, be a test function then integrating the left-hand side of (11) with $u \in \mathcal{H}^1_0(\kappa, D)$ against this function yields

$$\int_{\mathcal{S}'}\int_D \kappa(x,\omega)\nabla_x \left(\sum_{\alpha \in \mathcal{J}} u_\alpha(x) H_\alpha(\omega) \right) \nabla_x v_\beta(x) H_\beta(\omega) dx \, d\mu(\omega) \tag{26}$$

$$= \int_{\mathcal{S}'}\int_D f(x) v_\beta(x) H_\beta(\omega) dx \, d\mu(\omega)$$

Remark 14. *Notice that the variational formulation (26) induces the bilinear form associated to* $\mathcal{H}^1_0(\kappa, D)$*, as a consequence ellipticity and continuity are automatically satisfied. The assumption* $0 < \rho_0 < \kappa(x,\omega)$ *permits to consider* $\mathcal{H}^1_0(\kappa, D)$ *closed and separable since it is embedded into the closed and separable space* $L^2_\mu(\mathcal{S}', H^1_0(D))$*. In addition, that assumption makes the Poincaré inequality to hold on the space* $\mathcal{H}^1_0(\kappa, D)$ *with respect to the norms* $\mathcal{H}^1(\kappa, D)$ *and* $L^2_\mu(\mathcal{S}', L^2(D))$*, and hence the right hand side of (26) belongs to the dual of* $\mathcal{H}^1_0(\kappa, D)$*. The existence, uniqueness, and stability of the solution* u *satisfying (26) for all* $\beta \in \mathcal{J}$*, follows from the Lax-Milgram Theorem.*

4.1. **The Galerkin Approximation.** Now, consider

$$\mathcal{J}_{N,k} \doteq \{\alpha \in \mathcal{J} : d(\alpha) \leq N, \text{ and, } |\alpha| \leq k\} \tag{27}$$

Then to obtain a (N,k)-Galerkin approximation for $u(x,\omega)$ we restrict to $\alpha \in \mathcal{J}_{N,k}$. We will denote this approximation as $u^{(N,k)}(x,\cdot)$ and of the form:

$$u^{(N,k)}(x,\omega) \doteq \sum_{\alpha \in \mathcal{J}_{N,k}} u^{(N,k)}_\alpha(x) H_\alpha(\omega) \tag{28}$$

Definition 15. *The function* $u^{(N,k)} \in \mathcal{H}^1_0(\kappa, D)$ *given in (28) is the* **Galerkin approximation** *in* $\mathcal{H}^1(\kappa, D)$*, or the* $\mathcal{H}^1(\kappa, D)$-**orthogonal projection** *of the solution* u *if it satisfies for all* $\beta \in \mathcal{J}_{N,k}$ *and* $v_\beta(x)H_\beta(\omega) \in \mathcal{H}^1_0(\kappa, D)$

$$\int_{\mathcal{S}'}\int_D \kappa(x,\omega)\nabla_x \left(\sum_{\alpha\in\mathcal{J}_{N,k}} u_\alpha^{(N,k)}(x)H_\alpha(\omega)\right)\nabla_x v_\beta(x)H_\beta(\omega)dx\ d\mu(\omega) \tag{29}$$

$$= \int_{\mathcal{S}'}\int_D f(x)v_\beta(x)H_\beta(\omega)dx\ d\mu(\omega)$$

Remark 16. *Observe that our Galerkin approximation* $u^{(N,k)}$ *satisfies a variational equation with the original permeability tensor* κ*, as opposed to truncations of* κ*, and therefore the ellipticity is now maintained. As a consequence, Lax-Milgram Theorem can be applied to the Galerkin approximation problem and hence uniqueness, existence, and stability also follow.*

We point out that the left hand side of equation (26) results in

$$\sum_{\alpha\in\mathcal{J}_{N,k}} \int_{\mathcal{S}'}\int_D \kappa(x,\omega)\nabla_x u_\alpha^{(N,k)}(x)\nabla_x v_\beta(x)H_\alpha(\omega)H_\beta(\omega)dxd\mu(\omega)$$

$$= \int_{\mathcal{S}'}\int_D f(x)v_\beta(x)H_\beta(\omega)dxd\mu(\omega)$$

which can be written as

$$\sum_{\alpha\in\mathcal{J}_{N,k}} \int_D \nabla_x u_\alpha^{(N,k)}(x)\nabla_x v_\beta(x)E_\mu[\kappa(x,\cdot)H_\alpha H_\beta]dx = \int_D f(x)v_\beta(x)E_\mu[H_\beta]dx \tag{30}$$

Notice that the right hand side of (30) is zero unless $\beta = 0$, i.e. $H_\beta \equiv 1$. Also, in Section 3.1 we computed exactly the coefficients $\kappa_\gamma(x)$ in the Wiener-Itô-Chaos decomposition of $\kappa(x,\omega)$ for log-normal permeability functions. Then, if we substitute the Wiener-Itô-Chaos decomposition of $\kappa(x,\cdot)$ in $E_\mu[\kappa(x,\cdot)H_\alpha H_\beta]$ of (30) we have

$$\sum_{\alpha\in\mathcal{J}_{N,k}\gamma\in\mathcal{J}} \int_D \kappa_\gamma(x)\nabla_x u_\alpha^{(N,k)}(x)\nabla_x v_\beta(x)E_\mu[H_\gamma H_\alpha H_\beta]dx = \int_D f(x)v_\beta(x)E_\mu[H_\beta]dx \tag{31}$$

for each $\beta \in \mathcal{J}_{N,k}$. Notice that the later sum is actually a finite sum, since for $\alpha, \beta \in \mathcal{J}_{N,k}$, $H_\alpha H_\beta$ can be written in terms of a polynomial of degree at most $2k$, and therefore $E_\mu[H_\gamma H_\alpha H_\beta] = 0$ for $|\gamma| \geq 2k+1$, due to the orthogonality of the Hermite polynomials. Moreover, the terms $E_\mu[H_\gamma H_\alpha H_\beta]$ may be computed beforehand (see [10] and references therein). Therefore, (31) produces a linear system that can be re-written as

$$\sum_{\alpha\in\mathcal{J}_{N,k}} \int_D \kappa_{\alpha,\beta}(x)\nabla_x u_\alpha^{(N,k)}\nabla_x v_\beta(x)(x)dx = \int_D f(x)v_\beta(x)E_\mu[H_\beta]dx,$$

where

$$\kappa_{\alpha,\beta}(x) \doteq E_\mu[\kappa(x,\cdot)H_\alpha H_\beta] = \sum_{\gamma\in\mathcal{J}:|\gamma|\leq 2k} \kappa_\gamma(x)E_\mu[H_\gamma H_\alpha H_\beta].$$

After solving $u_\alpha^{(N,k)}$, for each $\alpha \in \mathcal{J}_{N,k}$ we compute

$$u^{(N,k)}(x,\omega) \doteq \sum_{\alpha\in\mathcal{J}_{N,k}} u_\alpha^{(N,k)}(x)H_\alpha(\omega).$$

5. **On the Choice of ϕ and the Orthonormal Functions $\eta_{j_s'}$.** In this section, we show another of the advantages of using the white noise probability space. Empirical studies in geophysical applications suggest specific forms of the covariance function. Some of these typical covariance functions are considered in [3, 4, 5]. On the other hand, different choices for the ϕ function will lead to different covariance functions. Our goal now is to show that the white noise probability space gives some flexibility to choose this function so that an specific form of a covariance function for C_κ or C_ξ is obtained. We complete the section showing a suitable choice of the orthonormal basis $\eta_{j_s'}$ in order to get simpler expressions than using (8) for the coefficients $a_j(x)$ defined on Lemma 12 as well as to explore the fact that a covariance function is given. The expressions are based on eigenvector space decomposition of an integral equation with $\phi(x-y)$ as its kernel, opposed to $C(x-y)$ as its kernel used for building the Karnunen-Loeve expansion.

5.1. **The Covariance Function $C_\kappa(x_1,x_2)$.** By definition, the covariance function of $\kappa(x,\cdot)$ is

$$C_\kappa(x_1,x_2) \doteq E_\mu\left[(\kappa(x_1,\cdot)-E_\mu[\kappa(x_1,\cdot)])(\kappa(x_2,\cdot)-E_\mu[\kappa(x_2,\cdot)])\right]$$

Also, since the covariance of $\kappa(x,\cdot)=\rho_0+e^{W_\phi(x,\omega)}$ is the same as the covariance of $e^{W_\phi(x,\omega)}$ it is enough to show our results considering only $e^{W_\phi(x,\omega)}$, and we have

Lemma 17. *The covariance of $e^{W_\phi(x,\omega)}$, where $W_\phi(x,\omega)$ is the smoothed white noise defined in Section 2, is given by:*

$$C_\kappa(x_1,x_2) \doteq e^{\|\phi\|^2}\left(e^{(\phi_{x_1},\phi_{x_2})}-1\right) \tag{32}$$

Proof: In order to show this we will choose η_1 and η_2 such that they are orthonormal, and span $\{\eta_1,\eta_2\}$ = span $\{\phi_{x_1},\phi_{x_2}\}$. This can be done by using the Gram-Schmidt orthogonalization procedure. Straightforward computations give $\eta_1=\phi_{x_1}/\|\phi\|$, and $\eta_2=a_1\phi_{x_1}+a_2\phi_{x_2}$, where

$$a_1=\frac{\mp(\phi_{x_1},\phi_{x_2})}{\|\phi\|\left(\|\phi\|^4-(\phi_{x_1},\phi_{x_2})^2\right)^{1/2}},\ \textit{and}\ a_2=\frac{\pm\|\phi\|}{\left(\|\phi\|^4-(\phi_{x_1},\phi_{x_2})^2\right)^{1/2}}$$

Then, using Lemma 3 we have that

$$\begin{aligned} E_\mu[e^{W_\phi(x_1,\cdot)}e^{W_\phi(x_2,\cdot)}] &= \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{\|\phi\|t}e^{\frac{(\phi_{x_1},\phi_{x_2})t}{\|\phi\|}}e^{\frac{\left(\|\phi\|^4-(\phi_{x_1},\phi_{x_2})^2\right)^{1/2}s}{\|\phi\|}}e^{-t^2/2}e^{-s^2/2}dtds \\ &= e^{\|\phi\|^2+(\phi_{x_1},\phi_{x_2})} \end{aligned} \tag{33}$$

Having computed this, we obtain

$$\begin{aligned} C_\kappa(x_1,x_2) &= E_\mu[e^{W_\phi(x_1,\cdot)}e^{W_\phi(x_2\cdot)}]-E_\mu(e^{W_\phi(x_1,\cdot)})E_\mu(e^{W_\phi(x_2,\cdot)}) \\ &= e^{\|\phi\|^2}\left(e^{(\phi_{x_1},\phi_{x_2})}-1\right) \end{aligned} \tag{34}$$

□

Remark 18. *Notice that if we assume that ϕ is symmetric with respect to the origin, i.e., $\phi(x) = \phi(-x)$, for all $x \in \mathbb{R}^d$, then in the expression above for the covariance of the function κ, we have that (ϕ_{x_1}, ϕ_{x_2}) can be written as*

$$\begin{aligned}(\phi_{x_1}, \phi_{x_2}) &= \int_{\mathbb{R}^d} \phi(y - x_1)\phi(y - x_2)dy = \int_{\mathbb{R}^d} \phi(z)\phi(z - (x_2 - x_1))dz \\ &= (\phi * \phi)(x_2 - x_1) = (\phi, \phi_{x_2 - x_1}) \\ &= \int_{\mathbb{R}^d} \phi(y)\phi(y + x_2 - x_1)dy = \int_{\mathbb{R}^d} \phi(y)\phi(y - (x_1 - x_2))dy \\ &= (\phi * \phi)(x_1 - x_2) = (\phi, \phi_{x_1 - x_2}) \end{aligned} \tag{35}$$

where we have used the symmetry of ϕ and made the change of variable $z = y - x_1$. Using this result in Lemma 17 we obtain that $C_\kappa(x_1, x_2) = C_\kappa(x_1 - x_2)$, where $C_\kappa(x)$ is an even function. We observe that if we want to match specific given values for the mean and variance of $\kappa(x, \cdot) = \rho_0 + k_1 e^{W_\phi(x,\cdot)}$, it is enough to match values for the stochastic part. Denote $\bar{\kappa} := E_\mu[\kappa(x, \cdot)]$ and $\sigma^2 = Var[\kappa(x, \cdot)]$. We can proceed as follows: define $k_2 = \|\phi\|$ and we need to have

$$\begin{aligned}\bar{\kappa} &= \rho_0 + k_1 e^{\|\phi\|^2/2} = \rho_0 + k_1 e^{k_2^2}/2 \\ \sigma^2 &= k_1^2(e^{2\|\phi\|^2} - e^{\|\phi\|^2}) = k_1^2(e^{2k_2^2} - e^{k_2^2}) \end{aligned} \tag{36}$$

hence straightforward computations yield

$$k_1 = \frac{\bar{\kappa} - \rho_0}{\sqrt{1 + \frac{\sigma^2}{(\bar{\kappa} - \rho_0)^2}}} \tag{37}$$

$$k_2 = \sqrt{\log\left(1 + \frac{\sigma^2}{(\bar{\kappa} - \rho_0)^2}\right)} \tag{38}$$

All the computations made in this section are done for a given ϕ. However, sometimes we might be interest in choosing ϕ so that a specific covariance function can be matched. We discuss this next.

5.1.1. *Choosing ϕ to Match a Covariance Function.* We will assume that ϕ is symmetric with respect to the origin, i.e., $\phi(x) = \phi(-x)$. Then, in our case and according to our results in the previous section, we need to consider covariance functions $C_\kappa(x_1, x_2)$ that depend only on the difference $|x_1 - x_2|$, so that they can be written as $C_\kappa(x_1 - x_2)$, where $C_\kappa(x)$ is an even function. Now, let $\bar{\kappa}$, and σ^2 be the mean and variance of the stochastic permeability field, then we find k_1, and k_2 from equations (37)-(38), and by Lemma 17, all we need to do is to choose ϕ such that $\|\phi\| = k_2$ and

$$C_\kappa(x) = k_1^2 e^{k_2^2}(e^{(\phi, \phi_x)} - 1)$$

hence

$$\ln\left(1 + \frac{C_\kappa(x)}{k_1^2 e^{k_2^2}}\right) = (\phi, \phi_x)$$

Let $g(x) := \ln\left(1 + \frac{C_\kappa(x)}{k_1^2 e^{k_2^2}}\right)$, and since (ϕ, ϕ_x) is actually a convolution, then taking the Fourier transform on both sides of the last equation, we arrive to

$$\hat{g}(\xi) = (\hat{\phi}(\xi))^2 \tag{39}$$

Equation (39) gives the condition on $g(x)$ for the existence of such ϕ, this condition is that the Fourier transform of $g(x)$ must be in $L^1(\mathbb{R}^d)$ and nonnegative, i.e. g

must be a positive definite function (see [16]). Hence, g real and even function such that $\hat{g} \in L^1(\mathbb{R}^d)$ and positive implies the existence of a real and even $\phi \in L^2(\mathbb{R}^d)$. Furthermore, by choosing $\hat{\phi}$ nonnegative, the operator associate to the convolution on ϕ is going to be nonnegative and of Hilbert-Schmidt type.

Below, we give two one-dimensional examples of functions $g(x)$ satisfying these properties.

Example 1.: $(d = 1)$ and $g(x) = \ln(1 + ae^{-b|x|})$. In fact,

$$\hat{g}(\xi) = 2\int_0^{+\infty} \ln(1 + ae^{-bx})\cos(2\pi\xi x)dx,$$

and using that $\int_0^{+\infty} \ln(1 + ae^{-bx})dx \leq C_1$, where C_1 is a constant, then $|\hat{\phi(\xi)}| \leq C_1$. Let us compute $\int_0^{+\infty} \ln(1 + ae^{-bx})\cos(2\pi\xi x)dx$. Integrating by parts, we have that

$$\begin{aligned}
&\int_0^{+\infty} \ln(1 + ae^{-bx})\cos(2\pi\xi x)dx \\
&= \int_0^{+\infty} \frac{abe^{-bx}}{1 + ae^{-bx}}\frac{\sin(2\pi\xi x)dx}{2\pi\xi} \\
&= \int_0^{+\infty} \frac{ab}{e^{bx} + a}\frac{\sin(2\pi\xi x)dx}{2\pi\xi} \\
&= -\int_0^{+\infty} \frac{ab^2e^bx}{(e^{bx} + a)^2}\frac{\cos(2\pi\xi x)dx}{4\pi^2\xi^2} - \frac{ab}{e^{bx} + a}\frac{\cos(2\pi\xi x)}{4\pi^2\xi^2}|_0^{+\infty}
\end{aligned}$$

and it is not hard to see that last terms are of order $\frac{1}{\xi^2}$, therefore $\hat{g} \in L^1$. In addition, using that $\frac{abe^{-bx}}{1+ae^{-bx}}$ is positive and strictly decreasing, it is easy to see that the second integral above is positive for any ξ, i.e. $\hat{g}(\xi)$ is positive.

Example 2.: $(d = 1)$ Let $g(x) = \ln\left(1 + (a + b|x|)^\beta\right)$, for $\beta < 0$. Let us compute the Fourier transform of ϕ. As in the previous example, since g is even then its Fourier transform is real and therefore

$$\hat{g}(\xi) = 2\int_0^{+\infty} \ln\left(1 + (a + bx)^\beta\right)\cos(2\pi\xi x)dx$$

Next, integrating by parts twice we have

$$\begin{aligned}
&\int_0^{+\infty} \ln\left(1 + (a + bx)^\beta\right)\cos(2\pi\xi x)dx \\
&= -\beta\int_0^{+\infty} \frac{b(a + bx)^{\beta-1}}{1 + (a + bx)^\beta}\frac{\sin(2\pi\xi x)dx}{2\pi\xi} \\
&= -\beta\int_0^{+\infty} \frac{b}{(a + bx)^{1-\beta} + (a + bx)}\frac{\sin(2\pi\xi x)dx}{2\pi\xi} \\
&= \beta b\frac{\cos(2\pi\xi x)}{[(a + bx)^{1-\beta} + (a + bx)](2\pi\xi)^2}|_0^{+\infty} \\
&- \beta b^2\int_0^{+\infty} \frac{1 + (1 - \beta)(a + bx)^{-\beta}}{[(a + bx)^{1-\beta} + (a + bx)]^2}\frac{\cos(2\pi\xi x)dx}{(2\pi\xi)^2}
\end{aligned}$$

Evaluating the last terms we obtain order $O(\frac{1}{\xi^2})$, i.e $\hat{g}(\xi) \in L^1[1,+\infty)$. For ξ small, we procedure as follows:

$$\begin{aligned} -\beta \int_0^{+\infty} \frac{b(a+bx)^{\beta-1}}{1+(a+bx)^\beta} \frac{\sin(2\pi\xi x)}{2\pi\xi} dx \\ = \quad -\beta \int_0^{\frac{1}{4\xi}} \frac{b(a+bx)^{\beta-1}}{1+(a+bx)^\beta} \frac{\sin(2\pi\xi x)}{2\pi\xi} dx \\ - \quad \beta \int_{\frac{1}{4\xi}}^{+\infty} \frac{b(a+bx)^{\beta-1}}{1+(a+bx)^\beta} \frac{\sin(2\pi\xi x)}{2\pi\xi} dx \end{aligned}$$

For the first term we use that $|\frac{\sin(2\pi\xi x)}{2\pi\xi}| \leq Cx$ to obtain $O(\frac{1}{\xi^{1+\beta}})$ and for the second term is bounded by $C\beta \int_{\frac{1}{4\xi}}^{+\infty} \frac{1}{\xi x^{1-\beta}} dx$ and gives $O(\frac{1}{\xi^{1+\beta}})$, therefore $\hat{g}(\xi) \in L^1[0,1]$. Finally, using that $-\beta\frac{b}{(a+bx)^{1-\beta}+(a+bx)}$ is positive and strictly decreasing, we have that $\hat{g}(\xi)$ is positive.

5.2. **The Covariance Function $C_\xi(x_1, x_2)$.** If one takes the approach of Furtado-Pereira in [3, 4, 5], where the type of stochastic fields are of the form $K(x,\cdot) = k_1 e^{\xi(x,\cdot)}$, and $\xi(x,\cdot)$ is a stationary Gaussian random field, with zero mean and covariance function $C_\xi(x,y)$, the result analogous to (32) that one needs is

$$E_\mu[W_\phi(x_1,\cdot)W_\phi(x_2,\cdot)] = (\phi_{x_1}, \phi_{x_2}),$$

and the ϕ to be found is such that

$$C_\xi(x_1,x_2) = (\phi_{x_1}, \phi_{x_2}),$$

and therefore, in accordance to the results in this section, the conditions to guarantee the existence of such a ϕ are that the Fourier transform of $C_\xi(x)$ is in $L^1(\mathbb{R}^d)$, and $C_\xi(x)$ is a positive definite function.

5.3. **On the Choice of the Orthonormal Functions $\eta_{j'_s}$.** The kernel $K(x,y) = \phi(x-y)$ is real and symmetric since ϕ is a real and even function. In addition, for fixed x or y, $K(x,y)$ is square integrable in $\mathbb{R}^d$ since $\hat{\phi} \in L^2(\mathbb{R}^d)$. Therefore, since the domain D is bounded, $K(x,y)$ is square integrable on $D \times D$. Hence, the operator $A_D : L^2(D) \to L^2(D)$ given by $h = A_D\vartheta$ where

$$h(x) = \int_D K(x,y)\vartheta(y)dy,$$

is compact of Hilbert-Schmidt type. We note that the operator A_D is nonnegative, i.e. $(A_D\vartheta, \vartheta)_D \geq 0$. Indeed, let us define the function $\theta \in L^2(\mathbb{R}^d)$ as equal to ϑ on D and equal to zero on D^c the complement of D. Hence, $(A_D\vartheta,\vartheta)_D = (A\theta,\theta)_{\mathbb{R}^d}$, where

$$A\theta(x) = \int_{\mathbb{R}^d} \phi(x-y)\theta(y)dy, x \in \mathbb{R}^d.$$

Using standard Fourier analysis, we obtain

$$(\widehat{A\theta}, \hat{\theta})_{\mathbb{R}^d} = (\hat{\phi}\hat{\theta}, \hat{\theta})_{\mathbb{R}^d}$$

which is nonnegative since $\hat{\phi}$ is assumed real and nonnegative. For the two examples considered in the paper, $\hat{g}(\xi)$ is always positive, hence it follows that $(A\vartheta,\vartheta)_D$ is always positive whenever $\|\vartheta\|_{L^2(D)} \neq 0$. Since A_D is a Hilbert-Schmidt operator, there exists a sequence of orthonormal basis ϑ_j for $L^2(D)$ which are eigenfunctions for A_D with corresponding nonnegative and nonincreasing eigenvalues $\mu_j \geq 0$. We define the $\eta_j \in L^2(\mathbb{R}^d)$ as equal to ϑ_j on D and zero on D^c. We next evaluate $a_j(x) = (\phi_x, \eta_j)_{\mathbb{R}^d}$. We note that $(\phi_x, \eta_j)_{\mathbb{R}^d} = (\phi_x, \vartheta_j)_D$. In addition, when $x \in D$, $(\phi_x, \vartheta_j)_D = \mu_j \vartheta_j(x)$, while when $x \in D^c$ we cannot explore the eigenfunctions properties since they are not defined outside D; fortunately the evaluation of $a_j(x)$ for $x \in D^c$ is not required in the proposed methods. An orthonormal basis for $L^2(\mathbb{R}^d)$ can be constructed by adding any orthonormal basis for $L^2(D^c)$ to the $\eta_{j'_s}$.

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