Constant mean curvature graphs in a class of warped product spaces

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Abstract

We give a new existence result for compact with boundary normal geodesic graphs of constant mean curvature in a class of warped product spaces. In particular, our result includes that of normal geodesic graphs with constant mean curvature in hyperbolic space $\mathbb{H}^{n+1}$ over a bounded domain in a totally geodesic $\mathbb{H}^n \subset \mathbb{H}^{n+1}$.
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1 Introduction

Let $\Omega \subset \mathbb{P}^n$ a mean convex bounded domain in a totally umbilical hypersurface $\mathbb{P}^n$ of standard hyperbolic space $\mathbb{H}^{n+1}$ of (normalized) constant sectional curvature -1. The Dirichlet problem for normal geodesic graphs with constant mean curvature and boundary $\Gamma = \partial\Omega$ was solved in [6] and [7] when $\mathbb{P}^n$ is a geodesic sphere and a horosphere, respectively. By a normal geodesic graph determined by $u \in \mathcal{C}^0(\Omega)$ we mean the hypersurface of points at distance $u(x)$ along the geodesic starting orthogonal to $\mathbb{P}^n$ at any point $x \in \Omega$.

Both of the above results are covered by the general theorem given in [2] on constant mean curvature normal geodesic graphs for a large class of warped product spaces. On the other hand, the case of a totally geodesic $\mathbb{P}^n = \mathbb{H}^n$ has neither been considered before nor follows from the results in [2]. Solving this basic problem was the starting point of this paper.

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In the general context of warped product ambient spaces (as considered in [1] and [2]) the setting of the latter problem goes as follows. Represent the hyperbolic space as the warped product manifold $\mathbb{H}^{n+1} = \mathbb{R} \times_{\cosh} \mathbb{H}^n$, and consider a bounded domain Ω in the totally geodesic hypersurface $\mathbb{P}^n = \{0\} \times \mathbb{H}^n$. Then the normal geodesic graph over Ω associated to a function $u \in \mathcal{C}^0(\Omega)$ is the hypersurface

$$\Sigma(u) = \{(u(x), x) : x \in \Omega\}.$$

In this paper, we solve the Dirichlet problem for constant mean curvature normal geodesic graphs in a class of warped product manifolds (includes $\mathbb{H}^{n+1}$) described in the sequel.

Let $(\mathbb{P}^n, \langle\ ,\ \rangle_{\mathbb{P}})$ denote an n-dimensional Riemannian manifold. Then let

$$M^{n+1} = \mathbb{R} \times_{\varrho} \mathbb{P}^n$$

be the product manifold $\mathbb{R} \times \mathbb{P}^n$ endowed with the warped product metric

$$\langle\ ,\ \rangle_M = \pi_{\mathbb{R}}^*(dt^2) + \varrho^2(\pi_{\mathbb{R}})\pi_{\mathbb{P}}^*(\langle\ ,\ \rangle_{\mathbb{P}}),$$

where $\varrho \in \mathcal{C}_+^\infty(\mathbb{R})$ and $\pi_{\mathbb{R}}$, $\pi_{\mathbb{P}}$ denote the corresponding projections. It is easy to see that $\mathcal{T} = \varrho T$ is a *closed conformal vector field* on M^{n+1} since

$$\bar{\nabla}_X \mathcal{T} = \varrho' X \quad \text{for any } \ X \in TM,$$

where $T = \partial/\partial t$ for $t \in \mathbb{R}$, and $\bar{\nabla}$ stands for the Levi-Civita connection in M^{n+1}. Moreover, each leaf of the foliation $\mathbb{P}_t = \{t\} \times \mathbb{P}^n$ is totally umbilical and has constant mean curvature

$$\mathcal{H}(t) = \varrho'(t)/\varrho(t). \tag{1}$$

Tashiro [9] called M^{n+1} a *pseudo-hyperbolic* space if $\mathbb{P}^n$ is complete and the warping function $\varrho \in \mathcal{C}_+^\infty(\mathbb{R})$ is a solution for some $c < 0$ of

$$\varrho'' + c\varrho = 0.$$

Thus either $\varrho(t) = \cosh(\sqrt{-c}t)$ or $\varrho(t) = e^{\sqrt{-c}t}$ up to changes of origin in $\mathbb{R}$. It is well known (cf. [8]) that the Riemannian product $\mathbb{R} \times \mathbb{P}^n$ together with the pseudo-hyperbolic manifolds can be characterized as being the universal cover of the complete manifolds carrying a closed conformal vector field without zeros and having constant Ricci curvature c in the direction of the field. Deck isometries is any subgroup of $\mathrm{Iso}(\mathbb{R}) \times \mathrm{Iso}(\mathbb{P})$, and quotients can also be characterized [5] as the complete manifolds supporting non-trivial solutions of the Obata type equation

$$\mathrm{Hess}\, \varphi\,(\ ,\ ) + c\varphi\,\langle\ ,\ \rangle = 0.$$

In this paper, we do not ask $\mathbb{P}^n$ to be complete since we work with graphs on a bounded domain. Moreover, we restrict ourselves to the case $\varrho(t) = \cosh t$ (for simplicity we take $c = -1$). The latter is because the case $\varrho(t) = e^t$ was already solved in [2] by different arguments that do *not* work in this case. Thus, for the remaining of the paper we denote

$$M^{n+1} = \mathbb{R} \times_{\cosh} \mathbb{P}^n$$

and $\Omega \subset \mathbb{P}_0 := \{0\} \times \mathbb{P}^n$ is a $\mathcal{C}^{2,\alpha}$ bounded domain. We also assume that Ω is mean convex, i.e., the mean curvature H_Γ of $\Gamma = \partial\Omega$ as a submanifold of $\mathbb{P}^n$ is positive with respect to the inner orientation.

By the *normal (geodesic) graph* $\Sigma^n = \Sigma^n(u)$ in M^{n+1} over Ω determined by a continuous function $u \colon \Omega \to \mathbb{R}$ vanishing at Γ we mean the compact hypersurface with boundary Γ defined as

$$\Sigma^n(u) = \{(u(x), x) : x \in \Omega\}.$$

In this paper we prove the following result.

Theorem 1. *Assume that the Ricci curvature of $\mathbb{P}^n$ satisfies that $\mathrm{Ric}_{\mathbb{P}} \geqslant -1$, and let $H > -1$ be given such that $-H_\Gamma < H \leqslant 0$. Then there exists a function $u \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$ whose normal graph $\Sigma^n(u)$ is a hypersurface in M^{n+1} with constant mean curvature H and boundary Γ.*

Finally, we refer to [1] and [3] for some global results.

2 The proof

A straightforward computation shows that $\Sigma^n(u)$ has constant mean curvature H (taken with respect to the downward pointing normal vector) and boundary Γ if $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ is a solution of the Dirichlet problem

$$\begin{cases} T(u) := \mathrm{Div}\left(\frac{Du}{\sqrt{\cosh^2 u + |Du|^2}}\right) + n\cosh u\left(H - \frac{\sinh u}{\sqrt{\cosh^2 u + |Du|^2}}\right) = 0 \\ u|_\Gamma = 0, \end{cases} \tag{2}$$

where Div and D denote the divergence and gradient operators in $\mathbb{P}^n$ respectively.

With the change of variable $s = s(t)$ given by

$$s = \phi(t) = \int_0^t \frac{1}{\cosh(r)} dr = \arctan(\sinh(t)).$$

the Dirichlet problem (2) becomes

$$\begin{cases} Q(v) := \mathrm{Div}\left(\frac{Dv}{\sqrt{1+|Dv|^2}}\right) + \frac{n}{\cos v}\left(H - \frac{\sin v}{\sqrt{1+|Dv|^2}}\right) = 0 \\ v|_\Gamma = 0. \end{cases} \tag{3}$$

Therefore, we have that

$$T(u) = 0 \quad \text{if and only if} \quad Q(v) = 0 \;\text{ where }\; v = \phi(u).$$

We have from (1) that the mean curvature of the hypersurface $\mathbb{P}_{t_0} = \{t_0\} \times \mathbb{H}^n$ is $\mathcal{H}(t_0) = \tanh t_0$. Thus, there is nothing to prove if $H = 0$ since $\Omega \subset \mathbb{P}_0$ itself is a minimal surface with boundary Γ.

We assume that $H < 0$. To prove the theorem we may apply to (3) the standard theory for quasilinear elliptic PDE's as given in [4]. We use the continuity method. Thus, we consider the family of Dirichlet problems

$$\begin{cases} Q_\tau(v_\tau) = 0 \quad \text{in} \quad \Omega \\ v_\tau|_\Gamma = 0 \end{cases} \tag{4}$$

where Q_τ equals Q except that we replace H by τH. Then, we prove that

$$J = \{\tau \in [0,1] : \text{ the problem (4) can be solved for } \tau\} \tag{5}$$

is nonempty, open and closed in $[0,1]$.

We have that J is not empty since $0 \in J$ with $v_0 = 0$ the trivial solution. We prove that J is open. Assuming that $\tau \in J$, we need to show that (4) can be solved in an open interval around τ. Let $\Sigma = \Sigma^n(u)$ denote the normal graph with constant mean curvature τH corresponding to $u = \phi^{-1}(v_\tau)$, where v_τ is the existing solution to (4). Recall that the linearized mean curvature operator about a normal geodesic graph in a Riemannian manifold M is

$$\mathcal{L} = \Delta + \|A\|^2 + \mathrm{Ric}_M(N, N)$$

where Δ is the Laplace-Beltrami operator on the graph, $\|A\|$ denotes the norm of its second fundamental form $A = A_N$ and N stands for a unit normal vector field along Σ. To prove that the operator Q is invertible, it suffices to show that $\mathcal{L}f \geqslant 0$ for some function f on Σ satisfying $f < 0$. We choose $f = \cosh(u)\,\Theta$ where

$$\Theta(p) = \langle N(p), T\rangle$$

and the orientation N of Σ has been taken such that $\Theta < 0$. Thus, we have $f < 0$.

Since Σ has constant mean curvature τH, we have from (20) and (21) in [1] that

$$\nabla f = -\cosh(u) A \nabla u \tag{6}$$

where ∇f is the gradient of f on Σ, and

$$\Delta f = \cosh(u)\mathrm{Ric}_M(N, \nabla u) - n\sinh(u)\tau H - \|A\|^2 f.$$

We use that

$$\nabla u = T - \Theta N, \tag{7}$$

and take into account that

$$\mathrm{Ric}_M(N, T) = -n\Theta$$

which follows from (22) in [1] using (1). We obtain that $\mathcal{L}f = -n\psi$ where

$$\psi = \sinh(u)\tau H + \cosh(u)\Theta.$$

But from Theorem 13 in [1] and because $\mathrm{Ric}_{\mathbb{P}} \geqslant -1$ we know that ψ is subharmonic on Σ. In particular, by the maximum principle and since $\psi|_{\partial\Sigma} = \Theta$, we obtain that $\psi \leqslant 0$ on Σ. Therefore, $\mathcal{L}f = -n\psi \geqslant 0$ on Σ with $f < 0$, and we conclude from the implicit function theorem that the Dirichlet problem (4) can be solved in an interval of τ.

To show that J is closed we have to obtain apriori $\mathcal{C}^{2,\alpha}$ estimates for any solution of the family of Dirichlet problems (4). Actually, standard theory for divergence type quasilinear elliptic equations and Schauder theory guarantee that it is sufficient to obtain apriori $\mathcal{C}^1$ estimation. In other words, it suffices to prove the existence of a constant $K = K(\Omega, H)$ independent of τ such that any solution $v_\tau \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$ of (4) satisfies

$$\|v_\tau\|_{\mathcal{C}^1(\bar{\Omega})} = \sup_{\Omega} |v_\tau| + \sup_{\Omega} |Dv_\tau| < K. \tag{8}$$

Let v_τ be a solution of (4), and let $\Sigma = \Sigma^n(u)$ denote the normal graph with constant mean curvature τH corresponding to $u = \phi^{-1}(v_\tau)$. Since $\partial\Sigma \subset \mathbb{P}_0$ and $\tau H \leqslant 0 = \inf_{[0,+\infty)} \mathcal{H}$, then from part (i) of Proposition 18 in [1] we have the apriori estimate

$$u \leqslant 0 \ \text{ on } \Omega. \tag{9}$$

We have seen above that ψ is a subharmonic function on Σ. Therefore, by the maximum principle,

$$\psi \leqslant \max_{\partial\Sigma} \psi = \Theta(q), \tag{10}$$

where $q \in \Gamma$ is a boundary point such that $\Theta(q) = \max_\Gamma \Theta$. First, we show that

$$\Theta(q) \leqslant -\frac{\sqrt{\kappa^2 - \tau^2 H^2}}{\kappa} < 0 \tag{11}$$

where $\kappa := \min_\Gamma H_\Gamma$. Observe that

$$\nabla\psi = \cosh(u)(\tau H\nabla u - A\nabla u).$$

From (7) and $u|_\Gamma = 0$, we obtain along Γ that

$$\nabla u = \langle \nabla u, \nu\rangle\nu = \langle T, \nu\rangle\nu,$$

and thus

$$\nabla\psi = \langle T, \nu\rangle(\tau H\nu - A\nu)$$

where ν denotes the inward pointing unit conormal vector field along Γ. Then, the maximum principle yields

$$\langle \nabla\psi(q), \nu_q\rangle = \langle T_q, \nu_q\rangle(\tau H - \langle A\nu_q, \nu_q\rangle) \leqslant 0. \tag{12}$$

Moreover, $\langle \nabla u, \nu\rangle = \langle T, \nu\rangle \leqslant 0$ along Γ since $u \leqslant u|_\Gamma$ on Σ. In fact, we may assume that $\langle T_q, \nu_q\rangle < 0$ since, otherwise, we obtain using (7) that $\Theta(q) = -1$, and (11) trivially holds. Thus (12) yields

$$\langle A\nu_q, \nu_q\rangle \leqslant \tau H.$$

Choosing an orthonormal basis $\{e_1, \ldots, e_{n-1}\}$ of $T_q\Gamma$, we obtain from

$$n\tau H = \sum_i \langle Ae_i, e_i\rangle + \langle A\nu_q, \nu_q\rangle$$

that

$$(n-1)\tau H \leqslant \sum_i \langle Ae_i, e_i\rangle. \tag{13}$$

Let η denote the inward pointing unit conormal η along Γ in $\mathbb{P}_0$. Then

$$\langle N, \eta\rangle = \langle T, \nu\rangle = -\sqrt{1-\Theta^2},$$

and hence

$$N = -\sqrt{1-\Theta^2}\,\eta + \Theta T. \tag{14}$$

Taking into account that $\mathbb{P}_0$ is totally geodesic in M^{n+1} and using (14), we have

$$\langle Ae_i, e_i\rangle = \langle \bar\nabla_{e_i} e_i, N\rangle = -\langle B_\eta e_i, e_i\rangle\sqrt{1-\Theta^2(q)} \tag{15}$$

where B_η stands for the second fundamental form of Γ in $\mathbb{P}_0$. We conclude from (13) and (15) that

$$\tau H + \kappa\sqrt{1-\Theta^2(q)} \leqslant 0. \tag{16}$$

In view of (16) consider the equation

$$P(x) := \tau H + \kappa\sqrt{1-x^2} = 0. \tag{17}$$

Our hypotheses yield

$$P(-1) = \tau H \leqslant 0 \quad \text{and} \quad P(0) = \tau H + \kappa \geqslant H + \kappa > 0.$$

It is easy to see that (17) has a unique root $-R(\tau) \in [-1, 0)$ where

$$R(\tau) = \frac{\sqrt{\kappa^2 - \tau^2 H^2}}{\kappa}.$$

Since $P(\Theta(q)) \leqslant 0$ by (16), then $\Theta(q) \leqslant -R(\tau)$ and (11) holds.

We are now ready to estimate $\sup_\Omega |v_\tau|$. Observe that (9) is equivalent to

$$v_\tau = \arctan(\sinh u) \leqslant 0 \ \text{ on } \Omega.$$

Thus, it suffices to estimate $\inf_\Omega v_\tau$ or, equivalently, $\underline{u} := \min_\Sigma u$.

Set $z = \sinh \underline{u}$ and recall that $z \leqslant 0$. Observe that the mean curvature vector of the totally umbilical slice $\mathbb{P}_{\underline{u}}$ is $-\tanh \underline{u}\, T$. Since the mean curvature vector of Σ is $-\tau H T$, it follows from the tangency principle that

$$z \geqslant \tau H \cosh \underline{u} = \tau H \sqrt{1+z^2}.$$

That is,

$$\tau H \leqslant \frac{z}{\sqrt{1+z^2}} \leqslant 0.$$

Taking into account that $-1 < H \leqslant \tau H \leqslant 0$ for every $0 \leqslant \tau \leqslant 1$, we get from here

$$z \geqslant \frac{\tau H}{\sqrt{1-\tau^2 H^2}} \geqslant \frac{H}{\sqrt{1-H^2}}$$

for every τ, concluding that

$$-C := \frac{H}{\sqrt{1-H^2}} \leqslant \sinh u \leqslant 0. \tag{18}$$

Therefore, taking $K_1 = K_1(H) =: \arctan C$ we have

$$\sup_\Omega |v_\tau| \leqslant K_1 \tag{19}$$

for every $\tau \in J$.

In order to estimate now $\sup_\Omega |Dv_\tau|$, first observe that

$$\Theta = \frac{-\cosh u}{\sqrt{\cosh^2 u + |Du|^2}} = \frac{-1}{\sqrt{1+|Dv_\tau|^2}}. \tag{20}$$

We have from (10) and (11) that

$$\tau H \sinh u + \Theta \cosh u \leqslant -\frac{\sqrt{\kappa^2 - \tau^2 H^2}}{\kappa} \leqslant -\frac{\sqrt{\kappa^2 - H^2}}{\kappa}.$$

Taking into account that $\tau H \sinh u \geqslant 0$ and using (18), we obtain

$$\Theta\sqrt{1+C^2} \leqslant \Theta \cosh u \leqslant -\frac{\sqrt{\kappa^2 - H^2}}{\kappa},$$

that is,

$$\Theta \leqslant -\frac{\sqrt{\kappa^2 - H^2}}{\kappa\sqrt{1+C^2}} < 0.$$

Therefore, from (20) we conclude for $K_2 = K_2(\Omega, H) := \frac{\sqrt{\kappa^2 C^2 + H^2}}{\sqrt{\kappa^2 - H^2}}$ that

$$\sup_\Omega |Dv_\tau| \leqslant K_2 \tag{21}$$

for any solution v_τ of (4). Then (19) and (21) yield (8), and the closeness of J follows from standard quasilinear elliptic PDE theory.

Finally, standard regularity results in the theory guarantee that any solution of $Q(v) = 0$ is smooth in Ω as required. This concludes the proof of Theorem 1.

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