Thermodynamical formalism for robust classes of potentials and non-uniformly hyperbolic maps*

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Abstract

We develop a Ruelle-Perron-Fröbenius transfer operator approach to the ergodic theory of a large class of non-uniformly expanding transformations on compact manifolds. For Hölder continuous potentials not too far from constant, we prove that the transfer operator has a positive eigenfunction, piecewise Hölder continuous, and use this fact to show that there is exactly one equilibrium state. Moreover, the equilibrium state is a non-lacunary Gibbs measure, a non-uniform version of the classical notion of Gibbs measure that we introduce here.

Dedicated to the memory of William Parry

1 Introduction

The theory of equilibrium states of uniformly hyperbolic (Axiom A) dynamical systems, developed by Sinai, Ruelle, Bowen, and Parry, is a major achievement in ergodic theory, and a spectacular application of ideas from statistical physics in the realm of smooth dynamics. Besides its intrinsic beauty, this theory yields a surprisingly complete picture of the behavior of such systems at the statistical level: finitely many invariant physical (or SRB) probability measures, which describe the asymptotic time averages of Lebesgue almost every point.

The strategy initiated by Sinai [Sin72] in the case of Anosov diffeomorphisms, and carried out in full generality by Ruelle and Bowen [Bow75, BR75, Rue76], may be briefly outlined as follows. Uniformly hyperbolic systems admit finite generating Markov partitions. Via the itinerary relative to such a partition, points in phase space are identified with configurations of a one-dimensional lattice gas. The Gibbs distributions of the gas correspond to the equilibrium states of the dynamical system. Later, Parry [Par88] proved that equilibrium states can also be obtained as weighted limits of orbital measures supported on periodic orbits.

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Extension of this approach and of the conclusions beyond the Axiom A context involves some fundamental difficulties, even restricted to non-uniformly hyperbolic systems, that is, systems such that almost every point admits an asymptotically hyperbolic splitting of the tangent space. For one thing, generating Markov partitions are not known to exist in general. Even when they do exist, Markov partitions usually have infinitely many atoms; this leads to considering gases with infinitely many states, a difficult subject not yet well understood.

Important contributions have been given recently by several authors: Bruin, Keller [BK98], Denker, Urbański [DU91, DU92, Urb98], Pesin, Senti [PS05], Wang, Young [WY01] for special classes of transformations, such as interval maps, rational functions of the sphere, and Hénon-like maps; Buzzi, Maume, Sarig [Buz99, BMD02, BS03], Sarig [Sar99, Sar01, Sar03], and Yuri [Yur99, Yur00, Yur03], for countable Markov shifts and piecewise expanding maps; and Leplaideur, Rios [LR06] for "horseshoes with tangencies" at the boundary of hyperbolic systems, to mention just a few of the most recent works. Many of these papers, and particularly [DU92, Sar03, Yur99, Yur00, Yur03], deal with systems having neutral periodic points, a setting of non-hyperbolic dynamics which attracted a great deal of attention over the last years. Also very recently, Buzzi [Buz05] introduced the important notion of entropy-expansiveness, which influenced other works such as [OV06] and [BR06].

There has also been substantial recent progress concerning physical measures. In particular, Alves, Bonatti, Viana [ABV00, BV00] proved existence and uniqueness of SRB measures for some large classes of non-uniformly hyperbolic maps. One important difficulty in this context lies in the very definition of non-uniform hyperbolicity: [ABV00] assume that *Lebesgue* almost every point has only non-zero Lyapunov exponents, but it is not clear how this kind of condition could be useful when considering more general potentials, since most equilibrium states should be singular relative to the Lebesgue measure.

In [Oli03] the first author tackled this difficulty and proved the existence of equilibrium states for open sets of non-uniformly expanding maps and of continuous potentials. Roughly, the map should be expanding on most of phase space, with possibly contracting behavior on the complement. Concerning the potential ϕ, its oscillation $\sup\phi - \inf\phi$ should not be too large. This ensures, *a priori*, that certain measures that are candidates to being equilibrium states accord a (uniformly) small weight to the possibly contracting regions. Using this fact, one can find a genuine equilibrium state within that set of candidates. Arbieto, Matheus [AM06] have recently shown that the equilibrium states one finds in this setting are exponentially mixing. Moreover, this approach has been extended to random non-uniformly expanding maps by Arbieto, Matheus, Oliveira [AMO04].

The results in the present paper are similar in flavor to those of [Oli03], but they improve that work in some important ways. To begin with, our hypotheses on the dynamical system are milder and more natural. In fact, they are quite close to conditions in the Appendix of [ABV00]. In addition, we develop a Ruelle-Perron-Fröbenius approach that provides a better understanding of the equilibrium states: for instance, we are able to prove that they are non-lacunary Gibbs states, a non-uniform variation of the classical notion, closely related to the weak Gibbs states in [Yur99]. Most important, this new approach allows us to prove, for the first time in this setting, uniqueness of the equilibrium state for every Hölder continuous potential whose oscillation is not too large.

Let us also mention that the special case of measures of maximal entropy, corresponding to constant potentials, has been treated before in [OV06], where we were able to give particularly short arguments for existence and uniqueness under some simple conditions.

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2 Setting and statements

Let $f : M \to M$ be a continuous transformation on a compact space M, and $\phi : M \to \mathbb{R}$ be a continuous function.

Definition 2.1. An f-invariant measure is an *equilibrium state* of f for the potential ϕ if it maximizes the functional

$$\eta \mapsto h_\eta(f) + \int \phi \, d\eta$$

among all f-invariant probabilities η.

By the variational principle [Wal82, Theorem 9.10], the supremum of this functional over the invariant probabilities coincides with the *topological pressure* $P(f, \phi)$ of f for ϕ.

2.1 The class of maps and potentials

Throughout this paper we take $f : M \to M$ to be a C^1 local diffeomorphism on a compact Riemannian manifold, satisfying conditions (H1) and (H2) below. A subset D of M has *finite inner diameter* if there exists $L > 0$ such that any two points in D may be joined by a curve of length less than L contained in D.

(H1) There are $p \geq 1$, $q \geq 0$, and a family $\mathcal{R} = \{R_1, \ldots, R_q, R_{q+1}, \ldots, R_{q+p}\}$ of pairwise disjoint open sets whose closures have finite inner diameter and cover the whole M, such that

- $f|(\bar{R}_i \cup \bar{R}_j)$ is injective whenever $\bar{R}_i \cap \bar{R}_j \neq \emptyset$
- if $f(R_i) \cap R_j \neq \emptyset$ then $f(R_i) \supset R_j$ and, hence, $f(\bar{R}_i) \supset \bar{R}_j$
- there is $N \geq 1$ such that $f^N(R_i) = M$ for every i.

Let us emphasize that *this Markov partition $\mathcal{R}$ needs not be generating:* different points may have the same itinerary relative to $\mathcal{R}$. We denote by $\partial\mathcal{R}$ the complement of the union of all $R_i \in \mathcal{R}$.

(H2) There exist σ_1, $\sigma_2 > 1$ such that

- $\|Df(x)^{-1}\|^{-1} \geq \sigma_1$ for every $x \in R_{q+1} \cup \cdots \cup R_{q+p}$
- $\|Df(x)^{-1}\|^{-1} \geq \sigma_2^{-1}$ for every $x \in R_1 \cup \cdots \cup R_q$

and σ_2 is close enough to 1: the precise conditions are stated in (3) below.

In other words, the map f is never very contracting, and it is quite expanding on the atoms $R_{q+1}, \ldots, R_{q+p}$. We also assume that the potential $\phi : M \to \mathbb{R}$ is not very far from being constant:

(H3) ϕ is Hölder continuous and $\sup \phi - \inf \phi < \log \deg(f) - \log q$.

This can happen only if $q < \deg(f)$. When $q = 0$ condition (H2) means that f is uniformly expanding. In this case (H3) is true for every ϕ. Closely related conditions have been considered by Hofbauer, Keller [HK82], for piecewise monotonic maps, and by Denker, Urbański [DU91], for rational maps of the sphere. See also Przytycki, Rivera-Letelier, Smirnov [PRLS] for a recent application in the latter context.

2.2 Non-lacunary Gibbs measures

An integer sequence $n_j \in \mathbb{N}$ is called *non-lacunary* if it is increasing and n_{j+1}/n_j converges to 1. For each $n \geq 1$, we call *cylinder* of length n any non-empty set of the form

$$R^n = R^n[i_0, \ldots, i_{n-1}] = \{y \in M : y \in R_{i_0}, f(y) \in R_{i_1}, \ldots, f^{n-1}(y) \in R_{i_{n-1}}\}.$$

Let $\mathcal{R}^n$ be the family of all cylinders of length n, and $\partial\mathcal{R}^n$ be the complement of the union of its elements.

The classical notion of Gibbs measure was brought from statistical mechanics by Sinai [Sin72] and Ruelle [Rue89]. In our setting it may be defined as follows. A probability η is a *Gibbs measure* of f for ϕ if $\eta(\partial\mathcal{R}) = 0$ and there exist $P \in \mathbb{R}$ and $K > 0$ such that

$$K^{-1} \leq \frac{\eta\big(R^n(x)\big)}{\exp\big(S_n\phi(x) - nP\big)} \leq K \quad \text{for every } x \notin \partial\mathcal{R}^n \text{ and } n \geq 1, \tag{1}$$

where $S_n\phi(x) = \sum_{j=0}^{n-1} \phi(f^j(x))$ and $R^n(x)$ denotes the cylinder of length n that contains x. More generally, we define

Definition 2.2. A probability η (not necessarily invariant) is a *non-lacunary Gibbs measure* of f for ϕ if $\eta(\partial\mathcal{R}) = 0$ and for η-almost every $x \in M$ there exists a non-lacunary sequence of values of n for which (1) is satisfied.

This notion of non-lacunary Gibbs measure is related to the notion of weak Gibbs measures in Yuri [Yur99]. See Proposition 3.17 and the observation following it. Here, non-lacunary sequences arise as sequences of hyperbolic times. This latter notion was introduced by Alves [Alv00] and further developed in [ABV00]:

Definition 2.3. Let $c > 0$ be fixed. We say that $n \in \mathbb{N}$ is a *hyperbolic time* for $x \in M$ if

$$\prod_{j=k}^{n-1} \|Df(f^j(x))^{-1}\|^{-1} \geq e^{2c(n-k)} \quad \text{for every } 0 \leq k \leq n-1. \tag{2}$$

We say $R^n \in \mathcal{R}^n$ is a *hyperbolic cylinder* if n is a hyperbolic time for every point $x \in R^n$. Denote by $\mathcal{R}_h^n \subset \mathcal{R}^n$ the subset of hyperbolic cylinders.

Our particular choice of the constant c will be given in (4) below. We denote by H the set of points $x \in M$ that belong to the closure $\bar{R}^n$ of some hyperbolic cylinder R^n for infinitely many values of n. In particular, since the inequality (2) extends to the closure, such points admit infinitely many hyperbolic times.

2.3 Statements of main results

The *Ruelle-Perron-Fröbenius transfer operator* $\mathcal{L}_\phi : C^0(M) \to C^0(M)$ associated to $f : M \to M$ and $\phi : M \to \mathbb{R}$ is the linear operator defined on the space $C^0(M)$ of continuous functions $g : M \to \mathbb{R}$ by

$$\mathcal{L}_\phi g(x) = \sum_{f(y)=x} e^{\phi(y)} g(y).$$

We also consider the dual operator $\mathcal{L}_\phi^* : \mathcal{M}(M) \to \mathcal{M}(M)$ acting on the space $\mathcal{M}(M)$ of Borel measures in M by

$$\int \xi \, d(\mathcal{L}_\phi^* \eta) = \int (\mathcal{L}_\phi \xi) \, d\eta$$

for every $\xi \in C^0(M)$. Let λ be the spectral radius of $\mathcal{L}_\phi$ and $P = \log \lambda$. For all our results, we assume f and ϕ satisfy conditions (H1), (H2), (H3) above.

Theorem A. *There exists some probability measure ν such that $\mathcal{L}_\phi^* \nu = \lambda \nu$. Moreover, $\nu(H) = 1$ and $\operatorname{supp} \nu = \bar{H}$.*

From the proofs of this and the next theorem we also get (Corollary 5.16) that ν is a non-lacunary Gibbs measure for ϕ. Notice ν is usually not invariant.

Theorem B. *There is a piecewise Hölder continuous function $h : M \to (0, \infty)$, bounded away from zero and infinity, such that $\mathcal{L}_\phi h = \lambda h$ and so $\mu = h\nu$ is an invariant ergodic non-lacunary Gibbs measure of f for ϕ.*

By piecewise Hölder continuous we mean there exists a finite partition of M such that that h is Hölder continuous on each of the atoms. From Theorems A and B we get that $\mu(H) = 1$, and so all the Lyapunov exponents of f are positive at μ-almost every point.

Theorem C. *The map f admits a unique equilibrium state for the potential ϕ, and the equilibrium state is an invariant ergodic non-lacunary Gibbs measure.*

This is our main result on existence and uniqueness of equilibrium states. From the proof we also get that the equilibrium state coincides with any of the non-lacunary Gibbs measures constructed in Theorem B.

Before closing this section, let us point out that we do not really need M to be a manifold or f to be smooth: with a little extra effort, the results stated above extend to local homeomorphisms on compact metric spaces, subject to a couple technical conditions that we are going to mention. Let us describe the few changes in our arguments needed to obtain this extension.

One should assume that each $f : \bar{R}_i \to f(\bar{R}_i)$ is a homeomorphism, and its inverse is Lipschitz continuous with respect to some choice of metrics $d_{f(\bar{R}_i)}$ on the $f(\bar{R}_i)$, larger than the metric in the ambient space yet uniformly bounded: compare (7) and (8). Then one should replace $\|Df^{-1}\|$ by the Lipschitz constant of the relevant inverse branch of f throughout the proof: see (H2), (2), (5), (6), Proposition 3.4, and the definitions of $b(i)$ and A in the proof of Proposition 3.2. The first paragraph of Section 3.3 should be omitted, as Lyapunov exponents are not defined in this non-smooth setting. Assume that every point has the same number of pre-images under f, so that $\deg(f)$ is well-defined. This is probably too strong an assumption. For Lemma 6.6 we need the metric space to satisfy the following technical condition (Lemma 6.5 means it holds for manifolds): there exists a sequence $\mathcal{T}_k$ of coverings with diameter going to zero and there exist $C > 0$ and $d > 0$ such that any set $E \subset M$ with $\operatorname{diam}(E) \le A \operatorname{diam}(\mathcal{T}_k)$ intersects at most CA^d elements of $\mathcal{T}_k$. Again, this is probably not optimal.

3 Preliminary results

Here we introduce several auxiliary facts to the proofs of the main results. The reader may choose to proceed directly to the next section, referring to the present one when necessary.

3.1 Combinatorics of orbits

Given $\gamma \in (0,1)$ and $n \ge 1$, let us consider the set $I(\gamma, n)$ of all itineraries $(i_0, \ldots, i_{n-1}) \in \{1, \ldots, q+p\}^n$ such that $\#\{0 \le j \le n-1 : i_j \le q\} > \gamma n$. Then let

$$c_\gamma = \limsup_{n\to\infty} \frac{1}{n} \log \# I(\gamma, n).$$

Lemma 3.1. *We have* $\limsup_{\gamma\to 1} c_\gamma \le \log q$.

Proof. Observe that

$$\# I(\gamma, n) \le \sum_{r>\gamma n} \binom{n}{r} p^{n-r} q^r \le \sum_{r>\gamma n} \binom{n}{r} p^{(1-\gamma)n} q^n.$$

Stirling's formula implies (see [BV00, Section 6.3]) that there exists a universal constant $B > 0$ such that

$$r \ge \frac{kn}{k+1} \quad \Rightarrow \quad \binom{n}{r} \le B\big((1+\frac{1}{k})(1+k)^{\frac{1}{k}}\big)^r \le B\big((1+\frac{1}{k})(1+k)^{\frac{1}{k}}\big)^n.$$

Assume that $\gamma \geq k/(k+1)$. Then

$$\#I(\gamma, n) \leq \sum_{r > \gamma n} B\big((1+\frac{1}{k})(1+k)^{\frac{1}{k}} p^{\frac{1}{k+1}} q\big)^n \leq Bn\big((1+\frac{1}{k})(1+k)^{\frac{1}{k}} p^{\frac{1}{k+1}} q\big)^n$$

and so

$$c_\gamma \leq \log(1+\frac{1}{k}) + \frac{1}{k}\log(1+k) + \frac{1}{k+1}\log p + \log q.$$

Now just note that the right hand side goes to $\log q$ when $k \to \infty$. □

We are in a position to state our first condition on the constant σ_2 in assumption (H2). By assumption (H3), we may find $\kappa > \log q$ such that

$$\sup \phi + \kappa < \inf \phi + \log \deg(f).$$

By Lemma 3.1, we may fix $\gamma < 1$ such that $c_\gamma < \kappa$. Assume σ_2 is close enough to 1 that (let d denote the dimension of the manifold M)

$$\sup \phi + \kappa + d \log \sigma_2 < \inf \phi + \log \deg(f) \quad \text{and} \quad \sigma_1^{-(1-\gamma)} \sigma_2^{\gamma} < 1. \tag{3}$$

We also fix the exponent $c > 0$ in Definition 2.3 once and for all, such that

$$\sigma_1^{-(1-\gamma)} \sigma_2^{\gamma} < e^{-4c} < 1. \tag{4}$$

This ensures that f^n is uniformly expanding, in a strong sense, on orbit segments with itineraries $(i_0, \ldots, i_{n-1})$ in the complement of $I(\gamma, n)$:

$$\|Df^n(x)^{-1}\| \leq \prod_{i=0}^{n-1} \|Df(f^i(x))^{-1}\| \leq \sigma_1^{-(1-\gamma)n} \sigma_2^{\gamma n} \leq e^{-4cn} \tag{5}$$

for every x in the closure of $R^n[i_0, \ldots, i_{n-1}]$.

Proposition 3.2. *There exists $\theta > 0$ such that, for any $n \geq 1$ and any itinerary $(i_0, \ldots, i_{n-1})$ in the complement of $I(\gamma, n)$, there exists $l > \theta n$ and integers $1 \leq n_1 < \cdots < n_l \leq n$ such that $R^{n_j}[i_0, \ldots, i_{n_j - 1}]$ is a hyperbolic cylinder for every $j = 1, \ldots, l$.*

Proof. The argument is based on the following result of Pliss [Pli72]:

Lemma 3.3. *Given $A \geq c_2 > c_1$ let $\theta = (c_2 - c_1)/(A - c_1)$. Assume $a_1, \ldots, a_n$ are such that $a_1 + \cdots + a_n \geq c_2 n$ and $a_s \leq A$ for all $s = 1, \ldots, n$. Then there are integer numbers $l > \theta n$ and $1 \leq n_1 < \cdots < n_l \leq n$ such that*

$$a_{k+1} + \cdots + a_{n_i} \geq c_1(n_i - k) \quad \textit{for every } 0 \leq k \leq n_i - 1 \textit{ and } i = 1, \ldots, l.$$

Denote $b(i) = \inf\{\log \|Df(x)^{-1}\|^{-1} : x \in \bar{R}_i\}$ for each $i = 1, \ldots, p+q$. By the hypothesis (H2),

$$b(i) \geq \begin{cases} \log \sigma_1 & \text{for } i = q+1, \ldots, p+q \\ -\log \sigma_2 & \text{for } i = 1, \ldots, q. \end{cases}$$

Let $a_s = b(i_s)$ for $s = 0, \dots, n-1$. The hypothesis $(i_0, \dots, i_{n-1}) \notin I(\gamma, n)$, together with (4), gives

$$\frac{1}{n} \sum_{s=0}^{n-1} a_s \geq (1-\gamma) \log \sigma_1 - \gamma \log \sigma_2 \geq 4c.$$

Take $A = \sup_M \{\log \|Df^{-1}\|^{-1}\}$ and $c_1 = 2c$ and $c_2 = 4c$. Then $\theta = 2c/(A - 2c)$ in Lemma 3.3. From the lemma we get that there exist $l > \theta n$ and integers $1 \leq n_1 < \dots < n_l \leq n$ such that

$$\log \prod_{s=k}^{n_j-1} \|Df(f^s(x))^{-1}\|^{-1} \geq \sum_{s=k}^{n_j-1} b(i_s) = \sum_{s=k}^{n_j-1} a_s \geq 2c(n_j - k) \tag{6}$$

for every $0 \leq k \leq n_j - 1$, every $j = 1, \dots, l$, and every $x \in R^{n_j}[i_0, \dots, i_{n_j-1}]$. This proves that $R^{n_j}[i_0, \dots, i_{n_j-1}]$ is hyperbolic, for every $j = 1, \dots, l$, as claimed. $\square$

3.2 Hyperbolic times

The following crucial property of hyperbolic times was proved in [ABV00]:

Proposition 3.4. *There exists $\delta = \delta(f, c) > 0$ such that, given any hyperbolic time $n \geq 1$ for a point $x \in M$ and given any $1 \leq j \leq n$, the inverse branch $f_{x,n}^{-j}$ of f^j that sends $f^n(x)$ to $f^{n-j}(x)$ is defined on the whole ball of radius δ around $f^n(x)$, and satisfies*

$$\|Df_{x,n}^{-j}(z)\| \leq e^{-jc} \quad \text{for every } z \text{ in the ball } B(f^n(x), \delta).$$

Here the ball of radius δ is meant with respect to the Riemann distance $d(x, y)$ on M. Given any path-connected domain $D \subset M$, we define the *inner distance* $d_D(x, y)$ between two points x and y in D to be the infimum of the lengths of all curves joining x to y inside D. Clearly,

$$d_D(x, y) \geq d(x, y) \quad \text{for every } x \text{ and } y \text{ in } D. \tag{7}$$

Assumption (H1) implies that the closure of $R^n[i_0, \dots, i_{n-1}]$ is also given by

$$\bar{R}^n[i_0, \dots, i_{n-1}] = \{y \in M : y \in \bar{R}_{i_0}, f(y) \in \bar{R}_{i_1}, \dots, f^{n-1}(y) \in \bar{R}_{i_{n-1}}\}$$

(recall cylinders are non-empty, by definition). Hence,

$$f^j(\bar{R}^n[i_0, \dots, i_{n-1}]) = \bar{R}^{n-j}[i_j, \dots, i_{n-1}] \quad \text{for any } 1 \leq j < n.$$

It follows that $f^n(\bar{R}^n[i_0, \dots, i_{n-1}]) = f(\bar{R}_{i_{n-1}})$, and so its inner diameter is bounded by the constant

$$K_2 = K_1 \max_{x \in M} \|Df(x)\| \tag{8}$$

where K_1 is the maximum inner diameter of $\bar{R}_i$ over all $i = 1, \dots, p+q$.

Corollary 3.5. *For every $1 \le j \le n$ and x, y in the closure of any $R^n \in \mathcal{R}_h^n$,*

$$d_{f^{n-j}(\bar{R}^n)}(f^{n-j}(x), f^{n-j}(y)) \le e^{-jc} d_{f^n(\bar{R}^n)}(f^n(x), f^n(y)) \le K_2 e^{-jc}.$$

Proof. Assumption (H1) implies that f^j is a homeomorphism from $f^{n-j}(\bar{R}^n)$ to $f^n(\bar{R}^n)$. Thus, any curve joining $f^n(x)$ to $f^n(y)$ inside $f^n(\bar{R}^n)$ lifts to a unique curve joining $f^{n-j}(x)$ to $f^{n-j}(y)$ inside $f^{n-j}(\bar{R}^n)$. By Proposition 3.4, the latter is shorter than the former by a factor e^{-cj}. This proves the claim. □

Corollary 3.6. *Given $\alpha > 0$ and any α-Hölder continuous function $\phi : M \to \mathbb{R}$, there exists $K_3 > 0$ such that for every x, y in the closure of any $R^n \in \mathcal{R}_h^n$,*

$$|S_n\phi(x) - S_n\phi(y)| \le K_3 d_{f^n(\bar{R}^n)}(f^n(x), f^n(y))^\alpha \le K_3 K_2^\alpha.$$

Proof. The assumption that ϕ is α-Hölder continuous means that there exists $C > 0$ such that, for every $1 \le j \le n$,

$$|\phi(f^{n-j}(x)) - \phi(f^{n-j}(y))| \le C d(f^{n-j}(x), f^{n-j}(y))^\alpha.$$

According to (7), we may replace the Riemann distance $d(\cdot,\cdot)$ by the inner distance $d_{f^{n-j}(\bar{R}^n)}(\cdot,\cdot)$ on the right hand side. Using Corollary 3.5 we conclude that

$$\begin{aligned}|S_n\phi(x) - S_n\phi(y)| &\le \sum_{j=1}^{n} C e^{-\alpha j c} d_{f^n(\bar{R}^n)}(f^n(x), f^n(y))^\alpha \\ &\le K_3 d_{f^n(\bar{R}^n)}(f^n(x), f^n(y))^\alpha,\end{aligned}$$

for some constant $K_3 > 0$ that depends only on C, α, and c. □

Remark 3.7. If m is a hyperbolic time for x then $m-s$ is a hyperbolic time for $f^s(x)$, for any $1 \le s < m$. The following converse is also a simple consequence of Definition 2.3: given $n < m$, if n is a hyperbolic time for x and there exists $1 \le s \le n$ such that $m - s$ is a hyperbolic time for $f^s(x)$ then m is a hyperbolic time for x. Consequently, if $R^n \in \mathcal{R}_h^n$ and $R^r \in \mathcal{R}_h^r$ then $R^n \cap f^{-s}(R^r) \in \mathcal{R}_h^{r+s}$ for any $1 \le s \le n$. Analogously, if $n_j(x)$, $j \ge 1$ denotes the sequence of values of n for which x belongs to the closure of some $R^n \in \mathcal{R}_h^n$ then, for every j and l, there exists $k \ge j + l$ such that

$$n_j(x) + n_l(f^{n_j(x)}(x)) = n_k(x) \ge n_{j+l}(x).$$

In principle, the inequality can be strict, because $n_{j+l}(x)$ is determined over a smaller cylinder than $n_l(f^{n_j(x)}(x))$.

3.3 Expanding measures

A probability measure η (not necessarily invariant) is *expanding* if $\eta(H) = 1$. Recall that H is the set of points $x \in M$ that belong to the closure of some hyperbolic cylinder R^n for infinitely many values $n_1(x) < \cdots < n_k(x) < \cdots$ of n. In particular, every $x \in H$ has infinitely many hyperbolic times, and so

$$\liminf_{n\to\infty} \frac{1}{n} \log \|Df^n(x)^{-1}\| \leq \liminf_{n\to\infty} \frac{1}{n} \log \prod_{j=0}^{n-1} \|Df(f^j(x))^{-1}\| \geq 2c > 0.$$

If η is invariant then, by the sub-additive ergodic theorem (Kingman [Kin68]), the limit exists at almost every points, and the previous inequality means that all the Lyapunov exponents of f are larger or equal than $2c > 0$ at η-almost every point.

For each $m \geq 1$, let H_m be the set of points $x \in M$ for which $n_1(x) = m$, that is, for which m is the smallest value of n such that x belongs to the closure of some hyperbolic cylinder R^n.

Proposition 3.8. *If η is an invariant expanding measure and $n_1(\cdot)$ is integrable with respect to η then the sequence $n_j(\cdot)$ is non-lacunary at η-almost every point.*

Proof. Let D be the set of points for which the sequence $n_j(\cdot)$ fails to be non-lacunary. For each $\theta > 0$, define $L_\theta(n) = \{x \in M : n_1(x) \geq \theta n\}$. If $x \in D$ then there exists a rational number $\theta > 0$, and there are infinitely many values of i such that $n_{i+1}(x) \geq (1+\theta)n_i(x)$. By Remark 3.7, the latter implies that

$$n_1(f^{n_i}(x)) \geq n_{i+1}(x) - n_i(x) \geq \theta n_i(x).$$

So, there are arbitrarily large values of n such that $x \in f^{-n}(L_\theta(n))$. In other words, D is contained in the set

$$L = \bigcup_{\theta\in\mathbb{Q}} \bigcap_{k=0}^{\infty} \bigcup_{n\geq k} f^{-n}(L_\theta(n)).$$

Since η is invariant, we have $\eta(f^{-n}(L_\theta(n))) = \eta(L_\theta(n))$ for all n. Then

$$\theta \sum_{n=1}^{\infty} \eta(L_\theta(n)) = \theta \sum_{n=1}^{\infty} \sum_{n_1\geq\theta n} \eta(H_{n_1}) = \theta \sum_{m=1}^{\infty} \sum_{n=1}^{m/\theta} \eta(H_m) \leq \sum_{m=1}^{\infty} m\eta(H_m).$$

Thus, using the hypothesis that $n_1(\cdot)$ is integrable,

$$\theta \sum_{n=1}^{\infty} \eta(L_\theta(n)) \leq \sum_{m=1}^{\infty} m\eta(H_m) = \int n_1(x) d\eta(x) < \infty.$$

By the Borel-Cantelli lemma, this implies that L has measure zero. It follows that $\eta(D) = \eta(L) = 0$, as claimed. □

Remark 3.9. The same argument proves a pointwise version of the lemma: if the first hyperbolic time is integrable, with respect to some invariant expanding probability, then the sequence of all hyperbolic times is non-lacunary.

For each $n \geq 1$, let $\mathcal{Q}^n$ be a covering of H by pairwise disjoint measurable sets such that every $Q^n \in \mathcal{Q}^n$ satisfies $R^m \subset Q^n \subset \bar{R}^m$ for some $R^m \in \mathcal{R}_h^m$ and $m \geq n$. For instance, we may consider the family formed by all cylinders $R^m \in \mathcal{R}_h^m$ with $m \geq n$ which are not contained in any $R^k \in \mathcal{R}_h^k$ with $m > k \geq n$. Since these maximal cylinders are pairwise disjoint and their closures cover H, we may then choose a set Q^n between each R^m and its closure, in such a way that these Q^n are pairwise disjoint and cover H.

Remark 3.10. In the situations we are interested in, for the measures μ and ν in our main theorems, every R^m differs from its closure by a zero measure set only, and so the distinction between Q^n and R^m is actually irrelevant. However, this fact will not be proven until much later.

We shall need the fact that, for any expanding measure, such a family $\mathcal{Q}^n$ is "generating", in the following sense:

Lemma 3.11. *Given any measurable set $E \subset M$ and any $\delta > 0$, there exist $n \geq 1$ and a subset $\{Q_i^n : i \in I\}$ of $\mathcal{Q}^n$ such that*

$$\eta\big(E \Delta \bigcup_{i \in I} Q_i^n\big) \leq \delta. \tag{9}$$

Proof. Let $L_1 \subset E$ and $L_2 \subset E^c$ be compact sets such $\eta(E \Delta L_1) \leq \delta/3$ and $\eta(E^c \Delta L_2) \leq \delta/3$. Then $r = \operatorname{dist}(L_1, L_2)$ is strictly positive. By Corollary 3.5, we have $\operatorname{diam}(\mathcal{Q}^n) \leq K_2 e^{-cn} < r$ if n is large enough. Fix such an n. Since $\mathcal{Q}^n$ covers the full measure set H, we may choose a subset $\{Q_i^n : i \in I\}$ of $\mathcal{Q}^n$ that almost covers L_1:

$$\eta\big(L_1 \setminus \bigcup_{i \in I} Q_i^n\big) \leq \frac{\delta}{3}.$$

We may assume all Q_i^n do intersect L_1, in which case they are disjoint from L_2. Hence,

$$\eta\big(E \Delta \bigcup_{i \in I} Q_i^n\big) \leq \eta(E \setminus L_1) + \eta\big(L_1 \setminus \bigcup_{i \in I} Q_i^n\big) + \eta(E^c \setminus L_2) \leq \delta.$$

This completes the proof of the lemma. □

Corollary 3.12. *Given any measurable set $E \subset M$ with $\eta(E) > 0$ and any $\varepsilon > 0$, there exists $n \geq 1$ and some $Q^n \in \mathcal{Q}^n$ such that*

$$\frac{\eta(E \cap Q^n)}{\eta(Q^n)} > 1 - \varepsilon.$$

Proof. Given E and ε, fix $\delta > 0$ such that $\varepsilon \eta(E) > 2\delta$. It is no restriction to suppose $\varepsilon < 1$, in which case this implies $\eta(E) > 2\delta$. Take n and $\{Q_i^n : i \in I\}$ as in Lemma 3.11. In particular,

$$\eta\big(\bigcup_{i \in I} Q_i^n\big) \geq \eta(E) - \delta > \frac{\eta(E)}{2}.$$

Suppose $\eta(Q_i^n \setminus E) \geq \varepsilon\eta(Q_i^n)$ for all $i \in I$. Adding these inequalities we would obtain

$$\eta(\bigcup_{i\in I} Q_i^n \setminus E) \geq \varepsilon\eta(\bigcup_{i\in I} Q_i^n) \geq \frac{\varepsilon}{2}\eta(E) \geq \delta,$$

contradicting (9). Thus, $\eta(Q_i^n \setminus E) < \varepsilon\eta(Q_i^n)$ for some $i \in I$ and this implies the conclusion of the lemma. □

3.4 Relative pressure

Here we recall some basic ideas related to the variational principle. Additional information can be found in Walters [Wal82] and Pesin [Pes97].

Let $f : M \to M$ be a continuous map on a compact space M, and let $\phi : M \to \mathbb{R}$ be continuous. Let Λ be any subset of M that is invariant under f, and let $\mathcal{U}$ be a cover of Λ. To each finite sequence $(U_1, \ldots, U_n)$ of elements of $\mathcal{U}$, associate the set

$$U = \{x \in M : x \in U_0, f(x) \in U_1, \ldots, f^{n-1}(x) \in U_{n-1}\} \tag{10}$$

and write $n(U) = n$ (a slight abuse of language). Given any $N \geq 1$, define $\mathcal{S}_N(\mathcal{U})$ to be the family of all sets U of this form, for all values of $n(U) \geq N$. We denote by $S_n\phi(V)$ the supremum of $S_n\phi$ over an arbitrary set V.

Given any $\alpha \in \mathbb{R}$, consider the number

$$m_\Lambda(f, \phi, \alpha, \mathcal{U}, N) = \inf_{\mathcal{G}} \sum_{U\in\mathcal{G}} \exp\big(S_{n(U)}\phi(U) - \alpha n(U)\big) \tag{11}$$

where the infimum is taken over all families $\mathcal{G} \subset \mathcal{S}_N(\mathcal{U})$ that cover Λ. Define

$$m_\Lambda(f, \phi, \alpha, \mathcal{U}) = \lim_{N\to\infty} m_\Lambda(f, \phi, \alpha, \mathcal{U}, N),$$

(the sequence is monotone increasing) and

$$P_\Lambda(f, \phi, \mathcal{U}) = \inf\{\alpha : m_\Lambda(f, \phi, \alpha, \mathcal{U}) = 0\}$$

Definition 3.13. The *pressure of f for ϕ relative to* Λ is

$$P_\Lambda(f, \phi) = \lim_{\operatorname{diam}\mathcal{U}\to 0} P_\Lambda(f, \phi, \mathcal{U}).$$

Theorem 11.1 in [Pes97] states that the limit does exist, that is, given any sequence of covers $\mathcal{U}_k$ of Λ with diameter going to zero, $P_\Lambda(f, \phi, \mathcal{U}_k)$ converges and the limit does not depend on the choice of the sequence.

Let $\mathcal{I}_\Lambda$ denote the set of invariant probability measures η such that $\eta(\Lambda) = 1$. If Λ is compact then (see [Wal82, Theorem 9.10] or [Pes97, Theorem A2.1])

$$P_\Lambda(f, \phi) = \sup\{h_\eta(f) + \int \phi \, d\eta : \eta \in \mathcal{I}_\Lambda\}.$$

This applies, in particular, when $\Lambda = M$. We just write $P(f, \phi)$ to mean $P_M(f, \phi)$. In the general non-compact case one inequality remains true:

$$P_\Lambda(f, \phi) \geq \sup\{h_\eta(f) + \int \phi \, d\eta : \eta \in \mathcal{I}_\Lambda\}. \tag{12}$$

In particular, if $\mathcal{I}_\Lambda$ contains some equilibrium state then the equality holds in (12), and $P_\Lambda(f, \phi) = P(f, \phi)$.

The next proposition is probably well-known but we could not find a proof in the literature. The one we give here was obtained jointly with Paulo Varandas. It uses the following alternative definition of the relative pressure, in terms of dynamical balls.

Fix $\varepsilon > 0$. Set $\mathcal{I}_n = M \times \{n\}$ and $\mathcal{I} = M \times \mathbb{N}$. For $\alpha \in \mathbb{R}$ and $N \geq 1$, define

$$m_\alpha(f, \phi, \Lambda, \varepsilon, N) = \inf_{\mathcal{G}} \Big\{ \sum_{(x,n) \in \mathcal{G}} e^{-\alpha n + S_n \phi(\mathrm{B}(x,n,\varepsilon))} \Big\}, \tag{13}$$

where the infimum is taken over all finite or countable families $\mathcal{G} \subset \cup_{n \geq N} \mathcal{I}_n$ such that the dynamical balls $\{B(x, n, \varepsilon) : (x, n) \in \mathcal{G}\}$ cover Λ. Then let

$$m_\alpha(f, \phi, \Lambda, \varepsilon) = \lim_{N \to \infty} m_\alpha(f, \phi, \Lambda, \mathcal{U}, N)$$

(once more, the sequence is monotone increasing) and

$$P_\Lambda(f, \phi, \varepsilon) = \inf\{\alpha : m_\alpha(f, \phi, \Lambda, \varepsilon) = 0\}.$$

By Remark 1 in [Pes97, Page 74] the limit when $\varepsilon \to 0$ exists and coincides with the relative pressure:

$$P_\Lambda(f, \phi) = \lim_{\varepsilon \to 0} P_\Lambda(f, \phi, \varepsilon).$$

Remark 3.14. Since ϕ is uniformly continuous, the definition of $P_\Lambda(f, \phi, \varepsilon)$ is not affected when one replaces in (13) the supremum $S_n\phi(B(x, n, \varepsilon))$ by the value $S_n\phi(x)$ at the center point.

Proposition 3.15. *Let M be a compact metric space, $f : M \to M$ be a continuous transformation, $\phi : M \to \mathbb{R}$ be a continuous function, and Λ be an f-invariant set. Then $P_\Lambda(f^\ell, S_\ell\phi) = \ell P_\Lambda(f, \phi)$ for every $\ell \geq 1$.*

Proof. Fix $\ell \geq 1$. By uniform continuity of f, given any $\rho > 0$ there exists $\varepsilon > 0$ such that $d(x, y) < \varepsilon$ implies $d(f^j(x), f^j(y)) < \rho$ for all $0 \leq j < \ell$. It follows that

$$B_f(x, \ell n, \varepsilon) \subset B_{f^\ell}(x, n, \varepsilon) \subset B_f(x, \ell n, \rho), \tag{14}$$

where $B_g(x, n, \varepsilon)$ denotes the dynamical ball of center x, length n, and radius ε, relative to a map g.

First, we prove the $\geq$ inequality. Given $N \geq 1$ and any family $\mathcal{G}_\ell \subset \cup_{n \geq N} \mathcal{I}_n$ such that the balls $B_{f^\ell}(x, j, \varepsilon)$ with $(x, j) \in \mathcal{G}_\ell$ cover Λ, denote

$$\mathcal{G} = \{(x, j\ell) : (x, j) \in \mathcal{G}_\ell\}.$$

The second inclusion in (14) ensures that the balls $B_f(x,k,\rho)$ with $(x,k)\in\mathcal{G}$ cover Λ. Clearly,

$$\sum_{(x,j)\in\mathcal{G}_\ell} e^{-\alpha\ell j+\sum_{i=0}^{j-1} S_\ell\phi(f^{i\ell}(x))} = \sum_{(x,k)\in\mathcal{G}} e^{-\alpha k+\sum_{i=0}^{k-1}\phi(f^i(x))}.$$

Since $\mathcal{G}_\ell$ is arbitrary, and recalling Remark 3.14, this proves that

$$m_{\alpha\ell}(f^\ell, S_\ell\phi, \Lambda, \varepsilon, N) \geq m_\alpha(f,\phi,\Lambda,\rho,N\ell).$$

So, $m_{\alpha\ell}(f^\ell, S_\ell\phi, \Lambda, \varepsilon) \geq m_\alpha(f,\phi,\Lambda,\rho)$. Then $P_\Lambda(f^\ell, S_\ell\phi,\varepsilon) \geq \ell P_\Lambda(f,\phi,\rho)$. Since $\varepsilon\to 0$ when $\rho\to 0$, it follows that $P_\Lambda(f^\ell, S_\ell\phi) \geq \ell P_\Lambda(f,\phi)$.

For the $\leq$ inequality, we observe that the definition of the relative pressure is not affected if one restricts the infimum in (13) to families $\mathcal{G}$ of pairs (x,k) such that k is always a multiple of ℓ. More precisely, let $m_\alpha^\ell(f,\phi,\Lambda,\varepsilon,N)$ be the infimum over this subclass of families, and let $m_\alpha^\ell(f,\phi,\Lambda,\varepsilon)$ be its limit as $N\to\infty$.

Lemma 3.16. *We have $m_\alpha^\ell(f,\phi,\Lambda,\varepsilon) \leq m_{\alpha-\rho}(f,\phi,\Lambda,\varepsilon)$ for every $\rho>0$.*

Proof. We only have to show that $m_\alpha^\ell(f,\phi,\Lambda,\varepsilon,N) \leq m_{\alpha-\rho}(f,\phi,\Lambda,\varepsilon,N)$ for any $\rho>0$ and any sufficiently large N. Let ρ be fixed and N be large enough so that $N\rho > \ell(\alpha+\sup|\phi|)$. Given any $\mathcal{G}\subset\cup_{n\geq N}\mathcal{I}_n$ such that the balls $B_f(x,k,\varepsilon)$ with $(x,k)\in\mathcal{G}$ cover Λ, define $\mathcal{G}'$ to be the family of all (x,k'), $k'=\ell[k/\ell]$ such that $(x,k)\in\mathcal{G}$. Notice that

$$-\alpha k' + S_{k'}\phi(x) \leq -\alpha k + \alpha\ell + S_k\phi(x) + \ell\sup|\phi| \leq (-\alpha+\rho)k + S_k\phi(x)$$

given that $k\geq N$. The claim follows immediately. □

Let $\mathcal{G}'$ be any family of pairs (x,k) with $k\geq N\ell$ and such that every k is a multiple of ℓ. Define $\mathcal{G}_\ell$ to be the family of pairs (x,j) such that $(x,j\ell)\in\mathcal{G}'$. The first inclusion in (14) ensures that if the balls $B_f(x,k,\varepsilon)$ with $(x,k)\in\mathcal{G}'$ cover Λ then so do the balls $B_{f^\ell}(x,j,\varepsilon)$ with $(x,j\in\mathcal{G}_\ell)$. Clearly,

$$\sum_{(x,k)\in\mathcal{G}'} e^{-\alpha k+\sum_{i=0}^{k-1}\phi(f^i(x))} = \sum_{(x,j)\in\mathcal{G}_\ell} e^{-\alpha\ell j+\sum_{i=0}^{j-1} S_\ell\phi(f^{i\ell}(x))}.$$

Since $\mathcal{G}_\ell$ is arbitrary, and recalling Remark 3.14, this proves that

$$m_\alpha^\ell(f,\phi,\Lambda,\varepsilon,N\ell) \geq m_{\alpha\ell}(f^\ell, S_\ell\phi,\Lambda,\varepsilon,N).$$

Taking the limit when $N\to\infty$ and using Lemma 3.16,

$$m_{\alpha-\rho}(f,\phi,\Lambda,\varepsilon) \geq m_\alpha^\ell(f,\phi,\Lambda,\varepsilon) \geq m_{\alpha\ell}(f^\ell, S_\ell\phi,\Lambda,\varepsilon).$$

It follows that $\ell\big(P_\Lambda(f,\phi,\varepsilon)+\rho\big) \geq P_\Lambda(f^\ell, S_\ell\phi,\varepsilon)$. Since ρ is arbitrary, we conclude that $\ell P_\Lambda(f,\phi,\varepsilon) \geq P_\Lambda(f^\ell, S_\ell\phi,\varepsilon)$ and so $P_\Lambda(f^\ell, S_\ell\phi) \geq \ell P_\Lambda(f,\phi)$. The proof of Proposition 3.15 is complete. □

3.5 Weak Gibbs measures

The next proposition is not used elsewhere in this paper. It is included to help clarify the relation between our non-lacunary Gibbs measures and the notion of weak Gibbs measure in [Yur99]: Yuri requires condition (15) at all points and K_n independent of x.

Proposition 3.17. *If η is a non-lacunary Gibbs measure then for η-almost every point $x \in M$ there is a sequence $K_n = K_n(x)$ such that $\lim \frac{1}{n} \log K_n = 0$ and*

$$K_n^{-1} \le \frac{\eta\big(R^n(x)\big)}{\exp\big(S_n\phi(x) - nP\big)} \le K_n \quad \textit{for every } n \ge 1. \tag{15}$$

Proof. By assumption, for almost every $x \in M$ there exists an increasing sequence $n_i \in \mathbb{N}$ such that $\varepsilon_i = (n_{i+1} - n_i)/n_i$ converges to zero and

$$K^{-1} \le \frac{\eta\big(R^n(x)\big)}{\exp\big(S_n\phi(x) - nP\big)} \le K \quad \text{whenever } n = n_i.$$

Given $n \ge n_1$ (clearly, we only need to consider large values of n), let $i = i(n)$ be such that $n_i \le n < n_{i+1}$. Then

$$\eta(R^{n_{i+1}}(x)) \le \eta(R^n(x)) \le \eta(R^{n_i}(x)).$$

Moreover,

$$|S_n\phi(x) - S_{n_i}\phi(x)| \le (n - n_i) \max|\phi| \le \varepsilon_i n \max|\phi|$$

and analogously for $|S_n\phi(x) - S_{n_{i+1}}\phi(x)|$. It follows that

$$K^{-1}e^{-\varepsilon_i n(\max|\phi|+P)} \le \frac{\eta\big(R^n(x)\big)}{\exp\big(S_n\phi(x) - nP\big)} \le Ke^{\varepsilon_i n(\max|\phi|+P)}.$$

Define $K_n = K \exp\big[\varepsilon_{i(n)} n(\max|\phi| + P)\big]$. Then

$$\lim_{n\to\infty} \frac{1}{n} \log K_n = \lim_{n\to\infty} \frac{1}{n} \log K + \varepsilon_{i(n)}(\max|\phi| + P) = 0,$$

and so the proof is complete. $\square$

4 Proof of Theorem A

Here we prove Theorem A and we also prepare the way toward proving that ν is a non-lacunary Gibbs measure (which will be done near the end of Section 5).

4.1 Eigenmeasures of the transfer operator

The *Jacobian* of a measure η with respect to f is the (essentially unique) function $J_\eta f$ satisfying

$$\eta(f(A)) = \int_A J_\eta f \, d\eta.$$

for any measurable set A such that $f \mid A$ is injective. A Jacobian needs not exist, in general.

Lemma 4.1. *Suppose ν is a Borel probability such that $\mathcal{L}_\phi^* \nu = \lambda \nu$ for some $\lambda > 0$. Then the Jacobian of ν with respect to f is given by $J_\nu f = \lambda e^{-\phi}$.*

Proof. Let A be any measurable set such that $f \mid A$ is injective. Take a sequence $(g_n)_n$ of continuous functions on M such that $g_n \to \chi_A$ at ν-almost every point and $\sup |g_n| \leq 2$ for all n. Then,

$$\mathcal{L}_\phi(e^{-\phi} g_n)(x) = \sum_{f(y)=x} e^{\phi(y)} \, e^{-\phi(y)} g_n(y) = \sum_{f(y)=x} g_n(y).$$

The last expression converges to $\chi_{f(A)}(x)$ at ν-almost every point, because $f \mid A$ is injective. Hence, by the dominated convergence theorem,

$$\int \lambda e^{-\phi} g_n \, d\nu = \int e^{-\phi} g_n \, d(\mathcal{L}_\phi^* \nu) = \int \mathcal{L}_\phi(e^{-\phi} g_n) \, d\nu \to \nu(f(A)).$$

Since the left hand side also converges to $\int_A \lambda e^{-\phi} d\nu$, we conclude that

$$\nu(f(A)) = \int_A \lambda e^{-\phi} d\nu,$$

which proves the lemma. $\square$

Lemma 4.2. *The spectral radius of the operator $\mathcal{L}_\phi$ is at least* $\deg(f) e^{\inf \phi}$ *and it is an eigenvalue for the dual operator $\mathcal{L}_\phi^*$.*

Proof. Let λ be the spectral radius of $\mathcal{L}_\phi$. Observe that

$$\mathcal{L}_\phi 1(x) = \sum_{f(y)=x} e^{\phi(y)} \geq \deg(f) \, e^{\inf \phi} = \deg(f) e^{\inf \phi}$$

for every $x \in M$. Since $\mathcal{L}$ is a positive operator, it follows that

$$\mathcal{L}_\phi^n 1 \geq \deg(f)^n e^{n \inf \phi} \quad \text{for every } n \geq 1,$$

and so the spectral radius is at least $\deg(f) e^{\inf \phi}$, as claimed in the first part of the lemma. The second part follows from general results in functional analysis. A quick argument goes as follows. Let C be the cone of positive continuous functions on M and

$$V = \{\lambda \varphi - \mathcal{L}(\varphi) : \varphi \in C^0(M)\}.$$

Clearly, V is a linear subspace and C is an open convex set. We claim that they are disjoint. Indeed, suppose $\psi = \lambda\varphi - \mathcal{L}(\varphi)$ is a positive function, for some $\varphi \in C^0(M)$. By compactness and continuity, there exists $\delta > 0$ such that $\delta \max(-\varphi) \leq \min \psi$. Then

$$\mathcal{L}(-\varphi) = -\lambda\varphi + \psi \geq (\lambda + \delta)(-\varphi).$$

Since $\mathcal{L}$ is a positive operator, it follows that $\mathcal{L}^n(-\varphi) \geq (\lambda+\delta)^n(-\varphi)$ for every $n \geq 1$. This implies that the spectral radius of $\mathcal{L}$ is at least $\lambda + \delta$, contradicting the definition of λ. This contradiction proves that $C \cap V = \emptyset$, as we claimed. Then, by Mazur's theorem (see [Dei85, Proposition 7.2]), there exists some continuous linear functional $\nu : C^0(M) \to \mathbb{R}$ such that

$$\int \varphi \, d\mu > 0 \text{ for every } \varphi \in C \quad \text{and} \quad \int \varphi \, d\mu = 0 \text{ for every } \varphi \in V.$$

The first property means that ν is a measure and so, up to normalization, we may suppose it is a probability. The second property means that

$$\int \varphi \, d(\mathcal{L}^*\nu) = \int \mathcal{L}(\varphi) \, d\nu = \int \lambda\varphi \, d\nu \quad \text{for every } \varphi \in C^0(M),$$

that is, $\mathcal{L}^*\nu = \lambda\nu$. Thus, λ is indeed an eigenvalue for the dual operator $\mathcal{L}^*$. $\square$

In all that follows, λ is an eigenvalue of $\mathcal{L}_\phi^*$ larger than $e^{\kappa+\sup\phi}$ and ν is an eigenmeasure associated to it. Lemma 4.2, combined with (H3), ensures that such an eigenvalue does exist. We shall see later, in Corollary 6.3 that it is also unique, that is, λ coincides with the spectral radius of $\mathcal{L}_\Phi$. From Lemma 4.1 we get that

$$J_\nu f(x) = \lambda e^{-\phi(x)} > e^\kappa > q \tag{16}$$

for all $x \in M$. This property allows us to prove that ν-almost every point spends at most a fraction γ of time inside the domain $\bar{R}_1 \cup \cdots \cup \bar{R}_q$ where f may fail to be expanding.

Let $B(n)$ be the union of all $\bar{R}[i_0, \ldots, i_{n-1}]$ corresponding to itineraries $(i_0, \ldots, i_{n-1}) \in I(\gamma, n)$, that is, such that

$$\#\{0 \leq j \leq n-1 : i_j \leq q\} > \gamma n,$$

and $G(n)$ be the union of those $\bar{R}[i_0, \ldots, i_{n-1}]$ corresponding to itineraries in the complement of $I(\gamma, n)$. Since the closures of all length n cylinders cover M, so does $\{B(n), G(n)\}$.

Proposition 4.3. *The measure $\nu(B(n))$ decreases exponentially fast when n goes to infinity. Consequently, ν-almost every $x \in M$ belongs to $G(n)$ for all but finitely many values of n.*

Proof. Since f is injective on the closure of every atom of $\mathcal{R}$, the map f^n is injective on the closure of every $R^n \in \mathcal{R}^n$. Then the inequality (16) implies

$$1 \geq \nu(f^n(\bar{R}^n)) = \int_{\bar{R}^n} J_\nu f^n \, d\nu = \int_{\bar{R}^n} \prod_{j=0}^{n-1} (J_\nu f \circ f^j) d\nu \geq e^{\kappa n} \nu(\bar{R}^n).$$

This proves that every $\nu(\bar{R}^n) \leq e^{-\kappa n}$. By definition, c_γ is the upper exponential rate of growth of the cardinality of $I(\gamma, n)$. Thus, given any $\varepsilon > 0$,

$$\nu(B(n)) = \sum_{I(\gamma,n)} \nu(\bar{R}^n) \leq e^{(c_\gamma + \varepsilon - \kappa)n}$$

for all large n. Since $c_\gamma < \kappa$, it follows that the right hand side decays exponentially fast with n, as claimed. The second statement in the lemma is a direct consequence, by the Borel-Cantelli lemma and the observation that $G(n)$ contains the complement of $B(n)$. □

4.2 Expanding property

Now we use Proposition 4.3 to prove that the measure ν is expanding:

Proposition 4.4. *The measure ν is expanding and satisfies $\int n_1 \, d\nu < \infty$.*

Proof. Proposition 3.2 implies that $n_1(x) \leq n$ for every $x \in G(n)$. Thus, the set of points x such that $n_1(x) > n$ must be contained in the union $B(n)$ of all $\bar{R}^n$ corresponding to itineraries in $I(\gamma, n)$. By Proposition 4.3, the measure of these sets decreases exponentially fast. Thus,

$$\int n_1 d\nu = \sum_{n=0}^{\infty} \nu(\{x : n_1(x) > n\}) \leq 1 + \sum_{n=1}^{\infty} \nu(B(n)) < \infty$$

and so $n_1(\cdot)$ is ν-integrable, as claimed. We have seen in Proposition 4.3 that ν-almost every point belongs to $G(n)$ for every large n. By Proposition 3.2, every point of $G(n)$ belongs to the closure of some $R^n \in \mathcal{R}_h^n$ for $l > \theta n$ values of n. It follows that ν-almost every $x \in M$ belongs to the closure of some $R^n \in \mathcal{R}_h^n$ for infinitely many values of n. In other words, H has full measure, which means that ν is expanding. □

From these arguments we also obtain, immediately,

Corollary 4.5. *For ν-almost any $x \in M$, the set $n_1(x) < \cdots < n_j(x) < \cdots$ of values of n such that x belongs to the closure of some $R^n \in \mathcal{R}_h^n$ has density at least θ at infinity: $\#\{j \geq 1 : n_j(x) \leq n\} \geq \theta n$ for every large n.*

4.3 Non-lacunary Gibbs property

The results in this subsection will be used in Section 5 to prove that ν is a non-lacunary Gibbs measure.

Lemma 4.6. *There is $K_4 > 0$ such for any x, y in the closure of any $R^n \in \mathcal{R}_h^n$,*

$$K_4^{-1} \le \frac{J_\nu f^n(x)}{J_\nu f^n(y)} \le K_4$$

Proof. Lemma 4.1 implies that $J_\nu f^n(x) = \lambda^n e^{-S_n\phi(x)}$ for every $n \ge 1$. Then, by Corollary 3.6,

$$\Big|\log \frac{J_\nu f^n(x)}{J_\nu f^n(y)}\Big| = \big|S_n\phi(y) - S_n\phi(x)\big| \le K_3 K_2^\alpha.$$

Thus, we may take $K_4 = \exp(K_3 K_2^\alpha)$. □

Proposition 4.7. *There exists $K_5 > 0$ such that for every $x \in R^n \in \mathcal{R}_h^n$,*

$$K_5^{-1} \le \frac{\nu(R^n)}{\exp(S_n\phi(x) - Pn)} \le K_5.$$

This remains true if one replaces $\nu(R^n)$ by $\nu(\bar{R}^n)$ and considers any $x \in \bar{R}^n$.

Proof. We prove only the first claim, as the version for the closure is analogous. By Lemma 4.1, for any $R^n = R^n[i_0, \ldots, i_{n-1}]$,

$$\nu(f(R_{i_{n-1}})) = \nu(f^n(R^n)) = \int_{R^n} J_\nu f^n d\nu = \int_{R^n} e^{nP - S_n\phi(x)}\, d\nu(x).$$

By Lemma 4.6, there exists K_4 independent of n such that for every $x, y \in R^n$

$$K_4^{-1} J_\nu f^n(y) \le J_\nu f^n(x) \le K_4 J_\nu f^n(y).$$

It follows that

$$K_4^{-1}\nu(f(R_{i_{n-1}})) \le \frac{\nu(R^n)}{\exp(S_n\phi(x) - Pn)} \le K_4 \nu(f(R_{i_{n-1}}))$$

for any $x \in R^n$. The right hand side is bounded above by K_4. On the other hand, according to (H1), there exists $N \ge 1$ such that every $f^N(R_i) = M$ and, consequently, has total ν-measure. Now, we may decompose $f(R_i)$ into finitely many subsets such that f^{N-1} is injective on each one of them. Using the fact that ν has a Jacobian, it follows that $\nu(f(R_i)) > 0$ for every i. To finish the proof, just take $K_5 = \max_i K_4/\nu(f(R_i)) \ge K_4$. □

To conclude that ν is a non-lacunary Gibbs state we only have to check that $\nu(\partial\mathcal{R}) = 0$ and hyperbolic times form a non-lacunary sequence at ν-almost every point. This will be done in the next section. In the meantime we deduce the following lemma, which completes the proof of Theorem A:

Corollary 4.8. *The support of ν coincides with the closure of H.*

Proof. Proposition 4.4 states that the set H has full ν-measure, and this implies the support of ν is contained in $\bar{H}$. To prove the converse, consider any $x \in H$. Then there is a sequence $n_i \to \infty$ of values of n such that x belongs to the closure of some hyperbolic cylinder $R^{n_i} \in \mathcal{R}^{n_i}$. By Proposition 4.7,

$$\nu(\bar{R}^{n_i}) > K_5^{-1} \exp(S_n\phi(x) - Pn) > 0.$$

By Corollary 3.5, the diameter of $\bar{R}^{n_i}$ converges to zero when $n_i \to \infty$. Thus, any neighborhood of x has positive ν measure, that is, $x \in \operatorname{supp} \nu$. □

5 Proof of Theorem B

In this section we prove Theorem B. The main idea is to introduce a sequence of linear operators $\mathcal{T}_{\phi,n}$ that are obtained from the iterates $\mathcal{L}_\phi^n$ of the transfer operator by considering only preimages for which n is a hyperbolic time. The main results (Corollaries 5.2 and 5.4) state that the sequence $\mathcal{T}_{\phi,n}1$ is uniformly bounded and piecewise Hölder continuous, with uniform Hölder constants. Thus, it admits Cesaro limits and they are piecewise Hölder continuous. We check that any such limit is an eigenfunction h of the transfer operator, bounded away from zero and infinity. Then $\mu = h\nu$ is invariant and ergodic, and it is equivalent to ν. Using also Proposition 4.7, we deduce that both μ and ν are non-lacunary Gibbs measures of f for ϕ.

5.1 Upper bound

Observe that the nth iterate of the transfer operator $\mathcal{L}_\phi$ is given by

$$\mathcal{L}_\phi^n g(x) = \sum_{y \in f^{-n}(x)} e^{S_n\phi(y)} g(y). \tag{17}$$

For each $n \geq 1$, we define a new operator $\mathcal{T}_{\phi,n}$ by restricting the sum in (17) to the pre-images belonging to the closure of some hyperbolic cylinder:

$$\mathcal{T}_{\phi,n} g(x) = \sum_{y \in f^{-n}(x) \cap \bar{R}^n \,:\, R^n \in \mathcal{R}_h^n} e^{S_n\phi(y)} g(y),$$

where $y = y(x, R^n)$ is the unique point in $f^{-n}(x) \cap \bar{R}^n$. We also define

$$T_n(x) = \mathcal{T}_{\phi,n}1(x) = \sum_{y \in f^{-n}(x) \cap \bar{R}^n \,:\, R^n \in \mathcal{R}_h^n} e^{S_n\phi(y)}$$
$$\text{and} \quad Z_n = \sum_{R^n \in \mathcal{R}_h^n} e^{S_n\phi(R^n)} = \sum_{R^n \in \mathcal{R}_h^n} e^{S_n\phi(\bar{R}^n)}.$$

It is clear that $T_n(x) \leq Z_n$ for all $x \in M$. There are two reasons why the inequality may be strict. Firstly, is that the definition of Z_n is in terms of the supremum $S_n\phi(\bar{R}^n)$ over all $y \in \bar{R}^n$. Secondly, x needs not have pre-images in every hyperbolic $\bar{R}^n$. But the converse inequality does hold up to a factor, as we shall see in Lemma 5.6. Right now we prove

Lemma 5.1. *There exists a constant $K_6 > 0$ such that for every $n \geq 1$*

$$\lambda^{-n} Z_n \leq K_6 \quad \textit{and} \quad K_6^{-1} \leq \frac{1}{n}\sum_{j=0}^{n-1} \lambda^{-j} Z_j \leq K_6 \,.$$

Proof. By Proposition 4.7, for every point $y \in R^n \in \mathcal{R}_h^n$ we have

$$K_5^{-1}\nu(R^n) \leq \lambda^{-n} e^{S_n\phi(y)} \leq K_5\nu(R^n).$$

Taking the supremum over $y \in R^n$ and then summing over all $R^n \in \mathcal{R}_h^n$, we get

$$K_5^{-1}\nu(E_n) \leq \lambda^{-n} Z_n \leq K_5\nu(E_n) \leq K_5,$$

where E_n denotes the union of all $R^n \in \mathcal{R}_h^n$. Both upper inequalities in the statement of the lemma are immediate consequences, as long as one chooses $K_6 \geq K_5$. We also get that

$$K_5^{-1}\frac{1}{n}\sum_{j=0}^{n-1}\nu(E_j) \leq \frac{1}{n}\sum_{j=0}^{n-1}\lambda^{-i} Z_j \,. \tag{18}$$

To prove the lower inequality, and finish the proof of the lemma, we only have to check that the left hand side of (18) is uniformly bounded away from zero. For that purpose, write

$$\frac{1}{n}\sum_{j=0}^{n-1}\nu(E_j) = \int\int \mathcal{X}_{E_j}(x)\, d\nu(x)\, dm_n(j).$$

where m_n is the normalized counting measure on the set $\{1, \ldots, n\}$. By Corollary 4.5, the sequence of hyperbolic times has density $\geq \theta > 0$ at infinity at ν-almost every point. Thus,

$$\sum_{j=0}^{n-1}\mathcal{X}_{E_j}(x) \geq \theta n$$

for ν-almost every x and every large n. Take n large enough so that the set X of points for which this holds has ν-measure at least 1/2. By Fubini's theorem, it follows that

$$\frac{1}{n}\sum_{j=0}^{n-1}\nu(E_j) = \int\int \mathcal{X}_{E_j}(x)\, dm_n(j)\, d\nu(x) \geq \int_X \theta\, d\nu(x) \geq \frac{\theta}{2} > 0.$$

Combining this with (18), and taking $K_6 \geq 2K_5/\theta$, one obtains the lower inequality in the statement. $\square$

Corollary 5.2. *The sequence $\lambda^{-n}T_n$ is uniformly bounded from above.*

Proof. Just notice that $\lambda^{-n}T_n(x) \leq \lambda^{-n}Z_n \leq K_6$ for all $x \in M$ and $n \geq 1$. $\square$

5.2 Equicontinuity

Here we show that the sequence $\lambda^{-n}T_n$ is uniformly piecewise Hölder continuous, at least for large enough n. More precisely, define

$$Q_I = \bigcap_{i\in I} f(\bar{R}_i) \cap \bigcap_{i\in I^c} f(\bar{R}_i)^c,$$

for each $I \subset \{1, \dots, p+q\}$. It is clear that $\{Q_I\}_I$ is a finite partition of M. Endow each Q_I with the metric

$$d_I(x_1, x_2) = \sum_{i\in I} d_{f(\bar{R}_i)}(x_1, x_2).$$

We have seen in (8) that the diameters of the $f(\bar{R}_i)$ are uniformly bounded. So, these metrics d_I are uniformly bounded. We are going to show that $\lambda^{-n}T_n$ is Hölder continuous on every Q_I, with uniform Hölder constants.

Lemma 5.3. *For every $I \subset \{1, \dots, p+q\}$ and $n \geq 1$ there exists $\mathcal{R}_I^n \subset \mathcal{R}^n$ such that*

$$T_n(x) = \sum_{R^n\in\mathcal{R}_I^n} e^{S_n\phi((f^n|\bar{R}^n)^{-1}(x))} \quad \textit{for every } x \in Q_I.$$

Proof. It follows from (H1) that the preimage $f^{-n}(x)$ of a point $x \in Q_I$ intersects the closure of a cylinder $R^n[i_0, \dots, i_{n-1}]$ if and only if $i_{n-1} \in I$. Moreover, in that case the intersection point is unique. Consider the following equivalence relation in the set of hyperbolic cylinders R^n whose last symbol belongs to I: $R^n[i_0, \dots, i_{n-1}]$ is equivalent to $R^n[j_0, \dots, j_{n-1}]$ if their closures contain exactly the same point of $f^{-n}(x)$. We claim that this relation is independent of the point $x \in Q_I$. Indeed, the injectivity condition in (H1) implies that f^n is injective on the union $\bar{R}^n[i_0, \dots, i_{n-1}] \cup \bar{R}^n[j_0, \dots, j_{n-1}]$. Then the preimages of any other point $x' \in Q_I$ in the closures of the two cylinders must also coincide, and this proves our claim. The conclusion of the lemma is then an immediate consequence: just take $\mathcal{R}_I^n \subset \mathcal{R}^n$ to contain exactly one cylinder in each equivalence class. $\square$

Corollary 5.4. *There is $K_8 > 0$ such that every $\lambda^{-n}T_n$ is Hölder continuous on every Q_I, with Hölder constants (K_8, α).*

Proof. Given any x_1, $x_2 \in Q_I$ and every $R^n \in \mathcal{R}_I^n$, denote $y_s = (f^n \mid \bar{R}^n)^{-1}(x_s)$ for $s = 1, 2$. By Corollary 3.6, there is a uniform constant K_3 such that

$$|S_n\phi(y_1) - S_n\phi(y_2)| \leq K_3 d_{f^n(\bar{R}^n)}(x_1, x_2)^\alpha \leq K_3 d_I(x_1, x_2)^\alpha$$

(the second inequality follows from the definition of the metric d_I). Then,

$$|e^{S_n\phi(y_1)} - e^{S_n\phi(y_2)}| = |e^{S_n\phi(y_1) - S_n\phi(y_2)} - 1|\, e^{S_n\phi(y_2)} \leq K_7\, d_I(x_1, x_2)^\alpha\, e^{S_n\phi(y_2)}$$

where the constant $K_7 > 0$ depends only on K_3 and a uniform upper bound on the diameter of Q_I. Adding over all $R^n \in \mathcal{R}^n_I$ and recalling Lemma 5.3, we get that

$$|\lambda^{-n} T_n(x_1) - \lambda^{-n} T_n(x_2)| \leq \lambda^{-n} K_7 \, d_I(x_1, x_2)^\alpha \, T_n(x_2).$$

By Corollary 5.2, the right hand side is bounded by $K_6 K_7 d_I(x_1, x_2)^\alpha$. Thus, it suffices to take $K_8 = K_6 K_7$. $\square$

Let us consider the sequence of functions $h_n : M \to \mathbb{R}$ defined by

$$h_n = \frac{1}{n} \sum_{i=0}^{n-1} \lambda^{-i} T_i. \tag{19}$$

Proposition 5.5. *Every subsequence of $(h_n)_n$ admits a subsequence converging uniformly to some piecewise Hölder continuous function h.*

Proof. By Corollaries 5.2 and 5.4, the sequence $(h_n)_n$ is uniformly bounded and equicontinuous on every Q_I. So, for every I, we may use the theorem of Ascoli-Arzela to find a subsequence that converges uniformly on Q_I. Up to successively refining the subsequence, we may suppose that it is the same for all I and, hence, converges uniformly on the whole M. Clearly, the limit h is Hölder continuous on each Q_I, with Hölder constants (K_8, α). $\square$

5.3 Lower bound

It is clear from Corollary 5.2 that any accumulation function of the sequence $(h_n)_n$ defined in (19) is bounded from infinity. We are going to show that it is also bounded from zero. The main step is

Lemma 5.6. *There exists a constant $K_9 > 0$ such that $Z_n \leq K_9 T_n(x)$ for every $x \in M$ and every $n \geq 1$.*

Proof. Fix $N \geq 1$ as in (H1), such that $f^N(R_i) = M$ for every $R_i \in \mathcal{R}$, and then fix $K \geq 1$ such that

$$\sigma_1^{-K} \sigma_2^{N-1} < e^{-2(K+N-1)c}. \tag{20}$$

Let $L = K + N - 1$. We claim that for every R_i and $R_j \in \mathcal{R}$ there exists a hyperbolic cylinder $R^L \in \mathcal{R}^L_h$ such that $R^L \subset f(R_i)$ and $f^L(R^L) \supset R_j$. Indeed, since $q < \deg(f)$, there exists $R^K \in \mathcal{R}^K$ such that $f^K(R^K)$ contains R_j and $f^k(R^K)$ is in the uniformly expanding region

$$f^k(R^K) \subset R_{q+1} \cup \cdots \cup R_{q+p}$$

for every $0 \leq k < K$. Moreover, by the choice of N, there exists $R^L \in \mathcal{R}^L$ contained in $f(R_i)$ such that $f^{N-1}(R^L) = R^K$. Condition (20) ensures that R^L is a hyperbolic cylinder, as claimed.

Now we use the previous observation to prove the lemma. Let $n \geq 1$ and $x \in M$ be fixed. Firstly, we have

$$\begin{aligned} T_{n+L}(x) = & \sum_{R^{n+L} \in \mathcal{R}_h^{n+L}} e^{S_n\phi(f^L(z))+S_L\phi(z)}, \qquad z \in f^{-(n+L)}(x) \cap \bar{R}^{n+L} \\ & \leq \deg(f^L) e^{L \sup \phi} \sum_{R^n \in \mathcal{R}_h^n} e^{S_n\phi(y)}, \qquad y \in f^{-n}(x) \cap \bar{R}^n : \end{aligned}$$

the inequality follows from the simple observation (see Remark 3.7) that if $z \in f^{-(n+L)}(x) \cap \bar{R}^{n+L}$ for some hyperbolic cylinder R^{n+L}, then $y = f^L(z)$ belongs to $f^{-n}(x) \cap \bar{R}^n$ where $R^n = f^L(R^{n+L})$ is a hyperbolic cylinder. This proves that

$$T_{n+L}(x) \leq K_{10} T_n(x), \qquad K_{10} = \deg(f^L) e^{L \sup \phi}. \tag{21}$$

Next, consider any hyperbolic cylinder $R^n \in \mathcal{R}_h^n$. The image $f^n(R^n)$ coincides with $f(R_i)$ for some $R_i \in \mathcal{R}$. Then, by the claim above (choose j such that x is in the closure of R_j), there exists a hyperbolic cylinder R^L contained in $f^n(R^n)$ and such that $f^L(\bar{R}^L)$ contains x. Observe that $R^{n+L} = R^n \cap f^{-n}(R^L)$ is a hyperbolic cylinder, by Remark 3.7. Let y be the unique point in the closure of R^{n+L} such that $f^{n+L}(y) = x$. By the bounded distortion Corollary 3.6, there exists a uniform constant $K_{11} > 0$ such that

$$e^{S_n\phi(R^n)} \leq K_{11}\, e^{S_n\phi(y)} \leq K_{11}\, e^{-L \inf \phi} e^{S_{n+L}\phi(y)}.$$

The correspondence $R^n \mapsto R^{n+L}$ thus constructed is injective, since $R^{n+L} \subset R^n$ and the cylinders R^n are pairwise disjoint. Combining these two observations, we find that

$$Z_n = \sum_{R^n} e^{S_n\phi(R^n)} \leq K_{12} \sum_{R^{n+L}} e^{S_{n+L}\phi(y)} = K_{12} T_{n+L}(x), \tag{22}$$

where $K_{12} = K_{11}\, e^{-L \inf \phi}$. Let $K_9 = K_{10} K_{12}$. Combining (21) and (22), we obtain $Z_n \leq K_9 T_n(x)$, as claimed. ☐

Remark 5.7. In particular, $T_n(x_1) \leq K_9 T_n(x_2)$ for any $x_1, x_2 \in M$ and $n \geq 1$.

Corollary 5.8. *Any accumulation function h of the sequence $(h_n)_n$ is uniformly bounded from zero and infinity.*

Proof. By Corollary 5.2, the function h is bounded from above by K_6. By Lemmas 5.1 and 5.6,

$$\frac{1}{n} \sum_{i=0}^{n-1} \lambda^{-i} T_i \geq K_9^{-1} \frac{1}{n} \sum_{i=0}^{n-1} \lambda^{-i} Z_i \geq (K_6 K_9)^{-1}.$$

and so h is bounded from below by $(K_6 K_9)^{-1}$. The proof is complete. ☐

5.4 Eigenfunction

Finally, we show that any accumulation function of the sequence $(h_n)_n$ defined in (19) is an eigenfunction of the transfer operator.

Lemma 5.9. *$\lambda^{-l}\big(\mathcal{L}_\phi T_l - T_{l+1}\big)$ converges uniformly to zero when $l \to \infty$.*

Proof. Let $l \geq 0$ and $x \in M$. By definition, $T_{l+1}(x)$ is the sum of $e^{S_{l+1}\phi(y)}$ over all $y \in f^{-(l+1)}(x)$ for which $l+1$ is a hyperbolic time. On the other hand,

$$\mathcal{L}_\phi T_l(x) = \sum_{z \in f^{-1}(x)} e^{\phi(z)} T_l(z) = \sum_{z \in f^{-1}(x)} e^{\phi(z)} \sum_{y \in f^{-l}(z) \cap \bar{R}^l \,:\, R^l \in \mathcal{R}_h^l} e^{S_l \phi(y)}$$

is the sum of $e^{S_{l+1}\phi(y)}$ over all $y \in f^{-(l+1)}(x)$ for which l is a hyperbolic time. Therefore,

$$\|(\mathcal{L}_\phi T_l - T_{l+1})(x)\| \leq \sum_{y \in f^{-(l+1)}(x) \cap \bar{R}^{l+1} : R^{l+1} \in E_{l+1}} e^{S_{l+1}\phi(y)}$$

where E_{l+1} is the collection of cylinders $R^{l+1}[i_0, \ldots, i_l] \in \mathcal{R}^{l+1} \setminus \mathcal{R}_h^{l+1}$ such that $R^l[i_0, \ldots, i_{l-1}] \in \mathcal{R}_h^l$. Observe that

$$E_{l+1} \subset \Big\{ R^{l+1}[i_0, \ldots, i_l] \in \mathcal{R}^{l+1} : \#\{0 \leq j < l : i_j \leq q\} \geq \gamma(l+1) \Big\}$$

and so $\#E_{l+1} \leq \text{const}\, e^{c_\gamma (l+1)} \leq e^{\kappa l}$ for every large l. Consequently,

$$\|\lambda^{-l}(\mathcal{L}_\phi T_l - T_{l+1})(x)\| \leq \lambda^{-l} \sum_{R^{l+1} \in E_{l+1}} e^{S_{l+1}\phi(R)} \leq \lambda^{-l} e^{\kappa l} e^{(l+1)\sup \phi}.$$

The right hand side converges to zero, since $\log \lambda = P$ is larger than $\sup \phi + \kappa$. Thus, the left hand side converges to 0 uniformly when $l \to \infty$, as claimed. □

Corollary 5.10. *Any accumulation function h of $(h_n)_n$ satisfies $\mathcal{L}_\phi h = \lambda h$.*

Proof. Let $(n_k)_k$ be any subsequence such that $(h_{n_k})_k$ converges to some h. Clearly,

$$\mathcal{L}_\phi h = \lim_k \frac{1}{n_k} \sum_{i=0}^{n_k - 1} \lambda^{-i} \big(\mathcal{L}_\phi T_i - T_{i+1}\big) - \frac{\lambda}{n_k} + \frac{\lambda^{-n_k} T_{n_k}}{n_k} + \frac{\lambda}{n_k} \sum_{i=0}^{n_k - 1} \lambda^{-i} T_i.$$

Lemma 5.9 implies that the first term on the right hand side converges to zero when $n \to \infty$. It is clear that the second term converges to zero. Since T_n is uniformly bounded (Corollary 5.2), the same is true about the third term. Therefore,

$$\mathcal{L}_\phi h = \lim_k \frac{\lambda}{n_k} \sum_{i=0}^{n_k - 1} \lambda^{-i} T_i = \lambda h,$$

as stated. The proof is complete. □

5.5 Invariant non-lacunary Gibbs measures

In the remainder of this section, $\mu = h\nu$ where h is an eigenfunction for the transfer operator bounded away from zero and infinity (recall Corollaries 5.8 and 5.10). Then μ is equivalent to ν, and it is easy to see it is invariant under f. Indeed, consider any integrable function $\psi : M \to \mathbb{R}$. Then

$$\int (\psi \circ f)\, d\mu = \int (\psi \circ f) h\, d\nu = \int (\psi \circ f) h \lambda^{-1}\, d(\mathcal{L}_\phi^* \nu).$$

The right hand side may be rewritten as

$$\int \lambda^{-1} \mathcal{L}_\phi((\psi \circ f)h)\, d\nu = \int \psi \lambda^{-1} \mathcal{L}_\phi(h)\, d\nu = \int \psi h\, d\nu = \int \psi\, d\nu,$$

and this proves invariance. Let $N \geq 1$ be as in condition (H1).

Lemma 5.11. *For any full μ-measure subset E of any $R_j \in \mathcal{R}$, the image $f^N(E)$ has full μ-measure in M. In particular, $\mu(R_j) > 0$ for every $R_j \in \mathcal{R}$.*

Proof. From the definition $\mu = h\nu$ and Lemma 4.1 we get that f admits a Jacobian with respect to ν, namely

$$J_\mu f = \frac{h \circ f}{h} J_\nu f = \lambda \frac{h \circ f}{h} e^{-\phi}.$$

Since f^N is locally injective, we may partition R_j into finitely many subsets $R_{i,j}$, $i = 1, \ldots, k(j)$ such that f^N is injective on each one of these sets. The assumption implies $\mu(R_{i,j} \cap E^c) = 0$ and so

$$\mu(f^N(R_{i,j} \cap E^c)) = \int_{R_{i,j} \cap E^c} J_\mu f\, d\mu = 0,$$

for every i. Observe that the $f^N(R_{i,j} \cap E^c)$ cover the complement of $f^N(E)$, because the $f^N(R_{i,j})$ cover the whole M. It follows that $f^N(E)^c$ has zero μ-measure. This proves the first claim in the lemma. The second one is an easy consequence: if R_j had zero measure then we could take E to be the empty set, which would immediately lead to a contradiction. □

Proposition 5.12. *The measure $\mu = h\nu$ is ergodic for f.*

Proof. Let A be any f-invariant set with positive μ-measure. By Corollary 3.12, one can find a sequence Q^n of measurable sets such that $R^{m_n} \subset Q^n \subset \bar{R}^{m_n}$ for some hyperbolic cylinder R^{m_n} with $m_n \geq n$ and

$$\frac{\mu(Q^n \cap A^c)}{\mu(Q^n)} \to 0 \quad \text{when } n \to \infty.$$

Then, using the distortion Lemma 4.6 and the assumption that A is invariant,

$$\frac{\mu(f^{m_n}(Q^n) \cap A^c)}{\mu(f^{m_n}(Q^n))} \to 0 \quad \text{when } n \to \infty.$$

Now, $f^{m_n}(Q^n) \supset f^{m_n}(R^{m_n}) \supset R_{j_n}$ for some $j_n \in \{1, \dots, p+q\}$. Fix any j such that $j_n = j$ for infinitely many values of n. By Lemma 5.11, the μ-measure of R_j is positive. Then the previous relation implies that $\mu(R_j \cap A^c) = 0$. Using Lemma 5.11 (with $E = A$) and the assumption that A is invariant, we conclude that $\mu(A) = 1$. □

Corollary 5.13. *We have $\mu(\partial\mathcal{R}) = 0$.*

Proof. The assumption (H1) implies that $f(\partial\mathcal{R}) \subset \partial\mathcal{R}$. Thus, by ergodicity, $\partial\mathcal{R}$ has either zero or full μ-measure. In the second case, $\mu(R_j)$ would be zero for every j, contradicting Lemma 5.11. Thus, the μ-measure of $\partial\mathcal{R}$ is indeed zero. □

Proposition 5.14. *The measure $\mu = h\nu$ is an invariant non-lacunary measure of f for the potential ϕ.*

Proof. We have already seen that μ is invariant under f. Corollary 5.13 gives that $\mu(\partial\mathcal{R}) = 0$. Since h is bounded from zero and infinity, Proposition 4.7 implies the Gibbs property at hyperbolic times: there exists $K_9 > 0$ such that

$$K_9^{-1} \le \frac{\mu(R^n)}{\exp(S_n\phi(x) - Pn)} \le K_9. \tag{23}$$

for every $x \in R^n$ and every hyperbolic cylinder R^n. Proposition 3.8 asserts that hyperbolic times constitute a non-lacunary sequence for μ-almost every point. The proof of the proposition is complete. □

We have finished the proof of Theorem B. Since μ and ν are equivalent measures, with density bounded from above and below (Corollary 5.8), Propositions 4.4 and 5.14 yield, respectively,

Corollary 5.15. *The measure μ is expanding, with integrable first hyperbolic time.*

Corollary 5.16. *The measure ν is a non-lacunary Gibbs measure of f for the potential ϕ.*

6 Proof of Theorem C

In this section, whenever we speak of non-lacunary Gibbs measures it is implicit that the corresponding non-lacunary sequence is the sequence of hyperbolic times. Note that this is so for all the cases in the previous sections.

The proof of Theorem C has two parts. First we check that any invariant non-lacunary Gibbs measure μ as in Theorem B is an equilibrium state (Proposition 6.1 and Corollary 6.7). This ensures that equilibrium states do exist. Then we show that every ergodic equilibrium state is a non-lacunary Gibbs measure and, moreover, any two non-lacunary Gibbs measures are equivalent (Corollary 6.12 and Lemma 6.13). This implies that there exists at most one ergodic equilibrium state, and so the equilibrium state is unique (and ergodic).

6.1 Existence of equilibrium states

Define $H_0 = H \setminus \bigcup_{n=1}^{\infty} \partial \mathcal{R}^n$. Observe that H_0 is a full μ-measure subset of M, by Corollaries 5.13 and 5.15. To each point $x \in H_0$ associate the largest hyperbolic cylinder $\mathcal{U}(x)$ that contains x (that is, the one with the smallest length). These $\mathcal{U}(x)$ form a partition $\mathcal{U}$ of the set H_0, in the sense that every $x \in H_0$ belongs to exactly one of them. By (23),

$$K_9^{-1} \mu(U) \le \exp(S_n \phi(U) - Pn) \le K_9 \mu(U) \tag{24}$$

for every $U \in \mathcal{U}$. Let $\mathcal{U}^n = S_n(\mathcal{U})$ be the family of sets of the form (10). Then $\mathcal{U}^n$ is an increasing sequence of partitions of H_0. By concatenation (Remark 3.7) every element of $\mathcal{U}^n$ is a hyperbolic cylinder, of length $\ge n$. Hence, (24) extends to every $U \in \mathcal{U}^n$ and every $n \ge 1$. Corollary 3.5 gives that

$$\operatorname{diam}(\mathcal{U}^n) \le K_2 e^{-cn} \tag{25}$$

goes to zero when $n \to \infty$. It follows that $\mathcal{U}$ is a generating partition for μ (see Lemma 3.11): every measurable set coincides, up to zero measure subsets, with some union of elements of $\mathcal{U}^n$, $n \ge 1$.

Proposition 6.1. *Every invariant expanding non-lacunary Gibbs measure μ satisfies $h_\mu(f) + \int \phi \, d\mu = P$.*

Proof. Since $\mathcal{U}$ is a generating partition, the theorem of Kolmogorov, Sinai (see § IV.4 in [Mañ87]) gives that $h_\mu(f) = h_\mu(f, \mathcal{U})$. Next, by the theorem of Shannon, McMillan, Breiman (see § IV.2 in [Mañ87]),

$$h_\mu(f,\mathcal{U}) = \int h(f,\mathcal{U},x)\, d\mu \quad \text{where} \quad h(f,\mathcal{U},x) = \lim_n -\frac{1}{n} \log \mu(\mathcal{U}^n(x))$$

and the limit exists at μ-almost every point. We can calculate the limit with the help of Proposition 3.17:

$$\lim_n -\frac{1}{n} \log \mu(\mathcal{U}^n(x)) = \lim_n \big(Pn - \frac{1}{n} S_n \phi(x)\big) = P - \lim_n \frac{1}{n} S_n \phi(x)$$

for μ-almost every $x \in M$. It follows that

$$h_\mu(f) = P - \int \lim \frac{1}{n} S_n \phi(x)\, d\mu = P - \int \phi \, d\mu,$$

where the last equality comes from the Birkhoff ergodic theorem. □

This implies that $P(f,\phi) \ge P$. In the sequel we prove that the two numbers are actually equal: they both coincide with the pressure $P_{H_0}(f,\phi)$ relative to the set H_0.

Proposition 6.2. *We have $P_{H_0}(f,\phi) = P$.*

Proof. Let $k, n \geq 1$ be fixed. As pointed out before, the inequalities (24) are valid for every $U \in \mathcal{U}^k$ and, for the same reason, for every $U \in S_n(\mathcal{U}^k)$. Since $S_n(\mathcal{U}^k)$ is a partition of the full μ-measure set H_0, adding these inequalities we obtain

$$K_9^{-1} \leq \sum_{U \in S_n(\mathcal{U}^k)} \exp(S_n\phi(U) - Pn) \leq K_9$$

and so, recalling the definition (11),

$$K_9^{-1} \leq m_{H_0}(f, \phi, P, \mathcal{U}^k, n) \leq K_9$$

for every $k, n \geq 1$. Taking the limit as $n \to \infty$, we find that $m_{H_0}(f, \phi, \alpha, \mathcal{U}^k)$ is equal to zero for $\alpha > P$ and is equal to infinity for $\alpha < P$. Consequently, $P_{H_0}(f, \phi, \mathcal{U}^k) = P$ for every $k \geq 1$. Finally, (25) implies that the diameter of $\mathcal{U}^k$ goes to zero when $k \to \infty$, and so $P_{H_0}(f, \phi) = \lim_{k\to\infty} P_{H_0}(f, \phi, \mathcal{U}^k) = P$ (recall Definition 3.13), as claimed. □

Corollary 6.3. *The spectral radius of $\mathcal{L}_\phi$ is the only real eigenvalue of the adjoint operator $\mathcal{L}_\phi^*$ larger than $e^{\kappa+\sup\phi}$.*

Proof. This is an immediate consequence of Lemma 4.2 and Proposition 6.2. □

The proof of the next proposition was obtained jointly with Paulo Varandas.

Proposition 6.4. *We have $P_{M\setminus H_0}(f, \phi) < P$.*

Proof. The key idea is akin to the Ruelle inequality [Rue78]. It can be outlined as follows, starting with the special case $\phi = 0$. The entropy associated to each inverse branch of f is bounded by $\log \|Df^{-1}\| \leq \log \sigma_2$ and, consequently, may be taken to be quite small. Thus, most of the entropy arises from the non-injectivity of f. In particular, the entropy relative to $M \setminus H_0$ is bounded above by the growth rate $\kappa < P$ of the number of non-hyperbolic inverse branches. That is the contents of the conclusion. Moreover, a similar estimate holds for more general potentials, with the relative pressure bounded by $\sup\phi + \kappa < P$.

For the detailed argument we need a couple auxiliary lemmas.

Lemma 6.5. *Let M be a compact manifold of dimension d. There exists a sequence $(\mathcal{T}_k)_k$ of finite triangulations of M and there exist positive constants C_1 and C_2 such that* $\operatorname{diam}(\mathcal{T}_k) \leq C_1 2^{-k}$ *and, given* $A \geq 1$, *any set* $E \subset M$ *such that* $\operatorname{diam}(E) \leq A \operatorname{diam}(\mathcal{T}_k)$ *intersects at most $C_2 A^d$ atoms of $\mathcal{T}_k$.*

Proof. Fix any finite triangulation $\mathcal{T}_0$ in M. For each $T \in \mathcal{T}_0$, let $\phi_T : T \to \Lambda^d$ be a diffeomorphism to the standard d-dimensional simplex $\Lambda^d \subset \mathbb{R}^d$ of size 1. For each $k \geq 1$, let $\mathcal{L}^{d,k}$ be the regular decomposition of Λ^d into simplices of size 2^{-k}. Then let $\mathcal{T}_k$ be the triangulation of M obtained by pulling $\mathcal{L}^{d,k}$ back under each ϕ_T. Clearly, $\operatorname{diam} \mathcal{T}_k \leq C_1 2^{-k}$ where the constant C_1 depends only on distortion bounds for the ϕ_T. This proves the first part of the lemma. To prove the second one, notice the hypothesis implies that $\operatorname{diam}(\phi_T(E \cap T)) \leq C_1' A 2^{-k}$ for every $T \in \mathcal{T}^0$ and some uniform constant C_1'. The whole point of the proof is to observe that this implies $\phi_T(E \cap T)$ intersects at most $C_2' A^d$ atoms of $\mathcal{L}^{d,k}$, for some uniform constant C_2'. The conclusion follows. □

Let $(\mathcal{T}_k)_{k\ge 1}$ be as in the previous lemma and, for each $j \ge 1$, denote

$$\mathcal{T}_k^{\ell,j} = \{T_0 \cap f^{-\ell}(T_1) \cap \cdots \cap f^{-\ell(j-1)}(T_{j-1}) : T_i \in \mathcal{T}_k \text{ for } 0 \le i < j\}.$$

The crucial estimate in the proof of the proposition is given in the next lemma.

Lemma 6.6. *For each ℓ, j, and k, there exists a family $\mathcal{G}_{\ell,j,k} \subset \mathcal{T}_k^{\ell,j}$ such that*

1. *for every ℓ, k, and L, the union $\cup_{j\ge L}\mathcal{G}_{\ell,j,k}$ covers the set $M \setminus H_0$*
2. *there is $L \ge 1$ and for each ℓ there is $L_\ell \ge 1$ such that*

$$\#\mathcal{G}_{\ell,j,k} \le C_2^j e^{(\kappa + d\log\sigma_2)\ell j} \# \mathcal{T}_k \quad \textit{for all } j \ge L \textit{ and every } k \ge L_\ell.$$

Proof. Recall we took γ such that $c_\gamma \le \kappa$. Fix $\varepsilon > 0$ such that $c_\gamma + 2\varepsilon \le \kappa$. Proposition 3.2 implies that $M \setminus H_0$ is covered by the closures of the cylinders $R^n[i_0, \ldots, i_{n-1}]$ associated to itineraries $(i_0, \ldots, i_{n-1}) \in I(\gamma, n)$ with large length n. Recall, from Lemma 3.1, that

$$\#I(\gamma, n) \le e^{(c_\gamma+\varepsilon)n} \quad \text{if } n \text{ is large.} \tag{26}$$

Define $\mathcal{G}_{\ell,j,k}$ to be the family of all elements of $\mathcal{T}_k^{\ell,j}$ that intersect $\bar{R}^n[i_0, \ldots, i_{n-1}]$ for some itinerary $(i_0, \ldots, i_{n-1}) \in I(\gamma, n)$ with $\ell(j-1) \le n < \ell j$. It is clear from the previous observations that, given any $L \ge 1$, the union of $\mathcal{G}_{\ell,j,k}$ over all $j \ge L$ covers $M \setminus H_0$, as stated in part 1 of the lemma.

Now we claim that, for large k and j, there are at most $C_2^j \sigma_2^{\ell j} \# \mathcal{T}_k$ elements of $\mathcal{T}_k^{\ell,j}$ that intersect any given $\bar{R}^n[i_0, \ldots, i_{n-1}]$ as before. Indeed, let

$$T_0 \cap f^{-\ell}(T_1) \cap \cdots \cap f^{-\ell(j-1)}(T_{j-1}) \in \mathcal{T}_k^{\ell,j}$$

be any such element. Then, $T_s \cap f^{-\ell}(T_{s+1})$ intersects $\bar{R}^{n-s\ell}[i_{s\ell}, \ldots, i_{n-1}]$ for every $s = 0, 1, \ldots, j-2$. Condition (H1) implies that f^ℓ is injective on every $\bar{R}_i$. Since $\bar{R}_i$ is compact, injectivity extends to some small neighborhood. Lemma 6.5 gives that the diameter of $\mathcal{T}_k$ goes to zero when $k \to \infty$. So, taking k larger than some function of ℓ, we can ensure that $f^{-\ell}(T_{s+1})$ has exactly $\deg(f)$ connected components and only one of them, that we denote C_{s+1}, intersects the neighborhood of radius $\operatorname{diam}(\mathcal{T}_k)$ around $R_{i_{s\ell}}$. Condition (H2) implies $\|Df^{-\ell}\| \le \sigma_2^\ell$, and so

$$\operatorname{diam}(C_{s+1}) \le \sigma_2^\ell \operatorname{diam}(T_{s+1}) \le \sigma_2^\ell \operatorname{diam}(\mathcal{T}_k).$$

Then, by Lemma 6.5, C_{s+1} intersects at most $C_2\sigma_2^{\ell d}$ atoms of $\mathcal{T}_k$. Applying this argument, successively, to $s = j-2, \ldots, 1, 0$, we conclude that there are at most $\#\mathcal{T}_k\big(C_2\sigma_2^{\ell d}\big)^{j-1}$ sequences $(T_0, \ldots, T_{j-1})$ as we have been considering. This is even slightly better than our claim.

Combining this claim with (26) we can easily deduce part 2 of the lemma:

$$\#\mathcal{G}_{\ell,j,k} \le C_2^j \sigma_2^{\ell j} \#\mathcal{T}_k \cdot \sum_{r=1}^{\ell} e^{(c_\gamma+\varepsilon)(\ell j - r)} \le C_2^j \sigma_2^{\ell j} \#\mathcal{T}_k \cdot C(\ell) e^{(c_\gamma+\varepsilon)\ell j}$$

for some constant $C(\ell) > 0$. Since j is large, the right hand side is bounded by

$$C_2^j \sigma_2^{\ell j} e^{(c_\gamma + 2\varepsilon)\ell j} \, \# \mathcal{T}_k \le C_2^j \sigma_2^{\ell j} e^{\kappa \ell j} \, \# \mathcal{T}_k.$$

This completes the proof of Lemma 6.6. $\square$

From the lemma and the definition (10),

$$\begin{aligned} m_\alpha(f^\ell, S_\ell\phi, M \setminus H_0, \mathcal{T}_k, L) &\le \sum_{j \ge L} \sum_{U \in \mathcal{G}_{\ell,j,k}} e^{-\alpha j + \sum_{i=0}^{j-1} (S_\ell \phi) \circ f^{i\ell}(U)} \\ &\le \sum_{j \ge L} e^{-\alpha j + \ell j \sup \phi} \, \# \mathcal{G}_{\ell,j,k} \end{aligned}$$

for every $k \ge L_\ell$ and $L \ge L_\ell$. In view of Lemma 6.6, the right hand side converges to zero when $L \to \infty$ for all $\alpha > (\sup \phi + \kappa + d \log \sigma_2)\, \ell + \log C_2$. Consequently,

$$P_{M \setminus H_0}(f^\ell, S_\ell \phi, \mathcal{T}_k) \le (\sup \phi + \kappa + d \log \sigma_2)\, \ell + \log C_2$$

for $k \ge L_\ell$. Taking the limit when $k \to \infty$, and recalling $\operatorname{diam}(\mathcal{T}_k)$ goes to zero,

$$P_{M \setminus H_0}(f^\ell, S_\ell \phi) \le (\sup \phi + \kappa + d \log \sigma_2)\, \ell + \log C_2.$$

According to Proposition 3.15, this means that

$$P_{M \setminus H_0}(f, \phi) \le \sup \phi + \kappa + d \log \sigma_2 + \frac{1}{\ell} \log C_2.$$

Taking the limit as $\ell \to \infty$ and then using the condition (3), we obtain the conclusion of Proposition 6.4. $\square$

Corollary 6.7. *We have $P = P_{H_0}(f, \phi) = P(f, \phi)$.*

Proof. Theorem 11.2 in [Pes97] gives that $P(f, \phi) = \sup\{P_{H_0}(\phi), P_{M \setminus H_0}(\phi)\}$. Thus, the statement is a direct consequence of Propositions 6.2 and 6.4. $\square$

At this point, Proposition 6.1 means that any invariant expanding non-lacunary Gibbs measure is an equilibrium state of f for the potential ϕ. In particular, equilibrium states do exist.

6.2 Uniqueness of the equilibrium state

Let η be an arbitrary equilibrium state of f for ϕ. Define $g : M \to (0, \infty)$ by

$$g(x) = \lambda^{-1} e^{\phi(x)} \frac{h(x)}{h(f(x))}.$$

Observe that, for every $x \in M$,

$$\sum_{f(y)=x} g(y) = \frac{\sum_{f(y)=x} e^{\phi(y)} h(y)}{\lambda h(x)} = \frac{\mathcal{L}_\phi h(x)}{\lambda h(x)} = 1 \qquad (27)$$

Lemma 6.8. *The measure η satisfies $\eta(H) = 1$, and so all the Lyapunov exponents of f are positive at η-almost every point.*

Proof. Suppose $\eta(M \setminus H_0) > 0$. Let ξ be the normalized restriction of η to $M \setminus H_0$. We may write $\eta = c\xi + (1-c)\zeta$ for some $0 < c \leq 1$ and some probability η. By (12) and Proposition 6.4,

$$h_\zeta(f) + \int \phi \, d\zeta \leq P(f,\phi) \quad \text{and} \quad h_\xi(f) + \int \phi \, d\xi \leq P_{M\setminus H_0}(f,\phi) < P(f,\phi).$$

Therefore,

$$h_\eta(f) + \int \phi \, d\eta = c\big(h_\xi(f) + \int \phi \, d\xi\big) + (1-c)\big(h_\zeta(f) + \int \phi \, d\zeta\big)$$

is strictly smaller than $P(f,\phi)$, that is, η is not an equilibrium state. This proves the first claim in the lemma. The second one is an immediate consequence, as explained at the beginning of Section 3.3. □

In the sequel we use the following elementary fact from Calculus, whose proof we omit:

Remark 6.9. Let $p_i > 0$ and $q_i > 0$, $i = 1, \ldots, n$ be such that $\sum_{i=1}^n p_i = 1$. Then $\sum_{i=1}^n p_i \log q_i \leq \log(\sum_{i=1}^n p_i q_i)$ and the equality holds if and only if the q_i are all equal.

The next proposition is a variation of a result in [BS03].

Proposition 6.10. *We have $J_\eta f(y) = 1/g(y)$ for η-almost every $y \in M$.*

Proof. From the assumption $h_\eta(f) + \int \phi \, d\eta = P(f,\phi) = P$ we get that

$$h_\eta(f) + \int \log g \, d\eta = h_\eta(f) - P + \int (\phi + \log h - \log h \circ f) \, d\eta \geq 0. \tag{28}$$

Let us write $g_\eta = 1/(J_\eta f)$. By Lemma 6.8, the measure η is expanding, and so we may use Rokhlin's formula (see [OV06])

$$h_\eta(f) = \int \log J_\eta f \, d\eta$$

Replacing this formula in the previous inequality we find

$$0 \leq \int \log \frac{g}{g_\eta} \, d\eta = \int \sum_{f(y)=x} g_\eta(y) \log \frac{g(y)}{g_\eta(y)} \, d\eta(x), \tag{29}$$

where the equality follows from the definitions of g_η and the Jacobian. Take $p_i = g_\eta(y_i)$ and $q_i = g(y_i)/g_\eta(y_i)$, where the y_i are the pre-images of x. The assumption that η is invariant means that

$$\sum_{i=1}^{\deg(f)} p_i = \sum_{f(y)=x} g_\eta(y) = 1 \tag{30}$$

for η-almost every $x \in M$. Thus, we may apply the first part of Remark 6.9, together with (27), to conclude that

$$\sum_{f(y)=x} g_\eta(y) \log \frac{g(y)}{g_\eta(y)} \le \log\Big(\sum_{f(y)=x} g_\eta(y) \frac{g(y)}{g_\eta(y)} \Big) = \log\Big(\sum_{f(y)=x} g(y) \Big) = 0$$

at η-almost every x. Integrating with respect to η and using (29), we conclude that the equality must hold at η-almost every point. Moreover, the equality must hold in (29) and, consequently, in (28). That is, $h_\eta(f) - P + \int \phi d\eta = 0$, as claimed in part (1) of the lemma.

Moreover, using the second part of Remark 6.9, the values of $g(y)/g_\eta(y)$ must be the same for all $y \in f^{-1}(x)$. In other words, for η-almost every $x \in M$ there exists a number $c(x)$ such that

$$\frac{g(y)}{g_\eta(y)} = c(x) \quad \text{for every } y \in f^{-1}(x).$$

Combining this with (27) and (30), we conclude that

$$c(x) = \frac{\sum_{f(y)=x} g(y)}{\sum_{f(y)=x} g_\eta(y)} = 1 \quad \text{for } \eta\text{-almost every } x.$$

This shows that $g = g_\eta$ for every point on the pre-image of a full η-measure set. Since η is invariant, this implies part (2) of the lemma. □

For notational convenience, in what follows we write h^{-1} to mean $1/h$.

Lemma 6.11. *The measure $\nu_\eta = h^{-1}\eta$ is a maximal eigenmeasure for the adjoint transfer operator: $\mathcal{L}_\phi^*(h^{-1}\eta) = \lambda(h^{-1}\eta)$.*

Proof. Given any continuous function ξ

$$\int \xi \, d(\mathcal{L}_\phi^* \nu_\eta) = \int (\mathcal{L}_\phi \xi) h^{-1} d\eta = \int \sum_{f(y)=x} e^{\phi(y)} h(f(y))^{-1} \xi(y) \, d\eta(x).$$

Using the definition of g and Proposition 6.10, we

$$e^{\phi(y)} h(f(y))^{-1} = \lambda g(y) h(y)^{-1} = \lambda J_\eta f(y)^{-1} h(y)^{-1} .$$

Replacing this in the previous formula,

$$\int \xi \, d(\mathcal{L}_\phi^* \nu_\eta)) = \lambda \int \Big(\sum_{f(y)=x} \xi(y) J_\eta f(y)^{-1} h(y)^{-1} \Big) d\eta(x) = \lambda \int \xi h^{-1} \, d\eta.$$

Since ξ is arbitrary, this implies $\mathcal{L}_\phi^*(\nu_\eta) = \lambda \nu_\eta$, as claimed. □

Corollary 6.12. *Any equilibrium state of ϕ is an invariant non-lacunary Gibbs measure.*

Proof. Let $\nu_\eta = \frac{1}{h}\eta$. We have just seen that ν_η is an eigenmeasure of $\mathcal{L}_\phi^*$, with eigenvalue λ. So, by Proposition 4.7 and Corollary 5.16, ν_η is a non-lacunary Gibbs measure. Since $\eta = h\nu_\eta$ and h is bounded from below and above, η is a non-lacunary Gibbs measure as well. □

Lemma 6.13. *Any two expanding non-lacunary Gibbs measures are equivalent.*

Proof. Consider expanding non-lacunary Gibbs measures ν_1 and ν_2. For each $R \in \mathcal{R}_h^n$ we have

$$K^{-1}e^{S_n\phi(x)-nP} \leq \nu_i(R) \leq Ke^{S_n\phi(x)-nP}, \quad \text{for } i = 1, 2.$$

This implies that $K^{-2}\nu_2(R) \leq \nu_1(R) \leq K^2\nu_2(R)$. Using Lemma 3.11 we conclude that the inequalities hold for every Borel set. Thus, ν_1 and ν_2 are equivalent measures. □

Now let η be any ergodic equilibrium state for ϕ, and let μ be any invariant non-lacunary Gibbs measure as constructed in Theorem B. Then η is an invariant non-lacunary Gibbs measure, according to Corollary 6.12. Using Lemma 6.13 we conclude that the measures μ and η are equivalent. Then, by ergodicity, $\eta = \mu$. Thus, there exists a unique ergodic equilibrium state for ϕ. Since any ergodic component of an equilibrium measure is also an equilibrium measure, this proves uniqueness of the equilibrium state. The proof of Theorem C is complete.

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