ON THE DEGREE OF POLAR TRANSFORMATIONS AN APPROACH THROUGH LOGARITHMIC FOLIATIONS

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ABSTRACT. We investigate the degree of the polar transformations associated to a certain class of multi-valued homogeneous functions. In particular we prove that the degree of the pre-image of generic linear spaces by a polar transformation associated to a homogeneous polynomial F is determined by the zero locus of F. For zero dimensional-dimensional linear spaces this was conjecture by Dolgachev and proved by Dimca-Papadima using topological arguments. Our methods are algebro-geometric and rely on the study of the Gauss map of naturally associated logarithmic foliations.

1. Introduction

Given a homogeneous polynomial $F \in \mathbb{C}[x_0, ..., x_n]$ one can naturally associate to it the rational map induced by its linear system of polars. Explicitly this map can be written as

$$\begin{aligned} \nabla F : \mathbb{P}^n \quad &\dashrightarrow \quad \mathbb{P}^n \\ x \quad &\mapsto \quad \left(\frac{\partial F}{\partial x_0}(x) : ... : \frac{\partial F}{\partial x_n}(x) \right) , \end{aligned}$$

and is the so called **polar transformation** or **polar map** of F.

The particular case when ∇F is a birational map is of particular interest [9, 10, 4] and in this situation the polynomial F is said to be **homaloidal**. The classification of reduced homaloidal polynomials in three variables was carried out by Dolgachev in [8]. It says that $F \in \mathbb{C}[x_0, x_1, x_2]$ is a reduced homaloidal polynomial if, and only if, its (set theoretical) zero locus $V(F) \subset \mathbb{P}^2$ has one of the following forms:

(1) a smooth conic;
(2) the union of three lines in general position;
(3) the union of a smooth conic and a line tangent to it.

In loc. cit. it is conjectured that the reduceness of F is not necessary to draw the same conclusion. More precisely it is conjectured that the degree of ∇F can be written as a function of $V(F)$.

Dimca and Papadima [6] settled Dolgachev's conjecture by proving that for a polynomial $F \in \mathbb{C}[x_0, \dots, x_n]$ the complement $D(F) = \mathbb{P}^n \backslash V(F)$ is homotopically equivalent to a CW-complex obtained from $D(F) \cap H$ by attaching $deg(\nabla F)$ cells of dimension n, where $H \subset \mathbb{P}^n$ is a generic hyperplane. In particular the degree of ∇F can be expressed as

$$deg(\nabla F) = (-1)^n \chi(D(F) \backslash H) .$$

Key words and phrases. Polar Transformations, Gauss Map, Foliations.

Their proof is topological and relies on complex Morse Theory. In [7], as well as in [4], the problem of giving an algebro-geometric proof of Dolgachev's conjecture is raised. Partial answers have been provided by [12] and [1].

The main goal of this paper is to provided one such algebro-geometric proof, cf. Theorem 3, by relating the degree of ∇F to the degree of the Gauss map of some naturally associated logarithmic foliations.

Our method allow us also to deal with the higher order degrees of ∇F — the degrees of the closure of pre-images of generic linear subspaces — and with more general functions than the polynomial ones, cf. §4.

The paper is organized as follows. In §2 we recall some basic definitions concerning holomorphic foliations and their Gauss map and prove Theorem 1 that express the higher order degrees of such Gauss maps in terms of the topological degree of the Gauss maps of generic linear sections of the corresponding foliations. In §3 we study the Gauss maps of logarithmic foliations and prove that their topological degrees — under suitable hypotheses — can expressed in terms of the top chern class of certain sheaves of logarithmic differentials. In §4 we prove Theorem 2 that relates the degrees of the polar map with the ones of a naturally associated logarithmic foliation. Finally, in §5 we prove Theorem 3 — our main result — and make a couple of side remarks.

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2. Foliations and Their Gauss Maps

A **codimension one singular holomorphic foliation**, from now on just foliation, $\mathcal{F}$ of a complex manifold M is determined by a line bundle $\mathcal{L}$ and an element $\omega \in \mathrm{H}^0(M, \Omega^1_M \otimes \mathcal{L})$ satisfying

(i) $\operatorname{codim} \operatorname{Sing}(\omega) \geq 2$ where $\operatorname{Sing}(\omega) = \{x \in M \,|\, \omega(x) = 0\}$;
(ii) $\omega \wedge d\omega = 0$ in $\mathrm{H}^0(M, \Omega^3_M \otimes \mathcal{L}^{\otimes 2})$.

If we drop condition (ii) we obtain the definition of a **codimension one singular holomorphic distribution** $\mathcal{D}$ on M. Although we will state the results of this section for foliations they can all be rephrased for distributions.

The **singular set** of $\mathcal{F}$, for short $\operatorname{Sing}(\mathcal{F})$, is by definition equal to $\operatorname{Sing}(\omega)$. The integrability condition (ii) determines in an analytic neighborhood of every point $p \in M \setminus \operatorname{Sing}(\mathcal{F})$ a holomorphic fibration with relative tangent sheaf coinciding with the subsheaf of TM determined by the kernel of ω. Analytic continuation of the fibers of this fibration describes the leaves of $\mathcal{F}$.

In our study the isolated singularities of $\mathcal{F}$ will play a key role. One of the most basic invariants attached to them is their **multiplicity** $m(\mathcal{F}, p)$ defined as the intersection multiplicity at p of the zero section of $\Omega^1_M \otimes \mathcal{L}$ with the graph of ω.

In this paper we will focus on the case $M = \mathbb{P}^n$. The **degree** of a foliation of $\mathbb{P}^n$ is geometrically defined as the number of tangencies of $\mathcal{F}$ with a generic line $\ell \subset \mathbb{P}^n$. If $\iota : \ell \to \mathbb{P}^n$ is the inclusion of such a line then the degree of $\mathcal{F}$ is the degree of the zero divisor of the twisted 1-form $\iota^*\omega \in \mathrm{H}^0(\ell, \Omega^1_\ell \otimes \mathcal{L}_{|\ell})$. Thus the degree of $\mathcal{F}$ is nothing more than $\deg(\mathcal{L}) - 2$.

2.1. **The Gauss Map.** The **Gauss map** of a foliation $\mathcal{F}$ of $\mathbb{P}^n$ is the rational map

$$\begin{array}{rcl} \mathcal{G}(\mathcal{F}) : \mathbb{P}^n & \dashrightarrow & \check{\mathbb{P}}^n \\ p & \mapsto & T_p\mathcal{F} \end{array}$$

where $T_p\mathcal{F}$ is the projective tangent space of the leaf of $\mathcal{F}$ through p.

It follows from Euler's sequence that a 1-form $\omega \in \mathrm{H}^0(\mathbb{P}^n, \Omega^1(\deg(\mathcal{F}) + 2))$ can be interpreted as a homogeneous 1-form on $\mathbb{C}^{n+1}$, still denoted by ω,

$$\omega = \sum_{i=0}^{n} a_i dx_i$$

with the coefficients a_i being homogenous polynomials of degree $\deg(\mathcal{F}) + 1$ and satisfying Euler's relation $i_R\omega = 0$, where i_R stands for the interior product with the radial (or Euler's) vector field $R = \sum_{i=0}^n x_i \frac{\partial}{\partial x_i}$.

If we interpret $[dx_0 : \ldots : dx_n]$ as projective coordinates of $\check{\mathbb{P}}^n$ then the Gauss map of the corresponding $\mathcal{F}$ can be written as $\mathcal{G}(\mathcal{F})(p) = [a_0(p) : \ldots : a_n(p)]$.

2.2. **Linear Sections of Foliations.** Assume that $1 \le k < n$ and let $\iota : \mathbb{P}^k \to \mathbb{P}^n$ be a linear embedding. If $\iota^*\omega = 0$ then we say that $\iota(\mathbb{P}^k)$ is **left invariant** by $\mathcal{F}$. Otherwise, after dividing $\iota^*\omega$ (here interpreted as a 1-form on $\mathbb{C}^{k+1}$) by the common factor of its coefficients, one obtains a foliation $i^*\mathcal{F} = \mathcal{F}_{|\mathbb{P}^k}$ on $\mathbb{P}^k$.

Notice that according to our definitions there is only one foliation of $\mathbb{P}^1$ and it is induced by the homogeneous 1-form $x_0dx_1 - x_1dx_0$ on $\mathbb{C}^2$. This odd remark will prove to be useful when we define the numbers $e_i^k(\mathcal{F})$ below. On the other hand if $k \ge 2$ and $\iota : \mathbb{P}^k \to \mathbb{P}^n$ is generic then there is no need to divide $\iota^*\omega$: one has just to apply the following well-known lemma $n - k$ times.

Lemma 1. *Let $n \ge 3$. If $H \subset \mathbb{P}^n$ is a generic hyperplane and $\mathcal{F}$ is a foliation of $\mathbb{P}^n$ then the degree of $\mathcal{F}_{|H}$ is equal to the degree of $\mathcal{F}$ and, moreover,*

$$\mathrm{Sing}(\mathcal{F}_{|H}) = (\mathrm{Sing}(\mathcal{F}) \cap H) \cup \mathcal{G}(\mathcal{F})^{-1}(H)$$

with $\mathcal{G}(\mathcal{F})^{-1}(H)$ being finite and all the corresponding singularities of $\mathcal{F}_{|H}$ have multiplicity one.

Proof. The proof follows from Bertini's Theorem applied to the linear system defining $\mathcal{G}(\mathcal{F})$, or equivalently, from Sard's Theorem applied to $\mathcal{G}(\mathcal{F})$. For the details see [2]. □

Notice that the conclusion of Lemma concerning the multiplicities can be rephrased by saying that H is a regular value of $\mathcal{G}(\mathcal{F})$ restricted to its domain of definition.

2.3. **Degrees of the Gauss Map.** For a rational map $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ recall that $\deg_i(\phi)$ is the cardinality of $\overline{\phi_{|U}^{-1}(L_i)} \cap \Sigma^{n-i}$, where $U \subset \mathbb{P}^n$ is a Zariski open set where ϕ is regular, $L_i \subset \mathbb{P}^n$ is a generic linear subspace of dimension i of the target and $\Sigma^{n-i} \subset \mathbb{P}^n$ is generic linear subspace of dimension $n - i$ of the domain.

On the remaining part of this section we will study the numbers $e_i^k(\mathcal{F})$, for pairs of natural numbers (k, i) satisfying $1 \le k \le n$ and $0 \le i \le k - 1$, defined as

$$e_i^k(\mathcal{F}) = \deg_i(\mathcal{G}(\mathcal{F}_{|\mathbb{P}^k})).$$

where $\mathbb{P}^k \subset \mathbb{P}^n$ is generic.

Notice that $e_0^n(\mathcal{F})$ is equal to the topological degree of $\mathcal{G}(\mathcal{F})$ and Lemma 1 implies that $e_0^2(\mathcal{F}) = \deg(\mathcal{F})$. More generally, for every $0 \le i \le n-1$, $e_i^n(\mathcal{F})$ coincides with the degree of the $(n-i)$-th polar class of $\mathcal{F}$ defined in [13] mimicking the corresponding definition for projective varieties, cf. for instance [14].

Our main result concerning the numbers $e_i^k(\mathcal{F})$ is the following.

Theorem 1. *If $\mathcal{F}$ is a foliation of $\mathbb{P}^n$ and (k,i) is a pair of natural numbers satisfying $2 \le k \le n$ and $1 \le i \le k-1$ then*

$$e_i^k(\mathcal{F}) = e_0^{k-i}(\mathcal{F}) + e_0^{k-i+1}(\mathcal{F}).$$

The corollary below follows immediately from Theorem 1.

Corollary 1. *For natural numbers s,k,i satisfying $s \ge 1$, $s+2 \le k \le n$ and $2 \le i \le k-1$ we have that*

$$e_i^k(\mathcal{F}) = e_{i-s}^{k-s}(\mathcal{F})\,.$$

Notice that this is as an analogous of the invariance of the polar classes of hypersurfaces under hyperplane sections — a particular case of [14, Theorem 4.2].

2.4. **Proof of Theorem 1.** It clearly suffices to consider the case $k = n$. Set $U = \mathbb{P}^n \setminus \mathrm{Sing}(\mathcal{F})$ and $\mathcal{G} = \mathcal{G}(\mathcal{F})_{|U}$.

Let $L^i \subset \check{\mathbb{P}}^n$ be a generic linear subspace of dimension i, $V^i = \mathcal{G}^{-1}(L^i) \subset U$ and $\Sigma^{n-i-1} = \check{L}^i$, i.e.,

$$\Sigma^{n-i-1} = \bigcap_{H \in L^i} H\,.$$

Thanks to Bertini's Theorem we can assume that V^i is empty or smooth of dimension i. Moreover, thanks to Lemma 1, we can also assume that all the singularities of $\mathcal{F}_{|\Sigma^{n-i-1}}$ contained in U have multiplicity one.

Lemma 2. *If Σ^{n-i} is a generic projective subspace of dimension $n-i$ ($i \ge 1$) containing Σ^{n-i-1} then*

$$V^i \cap \Sigma^{n-i} = U \cap \left(\mathrm{Sing}(\mathcal{F}_{|\Sigma^{n-i}}) \cup \mathrm{Sing}(\mathcal{F}_{|\Sigma^{n-i-1}})\right)\,.$$

Moreover Σ^{n-i} intersects V^i transversally.

Proof. By definition $V^i = \left\{p \in U \,|\, T_p\mathcal{F} \supseteq \Sigma^{n-i-1}\right\}$. Clearly the points $p \in \Sigma^{n-i-1}$ belonging to V^i coincides with $\mathrm{Sing}(\mathcal{F}_{|\Sigma^{n-i-1}})$. Similarly a point $p \in \Sigma^{n-i} \setminus \Sigma^{n-i-1}$ belongs to V^i if, and only if, $T_p\mathcal{F}$ contains the join of p and Σ^{n-i-1}. Since $\mathrm{Join}(p, \Sigma^{n-i-1}) = \Sigma^{n-i}$ the set theoretical description of $V^i \cap \Sigma^{n-i}$ follows.

It remains to prove the transversality statement. First take a point $p \in \Sigma^{n-i-1} \cap V_i$. If for every Σ^{n-i} containing Σ^{n-i-1} the intersection of V^i with Σ^{n-i} is not transverse then $T_pV^i \cap T_p\Sigma^{n-i-1} \neq 0$. Without loss of generality we can assume that $\Sigma^{n-i-1} = \{x_0 = \ldots = x_i = 0\}$. In this situation the variety V^i is defined by the projectivization of $\{a_{i+1} = \ldots = a_n = 0\}$ where $\omega = \sum_{i=0}^n a_i dx_i$ is a 1-form defining $\mathcal{F}$ on $\mathbb{C}^{n+1}$.

If $v \in T_pV^i$ then an arbitrary lift $\overline{v}$ to $\mathbb{C}^{n+1}$ satisfies $da_j(\overline{v}) = 0$ for every $i+1 \le j \le n$. Since $\mathcal{F}_{|\Sigma^{n-i-1}}$ is defined by the 1-form

$$\sum_{j=i+1}^{n} a_j(0,\ldots,0,x_{i+1},\ldots,x_n)dx_j$$

then it follows that $d\mathcal{G}(\mathcal{F}_{|\Sigma^{n-i-1}})_p \cdot (v) = 0$ for every $v \in T_p\Sigma^{n-i-1} \cap T_pV^i$. If this latter intersection has positive dimension then $m(\mathcal{F}_{|\Sigma^{n-i-1}}, p) > 1$ contrary to our assumptions. Therefore for a generic $\Sigma^{n-i} \supseteq \Sigma^{n-i-1}$ the intersection of V^i with Σ^{n-i} along Σ^{n-i-1} is transversal.

Let now $p \in \Sigma^{n-i} \setminus \Sigma^{n-i-1}$. If $G \subset \mathrm{aut}(\mathbb{P}^n)$ is the subgroup that preserves Σ^{n-i-1} then $\mathbb{P}^n \setminus \Sigma^{n-i-1}$ is G-homogeneous. It follows from the transversality of a generic G-translate (cf. [11]) that a generic $\Sigma^{n-i} \supseteq \Sigma^{n-i-1}$ intersects V^i transversally along $\Sigma^{n-i} \setminus \Sigma^{n-i-1}$. □

The Theorem will follow from the Lemma once we show that the closure of V^i in $\mathbb{P}^n$ cannot intersect $\Sigma^{n-i} \cap \mathrm{Sing}(\mathcal{F})$.

For a generic $\Sigma^{n-i} \supset \Sigma^{n-i-1}$ it is clear that $\overline{V^i} \cap (\Sigma^{n-i} \setminus \Sigma^{n-i-1}) \cap \mathrm{Sing}(\mathcal{F}) = \emptyset$. One has just to take a Σ^{n-i} transversal to V^i with the maximal number of isolated singularities contained in U.

Our argument to ensure that $\overline{V^i} \cap \Sigma^{n-i-1} \cap \mathrm{Sing}(\mathcal{F}) = \emptyset$ is more subtle. Let $\overline{\mathcal{G}} : X \to \mathbb{P}^n$ be a resolution of the rational map $\mathcal{G}(\mathcal{F})$, i.e, $\pi : X \to \mathbb{P}^n$ is a composition of smooth blow-ups and $\overline{\mathcal{G}}$ is define through the commutative diagram below.

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, bend left, "\overline{\mathcal{G}}"] & \\ \mathbb{P}^n \arrow[r, dashed, "\mathcal{G}(\mathcal{F})"] & \check{\mathbb{P}}^n \end{tikzcd}$$

Let also $\mathcal{I} \subset \mathbb{P}^n \times \check{\mathbb{P}}^n$ be the incidence variety, $\mathbb{G}_i(\check{\mathbb{P}}^n)$ be the Grassmanian of i-dimensional linear subspaces of $\check{\mathbb{P}}^n$ and

$$\mathcal{U} = \left\{ (L^i, x, H) \in \mathbb{G}_i(\check{\mathbb{P}}^n) \times \mathbb{P}^n \times \check{\mathbb{P}}^n \,\middle|\, H \in L^i, x \in \check{L}^i = \bigcap_{H \in L^i} H \right\}.$$

Notice that $\mathcal{U} \subset \mathbb{G}_i(\check{\mathbb{P}}^n) \times \mathcal{I}$.

If $E \subset X$ is an exceptional divisor then the set of i-dimensional linear subspaces $L^i \subset \check{\mathbb{P}}^n$ for which $\overline{\mathcal{G}^{-1}}(L^i) \cap \pi^{-1}(\check{L}^i) \cap E \neq \emptyset$ is given by the image of the morphism σ defined below, where the unlabeled arrows are the corresponding natural projections.

$$\begin{tikzcd} E \times_{\mathcal{I}} \mathcal{U} \arrow[r] \arrow[d] \arrow[rr, bend left, "\sigma"] & \mathcal{U} \arrow[r] \arrow[d] & \mathbb{G}_i(\check{\mathbb{P}}^n) \\ E \arrow[r, "\pi \times \overline{\mathcal{G}}"] & \mathcal{I} & \end{tikzcd}$$

Notice that $\mathcal{I}$ is a $\mathrm{aut}(\mathbb{P}^n)$-homogeneous space under the natural action and that the vertical arrow $\mathcal{U} \to \mathcal{I}$ is a $\mathrm{aut}(\mathbb{P}^n)$-equivariant morphism. The transversality of the general translate, cf. [11], implies that

$$\dim E \times_{\mathcal{I}} \mathcal{U} = \dim E + \dim \mathcal{U} - \dim \mathcal{I} = \dim \mathbb{G}_i(\check{\mathbb{P}}^n) - 1.$$

It follows that σ is not dominant. Repeating the argument for every exceptional divisor of π we obtain an open set contained in $\mathbb{G}_i(\check{\mathbb{P}}^n)$ with the desired property. This concludes the proof of Theorem 1. □

3. Degrees of the Gauss Map of Logarithmic Foliations

Let $F_1, \ldots, F_k \in \mathbb{C}[x_0, \ldots, x_n]$ be reduced homogeneous polynomials. If $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$ satisfies

$$\sum_{i=1}^{k} \lambda_i \deg(F_i) = 0$$

then the rational 1-form on $\mathbb{C}^{n+1}$

$$\omega_\lambda = \omega(F, \lambda) = \sum_{i=1}^{k} \lambda_i \frac{dF_i}{F_i}$$

induces a rational 1-form on $\mathbb{P}^n$. Formally it is equal to the logarithmic derivative of the degree 0 multi-valued function $F_1^{\lambda_1} \cdots F_k^{\lambda_k}$. Being ω_λ closed, and in particular integrable, it defines $\mathcal{F}_\lambda$ a singular holomorphic foliation of $\mathbb{P}^n$. The corresponding 1-form is obtained from $(\prod F_i)\omega_\lambda$ after clearing out the common divisors of its coefficients. The *level sets* of the multi-valued function $F_1^{\lambda_1} \cdots F_k^{\lambda_k}$ are union of leaves of $\mathcal{F}_\lambda$.

If the divisor D of $\mathbb{P}^n$ induced by the zero locus of the polynomial $\prod F_i$ has at most normal crossing singularities and all the complex numbers λ_i are non zero then the singular of $\mathcal{F}_\lambda$ has a fairly simple structure, cf. [3, 5], which we recall in the next few lines. It has a codimension two part corresponding to the singularities of D and a zero dimensional part away from the support of D. To obtain this description one has just to observe that under the hypothesis the sheaf $\Omega^1(\log D)$ is a locally free sheaf of rank n and that the rational 1-form ω_λ has no zeros on a neighborhood of $|D|$ when interpreted as an element of $\mathrm{H}^0(\mathbb{P}^n, \Omega^1(\log D))$. Moreover, under these hypotheses, the length of the zero dimensional part of the singular scheme of $\mathcal{F}_\lambda$ is measured by the top Chern class of $\Omega^1(\log D)$.

In order to extend the above description of $\mathrm{sing}(\mathcal{F}_\lambda)$ to a more general setup let

$$\pi : (X, \pi^* D) \to (\mathbb{P}^n, D),$$

be an embedded resolution of D, i.e., π is a composition of blow-ups along smooth centers contained in the total transforms of D and the support of $\pi^* D$ has at most normal crossings singularities.

Due to the functoriality of logarithmic 1-forms the pull-back $\pi^*\omega_\lambda$ is a global section of $\mathrm{H}^0(X, \Omega^1_X(\log \pi^* D))$. To each irreducible component E of $\pi^* D$ there is a naturally attached complex number — the **residue** of $\pi^*\omega_\lambda$ — that can be defined as

$$\lambda(E) = \lambda(E, \omega_\lambda) = \frac{1}{2\pi i} \int_{\gamma_i} \pi^*(\omega_\lambda)$$

where $\gamma : S^1 \to X \setminus |\pi^* D|$ is a naturally oriented closed path surrounding the support of E. If E is the strict transform of $V(F_i)$ then, clearly, $\lambda(E) = \lambda_i$. More generally one has the following lemma.

Lemma 3. *For every irreducible component $E \subset X$ of the exceptional divisor there exists natural numbers $m_1, \ldots, m_k \in \mathbb{N}$ such that*

$$\lambda(E) = \sum_{i=1}^{k} m_i \lambda_i \,.$$

Proof. Let $\pi_1 : (\mathcal{X}_1, \pi_1^* D) \to (\mathbb{P}^n, D)$ the first blow up in the resolution process of D with center $C_1 \subset D$ and let $E_1 = \pi^*(C_1)$ be the exceptional divisor.

If $D_i = V(F_i)$ and $\widetilde{D}_i$ denotes the strict transform of D_i then we can write

$$\pi_1^* D_i = n_i E_1 + \widetilde{D}_i$$

where n_i is the natural number measuring the multiplicity of $V(F_i)$ along C_1. Moreover if, over a generic point $p \in |E_1|$, we take t as a reduced germ of regular function cutting out E_1 then

$$\pi_1^*(\omega_\lambda) = \left(\sum_i \lambda_i n_i \right) \frac{dt}{t} + \alpha \, ,$$

for some closed regular 1-form α. The proof follows by induction on the number of blow ups necessary to resolve D. □

Definition 1. *The complex vector $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k$ is* **non resonant** *(with respect to π) if $\lambda(E) \neq 0$ for every irreducible component E of $\pi^* D$.*

The arguments of [3, 5] yields the following description of the singular set of $\mathcal{F}_\lambda$ for non resonant λ. We reproduce them below thinking on reader's ease.

Lemma 4. *If λ is non resonant then the restriction to the complement of D of the singular set of $\mathcal{F}_\lambda$ is zero-dimensional. Moreover the length of the corresponding part of the singular scheme is $c_n(\Omega^1_X(\log \pi^* D))$.*

Proof. Since λ is non resonant the 1-form $\pi^*\omega_\lambda$, seen as a section of $\Omega^1_X(\log \pi^* D)$, has no zeros on a neighborhood of $|\pi^* D|$.

Suppose that there exists a positive dimensional component of the singular set of $\mathcal{F}_\lambda$ not contained in $|\pi^* D|$. Being the divisor $\pi^* D$ ample this component has to intersect the support of $\pi^* D$. This leads to contradiction ensuring that the singular set of $\mathcal{F}_\lambda$ has no positive dimensional components in the complement of $|\pi^* D|$.

The assertion concerning the length of the singular scheme follows from the fact that $\Omega^1_X(\log \pi^* D)$ is a locally free sheaf of rank n. □

Let $\Sigma^s \subset \mathbb{P}^n$ be a generic linear subspace of dimension s and denote by $X_s = \pi^{-1}(\Sigma^s)$ and $D_s = (\pi^* D)_{|X_s}$. It follows from Bertini's Theorem that X_s is smooth and D_s is a divisor with at most normal crossings.

Proposition 1. *If λ is non resonant then*

$$\deg_0(\mathcal{G}(\mathcal{F}_\lambda)) = c_{n-1}(\Omega^1_{X_{n-1}}(\log D_{n-1}))$$

and, for $1 \leq i \leq n-1$

$$\deg_{n-i}(\mathcal{G}(\mathcal{F}_\lambda)) = c_{i-1}(\Omega^1_{X_{i-1}}(\log D_{i-1})) + c_i(\Omega^1_{X_i}(\log D_i)) \, .$$

Proof. If $H \subset \mathbb{P}^n$ is a generic hyperplane then, according to Lemma 1, $\mathcal{G}(\mathcal{F}_\lambda)^{-1}(H)$ coincides with the isolated singularities of $\mathcal{F}_{|H}$ that are not singularities of $\mathcal{F}$. By choosing H on the complement of the dual variety of the support of D we can assume that these isolated singularities are away from the support of D.

If $\pi_{n-1} : X_{n-1} \to H$ is the restriction of $\pi : X \to \mathbb{P}^n$ to X_{n-1} then π_{n-1} is an embedded resolution of D_{n-1} and, moreover, for every exceptional divisor of E intersecting $\pi^{-1}(H)$ we have that the residue of $\pi^*_{n-1}(\omega_{\lambda|H})$ along any irreducible component of $E \cap X_{n-1}$ is equal to the residue of $\pi^*\omega_\lambda$ along E. Therefore the logarithmic 1-form $\omega_{\lambda|H}$ is non resonant with respect to π_{n-1}.

It follows from Lemma 4 that the sought number of isolated singularities is $c_{n-1}(\Omega^1_{X_{n-1}}(\log D_{n-1}))$. Similar arguments shows that

$$e_0^k(\mathcal{F}_\lambda) = c_{k-1}(\Omega^1_{X_{k-1}}(\log D_{k-1})).$$

To conclude one has just to invoke Theorem 1.

□

4. A Logarithmic Foliation associated to a Polar Transformation

Consider the multivalued function

$$\mathbb{F}^\lambda = \prod_{i=1}^{k} F_i^{\lambda_i} : \mathbb{P}^n \dashrightarrow \mathbb{P}^1$$

where $F_i \in \mathbb{C}[x_0, ..., x_n]$ is a reduced homogeneous polynomial of degree d_i and $\lambda_i \in \mathbb{C}^*$. The function $\mathbb{F}^\lambda$ is a homogeneous function of degree $\deg(\mathbb{F}^\lambda) = \sum_{i=1}^k \lambda_i d_i$. If $\deg(\mathbb{F}^\lambda) = 0$ then the logarithmic derivative of $\mathbb{F}^\lambda$ defines a logarithmic foliation of $\mathbb{P}^n$ and the associated polar map (see below) coincides with the Gauss map of this foliation. Although one can in principle use the results of the previous section to control the degree of this polar map, in general, is rather difficult to control the singular set of the corresponding logarithmic foliation without further hypothesis. Therefore, from now on we will assume that $\deg(\mathbb{F}^\lambda) \neq 0$.

Although $\mathbb{F}^\lambda$ is not an algebraic function it is still possible to define its polar map as the rational map

$$\begin{array}{rcl} \nabla\mathbb{F}^\lambda : \mathbb{P}^n & \dashrightarrow & \mathbb{P}^n \\ x & \to & \left[\dfrac{\mathbb{F}_0^\lambda(x)}{\mathbb{F}^\lambda(x)} : \ldots : \dfrac{\mathbb{F}_n^\lambda(x)}{\mathbb{F}^\lambda(x)}\right] \end{array}$$

where $\mathbb{F}_i^\lambda$ denotes the partial derivative of $\mathbb{F}^\lambda$ with respect to x_i. Notice that when all the λ_i's are natural numbers this rational map coincides with the polar map defined in the introduction.

Consider the foliation of $\mathbb{C}^{n+1}$ defined by the polynomial 1-form

$$\left(\prod_{i=1}^{k} F_i\right)\frac{d\mathbb{F}^\lambda}{\mathbb{F}^\lambda} = \left(\prod_{i=1}^{k} F_i\right)\sum_{i=1}^{k}\lambda_i\frac{dF_i}{F_i}\,.$$

Notice that all the singularities of this foliation are contained in $V(\prod F_i)$ since Euler's formula implies that

$$i_R\left(\prod_{i=1}^{k} F_i\right)\frac{d\mathbb{F}^\lambda}{\mathbb{F}^\lambda} = \deg(\mathbb{F}^\lambda)\left(\prod F_i\right)\,.$$

This foliation of $\mathbb{C}^{n+1}$ naturally extends to a foliation of $\mathbb{P}^{n+1}$. If we consider $F_1, \ldots, F_k$ as polynomials in $\mathbb{C}[x_0, \ldots, x_n, x_{n+1}]$, $F_{k+1} = x_{n+1}$ and $\overline{\lambda} = (\lambda_0, \ldots, \lambda_n, -\deg(\mathbb{F}^\lambda))$ then it coincides with the foliation $\mathcal{F}_{\overline{\lambda}}$ of the previous section induced by the 1-form

$$\omega_{\overline{\lambda}} = \frac{d\mathbb{F}^\lambda}{\mathbb{F}^\lambda} - \deg(\mathbb{F}^\lambda)\frac{dx_{n+1}}{x_{n+1}}\,.$$

The degrees of the Gauss map of $\mathcal{F}_{\overline{\lambda}}$ are related with those of $\mathbb{F}^\lambda$ by means of the following Theorem.

Theorem 2. *If the degree of $\mathbb{F}^\lambda$ is not equal to zero then for $i = 0, \dots, n-1$,*

$$\deg_i(\mathcal{G}(\mathcal{F}_{\overline{\lambda}})) = \deg_i\left(\nabla\mathbb{F}^\lambda\right) + \deg_{i-1}\left(\nabla\mathbb{F}^\lambda\right),$$

where we are assuming that $\deg_{-1}\left(\nabla\mathbb{F}^\lambda\right) = 0$.

Proof. If we set $\hat{F}_j = \prod_{i\neq j, i=1}^{k} F_i$ then the Gauss map of the foliation $\mathcal{F}_{\overline{\lambda}}$ at the point $[x_0 : \dots : x_{n+1}]$ can be explicitly written as

$$\left[x_{n+1}\left(\sum_{j=1}^{k}\lambda_j\hat{F}_j\frac{\partial F_j}{\partial x_0}\right) : \dots : x_{n+1}\left(\sum_{j=1}^{k}\lambda_j\hat{F}_j\frac{\partial F_j}{\partial x_n}\right) : -\deg(\mathbb{F}^\lambda)\left(\prod_{j=1}^{k}F_j\right)\right].$$

Therefore if $p = [0 : \dots : 0 : 1]$ and $\pi : \mathrm{Bl}_p(\mathbb{P}^{n+1}) \to \mathbb{P}^{n+1}$ denotes the blow-up of $\mathbb{P}^{n+1}$ at p then the restriction of $\mathcal{G} = \mathcal{G}(\mathcal{F}_{\overline{\lambda}})\circ\pi^{-1}$ to the exceptional divisor $E \cong \mathbb{P}^n$ can be identified with $\nabla\mathbb{F}^\lambda$ as soon as we identify the target of $\nabla\mathbb{F}^\lambda$ with the set $\mathbb{P}^n_p \subset \check{\mathbb{P}}^{n+1}$ of hyperplanes containing p .

Consider the projection $\rho([x_0 : \dots : x_n : x_{n+1}]) = [x_0 : \dots : x_n]$ with center at p and notice that it lifts to a morphism $\widetilde{\rho} : \mathrm{Bl}_p(\mathbb{P}^{n+1}) \to \mathbb{P}^n$. If we write

$$\nabla\mathbb{F}^\lambda(x) = \left[\sum_{j=1}^{k}\lambda_j\hat{F}_j\frac{\partial F_j}{\partial x_0} : \dots : \sum_{j=1}^{k}\lambda_j\hat{F}_j\frac{\partial F_j}{\partial x_n}\right],$$

then it is clear that the rational maps $\mathcal{G}$ and $\nabla\mathbb{F}^\lambda$ fit in the commutative diagram below.

$$\begin{tikzcd}
\mathrm{Bl}_p(\mathbb{P}^{n+1}) \arrow[r, dashed, "\mathcal{G}"] \arrow[d, "\widetilde{\rho}"'] & \mathbb{P}^{n+1} \arrow[d, dashed, "\rho"'] \\
\mathbb{P}^n \arrow[r, dashed, "\nabla\mathbb{F}^\lambda"] & \mathbb{P}^n
\end{tikzcd}$$

Let $L^i \subset \check{\mathbb{P}}^{n+1}$ be a generic linear subspace of dimension i and set

$$W^i = \overline{\mathcal{G}(\mathcal{F}_{\overline{\lambda}})^{-1}(L^i)}, \quad \widetilde{W^i} = \overline{\mathcal{G}^{-1}(L^i)} \text{ and } V^i = \overline{\left(\nabla\mathbb{F}^\lambda\right)^{-1}(\rho(L^i))}.$$

If $U \subset \mathbb{P}^n$ is the complement of the hypersurface cut out by $\prod F_j$ then [14, lemma] implies that $V^i \cap U$ and $\widetilde{W^i} \cap \widetilde{\rho}^{-1}(U)$ are dense in V^i and $\widetilde{W^i}$.

It follows at once from the diagram above that $\widetilde{\rho}(\widetilde{W^i}) \subset V^i$. A simple computation shows that the restriction of $\mathcal{G}$ to a fiber of $\widetilde{\rho}$ over U induces an isomorphisms to the corresponding fiber of ρ. Combining this with the density of $V^i \cap U$ and $\widetilde{W^i} \cap \widetilde{\rho}^{-1}(U)$ in V^i and $\widetilde{W^i}$ respectively one promptly concludes that the i-cycle $\widetilde{\rho}_*\widetilde{W^i}$ is equal to the i-cycle V^i.

The i-th degree of the Gauss map of $\mathcal{F}_{\overline{\lambda}}$ can be expressed as

$$\deg_i\left(\mathcal{G}(\mathcal{F}_{\overline{\lambda}})\right) = c_1(\mathcal{O}_{\mathbb{P}^{n+1}}(1))^i \cdot W^i.$$

If $\widetilde{W^i} = \overline{\mathcal{G}^{-1}(L^i)}$, H denotes a generic hyperplane containing p and $\widetilde{H}$ is its strict transform then, thanks to the projection formula,

$$\begin{aligned}
\deg_i\left(\mathcal{G}_{\mathcal{F}_{\overline{\lambda}}}\right) &= c_1(\pi^*\mathcal{O}_{\mathbb{P}^{n+1}}(1))^i \cdot \widetilde{W^i} \\
&= c_1(\widetilde{\rho}^*\mathcal{O}_{\mathbb{P}^n}(1))^i \cdot \widetilde{W^i} + \left(\sum_{j=1}^{i} \binom{i}{j} \widetilde{H}^{i-j} \cdot E^j\right) \cdot \widetilde{W^i} \\
&= c_1(\mathcal{O}_{\mathbb{P}^n}(1))^i \cdot \widetilde{\rho}(\widetilde{W^i}) + \left(\left(\sum_{j=1}^{i} \binom{i}{j} \widetilde{H}^{i-j} \cdot E^{j-1}\right) \cap E\right) \cdot \left(\widetilde{W^i} \cap E\right) \\
&= c_1(\mathcal{O}_{\mathbb{P}^n}(1))^i \cdot V^i + c_1(\mathcal{O}_E(1))^{i-1} \cdot (\widetilde{W^i} \cap E)\,.
\end{aligned}$$

On the one hand $c_1(\mathcal{O}_{\mathbb{P}^n}(1))^i \cdot V^i$ is clearly equal to $\deg_i(\nabla\mathbb{F}^\lambda)$. On the other hand $c_1(\mathcal{O}_E(1))^{i-1} \cdot (\widetilde{W^i} \cap E) = \deg_{i-1}(\nabla\mathbb{F}^\lambda)$ since, for a generic L^i, $\widetilde{W^i} \cap E$ is equal to $\overline{\mathcal{G}^{-1}_{|E}(L^i \cap \mathbb{P}^n_p)}$ as an $(i-1)$-cycle on E. The Theorem follows. □

Corollary 2. *If the degree of $\mathbb{F}^\lambda$ is not equal to zero then*

$$\deg_i(\nabla\mathbb{F}^\lambda) = e_0^{n+1-i}(\mathcal{F}_{\overline{\lambda}})\,.$$

for $i = 0, \ldots, n-1$.

Proof. Follows at once when after comparing Theorem 1 with Theorem 2. □

5. The Main Result: Invariance of the Degrees

Theorem 3. *Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be an element of $\mathbb{C}^k$ such that $\mathfrak{H}(\lambda_j) > 0$ for some $\mathbb{R}$-linear map $\mathfrak{H} : \mathbb{C} \to \mathbb{R}$ and every $j = 1, \ldots, k$. Let also $F_1, \ldots, F_k$ be irreducible and homogeneous polynomials in $\mathbb{C}^{n+1}$. If $\mathbb{F}^\lambda = \prod F_j^{\lambda_j}$ then*

$$\deg_i\left(\nabla\mathbb{F}^\lambda\right) = \deg_i\left(\nabla\left(\prod F_j\right)\right)$$

for every $i = 0\ldots, n-1$.

Proof. Let $\mathcal{F} = \mathcal{F}_{\overline{\lambda}}$ be the foliation on $\mathbb{P}^{n+1}$ associated to $\mathbb{F}^\lambda$. Corollary 2 implies that $\deg_i(\nabla\mathbb{F}^\lambda)$ is equal to the degree of the Gauss map of $\mathcal{F}_{|\mathbb{P}^{n+1-i}}$ for a generic $\mathbb{P}^{n+1-i} \subset \mathbb{P}^{n+1}$.

If D is the divisor of $\mathbb{P}^n$ associated to $\prod F_j$ then the intersection in $\mathbb{P}^{n+1}$ of $V\left(x_{n+1}\left(\prod F_j\right)\right)$ and a generic $\mathbb{P}^{n-i}$ is isomorphic to the union of the intersection of $|D|$ with a generic $\mathbb{P}^{n-i} \subset \mathbb{P}^n$ and a generic hyperplane H in $\mathbb{P}^{n-i}$.

If $\pi : X \to \mathbb{P}^{n-i}$ is an embedded resolution of $|D| \cap \mathbb{P}^{n-i}$ then Bertini's Theorem implies that it is also an embedded resolution of the union of $|D| \cap \mathbb{P}^{n-i}$ with a generic H. Therefore in the computation of $\lambda(E)$ for an exceptional divisor of π the residue along H, $\lambda(H) = -\deg(\mathbb{F}^\lambda)$, plays no role since H and its strict transforms do not contain any of the blow-up centers. Thus the hypothesis on λ together with Lemma 3 implies that $\overline{\lambda}$ is non-resonant with respect to π. It follows from Proposition 1 that

$$\deg_0(\mathcal{G}(\mathcal{F}_{|\mathbb{P}^{n+1-i}})) = c_{n-i}(\Omega^1_X(\log(D \cap \mathbb{P}^{n-i} + H)))\,.$$

Since the same arguments implies that the same formula holds true for the foliation associated to $\mathbb{F} = \prod F_j$ the Theorem follows. □

The hypothesis on $\lambda \in \mathbb{C}^k$ can be of course weakened. Lemma 3 ensures that there exits finitely many subvarieties of $\mathbb{C}^k$ defined by linear equations with coefficients in $\mathbb{N}$ that have to be avoided. Outside these linear varieties the degree of $\nabla \mathbb{F}^\lambda$ is constant.

The example below shows, for resonant λ the degree of the associated polar map will in general decrease with respect to the non-resonant ones.

Example 1. *Let $F_1, \ldots, F_k, F_{k+1} \in \mathbb{C}[x,y,z]$ be linear forms such that $F_1, \ldots, F_k \in \mathbb{C}[x,y]$ and $F_{k+1} \notin \mathbb{C}[x,y]$. If $\lambda = (\lambda_1, \ldots, \lambda_k, \lambda_{k+1}) \in (\mathbb{C}^*)^{k+1}$ is such that*

$$\sum_{i=1}^{k} \lambda_i = 0$$

and $k \geq 2$ then the rational map $\nabla \mathbb{F}^\lambda$ is homaloidal, i.e, $\deg\left(\nabla \mathbb{F}^\lambda\right) = 1$.

Proof. If F_{k+2} is a generic linear form and $\lambda_{k+2} = -\sum_{j=1}^{k+1} \lambda_j = -\lambda_{k+1}$ then the proof of Theorem 3 shows that the degree of $\nabla \mathbb{F}^\lambda$ is equal to the number of singularities of the foliation $\mathcal{F}$ of $\mathbb{P}^2$ induced by the 1-form

$$\left(\prod_{j=1}^{k+2} F_j\right) \sum_{j=1}^{k+1} \lambda_j \frac{dF_j}{F_j}$$

outside $V\left(\prod_{j=1}^{k+2} F_j\right)$.

Notice that $\mathcal{F}$ has degree k and that

$$\sum_{p \in \operatorname{Sing}(\mathcal{F})} m(\mathcal{F}, p) = c_2(\Omega^1_{\mathbb{P}^2}(k+2)) = k^2 + k + 1\,.$$

On the curve cut out by $\prod F_j$ with $(2k+1)+1$ singularities. One of them at the confluence of k lines and the other $2k+1$ at the intersection of exactly two lines. The latter singularities have all multiplicity one as a simple local computation shows. The multiplicity of the latter can be computed using Van den Essen formula [15] and is equal to $k^2 - k - 1$. Summing up all these multiplicities one obtains $k^2 + k$. Thus $\deg(\nabla \mathbb{F}^\lambda) = 1$. □

In the example above if

$$\sum_{i=1}^{k} \lambda_i \neq 0 \quad \text{and} \quad \sum_{i=1}^{k+1} \lambda_i \neq 0$$

then Van den Essen Formula shows that the multiplicity of the singularity containing the k lines is $(k-1)^2$. Thus the degree of $\nabla \mathbb{F}^\lambda$ is, under these hypotheses, $k-1$. The first author have shown that all the homaloidal polar maps associated to a product of lines with complex weights are of the form above. A proof will appear elsewhere.

An easy consequence of Theorem 3 is the Corollary below. It would be interesting to replace the maximum on the left hand side of the inequality by a sum. Indeed [7, Proposition 5] does it for the topological degree under stronger hypothesis.

Corollary 3. *Let $F_1, F_2 \in \mathbb{C}[x_0, \ldots, x_n]$ be two homogeneous polynomials. If F_1 and F_2 are relatively prime then*

$$\deg_i(\nabla F_1 \cdot F_2) \geq \max\{\deg_i(\nabla F_1), \deg_i(\nabla F_2)\}$$

for $i = 0, \ldots, n-1$.

Proof. Let $\mathcal{F}_1$ be the foliation of $\mathbb{P}^{n+1}$ associated to F_1 and $\mathcal{F}_{12}$ the one associated to F_1F_2. They are, respectively, induced by the rational 1-forms on $\mathbb{P}^{n+1}$

$$\omega_1 = \frac{dF_1}{F_1} - \deg(F_1)\frac{dx_{n+1}}{x_{n+1}} \quad \text{and} \quad \omega_{12} = \frac{dF_1}{F_1} + \frac{dF_2}{F_2} - (\deg(F_1)\deg(F_2))\frac{dx_{n+1}}{x_{n+1}}.$$

Let $H \subset \check{\mathbb{P}}^{n+1}$ be a generic hyperplane and $\iota : H \to \mathbb{P}^{n+1}$ be the inclusion. Recall that $\mathcal{G}(\mathcal{F}_1)^{-1}(H)$ consists of $\deg_0(\mathcal{G}(\mathcal{F}_1))$ isolated points corresponding to the singularities of $\iota^*\omega_1$ contained in $H \setminus V(F_1)$. It follows from the proof of Theorem 3 that we can assume that $\iota^*\omega_{12}$ is non resonant (with respect to a certain resolution).

If H, seen as a point of $\check{\mathbb{P}}^{n+1}$, avoids the closure of the image of $V(F_2)$ under $\mathcal{G}(\mathcal{F}_1)$ then singularities of $\iota^*\omega_1$ contained in the complement of $V(F_1)$ are also contained in the complement of $V(F_1F_2)$. It follows that for $\epsilon > 0$ small enough the 1-form $\iota^*(\omega_1 + \epsilon\omega_{12})$ has at least $\deg_0(\mathcal{G}(\mathcal{F}_1))$ singularities contained in the complement of $V(F_1F_2)$. Since we can choose ϵ in such a way that $\iota^*(\omega_1 + \epsilon\omega_{12})$ is non resonant the induced foliation has Gauss map with the degree as the Gauss map of $\mathcal{F}_{12}$.

It follows from Theorem 3 that $\deg_0(\nabla F_1F_2) \geq \deg_0(\nabla F_1)$. Arguing exactly in the same way first with F_2 and then with linear sections of higher codimensions the Corollary follows. □

The Corollary above *essentially* reduces the problem of classification of homaloidal polynomials to the classification of irreducible homaloidal polynomials and irreducible polynomials with vanishing Hessian. Although, one should not be much optimistic about generalizing Dolgachev's Classification to higher dimensions. Already in $\mathbb{P}^3$ there are examples of irreducible homaloidal polynomials of arbitrarily high degree, cf. [4].

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