Reduction of Courant algebroids and generalized complex structures

Henrique Bursztyn* Gil R. Cavalcanti† Marco Gualtieri‡

Abstract

We present a theory of reduction for Courant algebroids as well as Dirac structures, generalized complex, and generalized Kähler structures which interpolates between holomorphic reduction of complex manifolds and symplectic reduction. The enhanced symmetry group of a Courant algebroid leads us to define *extended* actions and a generalized notion of moment map. Key examples of generalized Kähler reduced spaces include new explicit bi-Hermitian metrics on $\mathbb{C}P^2$.

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*henrique@impa.br
†gilrc@maths.ox.ac.uk
‡mgualt@math.mit.edu

1 Introduction

In the presence of a symmetry, a given geometrical structure may, under suitable conditions, pass to the quotient. Often, however, the quotient does not inherit the same type of geometry as the original space; it may be necessary to pass to a further *reduction* for this to occur. For example, a complex manifold M admitting a holomorphic S^1 action certainly does not induce a complex structure on M/S^1; rather, one considers the complexification of this action to a $\mathbb{C}^*$ action, whose quotient, under suitable conditions, inherits a complex structure. Similarly, the quotient of a symplectic manifold by a symplectic S^1 action is never symplectic; rather it is endowed with a natural Poisson structure, whose leaves are the symplectic reduced spaces one desires.

In this paper we consider the reduction of generalized geometrical structures such as Dirac structures and generalized complex structures. These are geometrical structures defined not on the tangent bundle of a manifold but on the sum $T \oplus T^*$ of the tangent and cotangent bundles (or, more generally, on an exact Courant algebroid). These structures interpolate between many of the classical geometries such as symplectic and Poisson geometry, the geometry of foliations, and complex geometry. As a result the quotient procedure described in this paper interpolates between the known methods of reduction in these cases.

The main conceptual advance required to understand the reduction of generalized geometries is the fact that one must extend the notion of action of a Lie group on a manifold. Traditional geometries are defined in terms of the Lie bracket of vector fields, whose symmetries are given precisely by diffeomorphisms. As a result, one considers reduction in the presence of a group homomorphism from a Lie group into the group of diffeomorphisms. The Courant bracket, on the other hand, has an enhanced symmetry group which is an abelian extension of a diffeomorphism group by the group of closed 2-forms. For this reason one must consider actions which may have components acting nontrivially on the Courant algebroid while leaving the underlying manifold fixed. To formalize this insight, we introduce the notion of a *Courant algebra*, and explain how it acts on a Courant algebroid in a way which extends the usual action of a Lie algebra by tangent vector fields.

A surprising benefit of this point of view is that the concept of *moment map* in symplectic geometry obtains a new interpretation as an object which controls the extended part of the action mentioned above, that is, the part of the action trivially represented in the diffeomorphism group.

In preparing this article, the authors drew from a wide variety of sources, all of which provided hints toward the proper framework for generalized reduction. First, the literature on holomorphic reduction of complex manifolds as well as the field of Hamiltonian reduction of symplectic manifolds in the style of Marsden-Weinstein [18]. Also, in the original work of Courant and Weinstein ([3], [4]) where the Courant bracket is introduced, some preliminary remarks about quotients can be found. Most influential, however, has been the work of physicists on the problem of finding gauged sigma models describing supersymmetric sigma models with isometries. The reason this is relevant is that the geometry of a general $N = (2, 2)$ supersymmetric sigma model is equivalent to generalized Kähler geometry [7], and so any insight into how to "gauge" or quotient such a model provides us with guidance for the geometrical reduction problem. Our sources for this material have been the work of Hull, Roček, de Wit, and Spence ([12],[13]), Witten [25], and Figueroa-O'Farill and Stanciu [6]. More recently in the physics literature, the gauging conditions have been re-interpreted in terms of the Courant bracket [5], a point of view which we develop and expand upon in this paper as well. Finally, in recent work of Hitchin [9], a natural generalized Kähler structure on the moduli space of instantons on a generalized Kähler 4-manifold is constructed by a method which amounts to an infinite-dimensional generalized Kähler quotient. This example as well as the questions it engenders was one of the guiding examples for this research.

The paper is organized as follows. In Section 2 we review the definition of Courant algebroid, describe its group of symmetries, and define the concept of extended action. This involves the definition of a Courant algebra, a particular kind of Lie 2-algebra. In this section we also define a moment map for an extended action. In Section 3 we describe how an extended action on an exact Courant algebroid gives rise to reduced spaces equipped with induced exact Courant algebroids. It turns out that, even if the original Courant algebroid has trivial 3-form curvature, its reduced spaces may have nontrivial curvature. In Section 4 we arrive at the reduction procedure for generalized geometries, introducing an operation which transports Dirac structures from a Courant algebroid to its reduced spaces. This operation generalizes both the operation of Dirac push-forward and pull-back outlined in [2]. In Section 5 we apply this procedure to reduce generalized complex structures and provide several examples, including some with interesting type change. Finally in Section 6 we study a way to transport a generalized Kähler structure to the reduced spaces. This is very much in the spirit of the usual Kähler reduction procedure of Hitchin, Karlhede, Lindstrom and Roček [10]. Finally we present two examples of generalized Kähler reduction: we produce generalized Kähler structures on $\mathbb{C}P^2$ with type change, first along a triple line (an example of which has been found in [9] using a different method) and second, along three distinct lines in the plane. These examples are particularly significant since they provide explicit bi-Hermitian metrics on $\mathbb{C}P^2$.

Recently there has been a great deal of interest in porting the techniques of Hamiltonian reduction to the setting of generalized geometry. The authors are aware of four other groups who have worked independently on this topic: Lin and Tolman [16], Stienon and Xu [21], Hu [11], and Vaisman [22].

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2 Symmetries of the Courant bracket

In this section we introduce an extended notion of group action on a manifold preserving twisted Courant brackets. We start by recalling the definition and basic properties of Courant algebroids.

2.1 Courant algebroids

The notion of Courant algebroid was introduced in [17] in order to axiomatize the properties of the Courant bracket, an operation on sections of $TM \oplus T^*M$ extending the Lie bracket of vector fields. The failure of this bracket to satisfy the Jacobi identity as well as the Leibniz rule is measured by a symmetric bilinear form, in a way which was generalized as follows.

A *Courant algebroid* over a manifold M is a vector bundle $E \longrightarrow M$ equipped with a skew-symmetric bracket $[\![\cdot,\cdot]\!]$ on $C^\infty(E)$, a nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$, and a bundle map $\pi : E \longrightarrow TM$ called the anchor, which satisfy the following conditions for all $e_1, e_2, e_3 \in C^\infty(E)$ and $f, g \in C^\infty(M)$:

C1) $\pi([\![e_1, e_2]\!]) = [\pi(e_1), \pi(e_2)]$,

C2) $[\![[\![e_1, e_2]\!], e_3]\!] + [\![[\![e_2, e_3]\!], e_1]\!] + [\![[\![e_3, e_1]\!], e_2]\!] = \mathcal{D}T(e_1, e_2, e_3)$,

C3) $[\![e_1, fe_2]\!] = f[\![e_1, e_2]\!] + (\pi(e_1)f)e_2 - \langle e_1, e_2\rangle\mathcal{D}f$,

C4) $\pi \circ \mathcal{D} = 0$, i.e. $\langle\mathcal{D}f, \mathcal{D}g\rangle = 0$,

C5) $\pi(e_1)\langle e_2, e_3\rangle = \langle e_1 \bullet e_2, e_3\rangle + \langle e_2, e_1 \bullet e_3\rangle$,

where $\mathcal{D} = \frac{1}{2}\pi^* \circ d : C^\infty(M) \longrightarrow C^\infty(E)$ (using $\langle\cdot,\cdot\rangle$ to identify E with E^*), T is given by

$$T(e_1, e_2, e_3) = \tfrac{1}{3}(\langle[\![e_1, e_2]\!], e_3\rangle + \langle[\![e_2, e_3]\!], e_1\rangle + \langle[\![e_3, e_1]\!], e_2\rangle),$$

and $\bullet$ denotes the combination

$$e_1 \bullet e_2 = [\![e_1, e_2]\!] + \mathcal{D}\langle e_1, e_2\rangle. \tag{1}$$

We see from axiom $C1$) that the Jacobi identity is "satisfied up to an exact term"; indeed as was shown in [19], a Courant algebroid is an example of an L_∞ algebra. We now briefly describe Ševera's classification of exact Courant algebroids.

Definition 2.1. A Courant algebroid is *exact* if the following sequence is exact:

$$0 \longrightarrow T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \longrightarrow 0 \tag{2}$$

Given an exact Courant algebroid, we may always choose a right splitting $\nabla : TM \longrightarrow E$ which is *isotropic*, i.e. whose image in E is isotropic with respect to $\langle\cdot,\cdot\rangle$. Such a splitting has a curvature 3-form $H \in \Omega^3_{cl}(M)$ defined as follows, for $X, Y \in C^\infty(TM)$:

$$i_Y i_X H = \tfrac{1}{2}s[\![\nabla(X), \nabla(Y)]\!], \tag{3}$$

where $s : E \longrightarrow T^*M$ is the induced left splitting. Using the bundle isomorphism $\nabla + \frac{1}{2}\pi^* : TM \oplus T^*M \longrightarrow E$, we transport the Courant algebroid structure onto $TM \oplus T^*M$. Given $X + \xi, Y + \eta \in C^\infty(TM \oplus TM^*)$, we obtain for the bilinear pairing

$$\langle X + \xi, Y + \eta\rangle = \frac{1}{2}(\eta(X) + \xi(Y)), \tag{4}$$

and the bracket becomes

$$[\![X + \xi, Y + \eta]\!] = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \tfrac{1}{2}d(i_X\eta - i_Y\xi) + i_Y i_X H, \tag{5}$$

which is the *H-twisted Courant bracket* on $TM \oplus T^*M$ [20]. Isotropic splittings of (2) differ by 2-forms $b \in \Omega^2(M)$, and a change of splitting modifies the curvature H by the exact form db. Hence the cohomology class $[H] \in H^3(M, \mathbb{R})$, called the *Ševera class*, is independent of the splitting and determines the exact Courant algebroid structure on E completely. When this class is integral, the exact Courant algebroid may be viewed as a generalized Atiyah sequence associated to a connection on an S^1 gerbe. In this sense, exact Courant algebroids arise naturally from the study of gerbes.

The symmetry group $\mathcal{C}$ of an exact Courant algebroid, that is, the group of orthogonal bundle automorphisms preserving the Courant bracket, can be easily described once an isotropic splitting is chosen [7]: it consists of the group of ordered pairs $(\varphi, B) \in \mathrm{Diff}(M) \times \Omega^2(M)$ such that $\varphi^*H - H = dB$. Diffeomorphisms act in the usual way on $TM \oplus T^*M$, and 2-forms B act via $X + \xi \mapsto X + \xi + i_X B$. As a result we see that $\mathcal{C}$ is an extension

$$0 \longrightarrow \Omega^2_{cl}(M) \longrightarrow \mathcal{C} \longrightarrow \mathrm{Diff}_{[H]}(M) \longrightarrow 0\ ,$$

where $\mathrm{Diff}_{[H]}(M)$ is the group of diffeomorphisms preserving the cohomology class $[H]$.

The Lie algebra $\mathfrak{c}$ of symmetries consists of pairs $(X, B) \in C^\infty(TM) \oplus \Omega^2(M)$ such that $\mathcal{L}_X H = dB$. For this reason, it is an extension of the form

$$0 \longrightarrow \Omega^2_{cl}(M) \longrightarrow \mathfrak{c} \longrightarrow C^\infty(TM) \longrightarrow 0\ .$$

Since H is closed, $\mathcal{L}_X H = d(i_X H)$ for any vector field X, and so we have a right splitting of the above sequence given by $Y \mapsto (Y, i_Y H)$. However, this is not a splitting that preserves the Lie bracket.

There is an adjoint, or interior, action of $C^\infty(E)$ as infinitesimal symmetries defined by $\mathrm{ad}_v(w) := v \bullet w$. We have the following consequences of (1):

$$\pi(e_1)\langle e_2, e_3\rangle = \langle e_1 \bullet e_2, e_3\rangle + \langle e_2, e_1 \bullet e_3\rangle \tag{6}$$

$$e_1 \bullet [\![e_2, e_3]\!] = [\![e_1 \bullet e_2, e_3]\!] + [\![e_2, e_1 \bullet e_3]\!], \tag{7}$$

showing that this adjoint action is an infinitesimal symmetry of the Courant algebroid. Unlike, however, the usual adjoint action of vector fields on the tangent bundle, the map $\mathrm{ad} : C^\infty(E) \longrightarrow \mathfrak{c}$ is neither surjective nor injective; indeed the Lie algebra $\mathfrak{c}$ fits into the following exact sequence:

$$0 \longrightarrow \Omega^1_{cl}(M) \longrightarrow C^\infty(E) \longrightarrow \mathfrak{c} \longrightarrow H^2(M, \mathbb{R}) \longrightarrow 0\ ,$$

where the map to cohomology can be written as $(X, B) \mapsto [i_X H - B]$ in a given splitting.

2.2 Extended actions

Let a Lie group G act on a manifold M, so that we have the Lie algebra homomorphism $\psi : \mathfrak{g} \longrightarrow C^\infty(TM)$. We wish to extend this action to a Courant algebroid E, making E into a G-equivariant vector bundle, in such a way that the Courant algebroid structure is preserved. In this section we show how this can be done by choosing an extension of $\mathfrak{g}$ equipped with a *Courant algebra* structure, and choosing a homomorphism from this extension to the Courant algebroid E.

We begin by introducing the concept of a Courant algebra.

Definition 2.2. A *Courant algebra* over the Lie algebra $\mathfrak{g}$ is a vector space $\mathfrak{a}$ equipped with a skew-symmetric bracket $[\cdot, \cdot] : \mathfrak{a} \times \mathfrak{a} \longrightarrow \mathfrak{a}$, a symmetric bilinear operation $\theta : \mathfrak{a} \times \mathfrak{a} \longrightarrow \mathfrak{a}$, and a map $\pi : \mathfrak{a} \longrightarrow \mathfrak{g}$, which satisfy the following conditions for all $a_1, a_2, a_3 \in \mathfrak{a}$:

c1) $\pi([a_1,a_2]) = [\pi(a_1),\pi(a_2)]$,

c2) $[[a_1,a_2],a_3]+[[a_2,a_3],a_1]+[[a_3,a_1],a_2] = T(a_1,a_2,a_3)$,

c3) $\theta(a_1,a_2)\bullet a_3 = 0$,

c4) $\pi\circ\theta = 0$,

c5) $a_1\bullet\theta(a_2,a_3) = \theta(a_1\bullet a_2,a_3)+\theta(a_2,a_1\bullet a_3)$,

where T is given by

$$T(a_1,a_2,a_3) = \tfrac{1}{3}(\theta([a_1,a_2],a_3)+\theta([a_2,a_3],a_1)+\theta([a_3,a_1],a_2))$$

and $\bullet$ denotes the combination

$$a_1\bullet a_2 = [a_1,a_2]+\theta(a_1,a_2).$$

A Courant algebroid gives an example of a Courant algebra over $\mathfrak{g} = C^\infty(TM)$, taking $\mathfrak{a} = C^\infty(E)$ and $\theta(e_1,e_2) = \mathcal{D}\langle e_1,e_2\rangle$. Following Roytenberg-Weinstein [19], we now indicate that any Courant algebra is an example of a L_∞-algebra [15, 1].

Proposition 2.3. *Let $(\mathfrak{a},\pi,[\cdot,\cdot],\theta)$ be a Courant algebra, and let $\mathfrak{h} = \ker\pi$. Then the following is a L_∞ algebra $(V_\bullet,l_\bullet)$: take $V_1 = \mathfrak{h}$, $V_0 = \mathfrak{a}$, $V_i = 0$ $\forall i>1$, and let l_1 be the inclusion of V_1 into V_0. Then define l_2,l_3 by*

$$l_2(a_1,a_2) = [a_1,a_2],\quad l_2(a_1,h_1) = [a_1,h],\quad l_2(h_1,h_2) = 0,$$
$$l_3(a_1,a_2,a_3) = -T(a_1,a_2,a_3),\quad l_3(h_1,\cdot,\cdot) = 0,$$

for all $a_i\in\mathfrak{a}$, $h_i\in\mathfrak{h}$, and set all higher brackets l_i, $\forall i>3$ to zero.

Proof. We must check the higher Jacobi identities $\sum_{i+j=n+1}(-1)^{i(j-1)}l_jl_i = 0$, where l_i are extended as coderivations on $\wedge V_\bullet$. The identities for $n\neq 4$ are trivially verified from the definitions, whereas the identity $l_3l_2 = l_2l_3$ follows from the following identity: define

$$\mathbf{J} = \theta(J(a_1,a_2,a_3),a_4)-\theta(J(a_1,a_2,a_4),a_3)+\theta(J(a_1,a_3,a_4),a_2)-\theta(J(a_2,a_3,a_4),a_1)$$
$$\mathbf{K} = \theta([a_1,a_2],[a_3,a_4])-\theta([a_1,a_3],[a_2,a_4])+\theta([a_1,a_4],[a_2,a_3]),$$

where $J(a_1,a_2,a_3) = [[a_1,a_2],a_3]+[[a_2,a_3],a_1]+[[a_3,a_1],a_2]$ is the Jacobiator. Then using axioms $c2$) and $c5$) one can prove $\mathbf{K}+2\mathbf{J} = 0$, which together with the identity $[a_1,\theta(a_2,a_3)] = \theta(a_1,\theta(a_2,a_3))$ coming from $c3$), implies the result. ☐

It is easily seen that the image $\tilde{\mathfrak{g}} = \pi(\mathfrak{a})$ of the Courant algebra anchor is itself a Lie subalgebra $\tilde{\mathfrak{g}}\subset\mathfrak{g}$. Indeed the projection map $\pi:\mathfrak{a}\longrightarrow\tilde{\mathfrak{g}}$ defines a L_∞ isomorphism between $\mathfrak{a}$ and the usual Lie algebra $\tilde{\mathfrak{g}}$. In this sense a Courant algebra is nothing but a particular 2-term L_∞ representative of the Lie algebra $\tilde{\mathfrak{g}}$. It will be useful to consider the notion of an *exact* Courant algebra, where π is surjective onto $\mathfrak{g}$ and $\mathfrak{h}$ is abelian and isotropic:

Definition 2.4. An *exact* Courant algebra is one for which

$$0\longrightarrow\mathfrak{h}\longrightarrow\mathfrak{a}\xrightarrow{\pi}\mathfrak{g}\longrightarrow 0$$

is an exact sequence and such that $[h_1,h_2] = \theta(h_1,h_2) = 0$ for all $h_i\in\mathfrak{h} = \ker\pi$.

For an exact Courant algebra, one obtains immediately an action of $\mathfrak{g}$ on $\mathfrak{h}$: $g\in\mathfrak{g}$ acts on $h\in\mathfrak{h}$ via $g\cdot h = a\bullet h$, for any a such that $\pi(a) = g$. This is well defined, and it determines an action because of the Leibniz property of $\bullet$: for all $a_i\in\mathfrak{a}$,

$$a_1\bullet(a_2\bullet a_3) = (a_1\bullet a_2)\bullet a_3+a_2\bullet(a_1\bullet a_3),$$

which implies that, given $g_i\in\mathfrak{g}$ and $a_i\in\mathfrak{a}$ such that $\pi(a_i) = g_i$,

$$\begin{aligned}g_1\cdot(g_2\cdot h)-g_2\cdot(g_1\cdot h) &= a_1\bullet(a_2\bullet h)-a_2\bullet(a_1\bullet h)\\ &= ([a_1,a_2]+\theta(a_1,a_2))\bullet h\\ &= [a_1,a_2]\bullet h = [g_1,g_2]\cdot h,\end{aligned}$$

for all $h\in\mathfrak{h}$, proving that $\mathfrak{g}$ acts on $\mathfrak{h}$ as required. In fact there is a natural nontrivial exact Courant algebra associated with any $\mathfrak{g}$-module, as we now explain.

Example 2.5 (Demisemidirect product). Let $\mathfrak{g}$ be a Lie algebra acting on the vector space $\mathfrak{h}$. Then $\mathfrak{a} = \mathfrak{g} \oplus \mathfrak{h}$ becomes a Courant algebra over $\mathfrak{g}$ via the bracket

$$[(g_1, h_1), (g_2, h_2)] = ([g_1, g_2], \tfrac{1}{2}(g_1 \cdot h_2 - g_2 \cdot h_1)), \tag{8}$$

and the bilinear operation

$$\theta((g_1, h_1), (g_2, h_2)) = (0, \tfrac{1}{2}(g_1 \cdot h_2 + g_2 \cdot h_1)), \tag{9}$$

where here $g \cdot h$ denotes the $\mathfrak{g}$-action. This bracket has appeared before in the context of Leibniz algebras [14], where it was called the *demisemidirect* product, due to the factor of $\frac{1}{2}$. Note that in [24], Weinstein studied the case where $\mathfrak{g} = \mathfrak{gl}(V)$ and $\mathfrak{h} = V$, and called it an *omni-Lie algebra* due to the fact that, when $\dim V = n$, any n-dimensional Lie algebra can be embedded inside $\mathfrak{g} \oplus \mathfrak{h}$ as an involutive subspace.

Morphisms between Courant algebras are simply structure-preserving chain homomorphisms, as we now describe.

Definition 2.6. A morphism of Courant algebras from $(\mathfrak{a} \xrightarrow{\pi} \mathfrak{g}, [\cdot,\cdot], \theta)$ to $(\mathfrak{a}' \xrightarrow{\pi'} \mathfrak{g}', [\cdot,\cdot]', \theta')$ is a commutative square

$$\begin{CD}
\mathfrak{a} @>{\pi}>> \mathfrak{g}\\
@V{\rho}VV @VV{\psi}V\\
\mathfrak{a}' @>{\pi'}>> \mathfrak{g}'
\end{CD}$$

where ψ is a Lie algebra homomorphism, $\rho([a_1, a_2]) = [\rho(a_1), \rho(a_2)]'$ and $\rho(\theta(a_1, a_2)) = \theta'(\rho(a_1), \rho(a_2))$ for all $a_i \in \mathfrak{a}$. Note that a morphism of Courant algebras induces a chain homomorphism of associated chain complexes $\mathfrak{h} \longrightarrow \mathfrak{a} \xrightarrow{\pi} \mathfrak{g}$.

Example 2.7 (The adjoint action). The adjoint action $\mathrm{ad}_a(b) = a \bullet b$ defines a morphism $a \mapsto \mathrm{ad}_a$ of Courant algebras from any Courant algebra $\mathfrak{a}$ over $\mathfrak{g}$ to its Lie algebra $\mathrm{Der}\,\mathfrak{a}$ of symmetries, the latter viewed as a Courant algebra over $\mathrm{Der}\,\mathfrak{g}$ with $\theta = 0$. The following diagram describes the morphism:

$$\begin{CD}
\mathfrak{a} @>{\pi}>> \mathfrak{g}\\
@V{\mathrm{ad}}VV @VV{\mathrm{ad}}V\\
\mathrm{Der}\,\mathfrak{a} @>{\pi_*}>> \mathrm{Der}\,\mathfrak{g}
\end{CD}$$

We now have all we need to define the extension of a G-action to a Courant algebroid E.

Definition 2.8 (Extended action). Let G be a connected Lie group acting on a manifold M with infinitesimal action $\psi : \mathfrak{g} \longrightarrow C^\infty(TM)$. An extension of this action to a Courant algebroid E over M is an exact Courant algebra $\mathfrak{a}$ over $\mathfrak{g}$ together with a Courant morphism $\rho : \mathfrak{a} \longrightarrow C^\infty(E)$:

$$\begin{CD}
0 @>>> \mathfrak{h} @>>> \mathfrak{a} @>>> \mathfrak{g} @>>> 0\\
@. @. @V{\rho}VV @VV{\psi}V @.\\
@. @. C^\infty(E) @>>> C^\infty(TM) @.
\end{CD}$$

which is such that $\mathfrak{h}$ acts trivially, i.e. $(\mathrm{ad} \circ \rho)(\mathfrak{h}) = 0$, and the induced action of $\mathfrak{g} = \mathfrak{a}/\mathfrak{h}$ on $C^\infty(E)$ integrates to a G-action on the total space of E.

Suppose now that the Courant algebroid in question is exact, as it will be in many cases of interest. Then an extended action is a chain homomorphism

$$\begin{CD}
0 @>>> \mathfrak{h} @>>> \mathfrak{a} @>>> \mathfrak{g} @>>> 0\\
@. @V{\nu}VV @V{\rho}VV @VV{\psi}V @.\\
0 @>>> C^\infty(T^*M) @>>> C^\infty(E) @>>> C^\infty(TM) @>>> 0
\end{CD}$$

such that $\mathfrak{h}$ acts trivially, which occurs precisely when it acts via *closed* 1-forms, i.e. $\nu(\mathfrak{h}) \subset \Omega^1_{cl}(M)$. Furthermore the induced $\mathfrak{g}$-action on E must integrate to a G-action (a priori, one has only the action of the universal cover of G). In order to make this condition more concrete, we observe that since we already know that the $\mathfrak{g}$-action on TM integrates to a G-action, one needs only to find a $\mathfrak{g}$-invariant splitting of E to guarantee that it is a G-bundle, as the splitting $E = TM \oplus T^*M$ carries a canonical G-equivariant structure.

Proposition 2.9. *Let the Lie group G act on the manifold M, and let $\mathfrak{a} \xrightarrow{\pi} \mathfrak{g}$ be an exact Courant algebra with a morphism ρ to an exact Courant algebroid E over M such that $\nu(\mathfrak{h}) \subset \Omega^1_{cl}(M)$.*

If E has a $\mathfrak{g}$-invariant splitting, then the $\mathfrak{g}$-action on E integrates to an action of G, and hence ρ is an extended action of G on E. Conversely, if G is compact and ρ is an extended action, then by averaging splittings one can always find a $\mathfrak{g}$-invariant splitting of E.

The condition that a splitting is $\mathfrak{g}$-invariant can be expressed more concretely as follows. As shown in Section 2.1, a split exact Courant algebroid is isomorphic to the direct sum $TM \oplus T^*M$, equipped with the H-twisted Courant bracket for a closed 3-form H. In this splitting, therefore, for each $a \in \mathfrak{a}$ the section $\rho(a)$ decomposes as $\rho(a) = X_a + \xi_a$, and it acts via $(X_a + \xi_a) \bullet (Y + \eta) = [X_a, Y] + \mathcal{L}_{X_a}\eta - i_Y d\xi_a + i_Y i_{X_a} H$, or as a matrix,

$$ad_{\rho(a)} = \begin{pmatrix} \mathcal{L}_{X_a} & 0 \\ i_{X_a}H - d\xi_a & \mathcal{L}_{X_a} \end{pmatrix}$$

We see immediately from this that the splitting is preserved by this action if and only if for each $a \in \mathfrak{a}$,

$$i_{X_a}H - d\xi_a = 0. \tag{10}$$

Example 2.10. Let G be compact, and consider the question of *trivially* extending a G-action, so that the extending Courant algebra is simply $\mathfrak{g}$ itself:

$$\begin{array}{ccccccccc} & & & & \mathfrak{g} & \xrightarrow{\mathrm{id}} & \mathfrak{g} & & \\ & & & & \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\psi} & & \\ 0 & \longrightarrow & C^\infty(T^*) & \longrightarrow & C^\infty(E) & \longrightarrow & C^\infty(TM) & \longrightarrow & 0 \end{array}$$

By Proposition 2.9, we can always express such an extension in terms of a $\mathfrak{g}$-invariant splitting, so finding such a morphism ρ is equivalent to finding 1-forms ξ_a such that $\rho : a \mapsto X_a + \xi_a$, where $X_a = \psi(a)$, preserves the Courant algebra structure: preserving the bracket yields

$$\xi_{[a,b]} = \mathcal{L}_{X_a}\xi_b - \mathcal{L}_{X_b}\xi_a - \tfrac{1}{2}d(i_{X_a}\xi_b - i_{X_b}\xi_a) + i_{X_b}i_{X_a}H, \tag{11}$$

whereas preserving the symmetric form θ yields

$$d\langle X_a + \xi_a, X_b + \xi_b\rangle = 0. \tag{12}$$

Also, by (10), we must have $i_{X_a}H - d\xi_a = 0$. These conditions can be phrased in terms of the Cartan model for G-equivariant cohomology. Recall that the Cartan complex of equivariant forms is the algebra of equivariant polynomial functions $\Phi : \mathfrak{g} \longrightarrow \Omega^\bullet(M)$:

$$\Omega^k_G(M) = \bigoplus_{2p+q=k} (S^p\mathfrak{g}^* \otimes \Omega^q(M))^G,$$

and the equivariant derivative d_G is defined by

$$(d_G\Phi)(a) = d(\Phi(a)) - i_{X_a}\Phi(a) \quad \forall a \in \mathfrak{g}.$$

Now consider the form $\Phi = H + \xi_a$, where the subscript in ξ_a is viewed as a variable so that $\xi_a \in \mathfrak{g}^* \otimes \Omega^1(M)$. Since the splitting is G-invariant, we have $\mathcal{L}_{X_a}H = 0$, and from (11) we see that $\xi_{[a,b]} = \mathcal{L}_{X_a}\xi_b$. Therefore Φ is an equivariant 3-form. Furthermore, equation (12) shows that $c_{ab} = \langle X_a + \xi_a, X_b + \xi_b\rangle$ is constant, i.e. a closed equivariant 4-form. Finally, computing $d_G\Phi$, we obtain

$$d_G\Phi = -c_{ab} = -\langle X_a + \xi_a, X_b + \xi_b\rangle.$$

That is, c_{ab} is exact. Note that this implies c_{ab} is an invariant symmetric form on the Lie algebra $\mathfrak{g}$.

Let ρ, ρ' be two trivially extended actions on the same exact Courant algebroid, with identical symmetric forms $\langle\rho(a), \rho(b)\rangle = \langle\rho'(a), \rho'(b)\rangle$. We call them *equivalent* if the closed form $\rho'(a) - \rho(a)$ is exact; more precisely, if $(\rho' - \rho)(a) = df_a$, for an equivariant map $f : M \longrightarrow \mathfrak{g}^*$. If ρ, ρ' are described by the forms Φ, Φ' in G-invariant splittings, then $d_G(\Phi' - \Phi) = 0$. If furthermore $\Phi' - \Phi = d_G\beta$ for $\beta = b + f_a \in \Omega^2_G(M)$, then the G-invariant 2-form b describes the difference between the splittings and the equivariant map f_a satisfies $\rho'(a) - \rho(a) = d(f_a)$. Hence we see that trivially extended actions are equivalent when their representative forms differ by exact forms.

In the case of trivially extended actions, we obtain conditions similar to those considered by physicists studying the gauging of sigma models with Wess-Zumino term [13], except that we do not require Φ to be equivariantly closed at this stage.

We summarize the results of the previous example in the following theorem.

Theorem 2.11. *Let G be a compact Lie group. Then trivially extended G-actions on a fixed exact Courant algebroid with prescribed symmetric form $c_{ab} = \langle\rho(a), \rho(b)\rangle$ are, up to equivalence, in bijection with solutions to $d_G(H + \xi_a) = c_{ab}$ modulo d_G-exact forms, where $[H] \in H^3(M, \mathbb{R})$ is the Ševera class of the Courant algebroid.*

To obtain similar conditions for more general extended actions where $\mathfrak{h} \neq 0$, we would need an extension of the Cartan model for equivariant cohomology, something which we leave for future work. In the remainder of this section we describe the notion of *moment map* for an extended action.

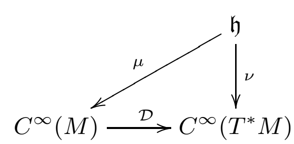
2.3 Moment maps for extended actions

Suppose that we have an extended G-action on an exact Courant algebroid as in the previous section, so that we have the map $\nu : \mathfrak{h} \longrightarrow \Omega^1_{cl}(M)$. Because the action is a Courant algebra morphism, this map is $\mathfrak{g}$-equivariant in the sense

$$\nu(g \cdot h) = \mathcal{L}_{\psi(g)}\nu(h). \tag{13}$$

Therefore we are led naturally to the definition of a moment map for this extended action, as an equivariant factorization of μ through the smooth functions.

Definition 2.12. A *moment map* for an extended $\mathfrak{g}$-action on an exact Courant algebroid is a $\mathfrak{g}$-equivariant map $\mu : \mathfrak{h} \longrightarrow C^\infty(M, \mathbb{R})$ satisfying $\mathcal{D} \circ \mu = \nu$, i.e. such that the following diagram commutes:



Note that μ may be alternatively viewed as an equivariant map $\mu : M \longrightarrow \mathfrak{h}^*$.

A moment map can be found only if two obstructions vanish. The first one is the induced map to cohomology $\nu_* : \mathfrak{h} \longrightarrow H^1(M, \mathbb{R})$. Since (13) implies that ν_* always vanishes on $\mathfrak{g} \cdot \mathfrak{h} \subset \mathfrak{h}$, the first obstruction may be defined as an element

$$o_1 \in H^0(\mathfrak{g}, \mathfrak{h}^*) \otimes H^1(M, \mathbb{R}),$$

where the first term denotes Lie algebra cohomology with values in the module $\mathfrak{h}^*$. When this obstruction vanishes we may choose a lift $\tilde{\mu} : \mathfrak{h} \longrightarrow C^\infty(M)$. The second obstruction results from the failure of this lift to be equivariant: consider the quantity $c(g, h) = \tilde{\mu}(g \cdot h) - \mathcal{L}_{\psi(g)}\tilde{\mu}(h)$ for $g \in \mathfrak{g}$, $h \in \mathfrak{h}$. From (13) we conclude that c is a constant function along M. It is easily shown that this discrepancy, modulo changes of lift, defines an obstruction class

$$o_2 \in H^1(\mathfrak{g}, \mathfrak{h}^*).$$

Proposition 2.13. *A moment map for an extended $\mathfrak{g}$-action exists if and only if the obstructions $o_1 \in H^0(\mathfrak{g}, \mathfrak{h}^*) \otimes H^1(M, \mathbb{R})$ and $o_2 \in H^1(\mathfrak{g}, \mathfrak{h}^*)$ vanish. When it exists, a moment map is unique up to the addition of an element $\lambda \in \mathrm{Ann}(\mathfrak{g} \cdot \mathfrak{h}) \subset \mathfrak{h}^*$.*

We now show how the usual notions of symplectic and Hamiltonian actions fit into the framework of extended actions of Courant algebras.

Example 2.14 (Symplectic actions). Let G be a Lie group acting on a symplectic manifold (M, ω) preserving the symplectic form, and let $\psi : \mathfrak{g} \longrightarrow C^\infty(TM)$ denote the infinitesimal action. We now show that there is a natural extended action of the Courant algebra associated to the adjoint action on the standard Courant algebroid $TM \oplus T^*M$ with zero twist $H = 0$. As described in Example 2.5, the Courant algebra is described by the sequence

$$0 \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g} \oplus \mathfrak{g} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

and is equipped with the bracket

$$[(g_1, h_1), (g_2, h_2)] = ([g_1, g_2], \tfrac{1}{2}([g_1, h_2] - [g_2, h_1])), \tag{14}$$

and the bilinear operation

$$\theta((g_1, h_1), (g_2, h_2)) = (0, \tfrac{1}{2}([g_1, h_2] + [g_2, h_1])). \tag{15}$$

We now claim that this Courant algebra acts naturally on $TM \oplus T^*M$. Let $X_g = \psi(g)$, for $g \in \mathfrak{g}$, denote the symplectic vector fields. Then we define the action $\rho : \mathfrak{g} \oplus \mathfrak{g} \longrightarrow C^\infty(TM \oplus T^*M)$ by

$$\rho(g, h) = X_g + i_{X_h}\omega,$$

where ω is the symplectic form. It is enough to verify that the pairing $\bullet$ is preserved; on the Courant algebra it is simply

$$(g_1, h_1) \bullet (g_2, h_2) = ([g_1, g_2], [g_1, h_2]),$$

whereas in $TM \oplus T^*M$ we have

$$(X_{g_1} + i_{X_{h_1}}\omega) \bullet (X_{g_2} + i_{X_{h_2}}\omega) = [X_{g_1}, X_{g_2}] + \mathcal{L}_{X_{g_1}} i_{X_{h_2}}\omega = X_{[g_1,g_2]} + i_{X_{[g_1,h_2]}}\omega,$$

showing that ρ is a Courant morphism.

The question of finding a moment map for this extended action then becomes one of finding an equivariant map $\mu : \mathfrak{g} \longrightarrow C^\infty(M)$ such that

$$d(\mu_g) = i_{X_g}\omega.$$

Hence we recover the usual notion of moment map for a Hamiltonian action on a symplectic manifold.

Note that in this formalism, the notion of moment map is no longer tied to the geometrical structure in question, such as the symplectic form. Instead, it is a constituent of the extended action. In fact, given an equivariant map $\mu : M \longrightarrow \mathfrak{h}^*$ for a $\mathfrak{g}$-module $\mathfrak{h}$, one can naturally construct an extended action for which μ is a moment map, as we now indicate.

Proposition 2.15. *Given a $\mathfrak{g}$-equivariant map $\mu : M \longrightarrow \mathfrak{h}^*$ where M is a G-space and $\mathfrak{h}$ a $\mathfrak{g}$-module, there is an induced extended action of the Courant algebra $\mathfrak{g} \oplus \mathfrak{h}$ with bracket (8) and symmetric product (9) on the exact Courant algebroid $TM \oplus T^*M$ with $H = 0$, given by*

$$\rho : (g, h) \mapsto X_g + d(\mu_h),$$

where as before $X_g = \psi(g)$ is the infinitesimal $\mathfrak{g}$-action.

More generally, given a trivially extended action $\rho : \mathfrak{g} \longrightarrow C^\infty(E)$ on an exact Courant algebroid, it can be extended to an action of $\mathfrak{g} \oplus \mathfrak{h}$ as above by any equivariant map $\mu : M \longrightarrow \mathfrak{h}^*$ via the same formula

$$\tilde{\rho} : (g, h) \mapsto \rho(g) + d(\mu_h).$$

3 Reduction of Courant algebroids

In this section we develop a reduction procedure for Courant algebroids which is analogous to the usual notion of symplectic reduction due to Marsden and Weinstein [18]. In fact the procedure we describe may be interpreted as an "odd" symplectic reduction, since an exact Courant algebroid, with its split-signature symmetric inner product, may be viewed as an odd symplectic bundle.

A characteristic feature of the reduction procedure is that an extended G-action on an exact Courant algebroid E over a manifold M does *not* necessarily induce an exact Courant algebroid on M/G, but rather one may need to pass to a submanifold $P \subset M$ which is suitably chosen. Then $P/G = M_{red}$, the *reduced space*, obtains naturally an exact Courant algebroid. This is directly analogous to the well-known fact that, for a symplectic manifold M, M/G inherits a Poisson structure whose leaves are the symplectic reduced spaces.

Also, we shall see that the reduction procedure is quite straightforward when the image of the extended action $\rho : \mathfrak{a} \longrightarrow C^\infty(E)$ is *isotropic*; if this is not the case, the procedure is more subtle and requires an additional condition to be satisfied by the action.

3.1 Reduction procedure

In the previous section we showed how a G-action on a manifold M could be extended to a Courant algebroid E, making it an equivariant G-bundle in such a way that the Courant structure is preserved by the G-action. Therefore, assuming the G-action on the base were appropriately well-behaved, E would descend to the quotient, yielding a Courant algebroid $E/G \longrightarrow M/G$. However, E/G would certainly not be an exact Courant algebroid, since its rank would be too large. We will show that the image of the extended action itself determines an equivariant sub-bundle whose quotient becomes an exact Courant algebroid when restricted to certain submanifolds of M/G, called the *reduced* manifolds. In this way we obtain a reduction of exact Courant algebroids.

The basic idea is to consider the two natural distributions in E determined by the extended action, which may be viewed as a bundle map $\rho : \mathfrak{a} \times M \longrightarrow E$. The image of this map is a distribution $K \subset E$, and its orthogonal complement is a second distribution $K^\perp \subset E$. These distributions are G-invariant, since the action of $g \in \mathfrak{g}$ on any generating section $\rho(a)$ of K is simply $\rho(\tilde{g}) \bullet \rho(a)$, for any lift $\tilde{g} \in \mathfrak{a}$, $\pi(\tilde{g}) = g$, and we have

$$g \cdot \rho(a) = \rho(\tilde{g}) \bullet \rho(a) = \rho(\tilde{g} \bullet a) \in K.$$

The other crucial observation is that the space of G-invariant sections of $K^\perp$ is closed under the Courant bracket, since for any sections $v_1, v_2 \in C^\infty(K^\perp)^G$, we have, for all $a \in \mathfrak{a}$,

$$\begin{aligned} \langle \rho(a), [\![v_1, v_2]\!] \rangle &= \tfrac{1}{2} \langle \rho(a), v_1 \bullet v_2 - v_2 \bullet v_1 \rangle \\ &= \tfrac{1}{2} (\pi(v_1)\langle \rho(a), v_2 \rangle - \langle v_1 \bullet \rho(a), v_2 \rangle - \pi(v_2)\langle \rho(a), v_1 \rangle + \langle v_2 \bullet \rho(a), v_1 \rangle) \\ &= \langle d\langle \rho(a), v_2 \rangle, v_1 \rangle - \langle d\langle \rho(a), v_1 \rangle, v_2 \rangle = 0. \end{aligned} \tag{16}$$

While the G-invariant sections of $K^\perp$ inherit a Courant bracket, the induced inner product may be degenerate, with kernel consisting of $C^\infty(K \cap K^\perp)^G$. The latter space actually forms a Courant ideal as a result of the following fact: for any $w = \sum f_i \rho(a_i) \in C^\infty(K)$ and $v \in C^\infty(K^\perp)^G$, we have

$$\begin{aligned} [\![w, v]\!] &= (\sum f_i \rho(a_i)) \bullet v - d\langle w, v \rangle \\ &= \sum f_i (\rho(a_i) \bullet v) + (\pi(v) f_i) \rho(a_i) \\ &= \sum (\pi(v) f_i) \rho(a_i) \end{aligned} \tag{17}$$

which clearly is in $C^\infty(K)$. Hence the quotient space

$$\frac{C^\infty(K^\perp)^G}{C^\infty(K \cap K^\perp)^G} \tag{18}$$

inherits both a bracket and a nondegenerate inner product with values in the G-invariant functions. Assuming that K as well as $K \cap K^\perp$ were of constant rank, (18) would define a Courant algebroid structure on $\frac{K^\perp}{K\cap K^\perp}$ over M/G. However the anchor would not be surjective since $K^\perp$ does not necessarily project surjectively to $T(M/G)$. Hence we are led to consider restricting to G-invariant submanifolds $P \subset M$ for which $K^\perp$ does project surjectively to $T(P/G)$. For this purpose we introduce two natural distributions in TM induced by the extended action.

Definition 3.1. Given an extended action with image distribution $\rho(\mathfrak{a}) = K \subset E$, define the *big distribution* $\Delta_b = \pi(K + K^\perp) \subset TM$ and the *small distribution* $\Delta_s = \pi(K^\perp) \subset TM$. These are G-invariant distributions.

In the presence of a moment map $\mu : M \longrightarrow \mathfrak{h}^*$, Δ_s is the distribution tangent to the level sets, whereas Δ_b is the distribution tangent to the G-orbits of the level sets.

Note that $\pi(K)$ is the distribution tangent to the G-orbits in M, and we wish to consider (18) restricted to submanifolds $P \subset M$ such that $TP = \pi(K + K^\perp)$, i.e., leaves of the big distribution Δ_b. We now describe the leaves of Δ_b. Observe first that the small distribution $\Delta_s = \pi(K^\perp)$ is such that

$$\mathrm{Ann}(\Delta_s) = K \cap T^* = \rho(\mathfrak{h}). \tag{19}$$

Therefore, wherever $\rho(\mathfrak{h})$ has locally constant rank, Δ_s is an integrable distribution. The G-orbit of any leaf of Δ_s (if smooth) is then a leaf of Δ_b. This observation allows us to prove the following useful lemma. Note that a *leaf* of a distribution is taken to mean a maximal connected integral submanifold.

Lemma 3.2. *Let $P \subset M$ be a leaf of the big distribution Δ_b on which G acts freely and properly, and suppose $\rho(\mathfrak{h})$ has constant rank along P. Then K and $K \cap K^\perp$ both have constant rank along P.*

Proof. Since G acts freely on P, $\psi(\mathfrak{g})$ has constant rank along P. Since $\rho(\mathfrak{h})$ also has constant rank along P, it follows that $\rho(\mathfrak{a}) = K$ has constant rank along P. Furthermore, by (19), the small distribution $\Delta_s \subset TP$ is integrable, and P is the G-orbit of a leaf S of Δ_s. However, because ρ is a Courant morphism, we have for all $a, b \in \mathfrak{a}$,

$$\rho(\theta(a, b)) = d\langle\rho(a), \rho(b)\rangle, \tag{20}$$

and since $\theta(a, b) \in \mathfrak{h}$, we see that $\langle\rho(a), \rho(b)\rangle$ is constant along S. Hence over this leaf we obtain an induced inner product on $\mathfrak{a}$ whose null space, modulo $\ker \rho|_{\mathfrak{h}}$, maps injectively to $K \cap K^\perp$. Hence $K \cap K^\perp$ has constant rank along S. But $K \cap K^\perp$ is G-invariant and hence it must have constant rank over the entire big leaf P. □

These arguments suggest that along a big leaf P over which $\rho(\mathfrak{h})$ has constant rank, and on which G acts freely and properly, the space

$$\mathcal{E}_{red} := \frac{C^\infty(K^\perp|_P)^G}{C^\infty(K \cap K^\perp|_P)^G} \tag{21}$$

could be identified with the space of sections $C^\infty(E_{red})$ of a reduced Courant algebroid defined over P/G. However we must explain why the Courant bracket remains well-defined for sections which are supported only along P.

Theorem 3.3. *Let $P \subset M$ be a leaf of Δ_b on which G acts freely and properly, and over which $\rho(\mathfrak{h})$ has constant rank. Then $\mathcal{E}_{red}$ defines a reduced Courant algebroid E_{red} over $M_{red} = P/G$ with surjective anchor. If K is isotropic then E_{red} is an exact Courant algebroid; in general, it is exact if and only if the following holds along P:*

$$\pi(K) \cap \pi(K^\perp) = \pi(K \cap K^\perp). \tag{22}$$

Proof. By Lemma 3.2, K and $K \cap K^\perp$ are G-invariant bundles over P, and hence $\mathcal{E}_{red} = C^\infty(E_{red})$ for the vector bundle

$$E_{red} = \left.\frac{K^\perp|_P}{K \cap K^\perp|_P}\right/ G,$$

defined over $M_{red} = P/G$. The bracket on sections $v_1, v_2 \in C^\infty(E_{red})$ is defined by choosing representative G-invariant sections of $K^\perp$ over P, choosing extensions $\tilde{v}_1, \tilde{v}_2 \in C^\infty(M, E)$ of these, and restricting $[\![\tilde{v}_1, \tilde{v}_2]\!]$ to P. This is a section of $K^\perp$ along P by a similar calculation to that in equation (16):

$$\langle\rho(a), [\![\tilde{v}_1, \tilde{v}_2]\!]\rangle|_P = \tfrac{1}{2}(\langle\rho(a) \bullet \tilde{v}_1, \tilde{v}_2\rangle - \langle\rho(a) \bullet \tilde{v}_2, \tilde{v}_1\rangle),$$

which vanishes along P since $\tilde{v}_i$ are invariant sections of $K^\perp$ there. To describe the dependence of $[\![\tilde{v}_1, \tilde{v}_2]\!]$ with respect to the extensions chosen, note that if $(\tilde{v}_2' - \tilde{v}_2)|_P = 0$, we can write $\tilde{v}_2' - \tilde{v}_2 = fs$ for $f \in C^\infty(M, \mathbb{R})$ with $f(P) = 0$ and $s \in C^\infty(M, E)$. Then the change in the bracket is

$$[\![\tilde{v}_1, fs]\!] = f[\![\tilde{v}_1, s]\!] + (\pi(\tilde{v}_1)f)s - \langle \tilde{v}_1, s\rangle df.$$

The first two terms vanish upon restriction to P, since $f(P) = 0$ and $\pi(\tilde{v}_1)$ is tangent to P there. Since $df \in \mathrm{Ann}(TP)$ and $\mathrm{Ann}(TP) = \mathrm{Ann}(\pi(K + K^\perp)) = K \cap K^\perp \cap T^*$, the third term is an invariant section of $K \cap K^\perp$ along P. On the other hand, a calculation just as in (17) shows that if $w \in C^\infty(K \cap K^\perp|_P)^G$ and $v \in C^\infty(K^\perp|_P)^G$, then

$$[\![\tilde{w}, \tilde{v}]\!]|_P \in C^\infty(K \cap K^\perp|_P)^G,$$

where $\tilde{w}, \tilde{v}$ are arbitrary extensions. As a result, the bracket on $\mathcal{E}_{red}$ is well-defined, and we obtain a Courant algebroid E_{red} over $M_{red} = P/G$ whose anchor is surjective by construction.

The Courant algebroid E_{red} is exact if and only if the kernel of its anchor is isotropic. Along P this can be expressed as the condition that $\{v \in K^\perp : \pi(v) \in \pi(K)\}$ be isotropic in E. This happens if and only if $\pi(K \cap K^\perp) = \pi(K) \cap \pi(K^\perp)$ in TP. If K itself was isotropic, then $K < K^\perp$, and hence the condition would be automatically satisfied. □

Since the tangent bundle of the reduced manifold M_{red} is identified with $(\Delta_b/\psi(\mathfrak{g}))/G$, we see that E_{red}, as constructed in the preceding theorem, can be expressed in the following sequence, assuming that condition (22) is satisfied:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T^*M_{red} & \longrightarrow & E_{red} & \longrightarrow & TM_{red} & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \frac{K^\perp \cap T^*}{K \cap K^\perp \cap T^*}\Big/G & \longrightarrow & \frac{K^\perp}{K \cap K^\perp}\Big/G & \longrightarrow & \frac{\pi(K + K^\perp)}{\pi(K)}\Big/G & \longrightarrow & 0 \end{array} \tag{23}$$

An important special case of the reduction procedure is that of a trivially extended action, as described in Example 2.10. We show that in this case, condition (22) is precisely the requirement that the action is isotropic, i.e. $K \subset K^\perp$.

Example 3.4. Let $\rho : \mathfrak{g} \longrightarrow C^\infty(E)$ be a trivially extended action of the free and proper action of G on the manifold M, so that $\mathfrak{h} = \{0\}$. Then by equation (19), we obtain $\pi(K^\perp) = TM$, and in particular, $\Delta_s = \Delta_b = TM$. Hence by Theorem 3.3, we obtain an exact reduced Courant algebroid E_{red} over $M_{red} = M/G$ if and only if $\pi(K) = \pi(K \cap K^\perp)$, which occurs if and only if $K \subset K^\perp$, since $K \cap T^* = \{0\}$. This provides an alternate motivation for the requirement in [13] that K be isotropic.

It is possible to use this last example to clarify condition (22) in the general case, essentially by restricting first to a small leaf $S \subset M$ where one obtains a restricted action of a smaller group, such that $\mathfrak{h} = \{0\}$. We now explain how this is done.

Exact Courant algebroids may always be *pulled back* to submanifolds; for any submanifold $f : S \hookrightarrow M$ of a manifold with exact Courant algebroid E, we have the isotropic subbundle $\mathrm{Ann}(TS) \subset T^* \subset E$. We may then form the quotient

$$f^*E = \frac{(\mathrm{Ann}(TS))^\perp}{\mathrm{Ann}(TS)} = \frac{\pi^{-1}(TS)}{\mathrm{Ann}(TS)}, \tag{24}$$

which becomes an exact Courant algebroid over S, inheriting a bracket by restriction just as in the proof of Theorem 3.3.

As we saw from (20), $\langle \rho(a), \rho(b)\rangle$ is constant along the small leaf S and induces a symmetric bilinear form on the Courant algebra $\mathfrak{a}$, for which $\mathfrak{h}$ is isotropic. Therefore we may define $\mathfrak{a}_s = \mathfrak{h}^\perp$ and $\mathfrak{g}_s = \pi(\mathfrak{a}_s)$, noting that $\mathfrak{a}_s$ is closed under the Courant bracket. This implies that $\mathfrak{g}_s$ is a Lie subalgebra of $\mathfrak{g}$, which we call the *isotropy subalgebra*, and it inherits a symmetric bilinear form $c_s \in S^2(\mathfrak{g}_s^*)$ by construction. Therefore we obtain the sub-Courant algebra

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{a}_s \xrightarrow{\pi} \mathfrak{g}_s \longrightarrow 0\ ,$$

which is mapped via the extended action ρ into $\pi^{-1}(TS)$. Quotienting by $\mathfrak{h}$, we obtain a trivially extended action ρ_s of the isotropy subalgebra on the pullback Courant algebra f^*E over S,

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & \mathfrak{g}_s & \xrightarrow{\pi} & \mathfrak{g}_s & \longrightarrow & 0 \\ & & & & \downarrow{\rho_s} & & \downarrow{\psi} & & \\ 0 & \longrightarrow & C^\infty(T^*S) & \longrightarrow & C^\infty(f^*E) & \longrightarrow & C^\infty(TS) & \longrightarrow & 0 \end{array}$$

which satisfies $\langle \rho_s(a), \rho_s(b) \rangle = c_s(a,b)$ by construction. Note that the underlying group action on S is by the subgroup $G_s \subset G$ stabilizing S, which we call the *isotropy subgroup*. Also there is a natural isomorphism $S/G_s \longrightarrow P/G$ if P is a leaf of Δ_b containing S and satisfying the conditions of Theorem 3.3.

These arguments show that after pullback to S, we obtain a trivially extended action as in Example 3.4. We now show that the quotient of this pullback is naturally isomorphic to the quotient Courant algebroid E_{red} constructed in Theorem 3.3. We may then conclude that E_{red} is exact if and only if the action ρ_s is isotropic, i.e. $\rho_s(\mathfrak{g}_s) \subset \rho_s(\mathfrak{g}_s)^\perp$.

Proposition 3.5. *Let P be as in Theorem 3.3, and let $f : S \hookrightarrow P$ be a leaf of Δ_s. Then the reduced Courant algebroid E_{red} over P/G is naturally isomorphic to the quotient of the pullback f^*E by the isotropy action ρ_s. In particular, E_{red} is exact if and only if ρ_s is isotropic, i.e. $c_s \in S^2(\mathfrak{g}_s^*)$ vanishes.*

Proof. The image of the isotropy action ρ_s in f^*E is given by

$$K_s = \frac{K \cap (K^\perp + T^*)}{K \cap T^*} \subset f^*E = \frac{K^\perp + T^*}{K \cap T^*}.$$

Then the reduced Courant algebroid over S/G_s is the G_s quotient of the bundle

$$\frac{K_s^\perp}{K_s \cap K_s^\perp} = \frac{(K^\perp + K \cap T^*)/K \cap T^*}{(K \cap K^\perp + K \cap T^*)/K \cap T^*},$$

which is canonically isomorphic to $E_{red} = (K^\perp / K \cap K^\perp)/G$ as a Courant algebroid. Since ρ_s is a trivially extended action, we conclude from Example 3.4 that E_{red} is exact if and only if K_s is isotropic in f^*E, a condition equivalent to the requirement that $\tilde{K} \subset E$ is isotropic along P, where

$$\tilde{K} = K \cap (K^\perp + T^*). \tag{25}$$

□

In the presence of a moment map $\mu : M \longrightarrow \mathfrak{h}^*$ for the generalized action, the moment map condition $d(\mu_h) = \nu(h)$ implies that

$$\ker(d\mu) = \mathrm{Ann}(\nu(\mathfrak{h})) = \Delta_s,$$

so that the leaves of the small distribution Δ_s are precisely the level sets $\mu^{-1}(\lambda)$ of the moment map. Similarly the leaves of the big distribution are inverse images $\mu^{-1}(\mathcal{O}_\lambda)$ of orbits $\mathcal{O}_\lambda \subset \mathfrak{h}^*$ of the action of G. The small leaf $S = \mu^{-1}(\lambda)$ then has isotropy Lie algebra $\mathfrak{g}_s = \mathfrak{g}_\lambda$, which is the Lie algebra of G_λ, the subgroup stabilizing λ under the action of G on $\mathfrak{h}^*$. Applying Theorem 3.3 together with Proposition 3.5, we obtain the following formulation of the reduction procedure:

Proposition 3.6 (Moment map reduction). *Let the extended action ρ on the Courant algebroid E have moment map μ. Then the reduced Courant algebroid associated to the regular value $\lambda \in \mathfrak{h}^*$ is obtained via pullback f^*E along $f : \mu^{-1}(\lambda) \hookrightarrow M$, followed by reduction by the isotropy action ρ_λ of G_λ on the level set, which we assume is free and proper. The result is an exact Courant algebroid if and only if ρ_λ is isotropic, i.e. the induced symmetric form $c_\lambda \in S^2(\mathfrak{g}_\lambda^*)$ vanishes.*

The reduced Courant algebroid E_{red} constructed in this section depends upon a choice of leaf $P \subset M$ of Δ_b or equivalently a leaf S of Δ_s. We remark here that if G acts freely and properly on the entire manifold M and $\rho(\mathfrak{h})$ has constant rank on M, then by (23), M/G has a singular foliation by smooth reduced manifolds M_{red}, given by the generalized distribution

$$\frac{\pi(K+K^\perp)}{\pi(K)}\Big/ G \subset T(M/G). \tag{26}$$

One sees that (26) is integrable since the leaves P of the big distribution are simply the G-orbits of the leaves S of Δ_s, and since $TS\cap\psi(\mathfrak{g}) = \psi(\mathfrak{g}_s)$ has constant rank along S, both P and its quotient $M_{red} = P/G$ are smooth manifolds. Therefore we obtain the singular foliation of the quotient M/G by submanifolds which support the reduced Courant algebroids.

3.2 Examples

In this section we will provide some examples of Courant algebroid reduction. Since Courant algebroids are often given together with a splitting, we describe the behaviour of splittings under reduction. This is then related to the way in which the Ševera class $[H]$ of an exact Courant algebroid is transported to the reduced space.

Example 3.7. Even a trivial group action may be extended by 1-forms; consider the extended action $\rho:\mathbb{R}\longrightarrow C^\infty(E)$ given by $\rho(1)=\xi$ for some closed 1-form ξ. Then $K=\langle\xi\rangle$ and $K^\perp=\{v\in E:\pi(v)\in\mathrm{Ann}(\xi)\}$ which induces the distribution $\Delta_b=\Delta_s=\mathrm{Ann}(\xi)\subset TM$, which is integrable wherever ξ is nonzero. Since the group action is trivial, a reduced Courant algebroid is simply a choice of integral submanifold $f:S\hookrightarrow M$ for ξ together with the pullback exact Courant algebroid $E_{red}=f^*E=K^\perp/K$, as in Equation (24).

Note that if a splitting $s:TM\longrightarrow E$ were chosen, rendering E isomorphic to $(TM\oplus T^*M,\langle\cdot,\cdot\rangle,[\![\cdot,\cdot]\!]_H)$ with $H\in\Omega^3_{cl}(M)$, then we would obtain a natural splitting

$$E_{red}=\mathrm{Ann}(\xi)\oplus T^*M/\langle\xi\rangle = TS\oplus T^*S$$

for the reduced algebroid. With this identification, the 3-form twisting the Courant algebroid structure on $TS\oplus T^*S$ is simply the pull-back f^*H.

Example 3.8. At another extreme, consider a free and proper action of G on M, with infinitesimal action $\psi:\mathfrak{g}\longrightarrow C^\infty(TM)$, and extend trivially by inclusion to a split Courant algebroid $(TM\oplus T^*M,\langle\cdot,\cdot\rangle,[\![\cdot,\cdot]\!]_H)$ such that the splitting is preserved by the action. By Equation (10), this is equivalent to the requirement that H is an invariant basic form.

Then $K=\psi(\mathfrak{g})$ and $K^\perp = TM\oplus\mathrm{Ann}(K)$, so that $\Delta_s=\Delta_b=TM$ and the reduced Courant algebroid is

$$TM/K\oplus \mathrm{Ann}K = TB\oplus T^*B,$$

where $B=M/G$ is the quotient and the 3-form twisting the Courant bracket on B is the push-down of the basic form H.

In the preceding examples, the reduced Courant algebroid inherited a natural splitting; this is not always possible. The next example demonstrates this as well as the phenomenon by which a trivial twisting $[H]=0$ may give rise to a cohomologically nontrivial reduced Courant algebroid.

Example 3.9. Let $M\xrightarrow{q}B$ be a principal S^1-bundle over B, and let $\rho:\mathfrak{s}^1\longrightarrow C^\infty(E)$ be a trivial extension ($\mathfrak{h}=0$) of this action such that $\rho(\mathfrak{s}^1)=K$ is isotropic, so that E_{red} will be exact. By Proposition 2.9, we may choose an invariant splitting so that $E=(TM\oplus T^*M,\langle\cdot,\cdot\rangle,[\![\cdot,\cdot]\!]_H)$, with $\rho(1)=\partial_\theta+\xi$ and $i_{\partial_\theta}H=d\xi$. As in Example 2.10, $H+\xi_a$ determines an equivariant closed 3-form, i.e.

$$[H+\xi_a]\in H^3_{S^1}(M,\mathbb{R}).$$

Now choose a connection $\theta\in\Omega^1(M)$ for the circle bundle, so that we obtain an identification

$$TB\oplus TB^*\longrightarrow (K^\perp/K)/S^1,$$

$$X+\eta\mapsto X^h+i_{X^h}(\theta\wedge\xi)+q^*\eta+K,$$

where the superscript of X^h denotes horizontal lift. To compute the reduced Courant bracket in this splitting, we use the decomposition $H=\alpha\wedge\theta+h$, where α and h are basic and invariant forms such that $\alpha = d\xi$. Also, let $F = d\theta$ be the curvature of the connection. Then we obtain the following expression for the curvature 3-form $\tilde{H}$ associated to the splitting of E_{red}:

$$\begin{aligned}\tilde{H}(X,Y,Z) &= 2\langle [\![X^h + i_{X^h}(\theta\wedge\xi), Y^h + i_{Y^h}(\theta\wedge\xi)]\!]_H, Z^h + i_{Z^h}(\theta\wedge\xi)\rangle\\ &= 2\langle [\![X^h, Y^h]\!]_{H+d(\theta\wedge\xi)}, Z^h\rangle\\ &= (h + F\wedge\xi)(X,Y,Z).\end{aligned}$$

The mapping obtained here, which sends $H + \xi_a$ to the closed form $h + F\wedge\xi \in \Omega^3(B,\mathbb{R})$ on the base, is exactly the pushdown isomorphism in equivariant cohomology:

$$H^3_{S^1}(M,\mathbb{R}) \xrightarrow{q_*} H^3(B,\mathbb{R}) .$$

Therefore we have shown that the curvature of the reduced exact Courant algebroid is precisely the pushforward of the equivariant extension of the original curvature induced by the extended action.

Note also that the splitting of E_{red} used to calculate $\tilde{H}$ depends on the choice of connection unless $\xi = 0$, in which case it is naturally induced from the original splitting of E.

The previous example can be carried out in the same way for a trivially extended, isotropic action of any Lie group G, as long as E admits invariant splitting. For example, for compact Lie groups we obtain the following result, which appeared in [6] in the context of gauging the Wess-Zumino term:

Proposition 3.10. *Let G be a compact Lie group acting freely and properly on M, and ρ be a trivially extended, isotropic action on the exact Courant algebroid E over M. Then if $[H] \in H^3(M,\mathbb{R})$ is the Ševera class of E, the reduced Courant algebroid has Ševera class $q_*[\Phi]$, where $\Phi = H + \xi_a$ is the equivariant extension induced by ρ, and q_* is the natural isomorphism*

$$H^3_G(M,\mathbb{R}) \xrightarrow{q_*} H^3(B,\mathbb{R}) .$$

Furthermore, a splitting $\nabla : T \longrightarrow E$ induces a splitting of E_{red} if and only if $\rho(\mathfrak{g}) \subset \nabla(T)$.

Combined with Example 3.7, this result indicates how to obtain the Ševera class of any exact reduced Courant algebroid: simply pull back to the leaf S of Δ_s and apply the previous result for the isotropy action ρ_s.

Example 3.11. One situation where E_{red} always inherits a splitting is when E is equipped with a G-invariant splitting ∇ and the action ρ is split, in the sense that there is a splitting s for $\pi : \mathfrak{a} \longrightarrow \mathfrak{g}$ making the diagram commutative:

$$\begin{array}{ccc} \mathfrak{a} & \xleftarrow{\;s\;} & \mathfrak{g} \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\psi} \\ C^\infty(E) & \xleftarrow[\nabla]{} & C^\infty(TM) \end{array} \tag{27}$$

In this case, the image distribution $\rho(\mathfrak{a}) = K$ decomposes as $K = K_T \oplus K_{T^*}$, with $K_T \subset TM$ and $K_{T^*} \subset T^*M$, and the reduced Courant algebroid decomposes as

$$\frac{K^\perp}{K\cap K^\perp} = \left(\frac{\mathrm{Ann}(K_T)}{\mathrm{Ann}(K_T)\cap K_{T^*}}\right) \oplus \left(\frac{\mathrm{Ann}(K_{T^*})}{\mathrm{Ann}(K_{T^*})\cap K_T}\right) = TM^*_{red} \oplus TM_{red}, \tag{28}$$

where the final equality is obtained since $\mathrm{Ann}(K_{T^*})/(\mathrm{Ann}(K_{T^*})\cap K_T) = \Delta_s/\rho(\mathfrak{g}_s)$. The curvature H of the given splitting for E is then basic, and the curvature for E_{red} is simply the pullback to S followed by pushdown to S/G_s.

If we are in the situation above, where the action is split, one has a natural trivially extended G-action on M coming from $\rho\circ s$. Assuming that G acts freely and properly on all of M, we may form the quotient Courant algebroid E^1_{red} over M/G, which is exact since $\rho\circ s$ is isotropic. Assuming that $\rho(\mathfrak{h})$ had constant rank on M, then as we saw in the previous section, M/G inherits a generalized foliation; the pullback of E^1_{red} to a leaf of this foliation would then recover the reduced Courant algebroid E_{red} over M_{red} constructed as before.

An example of such a split action, where the reduced Courant algebroid may be obtained in two equivalent ways, is the case of a symplectic action, as introduced in Example 2.14.

Example 3.12. Let (M, ω) be a symplectic manifold and consider the extended G-action $\rho : \mathfrak{g} \oplus \mathfrak{g} \longrightarrow C^\infty(TM \oplus T^*M)$ with curvature $H = 0$ defined in Example 2.14. This is clearly a split action in the above sense.

Let $\psi : \mathfrak{g} \longrightarrow C^\infty(TM)$ be the infinitesimal action and $\psi(\mathfrak{g})^\omega$ denote the symplectic orthogonal of the image distribution $\psi(\mathfrak{g})$. Then the extended action has image

$$K = \psi(\mathfrak{g}) \oplus \omega(\psi(\mathfrak{g})),$$

so that the orthogonal complement is

$$K^\perp = \psi(\mathfrak{g})^\omega \oplus \mathrm{Ann}(\psi(\mathfrak{g})).$$

Then the big and small distributions on M are

$$\begin{aligned}\Delta_s &= \psi(\mathfrak{g})^\omega, \\ \Delta_b &= \psi(\mathfrak{g})^\omega + \psi(\mathfrak{g}).\end{aligned}$$

If the action is Hamiltonian, with moment map $\mu : M \longrightarrow \mathfrak{g}^*$, then Δ_s is the tangent distribution to the level sets $\mu^{-1}(\lambda)$ while Δ_b is the tangent distribution to the sets $\mu^{-1}(\mathcal{O}_\lambda)$, for $\mathcal{O}_\lambda$ a coadjoint orbit containing λ. Therefore we see that the reduced Courant algebroid is simply $TM_{red} \oplus T^*M_{red}$ with $H = 0$, for the usual symplectic reduced space $M_{red} = \mu^{-1}(\mathcal{O}_\lambda)/G = \mu^{-1}(\lambda)/G_\lambda$.

Since the action is split, we may also observe, assuming that G acts freely and properly on M, that the quotient M/G is foliated via (26) by the possible reduced spaces. This generalized distribution is given in this case by

$$\left.\frac{\psi(\mathfrak{g})^\omega + \psi(\mathfrak{g})}{\psi(\mathfrak{g})}\right/ G \quad = dq(\psi(\mathfrak{g})^\omega) \subset T(M/G),$$

where $q : M \longrightarrow M/G$ is the quotient map. This is precisely the distribution defined by the image of the Poisson tensor $\Pi : T^*(M/G) \longrightarrow T(M/G)$ induced by ω (recall that $\Pi(df) = dq(X_{q^*f})$, where X_{q^*f} is the Hamiltonian vector field for q^*f). So a reduced manifold for the extended action is just a symplectic leaf of M/G.

Finally, we present an example of a reduced Courant algebroid which is not exact.

Example 3.13. Let $\rho : \mathfrak{s}^1 \longrightarrow E$ be a trivially extended S^1 action which is not isotropic, i.e. $\langle \rho(1), \rho(1) \rangle \neq 0$. Hence the reduced manifold for this action is just M/S^1 and the reduced algebroid is $E_{red} = (K^\perp/(K \cap K^\perp))/S^1$. However, $K \cap K^\perp = \{0\}$ and so E_{red} is odd dimensional; hence it is not an exact Courant algebroid.

4 Reduction of Dirac structures

A *Dirac structure* [3, 17] on a manifold M equipped with exact Courant algebroid E is a maximal isotropic subbundle $D \subset E$ whose sections are closed under the Courant bracket. This last requirement is referred to as the *integrability* condition for D. When the Courant algebroid is split, with curvature $H \in \Omega^3_{cl}(M)$, these are usually referred to as *H-twisted* Dirac structures [20].

For $H = 0$, examples of Dirac structures on M include closed 2-forms and Poisson bivector fields (in these cases D is simply the graph of the defining tensor, viewed either as a map $\omega : T \longrightarrow T^*$ or $\Pi : T^* \longrightarrow T$) as well as involutive regular distributions $F \subset T$, in which case $D = F \oplus \mathrm{Ann}(F)$.

In the presence of an extended action of a Lie group G on the Courant algebroid E, one may consider Dirac structures which are G-invariant subbundles of E, a condition equivalent to the following.

Definition 4.1. A Dirac structure $D \subset E$ is preserved by an extended action ρ if and only if $\rho(\mathfrak{a}) \bullet C^\infty(D) \subset C^\infty(D)$.

In this section we explain how a Dirac structure which is preserved by an extended action may be transported from a Courant algebroid E to its reduction E_{red}.

4.1 Odd symplectic category

Let E, F be real vector spaces with nondegenerate, symmetric bilinear forms of split signature. Linear Dirac structures on these are simply maximal isotropic subspaces, and they may be transported between E and F if there is a morphism between them in the sense of the odd symplectic category [2],[23]. Here "odd" indicates a parity reversal, whereby the symmetric inner product is viewed as an odd symplectic form and maximal isotropic subspaces are odd Lagrangians. Therefore, a morphism $Q : E \longrightarrow F$ is a maximal isotropic subspace

$$Q \subset \overline{E} \times F,$$

where $\overline{E}$ is obtained from E by multiplying the inner product by -1. This means that a Dirac structure $D \subset E$ may itself be viewed as a morphism $D : \{0\} \longrightarrow E$, which may then be composed as a relation with Q to yield $Q \circ D : \{0\} \longrightarrow F$, a Dirac structure in F. In this way, we obtain a map of linear Dirac structures:

$$Q : \mathrm{Dir}(E) \to \mathrm{Dir}(F).$$

An isotropic subspace $K \subset E$ determines not only another split-signature space $K^\perp/K$, but also a morphism

$$\varphi_K : E \longrightarrow K^\perp/K,$$

given by the following maximal isotropic:

$$\varphi_K = \left\{ (x, [x]) \in \overline{E} \times K^\perp/K \ : \ x \in K^\perp \right\}$$

Given a Dirac structure $D \subset E$, one obtains by composition with φ_K the Dirac structure

$$\varphi_K \circ D = \frac{D \cap K^\perp + K}{K} \subset K^\perp/K.$$

4.2 Reduction procedure

We now show how to use the morphism just described to transport an invariant Dirac structure in a Courant algebroid E to the reduced Courant algebroid E_{red} on P/G, in the notation of the previous section. We always assume that E_{red} is exact, and we use its expression in terms of the bundle $\rho(\mathfrak{a}) = K$:

$$E_{red} = \left. \frac{K^\perp}{K \cap K^\perp} \right/ G = \left. \frac{K^\perp + K}{K} \right/ G.$$

If the G-invariant bundle K is isotropic along P, then it defines a fibrewise morphism

$$\varphi_K : E \longrightarrow K^\perp/K \tag{29}$$

along P, as described above. Composition of relations then transports any G-invariant Dirac structure D to the following *reduced Dirac structure*, assuming the result is a smooth bundle:

$$D_{red} = \left. \frac{D \cap K^\perp + K}{K} \right/ G \ \subset E_{red}. \tag{30}$$

Note that D_{red} is smooth if $D \cap K^\perp$ has constant rank over P. For the proof that D_{red} is integrable, see Theorem 4.2.

If K is not isotropic, the procedure just described must be modified. In this case we use the result of Proposition 3.5 that E_{red} can be constructed by first pulling E back to a leaf $f : S \hookrightarrow M$ of Δ_s and then taking the quotient by the isotropy action. Over the leaf S, the isotropic subbundle $\rho(\mathfrak{h}) = K \cap T^*$ determines a morphism from $E|_S$ to the pullback Courant algebroid f^*E. This is a generalization of the pullback of Dirac structures discussed in [2]. After pullback, the isotropy action $\rho_s(\mathfrak{g}_s) \subset f^*E$ determines a morphism from f^*E to E_{red}. This morphism is a generalization of the Dirac pushforward [2]. Composing these morphisms yields a morphism

$$\varphi_{\widetilde{K}} : E \longrightarrow E_{red},$$

determined by the isotropic subbundle $\widetilde{K} = K \cap (K^\perp + T^*) \subset E$, as defined in (25). As a result, the reduced Dirac structure is obtained by the same procedure as in the isotropic case, applied to $\widetilde{K}$ instead of K:

Theorem 4.2. *Let $\rho : \mathfrak{a} \longrightarrow C^\infty(E)$ be an extended action preserving a Dirac structure $D \subset E$, and such that E_{red} is exact over $M_{red} = P/G$, i.e. the subbundle $\widetilde{K} = K \cap (K^\perp + T^*)$ is isotropic along P. Then if*

$$D_{red} = \frac{D \cap \widetilde{K}^\perp + \widetilde{K}}{\widetilde{K}} \Big/ G \subset E_{red} \tag{31}$$

is a smooth subbundle, it defines a Dirac structure on the reduction M_{red}.

Proof. The only property of D_{red} that remains to be checked is integrability. To do so, we first observe that the Courant bracket on $E_{red} = (\widetilde{K}^\perp/\widetilde{K})/G$ admits the following description, equivalent to the one given in Theorem 3.3. Given sections v_1, v_2 of E_{red}, let us consider representatives in $C^\infty(\widetilde{K}^\perp|_P)^G$, still denoted by v_1, v_2. Then extend them to sections $\tilde{v}_1, \tilde{v}_2$ of E over M, and define $[\![v_1, v_2]\!]$ as $[\![\tilde{v}_1, \tilde{v}_2]\!]|_P$. Similarly to Theorem 3.3, one can show that $[\![\tilde{v}_1, \tilde{v}_2]\!]|_P \in C^\infty(\widetilde{K}^\perp|_P)^G$, and that different choices of extensions change the bracket by invariant sections of $\widetilde{K}$ over P. Also, the bracket between elements in $C^\infty(\widetilde{K}|_P)^G$ and $C^\infty(\widetilde{K}^\perp|_P)^G$ remains in $C^\infty(\widetilde{K}|_P)^G$, so there is an induced bracket on E_{red}. This bracket agrees with the one defined in Theorem 3.3.

Let $v_1, v_2 \in C^\infty((D \cap \widetilde{K}^\perp + \widetilde{K})|_P)^G$, thought of as representing sections of D_{red}. We note that, around points of P where $D \cap \widetilde{K}^\perp|_P$ has locally constant rank, we can write $v_i = v_i' + v_i''$, where v_i' is an invariant local section of $D \cap \widetilde{K}^\perp|_P$, and v_i'' is an invariant local section of $\widetilde{K}|_P$. Then the bracket of v_1, v_2 is

$$[\![v_1' + v_1'', v_2' + v_2'']\!] = [\![v_1', v_2']\!] + [\![v_1'', v_2']\!] + [\![v_1', v_2'']\!] + [\![v_1'', v_2'']\!].$$

Note that the last three terms on the right-hand side are in $C^\infty(\tilde{K}|_P)^G$. As for the first term, we know that it lies in $\widetilde{K}^\perp|_P$. But since D is a vector bundle over M, we can locally extend v_i' to sections of D away of P and, using these extensions to compute the bracket, we see that $[\![v_1', v_2']\!] \in C^\infty(D|_P)$, since D is closed under the bracket. As a result, we conclude that $[\![v_1, v_2]\!]$ is in $(D \cap \widetilde{K}^\perp + \widetilde{K})|_P$ around points where $D \cap \widetilde{K}^\perp|_P$ is locally a bundle.

Since the points of P where $D \cap \widetilde{K}^\perp|_P$ has locally constant rank is an open dense set, the argument above shows that for $v_1, v_2 \in C^\infty(D_{red})$, $[\![v_1, v_2]\!]$ lies in D_{red} over all points in an open dense subset of P/G. But since D_{red} is smooth, this implies that $[\![v_1, v_2]\!] \in C^\infty(D_{red})$, hence D_{red} is integrable. □

The reduction of Dirac structures works in the same way for *complex* Dirac structures, provided one replaces K by its complexification $K_{\mathbb{C}} = K \otimes \mathbb{C}$.

Remark: As discussed in Example 3.11, the presence of a split action $\rho(\mathfrak{a}) = K_T \oplus K_{T^*}$ provides an alternative route in obtaining the reduced algebroid E_{red}; assuming that G acts freely and properly on all of M, we may first form the quotient by G and then restrict to a leaf of the generalized foliation on M/G. We note here that if this alternative route is used to reduce Dirac structures, one may obtain a different result $\widetilde{D}_{red}$. A useful criterion for the equality of D_{red} and $\widetilde{D}_{red}$ is obtained by observing that the isotropic space

$$D' = \frac{D \cap K^\perp + K \cap K^\perp}{K \cap K^\perp}$$

is contained in the intersection, i.e. $D' \subset D_{red} \cap \widetilde{D}_{red}$. As a result, $D_{red} = \widetilde{D}_{red}$ whenever D' is maximal isotropic, a condition equivalent to the equality

$$D \cap K^\perp + K \cap K^\perp = (D + K) \cap K^\perp. \tag{32}$$

5 Reduction of generalized complex structures

A *generalized complex structure* [8, 7] on a manifold M equipped with exact Courant algebroid E is a complex structure on the vector bundle E which is orthogonal with respect to the bilinear pairing and whose $+i$-eigenbundle is closed under the bracket. If the Courant algebroid is split, with curvature $H \in \Omega^3_{cl}(M)$, a generalized complex structure on E is called an *H-twisted* generalized complex structure on M.

Since a generalized complex structure is orthogonal, its $+i$-eigenbundle $L \subset E \otimes \mathbb{C} = E_{\mathbb{C}}$ is a maximal isotropic subbundle. Therefore a generalized complex structure on E is equivalent to a complex Dirac structure L satisfying

$$L \cap \overline{L} = \{0\}. \tag{33}$$

The *type* of a generalized complex structure at a point $p \in M$ is the complex dimension of the kernel of the projection $\pi : L \longrightarrow T_{\mathbb{C}}M$ at p. Two basic examples of generalized complex structures on a manifold M (with $H = 0$) arise as follows:

- Let $I : TM \longrightarrow TM$ be a complex structure on M. Then it induces a generalized complex structure on M by
$$\mathcal{J}_I = \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix}.$$
The associated Dirac structure is $L = T_{1,0} \oplus T^*_{0,1}$, which has type n.
- Let $\omega : TM \longrightarrow T^*M$ be a symplectic structure. The induced generalized complex structure is
$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$
The associated Dirac structure is $L = \{X - i\omega(X) \,:\, X \in T_{\mathbb{C}}M\}$, and the type is zero.

A structure on M^{2n} is of *complex type* if it has type n at all points, and it is of *symplectic type* if it has type zero at all points. The reader is referred to [7] for more details concerning generalized complex structures.

5.1 Reduction procedure

Throughout this section, $\rho : \mathfrak{a} \longrightarrow C^\infty(E)$ denotes an extended G-action on an exact Courant algebroid E over a manifold M. Let $K = \rho(\mathfrak{a})$, and let $K_{\mathbb{C}} = K \otimes \mathbb{C}$. We fix a leaf $P \hookrightarrow M$ of the distribution Δ_b and assume that the reduced Courant algebroid E_{red} over P/G is exact, which amounts to the assumption that $\tilde{K} = K \cap (K^\perp + T^*)$ is isotropic along P.

Suppose that the extended action ρ preserves a generalized complex structure $\mathcal{J}$ on E, i.e., that the associated Dirac structure $L \subset E_{\mathbb{C}}$ is invariant. We consider its reduction to E_{red}:

$$L_{red} = \frac{L \cap \widetilde{K}_{\mathbb{C}}^\perp + \widetilde{K}_{\mathbb{C}}}{\widetilde{K}_{\mathbb{C}}} \Big/ G \tag{34}$$

If L_{red} is smooth, then it determines a generalized complex structure on E_{red} if and only if it satisfies $L_{red} \cap \overline{L}_{red} = \{0\}$.

Lemma 5.1. *The distribution L_{red} satisfies $L_{red} \cap \overline{L}_{red} = \{0\}$ if and only if*

$$\mathcal{J}\widetilde{K} \cap \widetilde{K}^\perp \subset \widetilde{K} \text{ over } P. \tag{35}$$

Proof. It is clear from (34) that $L_{red} \cap \overline{L}_{red} = \{0\}$ over the reduced manifold if and only if

$$(L \cap \widetilde{K}_{\mathbb{C}}^\perp + \widetilde{K}_{\mathbb{C}}) \cap (\overline{L} \cap \widetilde{K}_{\mathbb{C}}^\perp + \widetilde{K}_{\mathbb{C}}) \subset \widetilde{K}_{\mathbb{C}} \quad \text{over } P. \tag{36}$$

Hence, we must prove that conditions (35) and (36) are equivalent.

We first prove that (35) implies (36). Let $v \in (L \cap \widetilde{K}_{\mathbb{C}}^\perp + \widetilde{K}_{\mathbb{C}}) \cap (\overline{L} \cap \widetilde{K}_{\mathbb{C}}^\perp + \widetilde{K}_{\mathbb{C}})$ over a given point. Without loss of generality we can assume that v is real. Since $v \in L \cap \widetilde{K}_{\mathbb{C}}^\perp + \widetilde{K}_{\mathbb{C}}$, we can find $v_L \in L \cap \widetilde{K}_{\mathbb{C}}^\perp$ and $v_{\widetilde{K}} \in \widetilde{K}_{\mathbb{C}}$ such that $v = v_L + v_{\widetilde{K}}$. Taking conjugates, we get that $v = \overline{v_L} + \overline{v_{\widetilde{K}}}$, hence

$$v_L - \overline{v_L} = \overline{v_{\widetilde{K}}} - v_{\widetilde{K}}.$$

Applying $-i\mathcal{J}$, we obtain

$$v_L + \overline{v_L} = -i\mathcal{J}(\overline{v_{\widetilde{K}}} - v_{\widetilde{K}}).$$

The left hand side lies in $\widetilde{K}^\perp$ while the right hand side lies in $\mathcal{J}\widetilde{K}$. Therefore, according to (35), $2v - v_{\widetilde{K}} - \overline{v_{\widetilde{K}}} = v_L + \overline{v_L} \in \widetilde{K}$ and $v \in \widetilde{K}$, as desired.

Conversely, if (35) does not hold, i.e., there is $v \in \mathcal{J}\widetilde{K} \cap \widetilde{K}^\perp$ with $v \notin \widetilde{K}$, then $v - i\mathcal{J}v \in L \cap \widetilde{K}_{\mathbb{C}}^\perp$ and $v + i\mathcal{J}v \in \overline{L} \cap \widetilde{K}_{\mathbb{C}}^\perp$. Since $v \in \mathcal{J}\widetilde{K}$ and $\mathcal{J}v \in \widetilde{K}$, it follows that $v \in L \cap \widetilde{K}_{\mathbb{C}}^\perp + \widetilde{K}$ and $v \in \overline{L} \cap \widetilde{K}_{\mathbb{C}}^\perp + \widetilde{K}$, showing that $(L \cap \widetilde{K}_{\mathbb{C}}^\perp + \widetilde{K}_{\mathbb{C}}) \cap (\overline{L} \cap \widetilde{K}_{\mathbb{C}}^\perp + \widetilde{K}_{\mathbb{C}}) \not\subset \widetilde{K}_{\mathbb{C}}$. This concludes the proof. ∎

If the Dirac reduction of the $+i$-eigenbundle of a generalized complex structure $\mathcal{J}$ on E defines a generalized complex structure on E_{red}, then we denote it by $\mathcal{J}^{red}$. We now present a situation where this occurs.

Theorem 5.2. *Let ρ be an extended G-action on the exact Courant algebroid E. Let P be a leaf of the distribution Δ_b where G acts freely and properly with exact quotient E_{red}. If the action preserves a generalized complex structure $\mathcal{J}$ on E and $\mathcal{J}K = K$ over P then $\mathcal{J}$ reduces to E_{red}.*

Proof. We start with a general observation: given a Dirac structure D invariant under an extended action, let us consider in the reduced Courant algebroid the isotropic distribution

$$D' := \frac{D \cap K^\perp + K \cap K^\perp}{K \cap K^\perp} \Big/ G.$$

One can check that $D' \subset D_{red}$, so, if D' is maximal isotropic, then it agrees with D_{red}.

In our case, we have

$$L' = \frac{L \cap K_{\mathbb{C}}^\perp + K_{\mathbb{C}} \cap K_{\mathbb{C}}^\perp}{K_{\mathbb{C}} \cap K_{\mathbb{C}}^\perp}. \tag{37}$$

Since $\mathcal{J}K^\perp = K^\perp$, it follows that $K_{\mathbb{C}}^\perp = L \cap K_{\mathbb{C}}^\perp + \overline{L} \cap K_{\mathbb{C}}^\perp$. Hence

$$L' + \overline{L'} = \frac{L \cap K_{\mathbb{C}}^\perp + \overline{L} \cap K_{\mathbb{C}}^\perp + K_{\mathbb{C}} \cap K_{\mathbb{C}}^\perp}{K_{\mathbb{C}} \cap K_{\mathbb{C}}^\perp} = \frac{K_{\mathbb{C}}^\perp}{K_{\mathbb{C}} \cap K_{\mathbb{C}}^\perp} = E_{red} \otimes \mathbb{C},$$

showing that L' is maximal and therefore agrees with L_{red}. The argument above also shows that $L \cap K_{\mathbb{C}}^\perp$ is a bundle and, since $K \cap K^\perp$ is a bundle over P, this implies that L' as defined in (37) is smooth.

Finally, in order to conclude that L_{red} induces a generalized complex structure we must check that condition (35) in Lemma 5.1 holds:

$$\mathcal{J}\widetilde{K} \cap \widetilde{K}^\perp = K \cap (K^\perp + \mathcal{J}T^*) \cap (K^\perp + K \cap T^*) \subset K \cap (K^\perp + K \cap T^*) = \widetilde{K},$$

as desired. □

Corollary 5.3. *If the hypotheses of the previous theorem hold and the extended action has a moment map $\mu : M \longrightarrow \mathfrak{h}^*$, then the reduced Courant algebroid over $\mu^{-1}(O_\lambda)/G$ has a reduced generalized complex structure.*

It is easy to check that the reduced generalized complex structure $\mathcal{J}^{red}$ constructed in Theorem 5.2 is characterized by the following commutative diagram:

$$\begin{array}{ccc} K^\perp & \xrightarrow{\mathcal{J}} & K^\perp \\ \downarrow & & \downarrow \\ \frac{K^\perp}{K \cap K^\perp} & \xrightarrow{\mathcal{J}^{red}} & \frac{K^\perp}{K \cap K^\perp} \end{array} \tag{38}$$

Theorem 5.2 uses the compatibility condition $\mathcal{J}K = K$ for the reduction of $\mathcal{J}$. We now observe that the reduction procedure also works in an extreme opposite situation.

Proposition 5.4. *Consider an extended G-action ρ on an exact Courant algebroid E. Let P be a leaf of the distribution Δ_b where G acts freely and properly. If K is isotropic over P and $\langle \cdot, \cdot \rangle : K \times \mathcal{J}K \longrightarrow \mathbb{R}$ is nondegenerate then $\mathcal{J}$ reduces.*

Proof. As K is isotropic over P, the reduced Courant algebroid is exact and $\widetilde{K} = K$. The nondegeneracy assumption implies that $\mathcal{J}K \cap K^\perp = \{0\}$, and it follows that $L \cap K_{\mathbb{C}}^\perp$ is a bundle and the Dirac reduction of L is smooth. Finally, (35) holds trivially. □

5.2 Symplectic structures

We now present two examples of reduction obtained from a symplectic manifold (M, ω): First, we show that ordinary symplectic reduction is a particular case of our construction; the second example illustrates how one can obtain a type 1 generalized complex structure as the reduction of an ordinary symplectic structure. In both examples, the initial Courant algebroid is just $TM \oplus T^*M$ with $H = 0$.

Example 5.5. *(Ordinary symplectic reduction)* Let (M, ω) be a symplectic manifold, and let $\mathcal{J}_\omega$ be the generalized complex structure associated with ω. Following Example 2.14 and keeping the same notation, consider a symplectic G-action on M, regarded as an extended action. It is clear that $\mathcal{J}_\omega K = K$, so we are in the situation of Theorem 5.2.

Following Example 3.12, let S be a leaf of the distribution $\Delta_s = \psi(\mathfrak{g})^\omega$. Since K splits as $K_T \oplus K_{T^*}$, the reduction procedure of Theorem 4.2 in this case amounts to the usual pull-back of ω to S, followed by a Dirac push-forward to $S/G_s = M_{red}$. If the symplectic action admits a moment map $\mu : M \longrightarrow \mathfrak{g}^*$, then the leaves of Δ_s are level sets $\mu^{-1}(\lambda)$, and Theorem 5.2 simply reproduces the usual Marsden-Weinstein quotient $\mu^{-1}(\lambda)/G_\lambda$.

If the symplectic G-action on M is free and proper, then ω induces a Poisson structure Π on M/G. We saw in Example 3.12 that the reduced manifolds fit into a singular foliation of M/G, which coincides with the symplectic foliation of Π. Following the remark at the end of Section 4, the reduction of $\mathcal{J}_\omega$ to each leaf can be obtained by the Dirac push-forward of ω to M/G, which is just Π, followed by the Dirac pull-back of Π to the leaf, which is the symplectic structure induced by Π on that leaf.

Next, we show that by allowing the projection $\pi : K \longrightarrow TM$ to be injective, one can reduce a symplectic structure (type 0) to a generalized complex structure with nonzero type.

Example 5.6. Assume that X and Y are linearly independent symplectic vector fields generating a T^2-action on M. Assume further that $\omega(X, Y) = 0$ and consider the extended T^2-action on $TM \oplus T^*M$ defined by

$$\rho(\alpha_1) = X + \omega(Y); \qquad \rho(\alpha_2) = -Y + \omega(X),$$

where $\{\alpha_1, \alpha_2\}$ is the standard basis of $\mathfrak{t}^2 = \mathbb{R}^2$. It follows from $\omega(X, Y) = 0$ and the fact that the vector fields X and Y are symplectic that this is an extended action with isotropic K.

Since $\mathcal{J}_\omega K = K$, Theorem 5.2 implies that the quotient M/T^2 has an induced generalized complex structure. Note that

$$L \cap K_{\mathbb{C}}^\perp = \{Z - i\omega(Z) : Z \in \mathrm{Ann}(\omega(X) \wedge \omega(Y))\},$$

and it is simple to check that $X - i\omega(X) \in L \cap K_{\mathbb{C}}^\perp$ represents a nonzero element in $L_{red} = ((L \cap K_{\mathbb{C}}^\perp + K_{\mathbb{C}})/K_{\mathbb{C}})/G$ which lies in the kernel of the projection $L_{red} \longrightarrow T(M/T^2)$. As a result, this reduced generalized complex structure has type 1.

One can find concrete examples illustrating this construction by considering symplectic manifolds which are T^2-principal bundles with lagrangian fibres, such as $T^2 \times T^2$, or the Kodaira–Thurston manifold. In these cases, the reduced generalized complex structure determines a complex structure on the base 2-torus.

5.3 Complex structures

In this section we show how a complex manifold (M, I) may have different types of generalized complex reductions.

Example 5.7. *(Holomorphic quotient)* Let G be a complex Lie group acting holomorphically on (M, I), so that the induced infinitesimal map $\rho : \mathfrak{g} \longrightarrow C^\infty(TM)$ is a holomorphic map. Since $K = \rho(\mathfrak{g}) < TM$, it is clear that K is isotropic and the reduced Courant algebroid is exact. Furthermore, as ρ is holomorphic, it follows that $\mathcal{J}_I K = K$. By Theorem 5.2, the complex structure descends to a generalized complex structure in the reduced manifold M/G. The reduced generalized complex structure is nothing but the quotient complex structure obtained from holomorphic quotient.

The previous example is a particular case of a more general fact: if (M, I) is a complex manifold, then any reduction of $\mathcal{J}_I$ by an extended action satisfying $\mathcal{J}_I K = K$ results in a generalized complex structure of complex type. Indeed, T^*M_{red} can be identified with

$$\frac{K^\perp \cap T^* + K \cap K^\perp}{K \cap K^\perp} \Big/ G \subset E_{red}$$

and using that $\mathcal{J}_I(T^*M) = T^*M$, one sees that $\mathcal{J}^{red}(T^*M_{red}) = T^*M_{red}$, i.e., $\mathcal{J}^{red}$ is of complex type. However, using Proposition 5.4, one can produce reductions of complex structures which are not of complex type.

Example 5.8. Consider $\mathbb{C}^2$ equipped with its standard holomorphic coordinates $(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2)$, and let ρ be the extended $\mathbb{R}^2$-action on $\mathbb{C}^2$ defined by

$$\rho(\alpha_1) = \partial_{x_1} + dx_2, \qquad \rho(\alpha_2) = \partial_{y_2} + dy_1,$$

where $\{\alpha_1, \alpha_2\}$ is the standard basis for $\mathbb{R}^2$. Note that $K = \rho(\mathbb{R}^2)$ is isotropic, so the reduced Courant algebroid over $\mathbb{C}/\mathbb{R}^2$ is exact. Since the natural pairing between K and $\mathcal{J}_I K$ is nondegenerate, Proposition 5.4 implies that one can reduce $\mathcal{J}_I$ by this extended action. In this example, one computes

$$K_{\mathbb{C}}^\perp \cap L = \mathrm{span}\{\partial_{x_1} - i\partial_{x_2} - dy_1 + idx_1, \partial_{y_1} - i\partial_{y_2} - dy_2 + idx_2\}$$

and $K_{\mathbb{C}}^\perp \cap L \cap K_{\mathbb{C}} = \{0\}$. As a result, $L_{red} \cong K_{\mathbb{C}}^\perp \cap L$. So $\pi : L_{red} \longrightarrow \mathbb{C}^2/\mathbb{R}^2$ is an injection, and $\mathcal{J}^{red}$ has zero type, i.e., it is of symplectic type.

5.4 Extended Hamiltonian actions

In order to reduce a generalized complex structure $\mathcal{J}$ preserved by an extended action, we saw in Theorem 5.2 that a sufficient condition is the compatibility $\mathcal{J}K = K$. Natural examples where this condition holds arise as follows: one starts with an action generated by sections $v_i \in C^\infty(E)$, and then enlarges it to a new extended action generated by sections

$$\{v_i, \mathcal{J}v_j\}. \tag{39}$$

Examples where this construction works are the extended actions associated with symplectic and holomorphic actions: in the symplectic case (see Example 2.14), one starts with symplectic vector fields X_i and defines an extended action of the demisemidirect Courant algebra, by adding new generators $\mathcal{J}_\omega(X_j) = \omega(X_j)$, which act as closed 1-forms; in the holomorphic case, one starts with an action generated by X_i preserving a complex structure I, and then forms the (trivially) extended action of the complexified Lie algebra, generated by $\{X_i, \mathcal{J}_I X_j\}$, where now $\mathcal{J}_I X_j = IX_j$ are new vector fields.

The "complexification" (39) does not always define an extended action, as we will show. However, in the case of a *Hamiltonian* action we show that it does produce an example of an extended action satisfying $\mathcal{J}K = K$.

It is familiar in the case of a complex manifold that a real vector field X preserves the complex structure I if and only if its $(1,0)$ component $X^{1,0} \in T_{1,0}M$ is a holomorphic vector field. Therefore $IX = iX^{1,0} - iX^{0,1}$ also preserves the complex structure. In paritcular, if X generates an S^1 action then $\{X, IX\}$ defines a holomorphic $\mathbb{C}^*$ action on the complex manifold.

For generalized complex structures a similar phenomenon occurs, except that symmetries are governed by the differential complex $(\Omega^\bullet(L) = C^\infty(\wedge^\bullet L^*), d_L)$ associated to the complex Lie algebroid L defined by the $+i$–eigenbundle of $\mathcal{J}$.

Lemma 5.9. *A real section $v \in C^\infty(E)$ preserves the generalized complex structure $\mathcal{J}$ under the adjoint action if and only if $d_L v^{0,1} = 0$, where $v = v^{1,0} + v^{0,1} \in L \oplus \overline{L} = E \otimes \mathbb{C}$ and we use the inner product to identify $\overline{L} = L^*$.*

Proof. A real section $v \in C^\infty(E)$ preserves $\mathcal{J}$ if and only if $v \bullet C^\infty(L) \subset C^\infty(L)$. Since L is maximal isotropic, it suffices to check that $\langle v^{0,1} \bullet w_1, w_2\rangle = 0$ for all $w_1, w_2 \in C^\infty(L)$. By definition of the Lie algebroid differential d_L, and using the basic properties of $\bullet$, we have

$$\begin{aligned} d_L v^{0,1}(w_1, w_2) &= \pi(w_1)\langle v^{0,1}, w_2\rangle - \pi(w_2)\langle v^{0,1}, w_1\rangle - \langle v^{0,1}, w_1 \bullet w_2\rangle \\ &= 2\langle v^{0,1} \bullet w_2, w_1\rangle + \langle v^{0,1}, w_1 \bullet w_2\rangle + \pi(w_2)\langle v^{0,1}, w_1\rangle - \pi(w_1)\langle v^{0,1}, w_2\rangle \\ &= 2\langle v^{0,1} \bullet w_2, w_1\rangle - d_L v^{0,1}(w_1, w_2), \end{aligned}$$

so $d_L v^{0,1}(w_1, w_2) = \langle v^{0,1} \bullet w_2, w_1\rangle$, which immediately implies the result. ∎

We obtain the following exact sequence describing infinitesimal symmetries of $\mathcal{J}$ [7]:

$$C^\infty(M,\mathbb{C}) \xrightarrow{D} \mathrm{sym}(\mathcal{J}) \longrightarrow H^1(L) \longrightarrow 0 ,$$

where $D(f) = d_L f + \overline{d_L f} \in C^\infty(E)$, and the final term denotes the first Lie algebroid cohomology of L. Infinitesimal symmetries which lie in the image of D are called *Hamiltonian* symmetries [7], in direct analogy with the symplectic case. Note that for $f \in C^\infty(M,\mathbb{C})$ we have by definition

$$d_L f = \tfrac{1}{2}(df + i\mathcal{J}df),$$

so that the operator D may be expressed as

$$Df = d(\mathrm{Re}f) - \mathcal{J}d(\mathrm{Im}f).$$

Also note that the projection $\pi(Df) \in C^\infty(TM)$ lies in the projection $\pi(\mathcal{J}(T^*M))$ of the Dirac structure $\mathcal{J}(T^*M) \subset E$ and hence is tangent to the symplectic leaves of the Poisson structure induced by $\mathcal{J}$. This places a strong constraint on Hamiltonian symmetries which is familiar from the situation in Poisson geometry.

Example 5.10. In the symplectic case, a section $X + \xi \in C^\infty(TM \oplus T^*M)$ is a symmetry precisely when X is a symplectic vector field and $d\xi = 0$, whereas it is Hamiltonian if and only if X is Hamiltonian in the usual sense and ξ is exact. In the complex case, $X + \xi$ is a symmetry when $X^{1,0}$ is holomorphic and $\overline{\partial}\xi^{0,1} = 0$, whereas it is Hamiltonian if and only if $X = 0$ and $\xi = \overline{\partial} f + \partial\overline{f}$ for $f \in C^\infty(M,\mathbb{C})$.

We have the following immediate consequence of Lemma 5.9.

Corollary 5.11. *If $v \in C^\infty(E)$ preserves $\mathcal{J}$ then so does $\mathcal{J}v = iv^{1,0} - iv^{0,1}$.*

However, if the infinitesimal action of v integrates to an extended action on E, then this does not guarantee that $\mathcal{J}v$ also does, as we now show.

Example 5.12. Let $\mathfrak{h} = \mathfrak{a} = \mathbb{R}$ be a Courant algebra over the trivial Lie algebra $\mathfrak{g} = \{0\}$ and consider an action by covectors $\rho : \mathfrak{a} \longrightarrow T^*M \subset TM \oplus T^*M$. In order that ρ define an extended action we need $\xi = \rho(1) \in \Omega^1_{cl}(M)$. If M is endowed with a complex structure I, then the complexification of ρ satisfies $\rho_{\mathbb{C}}(i) = I^*\xi$, which is closed only if $d^c\xi = 0$.

While the "complexification" proposed in (39) may be obstructed because of the fact that $\mathcal{J}v$ may not define an extended action even if v does, we now show that if the given action is *Hamiltonian*, then it is equivalent, in the sense of Example 2.10, to an action which can be extended so that $\mathcal{J}K = K$.

Theorem 5.13. *Let $\rho : \mathfrak{g} \longrightarrow C^\infty(E)$ be a trivially extended, isotropic, Hamiltonian action on a generalized complex manifold, i.e. $\rho(a) = D(f_a)$ for a $\mathfrak{g}$-equivariant function $f : M \longrightarrow \mathfrak{g}^*_{\mathbb{C}}$. Then the equivalent action $\tilde{\rho}(a) = \rho(a) - d(\mathrm{Re}f_a)$ may be extended to an action of the demisemidirect Courant algebra $\mathfrak{g} \oplus \mathfrak{g}$, with moment map $\mathrm{Im}f$, and which satisfies the condition $\mathcal{J}K = K$.*

Proof. Since $\rho(a) = D(f_a) = d(\mathrm{Re}f_a) - \mathcal{J}d(\mathrm{Im}f_a)$, we see that

$$\mathcal{J}\tilde{\rho}(a) = d(\mathrm{Im}f_a),$$

which shows that the map $\rho' : \mathfrak{g} \oplus \mathfrak{g} \longrightarrow C^\infty(E)$ given by

$$\rho' : (g,h) \longmapsto \tilde{\rho}(g) + d(\mathrm{Im}f_h)$$

defines an extended action, as we saw in Proposition 2.15, and by construction satisfies $\mathcal{J}K = K$. □

Although this theorem concerns only Hamiltonian actions, which for generalized complex structures is increasingly restrictive as the type grows, we will use it to construct new examples of generalized Kähler structures (see Section 6). Also note that Examples 5.6, 5.7 and 5.8 are not Hamiltonian. We remark that the actions which are independently described by Lin and Tolman [16], as well as Hu [11], can be seen to be of this Hamiltonian type.

Finally, we provide a cohomological criterion which determines if a given action is Hamiltonian. If a trivially extended, isotropic action $\rho : \mathfrak{g} \longrightarrow C^\infty(E)$ preserving $\mathcal{J}$ is given, then we may decompose

$\rho(a) = Z_a + \zeta_a \in L \oplus \overline{L}$ for all $a \in \mathfrak{g}_{\mathbb{C}}$. Since this is a Courant morphism, we may then define an equivariant Cartan model for the differential complex $(\Omega^\bullet(L), d_L)$, by considering equivariant polynomial functions $\Phi : \mathfrak{g}_{\mathbb{C}} \longrightarrow \Omega^\bullet(L)$, and equivariant derivative

$$(d_{\mathfrak{g}_{\mathbb{C}}}\Phi)(a) = d(\Phi(a)) - i_{Z_a}\Phi(a), \quad \forall a \in \mathfrak{g}_{\mathbb{C}}.$$

Since $\rho(a)$ preserves $\mathcal{J}$, we see that $\zeta_a \in \mathfrak{g}_{\mathbb{C}}^* \otimes \Omega^1(L)$ defines an equivariant closed 3-form. Supposing that $[\zeta_a] = 0$ in $H^3_{\mathfrak{g}_{\mathbb{C}}}(L)$, we then have

$$\zeta_a = d_{\mathfrak{g}_{\mathbb{C}}}(\varepsilon + h_a),$$

for $\varepsilon \in \Omega^2(L)$ an invariant d_L-closed form and $h_a \in \mathfrak{g}_{\mathbb{C}}^* \otimes \Omega^0(L)$ an equivariant function. Supposing further that $[\varepsilon] = 0$ in the invariant cohomology $H^2(L)^{\mathfrak{g}_{\mathbb{C}}}$, then $\varepsilon = d_L\eta$ for an invariant 1-form η, and

$$\zeta_a = d_{\mathfrak{g}_{\mathbb{C}}}(h_a + i_{Z_a}\eta),$$

implying that $\rho(a) = D(f_a)\ \ \forall a \in \mathfrak{g}$, where $f_a = h_a + i_{z_a}\eta$. This provides the following result.

Proposition 5.14. *Let ρ be a trivially extended, isotropic action preserving a generalized complex structure. Then it is Hamiltonian if and only if the classes $[\zeta_a] \in H^3_{\mathfrak{g}_{\mathbb{C}}}(L)$ and $[\varepsilon] \in H^2(L)^{\mathfrak{g}_{\mathbb{C}}}$, defined above, vanish.*

6 Generalized Kähler reduction

A *generalized Kähler structure* [7] on an exact Courant algebroid E is a pair of commuting generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ such that

$$\langle \mathcal{J}_1\mathcal{J}_2 v, v\rangle > 0 \qquad \text{for all } v \in E.$$

The symmetric endomorphism $\mathcal{G} = \mathcal{J}_1\mathcal{J}_2$ therefore defines a positive-definite metric on E, called the *generalized Kähler metric.* In this section we follow the original treatment [10] of Kähler reduction and extend it to the generalized setting.

6.1 Reduction procedure

We consider an extended action $\rho : \mathfrak{a} \times M \longrightarrow E$ on an exact Courant algebroid E over a manifold M with image distribution $K = \rho(\mathfrak{a})$.

Theorem 6.1. (Generalized Kähler reduction): *Let $(M, \mathcal{J}_1, \mathcal{J}_2)$ be a generalized Kähler structure preserved by the extended action. Let $P \subset M$ be a leaf of Δ_b on which G acts freely and properly, and over which $\mathcal{J}_1 K = K$ and K is isotropic. Then $\mathcal{J}_1$ and $\mathcal{J}_2$ can be reduced and define a generalized Kähler structure on the reduced Courant algebroid.*

Proof. Since K is isotropic, the reduced Courant algebroid is exact, and by Theorem 5.2, $\mathcal{J}_1$ descends to E_{red}. In order to show that $\mathcal{J}_2$ also descends, we will find an identification of E_{red} with a subbundle of $K^\perp$ which is invariant by $\mathcal{J}_2$.

Let $K^{\mathcal{G}}$ denote the orthogonal of K with respect to the metric $\mathcal{G}$. Since $\mathcal{J}_1 K = K$,

$$K^{\mathcal{G}} = (\mathcal{J}_2\mathcal{J}_1 K)^\perp = (\mathcal{J}_2 K)^\perp = \mathcal{J}_2 K^\perp. \tag{40}$$

Since $K \subset K^\perp$ over P, we have the $\mathcal{G}$-orthogonal decomposition of $K^\perp$ over P as

$$K^\perp = K \oplus (K^{\mathcal{G}} \cap K^\perp).$$

It follows from (40) that $K^{\mathcal{G}} \cap K^\perp$ is $\mathcal{J}_2$-invariant. Using the natural identification

$$E_{red} = (K^\perp/K)/G \cong (K^{\mathcal{G}} \cap K^\perp)/G$$

and the fact that $\mathcal{J}_2$ is G-invariant, we obtain an induced orthogonal endomorphism $\mathcal{J}_2^{red} : E_{red} \longrightarrow E_{red}$ satisfying $(\mathcal{J}_2^{red})^2 = -1$. It remains to check that $\mathcal{J}_2^{red}$ is integrable.

In order to verify integrability, we first describe the $+i$-eigenbundle of $\mathcal{J}_2^{red}$. Let L_2 be the $+i$-eigenbundle of $\mathcal{J}_2$. The $+i$-eigenbundle of $\mathcal{J}_2^{red}$ is the image under the projection $p : K_{\mathbb{C}}^{\perp} \longrightarrow E_{red} \otimes \mathbb{C}$ of $L_2 \cap (K_{\mathbb{C}}^{\perp} \cap K_{\mathbb{C}}^{\mathcal{G}})$. But since

$$L_2 \cap K_{\mathbb{C}}^{\perp} = \mathcal{J}_2(L_2 \cap K_{\mathbb{C}}^{\perp}) = L_2 \cap \mathcal{J}_2(K_{\mathbb{C}}^{\perp}) = L_2 \cap K_{\mathbb{C}}^{\mathcal{G}},$$

it follows that the $+i$-eigenbundle of $\mathcal{J}_2^{red}$ is

$$p(L \cap (K_{\mathbb{C}}^{\perp} \cap K_{\mathbb{C}}^{\mathcal{G}})) = p(L \cap K_{\mathbb{C}}^{\perp}) = (L_2)_{red},$$

the reduction of the Dirac structure L_2. It follows that $(L_2)_{red}$ is a smooth and maximal isotropic subbundle of $E_{red} \otimes \mathbb{C}$, and by Theorem 4.2 we know that it is integrable. So $\mathcal{J}_2^{red}$ is integrable.

Finally, we need to show that $(\mathcal{J}_1^{red}, \mathcal{J}_2^{red})$, where $\mathcal{J}_1^{red}$ is the reduction of L_1, is a generalized Kähler pair in E_{red}. For that, we note that $K^{\mathcal{G}} \cap K^{\perp}$ is $\mathcal{J}_1$-invariant, since $\mathcal{J}_1(K^{\perp}) = K^{\perp}$ and $\mathcal{J}_1(K^{\mathcal{G}}) = K^{\mathcal{G}}$. So $\mathcal{J}_1$ induces an endomorphism of E_{red}, which coincides with the Dirac reduction $\mathcal{J}_1^{red}$ since they have the same $+i$-eigenbundle:

$$(L_1)_{red} = \frac{L_1 \cap K_{\mathbb{C}}^{\perp} + K_{\mathbb{C}}}{K_{\mathbb{C}}} = \frac{(L_1 \cap K_{\mathbb{C}}) \oplus (L_1 \cap K_{\mathbb{C}}^{\mathcal{G}} \cap K_{\mathbb{C}}^{\perp}) + K_{\mathbb{C}}}{K_{\mathbb{C}}} = p(L_1 \cap K_{\mathbb{C}}^{\mathcal{G}} \cap K_{\mathbb{C}}^{\perp}).$$

The fact that $\mathcal{J}_1^{red}$ and $\mathcal{J}_2^{red}$ form a generalized Kähler pair is now a direct consequence of the fact that the restrictions of $\mathcal{J}_1$ and $\mathcal{J}_2$ to $K^{\mathcal{G}} \cap K^{\perp}$ commute and their product is positive definite. □

An important particular case of Theorem 6.1 is when the extended action admits a moment map.

Corollary 6.2. *Let $(M, \mathcal{J}_1, \mathcal{J}_2)$ be a generalized Kähler structure preserved by an extended action admitting a moment map $\mu : M \longrightarrow \mathfrak{h}^*$. Assume that the G-action on $\mu^{-1}(0)$ is free and proper. If $\mathcal{J}_1(K) = K$ over $\mu^{-1}(0)$, and the induced symmetric form $c_0 \in S^2\mathfrak{g}^*$ vanishes, then $\mathcal{J}_1$ and $\mathcal{J}_2$ can be reduced to M_{red} and define a generalized Kähler structure.*

This corollary follows from the fact that if c_0 vanishes, then both the isotropy action and the full action along $\mu^{-1}(0)$ are isotropic, i.e. $K \subset K^{\perp}$ on the level set. Of course these hypotheses are all fulfilled for a complexified Hamiltonian action as in Theorem 5.13. We now state the particular case when $\mathcal{J}_1$ is a symplectic structure since we use it in the next section.

Corollary 6.3. *Let $(\mathcal{J}_1, \mathcal{J}_2)$ be a generalized Kähler structure on $E = TM \oplus T^*M$ with $H = 0$, such that $\mathcal{J}_1$ is an ordinary symplectic structure. Assume that there is a Hamiltonian action on $(M, \mathcal{J}_1)$, with moment map $\mu : M \longrightarrow \mathfrak{g}^*$, and preserving $\mathcal{J}_2$. If the action of G on $\mu^{-1}(0)$ is free and proper, then the symplectic reduced space $M_{red} = \mu^{-1}(0)/G$ carries a generalized Kähler structure given by $(\mathcal{J}_1^{red}, \mathcal{J}_2^{red})$.*

We indicate that this result was independently obtained in [16]. Also, when $\mathcal{J}_2$ is a complex structure, then $\mathcal{J}_2^{red}$ is as well, and we recover the original Kähler reduction of [10].

Example 6.4. (Symplectic cut): Let $(M, \mathcal{J}_1, \mathcal{J}_2)$ be a generalized Kahler manifold as in Corollary 6.3. Assume that there is a Hamiltonian S^1-action on M preserving $\mathcal{J}_2$, and let $f : M \longrightarrow \mathbb{R}$ be its moment map. Consider $\mathbb{C}$ with its natural Kähler structure (ω, I), and equipped with the S^1-action $\theta \cdot z := e^{i\theta} z$. Then $N = M \times \mathbb{C}$ has a generalized Kähler structure $(\mathcal{J}_1', \mathcal{J}_2')$, where $\mathcal{J}_1'$ is the product symplectic structure and $\mathcal{J}_2' = \mathcal{J}_2 \times I$, and

$$\mu : N \longrightarrow \mathbb{R}; \qquad \mu(p, z) = f(p) + |z|^2$$

is a moment map for the diagonal S^1-action on N. This action preserves the generalized Kähler structure so, by Corollary 6.3, the symplectic quotient of N inherits a generalized Kähler structure.

6.2 Examples of generalized Kähler structures on $\mathbb{C}P^2$

Now we apply the results from the last section to produce new examples of generalized Kähler structure on $\mathbb{C}P^2$ with type change. The method consists of deforming the standard Kähler structure in $\mathbb{C}^3$ so that the deformed structure is still preserved by the circle action

$$e^{i\theta} : (z_1, z_2, z_3) \mapsto (e^{i\theta} z_1, e^{i\theta} z_2, e^{i\theta} z_3). \tag{41}$$

Then Corollary 6.3 implies that $\mathbb{C}P^2$, regarded as a symplectic reduction of $\mathbb{C}^3$, inherits a reduced generalized Kähler structure.

In the computations that follow, it will be convenient to use differential forms to describe a generalized complex structure $\mathcal{J}$ on a manifold M. So we recall from [7] that $\mathcal{J}$ is completely determined by its *canonical line bundle*, $C \subset \wedge^\bullet T^*_{\mathbb{C}}M$. This bundle is defined as the Clifford annihilator of L, the $+i$-eigenspace of $\mathcal{J}$. The fact that L is a Dirac structure of real index zero ($L \cap \overline{L} = \{0\}$) translates into properties for C: if φ is a nonvanishing local section of C, then

- At each point, $\varphi = e^{B+i\omega} \wedge \Omega$, where B and ω are real 2-forms and Ω is a decomposable complex k-form;
- There is a local section $X + \xi \in C^\infty(TM \oplus T^*M)$ such that
$$d\varphi = (X + \xi) \cdot \varphi = i_X\varphi + \xi \wedge \varphi;$$
- If σ is the linear map which acts on k-forms by $\sigma(a) = (-1)^{\frac{k(k-1)}{2}} a$, then the Mukai pairing $(\varphi, \overline{\varphi})$ must be nonzero, where
$$(\varphi, \overline{\varphi}) := (\varphi \wedge \sigma(\overline{\varphi}))_{top}.$$

The subscript *top* indicates a projection to the volume form component.

We begin with the standard Kähler structure on $(\mathbb{C}^3, \mathcal{J}_\omega, \mathcal{J}_I)$, defined by the following differential forms:

$$\begin{aligned}\Omega &= dz_0 \wedge dz_1 \wedge dz_2\\ \omega &= \tfrac{i}{2}(dz_0 \wedge d\bar{z}_0 + dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)\end{aligned}$$

As explained in [7], it is possible to deform this Kähler structure as a generalized Kähler structure in such a way that ω is unchanged whereas the complex structure Ω becomes a generalized complex structure of generic type 1. To achieve this, we must select a deformation $\varepsilon \in C^\infty(L_+^* \otimes L_-^*)$, where

$$L_\pm^* = \{X \pm i\omega(X) \ : \ X \in T_{1,0}\},$$

which satisfies the Maurer-Cartan equation $\overline{\partial}\varepsilon + \frac{1}{2}[\varepsilon, \varepsilon] = 0$. Then in regions where ε does not invalidate the open condition that $e^\varepsilon\Omega$ be of real index zero, $(e^\varepsilon\Omega, e^{i\omega})$ will be a generalized Kähler pair.

Example 6.5. In this example we deform the structure in $\mathbb{C}^3$ so that the reduced structure in $\mathbb{C}P^2$ has type change along a triple line. A similar deformation and quotient has been considered independently by Lin and Tolman [16], and a generalized Kähler structure on $\mathbb{C}P^2$ with type change along a triple line has recently been constructed by Hitchin [9] using a different method.

The deformation. We select the decomposable element

$$\varepsilon = \tfrac{1}{2}z_0^2(\partial_1 + \tfrac{1}{2}dz_1) \wedge (\partial_2 - \tfrac{1}{2}dz_2),$$

whose bivector component $\frac{1}{2}z_0^2\partial_1 \wedge \partial_2$ is a quadratic holomorphic Poisson structure. The projectivization of this structure is a Poisson structure on $\mathbb{C}P^2$ vanishing to order 3 along the line $z_0 = 0$. The deformed complex structure in $\mathbb{C}^3$ can be written explicitly (we omit the wedge symbol):

$$\begin{aligned}\varphi = e^\varepsilon dz_0dz_1dz_2 &= (1+\varepsilon)dz_0dz_1dz_2\\ &= dz_0dz_1dz_2 - \tfrac{1}{2}z_0^2dz_0 - \tfrac{1}{4}z_0^2dz_0dz_2d\bar{z}_2 + \tfrac{1}{4}z_0^2dz_0dz_1d\bar{z}_1 + \tfrac{1}{8}z_0^2dz_0dz_1dz_2d\bar{z}_2d\bar{z}_1\\ &= -\tfrac{1}{2}z_0^2dz_0\exp(-\tfrac{2}{z_0^2}dz_1dz_2 + \tfrac{1}{2}(dz_2d\bar{z}_2 - dz_1d\bar{z}_1)) \qquad (42)\end{aligned}$$

Let $\zeta = -\frac{1}{2}z_0^2dz_0$ and $b + i\sigma = -\frac{2}{z_0^2}dz_1dz_2 + \frac{1}{2}(dz_2d\bar{z}_2 - dz_1d\bar{z}_1)$. Then the pure differential form φ is of real index zero as long as the Mukai pairing of φ with its complex conjugate satisfies

$$(\varphi, \overline{\varphi}) = \sigma^2 \wedge \zeta \wedge \overline{\zeta} \neq 0.$$

Calculating this quantity, we obtain:

$$\sigma^2 \wedge \zeta \wedge \overline{\zeta} = \tfrac{1}{2}(4 - |z_0|^4)dz_0dz_1dz_2d\bar{z}_0d\bar{z}_1d\bar{z}_2,$$

proving that $(\varphi, e^{i\omega})$ defines a generalized Kähler structure in $\mathbb{C}^3$ away from the cylinder $|z_0| = \sqrt{2}$.

The reduction. Notice that the line generated by φ, and hence the generalized complex structure it defines, is invariant by the S^1-action given by (41). Hence, by Corollary 6.3, the symplectic reduction of $\mathbb{C}^3$ will have a reduced generalized Kähler structure induced by the deformed structure above. We spend the rest of this example describing this structure. The particular reduction we wish to calculate is the quotient of the unit sphere $\sum_i z_i \bar{z}_i = 1$ by the S^1-action give by (41).

We begin with the generalized complex structure φ given by equation (42). The induced Dirac structure on the reduced Courant algebroid may be calculated by pulling back to the unit sphere in $\mathbb{C}^3$ and pushing forward to the quotient. The latter operation on differential forms may be expressed simply as interior product with ∂_θ, the generator of the circle action

$$\partial_\theta = i(z_0\partial_0 - \bar{z}_0\overline{\partial}_0 + z_1\partial_1 - \bar{z}_1\overline{\partial}_1 + z_2\partial_2 - \bar{z}_2\overline{\partial}_2),$$

and this commutes with pull-back to the sphere. So let us first take interior product:

$$\begin{aligned} i_{\partial_\theta}\varphi &= (i_{\partial_\theta}\zeta)\exp(\tfrac{-\zeta\wedge i_{\partial_\theta}(b+i\sigma)}{i_{\partial_\theta}\zeta} + b + i\sigma) \\ &= -\tfrac{i}{2}z_0^3\exp(-\tfrac{dz_0}{z_0}(\tfrac{2(z_2dz_1 - z_1dz_2)}{z_0^2} + \tfrac{z_2d\bar{z}_2 + \bar{z}_2dz_2 - z_1d\bar{z}_1 - \bar{z}_1dz_1}{2}) - \tfrac{2dz_1dz_2}{z_0^2} + \tfrac{dz_2d\bar{z}_2 - dz_1d\bar{z}_1}{2}) \end{aligned}$$

Now we pull back to S^5 by imposing $1 = R^2 = \sum_i z_i\bar{z}_i$ and obtain a homogeneous differential form after rescaling:

$$\tilde{\varphi} = \exp(-\tfrac{dz_0}{z_0}(\tfrac{2(z_2dz_1 - z_1dz_2)}{z_0^2} + \tfrac{z_2d\bar{z}_2 + \bar{z}_2dz_2 - z_1d\bar{z}_1 - \bar{z}_1dz_1}{2R^2}) - \tfrac{2dz_1dz_2}{z_0^2} + \tfrac{dz_2d\bar{z}_2 - dz_1d\bar{z}_1}{2R^2}).$$

The holomorphic Euler vector field is $\mathsf{e} = \sum_i z_i\partial_i$ and $\partial_\theta = i(\mathsf{e} - \bar{\mathsf{e}})$. The radial vector field is $\partial_r = \mathsf{e} + \bar{\mathsf{e}}$. In order to be the pull-back of a form on $\mathbb{C}P^2$, a differential form α on $\mathbb{C}^3$ must satisfy $\mathcal{L}_{\mathsf{e}}\alpha = \mathcal{L}_{\bar{\mathsf{e}}}\alpha = i_{\mathsf{e}}\alpha = i_{\bar{\mathsf{e}}}\alpha = 0$. We have already ensured that $\mathcal{L}_{\mathsf{e}}\tilde{\varphi} = \mathcal{L}_{\bar{\mathsf{e}}}\tilde{\varphi} = 0$ and $i_{\mathsf{e}-\bar{\mathsf{e}}}\tilde{\varphi} = 0$, so now we may add a multiple of dR to ensure $i_{\mathsf{e}+\bar{\mathsf{e}}}\tilde{\varphi} = 0$. Since dR vanishes on the sphere, this is a trivial modification.

Recall that $i_{\mathsf{e}+\bar{\mathsf{e}}}\frac{dR}{R} = 1$, so we shall subtract

$$\tfrac{dR}{R}\wedge i_{\mathsf{e}+\bar{\mathsf{e}}}\tilde{\varphi} = \tfrac{dR}{R}(\tfrac{dz_0}{z_0}(\tfrac{z_2\bar{z}_2 - z_1\bar{z}_1}{R^2}) + \tfrac{\bar{z}_1dz_1 - \bar{z}_2dz_2}{R^2})\tilde{\varphi}$$

Finally we get a manifestly projective representative for the generator of the canonical bundle:

$$\begin{aligned} \varphi_B = \exp(&-\tfrac{dz_0}{z_0}(\tfrac{2(z_2dz_1 - z_1dz_2)}{z_0^2} + \tfrac{z_2d\bar{z}_2 + \bar{z}_2dz_2 - z_1d\bar{z}_1 - \bar{z}_1dz_1}{2R^2}) - \tfrac{2dz_1dz_2}{z_0^2} + \tfrac{dz_2d\bar{z}_2 - dz_1d\bar{z}_1}{2R^2} \\ &- \tfrac{dR}{R}(\tfrac{dz_0}{z_0}(\tfrac{z_2\bar{z}_2 - z_1\bar{z}_1}{R^2}) + \tfrac{\bar{z}_1dz_1 - \bar{z}_2dz_2}{R^2})) \end{aligned}$$

This differential form is closed, but blows up along the type change locus, where one can see by rescaling that it defines a complex structure. This generalized complex structure, together with the Fubini-Study symplectic structure, forms a generalized Kähler structure on $\mathbb{C}P^2$.

It may be of interest to express this generalized Kähler structure in affine coordinates (z_1, z_2) where $z_0 = 1$. Then the type change locus is the line at infinity. Define $r^2 = z_1\bar{z}_1 + z_2\bar{z}_2$:

$$\varphi_B = \exp(-2dz_1dz_2 + \tfrac{dz_2d\bar{z}_2 - dz_1d\bar{z}_1}{2(1+r^2)} - \tfrac{1}{2}\tfrac{d(r^2)(\bar{z}_1dz_1 - \bar{z}_2dz_2)}{(1+r^2)^2}))$$

The form defining the Fubini-Study symplectic form in these coordinates is, as usual:

$$\varphi_A = \exp(-\tfrac{1}{2}\tfrac{(1+r^2)(dz_1d\bar{z}_1 + dz_2d\bar{z}_2) - (\bar{z}_1dz_1 + \bar{z}_2dz_2)(z_1d\bar{z}_1 + z_2d\bar{z}_2))}{(1+r^2)^2})$$

An important constituent of a generalized Kähler structure is its associated bi-Hermitian metric; this can be derived from the above forms as follows. Define real 2-forms ω_1, ω_2, b such that $\varphi_A = e^{i\omega_1}$ and $\varphi_B = e^{b+i\omega_2}$. Then the bi-Hermitian metric g is simply

$$g = -\omega_2 b^{-1}\omega_1.$$

Example 6.6. To demonstrate the versatility of the quotient construction we now construct a generalized Kähler structure on $\mathbb{C}P^2$ with type change along a slightly more general cubic: the union of three distinct lines forming a triangle. We postpone the discussion of the general cubic curve to a future paper.

The deformation. In this example we select a deformation ε given by the following decomposable section of $L_+^* \otimes L_-^*$:

$$\varepsilon = \tfrac{1}{2}(z_0(\partial_1 + \tfrac{1}{2}d\bar{z}_1) + z_1(\partial_2 + \tfrac{1}{2}d\bar{z}_2) + z_2(\partial_0 + \tfrac{1}{2}d\bar{z}_0)) \wedge (z_0(\partial_2 - \tfrac{1}{2}d\bar{z}_2) + z_1(\partial_0 - \tfrac{1}{2}d\bar{z}_0) + z_2(\partial_1 - \tfrac{1}{2}d\bar{z}_1))$$

whose bivector component $\beta = (z_0^2 - z_1z_2)\partial_1\partial_2 + (z_1^2 - z_2z_0)\partial_2\partial_0 + (z_2^2 - z_0z_1)\partial_0\partial_1$ is a quadratic holomorphic Poisson structure on $\mathbb{C}^3$. This induces a Poisson structure on $\mathbb{C}P^2$ vanishing on the zero set of the following cubic polynomial:

$$\begin{aligned}\mathsf{e} \wedge \beta &= (z_0^3 + z_1^3 + z_2^3 - 3z_0z_1z_2)\partial_0\partial_1\partial_2 \\ &= (z_0 + z_1 + z_2)(z_0 + \lambda z_1 + \lambda^2 z_2)(z_0 + \lambda^2 z_1 + \lambda z_2)\partial_0\partial_1\partial_2,\end{aligned}$$

where $\mathsf{e} = \sum z_i\partial_i$ is the holomorphic Euler vector field and λ is a third root of unity. We see that the vanishing set of this Fermat cubic is the union of three distinct lines in the plane which intersect at the points $\{[1:1:1], [1:\lambda:\lambda^2], [1:\lambda^2:\lambda]\}$.

The deformed complex structure can be written explicitly:

$$\begin{aligned}\varphi &= e^{\varepsilon} dz_0dz_1dz_2 = (1+\varepsilon)dz_0dz_1dz_2 \\ &= (\tfrac{1}{2}(-z_0^2 + z_1z_2)dz_0 + c.p.)\exp(-\tfrac{1}{2}\tfrac{z_1^2+z_0z_2}{-z_2^2+z_0z_1}dz_1d\bar{z}_2 + \tfrac{1}{2}\tfrac{z_0^2+z_1z_2}{-z_2^2+z_0z_1}dz_0d\bar{z}_2 + c.p.),\end{aligned}$$

where "c.p." denotes cyclic permutations of $\{0,1,2\}$. The pure differential form φ is of real index zero as long as it has nonvanishing Mukai pairing with its complex conjugate:

$$\langle \varphi, \overline{\varphi} \rangle = (\tfrac{R^4}{4} - 1)dz_0dz_1dz_2d\bar{z}_0d\bar{z}_1d\bar{z}_2,$$

where $R^2 = |z_0|^2 + |z_1|^2 + |z_2|^2$. The generalized almost complex structure determined by φ on the ball of radius $\sqrt{2}$ is not integrable, however, since

$$d\varphi = \tfrac{1}{2}(z_0d\bar{z}_0 + z_1d\bar{z}_1 + z_2d\bar{z}_2)dz_0dz_1dz_2.$$

Nonetheless, when pulled back to the unit sphere in $\mathbb{C}^3$ this derivative vanishes, and hence we may proceed as before, quotienting by the S^1-action (41), as we do next.

The reduction. We begin with the generalized complex structure φ:

$$\varphi = (\tfrac{1}{2}(-z_0^2 + z_1z_2)dz_0 + c.p.)\exp(-\tfrac{1}{2}\tfrac{z_1^2+z_0z_2}{-z_2^2+z_0z_1}dz_1d\bar{z}_2 + \tfrac{1}{2}\tfrac{z_0^2+z_1z_2}{-z_2^2+z_0z_1}dz_0d\bar{z}_2 + c.p.).$$

As in Example 6.5, we calculate the interior product by ∂_θ:

$$\begin{aligned}i_{\partial_\theta}\varphi = -\tfrac{i(z_0^3+z_1^3+z_2^3-3z_0z_1z_2)}{2}\exp(&\tfrac{(z_2|z_0|^2+z_2|z_1|^2+z_0^2\bar{z}_1+2z_0z_1\bar{z}_2+z_1^2\bar{z}_0)dz_0dz_1+(z_1^3-z_2^3)dz_0d\bar{z}_0)}{2(z_0^3+z_1^3+z_2^3-3z_0z_1z_2)} \\ &+ \tfrac{(z_0^2z_1-2z_0z_2^2+z_1^2z_2)dz_0d\bar{z}_1-(z_0^2z_2-2z_0z_1^2+z_1z_2^2)dz_0d\bar{z}_2}{2(z_0^3+z_1^3+z_2^3-3z_0z_1z_2)} + c.p.)\end{aligned}$$

Now we pull back to S^5 by imposing $1 = R^2 = \sum_i z_i\bar{z}_i$ and obtain a homogeneous differential form after rescaling:

$$\begin{aligned}\tilde{\varphi} = \exp(&\tfrac{(z_2|z_0|^2+z_2|z_1|^2+z_0^2\bar{z}_1+2z_0z_1\bar{z}_2+z_1^2\bar{z}_0)dz_0dz_1+(z_1^3-z_2^3)dz_0d\bar{z}_0}{2R^2(z_0^3+z_1^3+z_2^3-3z_0z_1z_2)} \\ &+ \tfrac{(z_0^2z_1-2z_0z_2^2+z_1^2z_2)dz_0d\bar{z}_1-(z_0^2z_2-2z_0z_1^2+z_1z_2^2)dz_0d\bar{z}_2}{2R^2(z_0^3+z_1^3+z_2^3-3z_0z_1z_2)} + c.p.)\end{aligned}$$

As in the previous example, we subtract from this the quantity $\frac{dR}{R}\wedge i_{\mathsf{e}+\bar{\mathsf{e}}}\tilde{\varphi}$, obtaining finally a manifestly projective representative for the generator for the canonical bundle:

$$\begin{aligned}\varphi_B = \exp(&\tfrac{((z_1^3-z_2^3-z_0^3-z_0z_1z_2)|z_1|^2-(z_2^3-z_0^3-z_1^3-z_0z_1z_2)|z_2|^2+2z_0^2(z_2^2\bar{z}_1-z_1^2\bar{z}_2))dz_0d\bar{z}_0}{2R^2(z_0^3+z_1^3+z_2^3-3z_0z_1z_2)}\\ &+\tfrac{(z_1\bar{z}_0(z_0^3-z_1^3+z_2^3+z_0z_1z_2)-2z_0\bar{z}_2(z_1^3+z_2^3-z_0z_1z_2)-2|z_0|^2z_0z_2^2+2|z_2|^2z_2z_1^2)dz_0d\bar{z}_1}{2R^2(z_0^3+z_1^3+z_2^3-3z_0z_1z_2)}\\ &+\tfrac{(z_2\bar{z}_0(-z_0^3-z_1^3+z_2^3-z_0z_1z_2)+2z_0\bar{z}_1(z_1^3+z_2^3-z_0z_1z_2)+2|z_0|^2z_0z_1^2-2|z_1|^2z_1z_2^2)dz_0d\bar{z}_2}{2R^2(z_0^3+z_1^3+z_2^3-3z_0z_1z_2)}\\ &+\tfrac{(z_2(|z_0|^2|z_1|^2+z_0^2\bar{z}_1\bar{z}_2+\bar{z}_0^2z_1z_2+c.p.))dz_0dz_1}{2R^2(z_0^3+z_1^3+z_2^3-3z_0z_1z_2)}\\ &+c.p.)\end{aligned}$$

This differential form is closed, but blows up along the three distinct lines of the type change locus, where one can verify by rescaling that it defines a complex structure. This generalized complex structure, together with the Fubini-Study symplectic structure, forms a generalized Kähler structure on $\mathbb{C}P^2$.

In affine coordinates (z_1, z_2) for $\mathbb{C}P^2$, the type change locus consists of three lines intersecting at $\{(1,1),(\lambda,\lambda^2),(\lambda^2,\lambda)\}$. Define $r^2 = z_1\bar{z}_1 + z_2\bar{z}_2$. We may now write φ_B in these coordinates:

$$\begin{aligned}\varphi_B = \exp(&\tfrac{((z_2^3-1-z_1^3-z_1z_2)|z_2|^2-(1-z_1^3-z_2^3-z_1z_2)+2z_1^2(\bar{z}_2-z_2^2))dz_1d\bar{z}_1}{2(1+r^2)(1+z_1^3+z_2^3-3z_1z_2)}\\ &+\tfrac{((1-z_1^3-z_2^3-z_1z_2)-(z_1^3-z_2^3-1-z_1z_2)|z_1|^2+2z_2^2(z_1^2-\bar{z}_1))dz_2d\bar{z}_2}{2(1+r^2)(1+z_1^3+z_2^3-3z_1z_2)}\\ &+\tfrac{(z_2\bar{z}_1(z_1^3-z_2^3+1+z_1z_2)-2z_1(z_2^3+1-z_1z_2)-2|z_1|^2z_1+2z_2^2)dz_1d\bar{z}_2}{2(1+r^2)(1+z_1^3+z_2^3-3z_1z_2)}\\ &+\tfrac{(z_1\bar{z}_2(-z_2^3-1+z_1^3-z_1z_2)+2z_2(1+z_1^3-z_1z_2)+2|z_2|^2z_2-2z_1^2)dz_2d\bar{z}_1}{2(1+r^2)(1+z_1^3+z_2^3-3z_1z_2)}\\ &+\tfrac{(|z_1|^2|z_2|^2+z_1^2\bar{z}_2+\bar{z}_1^2z_2+z_1(|z_2|^2+z_2^2\bar{z}_1+\bar{z}_2^2z_1)+z_2(|z_1|^2+\bar{z}_1\bar{z}_2+z_1z_2))dz_1dz_2}{2(1+r^2)(1+z_1^3+z_2^3-3z_1z_2)})\end{aligned}$$

This form, together with the Fubini-Study symplectic structure

$$\varphi_A = \exp(-\tfrac{1}{2}\tfrac{(1+r^2)(dz_1d\bar{z}_1+dz_2d\bar{z}_2)-(\bar{z}_1dz_1+\bar{z}_2dz_2)(z_1d\bar{z}_1+z_2d\bar{z}_2))}{(1+r^2)^2}),$$

defines explicitly a generalized Kähler structure on $\mathbb{C}P^2$ with type change along a triangle as described above.

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Henrique Bursztyn, Instituto de Matemática Pura e Aplicada, Estrada Dona Castorina 110, Rio de Janeiro, 22460-320, Brasil.

Gil R. Cavalcanti, Mathematical Institute, 24-29 St. Giles, Oxford, OX1 3LB, UK.

Marco Gualtieri, Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA.