Affine Skeletons and Monge-Ampère Equations

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Abstract

An important question about affine skeletons is the existence of differential equations that related to the "Affine Distance" and the "Area Distance" (hence to affine skeletons) the same way the Eikonal equation is related to the "Euclidean Distance" (and the medial axis). We show a surprisingly simple nonlinear second order PDE of Monge-Ampère type that relates to the affine skeletons (and extends the Eikonal equation for the medial axis). We also discuss some consequences and ideas that this new PDE formulation suggests.

Keywords: affine distance, medial axis, skeleton, affine geometry, monge-ampère equation, differential propagation.

1 Introduction

The medial axis is the most famous shape skeleton and it has been used in a wide range of applications. One of its attractive properties is the covariance by rigid transformations. In a series of papers ([6], [7], [5] and [1]), Giblin, Sapiro et al, introduced new skeletons inspired in the medial axis transformation, but having the nice property of being covariant by all affine transformations. They used two new definitions of distance (the Affine Distance and the Area Distance) to introduce two new affine symmetry sets: the Affine Distance Symmetry Set (ADSS) and the Area Distance Symmetry Set (AASS). As in the connection between the (euclidean) symmetry set and the medial axis, they selected a particular subset of the ADSS and AASS to be affine skeletons, namely, the Affine Distance Skeleton (ADS) and Affine Area Skeleton (AAS). They introduced also a third object, the Affine Envelope Symmetry Set (AESS), that does not come from a distance; besides, it's not obvious whether the AESS has a subset that can be properly called the "Affine Envelope Skeleton".

The Eikonal equation plays a central role on the research of distance functions and the medial axis. Thus, as pointed out by Giblin, Sapiro et al., a natural and important question is the existence of an analogous of the Eikonal for the affine-invariant skeletons. In this paper, we find surprisingly simple Monge-Ampère equations that play such role for the medial axis and these new skeletons.

More explicitly, using the only affine-invariant differential operators of second order $H = \det D^2 f$ and $J = k \cdot |D f|^3$, where k is the curvature of the level curve of f (as described in [3]), we find:

- The Medial Axis is the set of shocks of the solution to the homogeneous Monge-Ampère equation

$$\begin{cases} H = 0 \\ J = k, \text{if } x \in \Gamma \\ f(x) = 0, \text{if } x \in \Gamma \end{cases}$$

- The Affine Distance Skeleton is the set of shocks of the solution of the homogeneous Monge-Ampère equation

$$\begin{cases} H = 0 \\ J = 1, \text{if } x \in \Gamma \\ f(x) = 0, \text{if } x \in \Gamma \end{cases}$$

- The Affine Area Skeleton is the set of shocks of the solution the Monge-Ampère equation

$$\begin{cases} H = -4 \\ J = 0, \text{if } x \in \Gamma \\ f(x) = 0, \text{if } x \in \Gamma \end{cases}$$

The Monge-Ampère equations open the toolbox of PDEs for affine distances and skeletons. As with the Eikonal equation for Euclidean Distance and medial axis, the Monge-Ampère equations make possible the development of fast algorithms (similar to the Fast Marching Method) to compute the affine distances and its skeletons. Also, the new approach suggests a plethora of new ideas. As examples, we propose a definition of an Affine Distance for non-convex curves; an outer Affine Area Skeleton (or the inner skeleton of a non-convex curve Γ). The properties and merits (if any) of these proposals must be further investigated (as many other issues, like an appropriate treatment of shocks and the development of good numerical methods to solve these PDEs).

The structure of this paper is as follows: first, we briefly review the main properties of medial axis and the definitions of the affine skeletons as in [6], [7], [5] and [1]. Next, we show how the Affine Distance (and the old Euclidean Distance) can be thought of as solutions to homogeneous Monge-Ampère equations (which distance is obtained depends on specific boundary conditions). The following section adjusts the Monge-Ampère equation so that its solution is now the Area Distance, and we briefly suggest an extension of this distance for non-convex curves.

Many ideas in this paper come from Silva's Ph.D. thesis [8].

2 Medial Axis

In this section we list some medial axis topics that will be useful in the comparison with the affine skeletons.

Let Ω be a connected shape (a connected open set of the plane R^2) and Γ the boundary of Ω. The medial axis is (the closure of) the locus of the points X of R^2 where the distance of X to the a point P on the boundary Γ is reached by another point Q of Γ, provided that the distance is a global minimum at X.

$$d(X,P) = d(X,Q) = \min_{T \in \Gamma} d(X,T)$$

The set of pairs $(X, d(X,\Gamma))$ - the medial axis points with the distance function to the curve Γ - is usually called Medial Axis Transform (MAT).

In order to compare the medial axis skeleton with the affine ones we point out some alternative definitions:

- (closure of) the centers of maximal disks inside Ω. The distance function is the radius of each disk.
- (closure of) the center of disks inside Ω that are tangents to the curve Γ at least at two points. The distance function is the radius of each disk.
- the singularities of the "distance function of Γ ".
- Evolving each point of Γ along the normals with constant speed (say, 1); the medial axis is the locus of the evolution shocks. The distance function is the time when the shock takes place.

The last two items relate the medial axis with the Eikonal PDE (boundary value problem):

$$\begin{cases} |Df| = 1 \\ f(x) = 0, \text{if } x \in \Gamma \end{cases}$$

and its formulation as a initial value problem

$$\begin{cases} C(s,t)_t = N(s,t) \\ C(s,0) = \gamma(s) \end{cases}$$

where $\gamma(s)$ is a parametrization of Γ and $N(s)$ is the normal vector at $\varphi(s)$ (pointing inwards).

3 Affine Skeletons

In this section we briefly review the definition and main properties of the affine skeletons introduced by Sapiro, Giblin et al ([6], [7] and [1]). We assume, throughout this section, that Γ is a simple convex curve.

3.1 Affine Distance Skeleton (ADS)

Replacing the "distance" by "affine distance" in the definition of medial axis we get the affine distance skeleton - ADS. The ADS was introduced by Giblin and Sapiro in [6] and [7]. Let's review the definition of affine distance.

Let $\gamma(s)$ be a affine parametrization of the curve Γ. The affine distance from a point X to the point $\gamma(s)$ on the curve Γ is the area of the parallelogram defined by the vectors $\gamma_s(s)$ and $X - \gamma(s)$, that is (figure 1),

$$d(X, \gamma(s)) = \frac{1}{2}[\gamma_s(s), X - \gamma(s)]$$

where $[u, v]$ denotes the determinant of the matrix whose columns are the vectors u and v.

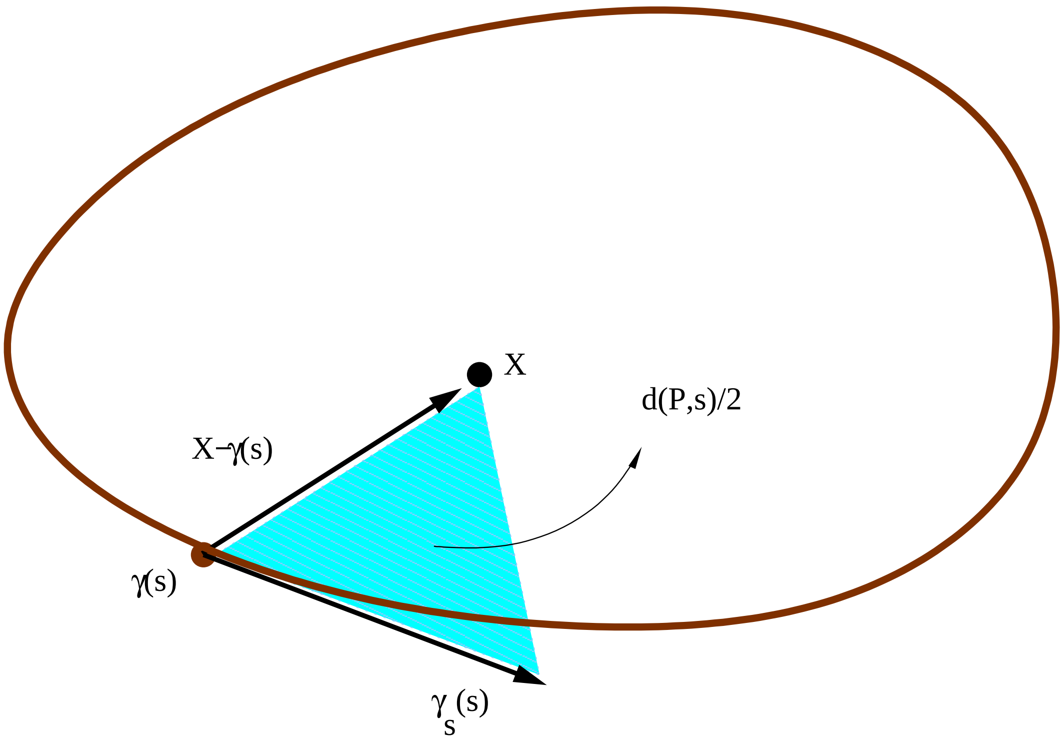

Figure 1. The affine distance

The affine distance from a point X to the convex curve Γ is the minimum distance of X to all points of Γ:

$$d(X, \Gamma) = \min_s d(X, \gamma(s))$$

Now, the affine distance skeleton (ADS) of Γ is the locus of points $X \in R^2$ such that the affine distance of X to the curve Γ is reached at two distinct points $P, Q \in \Gamma$:

$$d(X, P) = d(X, Q) = \min_{T \in \Gamma} d(X, T) = d(X, \Gamma)$$

Like the Euclidean Distance, the affine distance is zero on Γ and increases linearly along straight lines whose directions are given by the affine normals. In other words, the graph of the affine distance of Γ is a ruled surface. We can formulate the ADS as the shock points of the evolution

$$\begin{cases} C(s,t)_t = N(s,0) \\ C(s,0) = \gamma(s) \end{cases}$$

where $\gamma(s)$ is a parametrization of Γ and $N(s,t)$ is the affine normal vector at the point $C(s,t)$. But this is not quite a "front propagation" PDE, because the affine normal vector $N(s,0)$ that gives the velocity of each point is calculated at the starting point on Γ, not the affine normal vector at the point $C(s,t)$.

3.2 Affine Area Skeleton (AAS)

The affine area skeleton is based on another "distance", namely, the area distance. The area distance was introduced by Moisan [4], and used, with a slight modification, by Giblin and Sapiro et al [5]. The process is the following: take a point $\gamma(s)$ on Γ and a point X inside Γ; consider the chord that starts at $\gamma(s)$ and goes through X – this chord meets the curve Γ at one other point $\gamma(r)$ (remember, Γ is convex). The smallest area bounded by Γ and such chord is the area distance from X to the point $\gamma(s)$ on Γ (figure 2).

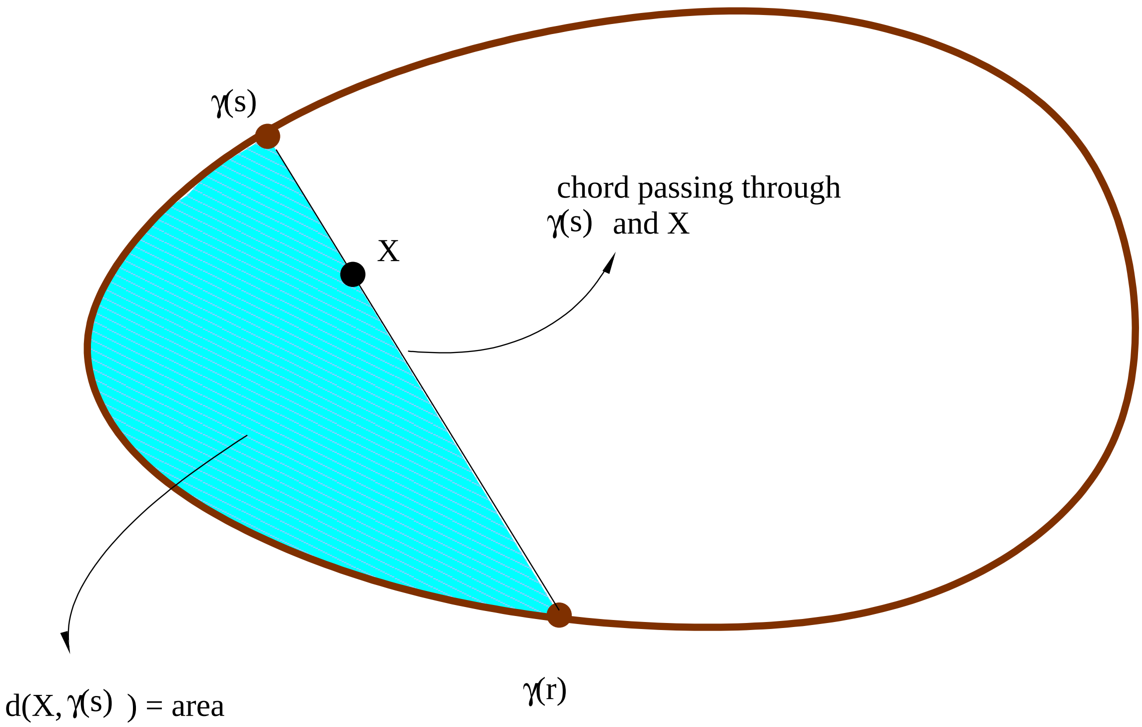

Figure 2. The area distance

The area distance of X to Γ is, as expected, the minimum of the distances of X to all the points of Γ

$$d(X,\Gamma) = \min_s d(X,\gamma(s))$$

The Affine Area Skeleton (AAS) of Γ is the locus of points $X \in R^2$ where the affine distance of X to Γ is reached at two (P and Q) or more points of Γ

$$d(X,P) = d(X,Q) = \min_{T\in\Gamma} d(X,T) = d(X,\Gamma)$$

The level curves of the area distance could be considered as the successive evolutions of Γ and the AAS would be the shock points of this evolution. This evolution of Γ has been called the "affine erosion" of Γ.

3.3 Affine Envelope Symmetry Set (AESS)

For the sake of completeness, we review the third symmetry set introduced by Giblin and Sapiro. The affine envelope symmetry set (AESS) is not based on a distance function. Instead, it is a variation of the definition of the medial axis by its bi-tangential circles. The AESS is the locus of centers of conics that have contact of order at least 3 at two or more points of Γ (an order 1 contact is just an intersection where the tangents do not coincide; order 2 means that the curves are tangent at the point of contact; finally, order 3 contact means that the curves are tangent and have the same curvature at the point of contact).

Since there is no "distance function" nor level curves involved, there is no mention to "curve evolution" or "shock points". Besides, the definition gives only a symmetry set. It's not clear how to select a subset of AESS as the natural candidate to be the "affine envelope skeleton".

4 $H = 0$: Medial Axis and Affine Distance Skeleton

4.1 Ruled Distances

The Euclidean Distance and the affine distance to a curve Γ share a couple of properties: both are zero on Γ and both increase linearly along straight lines. In other words, both can be thought as f in the equation

$$f(\gamma(s) + t.w(s)) = t$$

where $\gamma(s)$ is a parametrization of Γ and $w(s)$ is a continuous vector field over Γ (the propagation velocity vectors). In the Euclidean Distance, the vector $w(s)$ is the unitary normal vector at $\gamma(s)$; in the affine distance, the vector $w(s)$ is the affine normal vector at $\gamma(s)$. On both cases we have that $w_s(s)$ is colinear with $\gamma_s(s)$.

A simple calculation shows the following proposition:

Proposition 1. $D^2 f$ is singular at all points X where f is differentiable.

Proof. Since X is reached by the propagation, we write $X = \gamma(s) + t.w(s)$ for some s and t. Differentiating $f(X = \gamma(s) + t.w(s)) = t$ with respect to t and s we obtain

$$D f(X) \cdot (w) = 1 \tag{1}$$

and

$$D f(X)(\gamma_s + t.w_s) = 0 \tag{2}$$

Differentiating equation (1) with respect to t and equation (2) with respect to s) we have

$$D^2 f(X)(w)^2 = 0 \tag{3}$$

and

$$D^2 f(X)(\gamma_s + t.w_s)^2 + D f(X)(\gamma_{ss} + t.w_{ss}) = 0 \tag{4}$$

Finally, differentiating equation (1) with respect to s (or equation (2) in t) we have

$$D^2 f(X)(w, \gamma_s + t.w_s) + D f(X) \cdot (w_s) = 0 \tag{5}$$

Since w_s is colinear with γ_s we can write $w_s = -\lambda\gamma_s$ and then equation (2) shows that the gradient $D f$ is orthogonal to γ_s not only at Γ but at all the points on the line $X = \gamma(s) + t \cdot w(s)$. Together with equation (1) we have that $D f$ is constant over the line $X = \gamma(s) + t.w(s)$. We say that the gradient is transported over the characteristics curves (which are lines).

Since the gradient is constant along the direction w, this direction is a null-space of the Hessian. Indeed, using equation (4) and (5), the colinearity $w_s = -\lambda.\gamma_s$ implies that

$$(1+t\lambda)^2 D^2 f(X)(\gamma_s)^2 = -Df(X)(\gamma_{ss} + t.w_{ss}) \tag{6}$$

so

$$D^2 f(X)(w, \gamma_s) = 0 \tag{7}$$

Putting together equations (3) and (7) we conclude that $D^2 f(X) \cdot w = 0$. □

Equation (6) also gives a nice relation between the curvature of Γ and the curvature of the subsequent level curves. Let's use the $u\,v$ coordinate system (where u is the unitary vector tangent to the level curve at X, v is the normal to the same level curve and, therefore, $[u, v] = 1$). We saw that the gradient Df at X is the same as the gradient Df at $\gamma(s)$. So, if $\gamma(s)$ is a parametrization by arc-length, then $\gamma_s = u$ and $\gamma_{ss} = k(P)v$ (k is the curvature of Γ at $P = \gamma(s)$). On the other hand, we have that $w_s = -\lambda\gamma_s$, so $w_{ss} = -\lambda_s\gamma_s - \lambda\gamma_{ss} = -\lambda_s u - \lambda k v$. Equation (6) may be rewritten as

$$(1-t\lambda)^2 f_{uu} = -f_v k(P)(1-t\lambda)$$

$$-\frac{f_{uu}}{f_v} = \frac{k(P)}{1-t\lambda} \tag{8}$$

But $-f_{uu}/f_v = k(X)$, the curvature of the level curve of f at X. Thus, (8) says that

$$k(X) = \frac{k(P)}{1 - f(X)\lambda} \tag{9}$$

Summarizing, we have that both Euclidean and Affine Distances functions of Γ share the property that the Hessian is singular at differentiable points X, i.e., they are solutions of the PDE $\det(D^2 f) = 0$. The functions differ because their initital gradients (at Γ) are different:

- For the Euclidean Distance, $w(s) = N(s)$ ($N(s)$ is the unitary normal vector at $\gamma(s)$). Equation (1) is $Df(X)\cdot N = 1$, so the gradient vector Df (that is also orthogonal to Γ at $P = \gamma(s)$) must satisfy $||Df(P)|| = 1$.
- For the affine distance, we have that $w(s)$ is the affine normal vector $w(s) = -\frac{1}{3}k^{-\frac{5}{3}}(P)T(s) + k^{\frac{1}{3}}(P)N(s)$ ($T(s)$ and $N(s)$ are the unitary tangent and normal vectors at $P = \gamma(s)$). From equation (1), the gradient vector must satisfy $|Df(P)| = k^{-\frac{1}{3}}(P)$.

The Euclidean Distance is then a solution of the Monge-Ampère equation

$$\begin{cases} \det(D^2 f) = 0 \\ ||Df(x)|| = 1, \text{if } x \in \Gamma \\ f(x) = 0, \text{if } x \in \Gamma \end{cases}$$

while the affine distance is a solution of

$$\begin{cases} \det(D^2 f) = 0 \\ ||Df(x)|| = k^{-1/3}, \text{if } x \in \Gamma \\ f(x) = 0, \text{if } x \in \Gamma \end{cases}$$

These are particular cases of the "ruled distances" that satisfy $f(\gamma(s) + t\cdot w(s)) = t$, where $w(s)$ is a smooth vector field over Γ and $w_s(s)$ is colinear with $\gamma(s)$. The calculations made above just show that any such "ruled distance" will be the solution of the following equation with double boundary value conditions:

$$\begin{cases} \det(D^2 f) = 0 \\ \langle Df(x), w(x)\rangle = 1, x \in \Gamma \\ f(x) = 0, \text{if } x \in \Gamma \end{cases}$$

We must be careful with the meaning of "solutions" of such a PDE, since we expect the occurrence of shock points, where the Hessian $D^2 f$ is not defined in the classical sense. A good choice is to define an entropy solution: the values of f evolve along each characteristic curve $\varphi(t) = \gamma(s) + t \cdot w(s)$ from $t = 0$ until the characteristic curves themselves first intersect at a shock point (both with the same t). At such shock points, the evolutions of the colliding characteristic curves cease and thus they don't contribute to create any other shock points.

At this point, it is natural to ask if the homogenous Monge-Ampère equation with double boundary condition

$$\begin{cases} \det(D^2 f) = 0 \\ |D\,f(x)| = h(x), \text{if } x \in \Gamma \\ f(x) = 0, \text{if } x \in \Gamma \end{cases} \tag{10}$$

has a unique (entropy) solution, and if this would be always a "ruled distance".

We can exhibit an entropy solution when the function $h(x)$ is smooth and never zero. In this case, taking $f(\gamma(s) + t \cdot w(s)) = t$ provides such a solution, with $w(s) = -\frac{b_s}{k(s)} T(s) + b\,N(s)$ (where $b(s) = 1/h(s)$; T and N are the unitary tangent and normal vectors at $\gamma(s)$; $k(s)$ is the curvature of Γ at $\gamma(s)$; and s is the arc-length parameter). In fact, one can check that this actually satisfies the boundary conditions and that w_s is colinear with γ_s. The previous calculations of this section then shows that $\det\left(D^2 f\right) = 0$.

If we impose that the solution must be sufficiently smooth on an open set containing points of the curve Γ, then it can be proven that the solution is unique on that set. The proof is sketched in Silva's thesis [8].

Summarizing, this section shows that the homogeneous Monge-Ampère equation is the "local" description of the ruled distances such as the Euclidean Distance and the Affine Distance; therefore, this second order PDE plays a role similar to the Eikonal's role in relation to the Euclidean Distance – in order to gain generality, we were forced to increase the order of the PDE by one.

4.2 Front propagation of Γ

We can also describe such ruled distances by an evolution of the curve Γ using (euclidean) normal vectors. Let $C(s,\,t)$ be a parametrization of the level curve t of f, where s is the arc-length parameter for a fixed t. Then, differentiating $f(C(s,\,t)) = t$ with respect to s and t, and using that f is a solution of equation (10), we conclude that $C(s,t)$ evolves according to

$$\begin{cases} C_t(s,t) = v(s,t) \cdot N(s,t) \\ v_t(s,t) = \dfrac{v_s(s,t)}{k} \end{cases}$$

subject the initial conditions

$$\begin{cases} C(s,0) = \gamma(s) \\ v(s,t) = \dfrac{1}{h(\gamma(s))} \end{cases}$$

Here, $v(s,t)$ is the speed of the point $C(s,t)$ in the direction of the normal $N(s,t)$. So, the evolution of Γ that represents the successive level curves of these ruled distances is a second order differential equation, written above as a coupled system of first order equations.

4.3 Proposal: Affine Distance for non-convex curves

When Γ is a smooth non-convex curve, it has inflexion points, i.e., points where the curvature is zero. Since the boundary condition for the Affine Distance is $||D\;f(x)|| = k^{-1/3}$, we face a problem. Even if somehow we worked around the ill-defined gradient at the inflexion point, we would still have points very close to it evolving with arbitrarily large speeds. So, the values of the Affine Distance along characteristic curves emanating from a neighborhood of the inflexion point would be close to zero, even at points "far away" from Γ. As a consequence of this, some "parts" of the ADS, as defined previously, would seem very counter intuitive.

It is possible to tackle this problem using the idea of an entropy solution. Although the characteristic curves "run very fast" when the points are near the inflexion points, they soon quench each other. We illustrate this idea with the cubic curve $(t,\ t^3)$, where the inflexion point is at (0,0) (figure 3). We show the piece of the curve with parameter $t \in [-1, 0.1]$. The cubic curve is the solid thick line at the third quadrant. The characteristics are the thin straight lines. Note that at the second quadrant, the characteristics originated at the first quadrant meet the ones originated at the third quadrant. The locus of the shock points can be visualized easily.

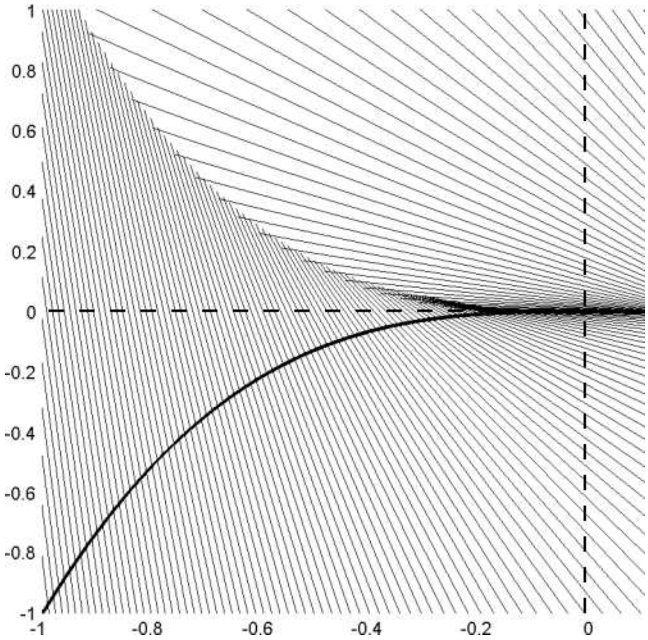

Figure 3. Schocks of the characteristics emanating from the cubic curve

This suggests a slight modification of the definition of the ADS: instead of "the set of points X where the global minimum of the Affine Distance from X to $P \in \Gamma$ is attained at two points P and Q", we propose "the set of points where two distinct characteristics intersect, provided that each characteristic has no previous intersection with any other 'active' characteristic". Doing so, it is possible to extend the idea of Affine Distance and ADS to non-convex curves.

5 $H = -4$: Affine Area Skeleton

The definition of Affine Area Skeleton of a curve Γ is based on the "area distance function" of Γ. The aim of this section is to show that the area distance function is a solution of the following Monge-Ampère equation with double boundary conditions

$$\begin{cases} \det(D^2 f) = -4 \\ |D f(x)| = 0, \text{if } x \in \Gamma \\ f(x) = 0, \text{if } x \in \Gamma \end{cases} \tag{11}$$

We assume, throughout this section, that Γ is a simple and strictly convex curve. The treatment of non-convex curves is subtle and deserves special care.

5.1 The gradient and the Hessian at regular points

As presented in [4] and [5], it's not difficult to show that the k-level curve of the area distance is the envelope of the chords that bound a region of area k with Γ; it is also the set of the mid-points of such chords. So, the gradient of the area distance at a point X is orthogonal to the "area minimizing" chord that has X as its midpoint (figure 4).

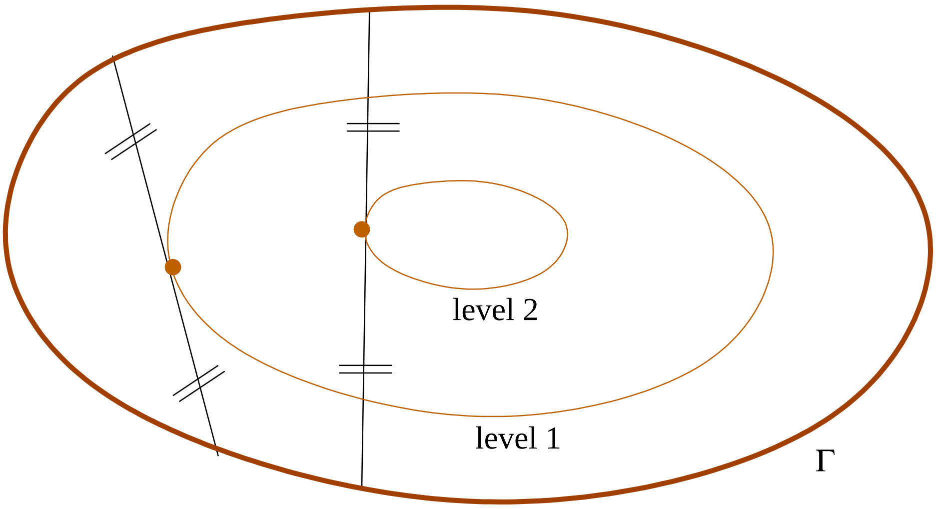

Figure 4. The midpoint property

The generator points of X are the two extremes of the chord that has X at its midpoint. Let us see how the generator points vary when we move the point X (figure 5).

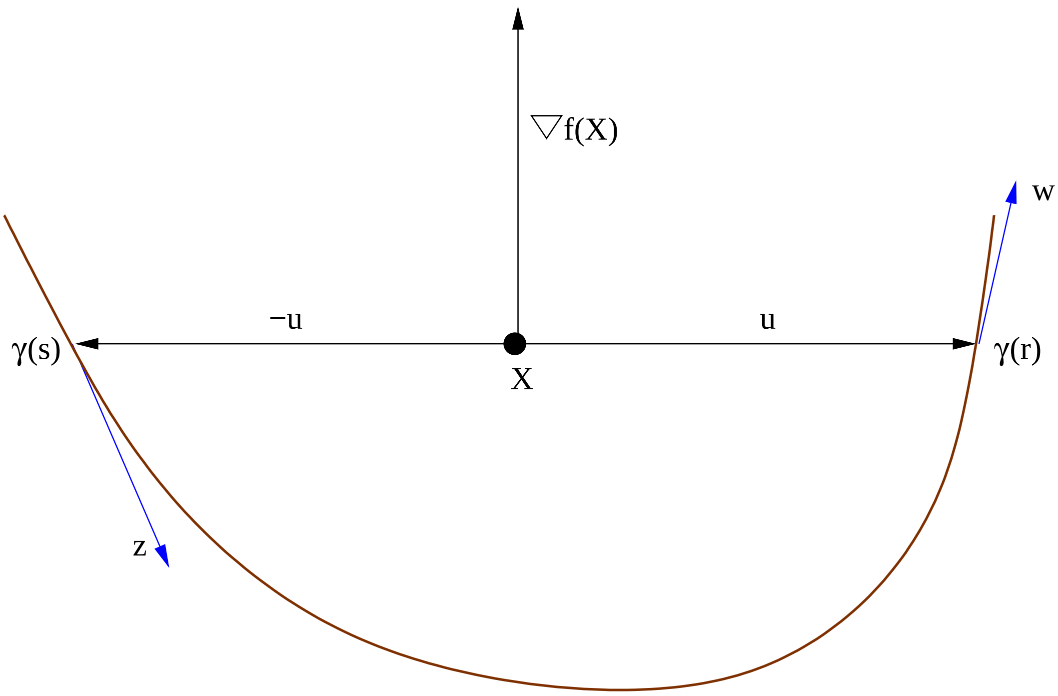

Figure 5. The gradient of the area distance at a regular point

Let $\gamma(s)$ and $\gamma(r)$ be the generator points of X (where r and s are parameters along Γ). Let $z = \gamma'(s)$ and $w = \gamma'(r)$ be the tangent vectors at those points. Assume that z and w are linearly independent. So, we can write the vector $u = \gamma(r) - X$ as a linear combination of z and w as

$$u = \frac{\langle R u, w \rangle}{\langle R z, w \rangle} z - \frac{\langle R u, z \rangle}{\langle R z, w \rangle} w$$

where R is the anti-clockwise rotation by $\pi/2$.

We know that X is the midpoint of the chord joining $\gamma(s)$ and $\gamma(r)$, so $X(s,r)=\frac{\gamma(r)+\gamma(s)}{2}$. Hence, the Jacobian matrix of X with relation to s and r is $DX=\frac{1}{2}\cdot(z,w)$, putting the vectors z and w as the columns. As the vectors z and w are independent, the Jacobian is invertible. Thus, by the inverse function theorem, we have functions $s(Y)$ and $r(Y)$ that give the parameters r and s of the points $\gamma(r)$ and $\gamma(s)$ for a point Y in a neighborhood of X. Differentiating $\frac{\gamma(s(X))+\gamma(r(X))}{2}=X$ with respect to X, we get the following equations for ∇s and ∇r

$$DX\cdot\begin{pmatrix}\nabla s\\ \nabla r\end{pmatrix}=I$$

$$\frac{1}{2}(z,w)\cdot\begin{pmatrix}\nabla s\\ \nabla r\end{pmatrix}=I \tag{12}$$

where I is the identity matrix and $\begin{pmatrix}\nabla s\\ \nabla r\end{pmatrix}$ is the matrix whose lines are the gradient vectors of $s(X)$ and $r(X)$. Solving this equation for ∇s and ∇r, we get

$$\nabla s=\frac{-2Rw}{\langle Rz,w\rangle}\text{ and }\nabla r=\frac{2Rz}{\langle Rz,w\rangle} \tag{13}$$

Therefore, we can evaluate the gradient $Df(X)$ at X. The value of f is the area calculated by the integral

$$f(X)=\frac{1}{2}\int_{s(X)}^{r(X)}\langle R(\gamma(m)-X),\gamma'(m)\rangle\,dm$$

Differentiating we get

$$2Df(X)=\langle R(\gamma(r)-X,w\rangle\cdot\nabla r-\langle R(\gamma(s)-X,z\rangle\cdot\nabla s+\int_s^r R\cdot\gamma'(m)\,dm$$

Simplifying this expression and using (13) we get $2Df(X)=4\cdot R\cdot u$. That is, the gradient is

$$Df(X)=R\cdot(\gamma(r)-\gamma(s)) \tag{14}$$

Thus, the size of the gradient of f is exactly the length of the chord! It is easy to see that the length of the chord approaches zero as X gets close to the strictly convex boundary Γ. So, assuming the continuity of the gradient of f on the boundary curve Γ, the gradient must be zero on Γ.

Differentiating (14) again, we get

$$D^2f(X)=R\cdot(z,-w)\cdot\begin{pmatrix}\nabla s\\ \nabla r\end{pmatrix}$$

but equation (12) says that $(z,w)\cdot\begin{pmatrix}\nabla s\\ \nabla r\end{pmatrix}=2\cdot I$. Hence,

$$\det(D^2f)=-4$$

Thus, we have the following theorem

Theorem 2. Let $f\colon\Omega\to R$ be the area distance function to a simple and strictly convex curve $\Gamma(=\partial\Omega)$. Then, the function f is a solution of the Monge-Ampre equation

$$\begin{cases}\det(D^2f)=-4,\text{if } x\in\Omega\\ Df(x)=0,\text{if } x\in\Gamma\\ f(x)=0,\text{if } x\in\Gamma\end{cases}$$

As with the Affine Distance, we expect to find singular points (the skeleton) and then we must be careful with what we mean by "a solution" of this equation. What we can say for sure at this point is that, if f is twice differentiable at $x\in\Omega$, then $\det\left(D^2f\right)=-4$ there.

5.2 Example - Circle and Ellipse

By a simple calculation we can find the expression of the PDE solution when the curve Γ is a circle (if Γ were an ellipsis, the solution would be found by an affine transformation). When the radius of the circle is one, the equation is

$$\begin{cases} \det(D^2 f) = -4, & \text{if } x^2 + y^2 < 1 \\ D f(x,y) = 0, & \text{if } x^2 + y^2 = 1 \\ f(x) = 0, & \text{if } x^2 + y^2 = 1 \end{cases} \tag{15}$$

Using polar coordinates (r, θ), the solution is, by rotational symmetry, a function of the radius only, that is, $f(x, y) = g(r)$, where $r = \sqrt{x^2 + y^2}$. The gradient vector and Hessian matrix of f are

$$D f(x,y) = (g' \cdot r_x, g' \cdot r_y)$$

$$D^2 f(x,y) = \begin{pmatrix} g'' \cdot r_x^2 + g' \cdot r_{xx} & g'' \cdot r_x r_y + g' \cdot r_{xy} \\ g'' \cdot r_x r_y + g' \cdot r_{xy} & g'' \cdot r_y^2 + g' \cdot r_{yy} \end{pmatrix}$$

choosing the point $(r = 1, \theta = 0)$, we have $r_x = -1$, $r_{xx} = 0$, $r_y = 0$, $r_{xy} = 0$ and $r_{yy} = 1/r$. Thus,

$$D f(x,y) = (-g'_x, 0)$$

$$D^2 f(x,y) = \begin{pmatrix} g'' & 0 \\ 0 & \frac{g'}{r} \end{pmatrix}$$

The partial differential equation (15) translates into the ordinary differential equation

$$\begin{cases} g''(r) g'(r) = -4r \\ g'(1) = 0 \\ g(1) = 0 \end{cases}$$

Observing that $g''(r) g'(r) = \frac{1}{2}((g')^2)'$, it's easy to solve the ODE. Indeed, the explicit solution is

$$g(r) = \pi/2 - r\sqrt{1 - r^2} - \arcsin(r)$$

It should be noted that g assume real values only inside the circle Γ, and the corresponding solution $f(x, y) = g(r)$ is not differentiable at $(0, 0)$ (see figure in the next section).

5.3 Proposal: Area Distance for non-convex curves

The selection of successive level sets of the area distance function is an operator called "affine erosion", in analogy with the "erosion operator" commonly dicussed in mathematical morphology. An area distance function for non-convex curves applied to the exterior of a shape Ω should therefore behave like an "affine dilation" operator.

Taking the example of the circle from the previous section, we can try a solution of the equation

$$\begin{cases} \det(D^2 f) = 4 \\ D f(x) = 0, \text{if } x \in \Omega \\ f(x) = 0, \text{if } x \in \Gamma \end{cases}$$

where we changed the constant -4 to $+4$ (remember that the solution g we found in the previous section did not assume real values outside of Γ). This new equation is well defined for the region outside the convex shape Ω. Using calculations similar to the ones in the previous section, we find a solution $f(x, y) = h(r)$ for this new equation, where

$$h(r) = -r\sqrt{r^2 - 1} + \log(r + \sqrt{r^2 - 1})$$

has real values only outside the circle. Gluing the functions g and h we get the following function (figure 6).

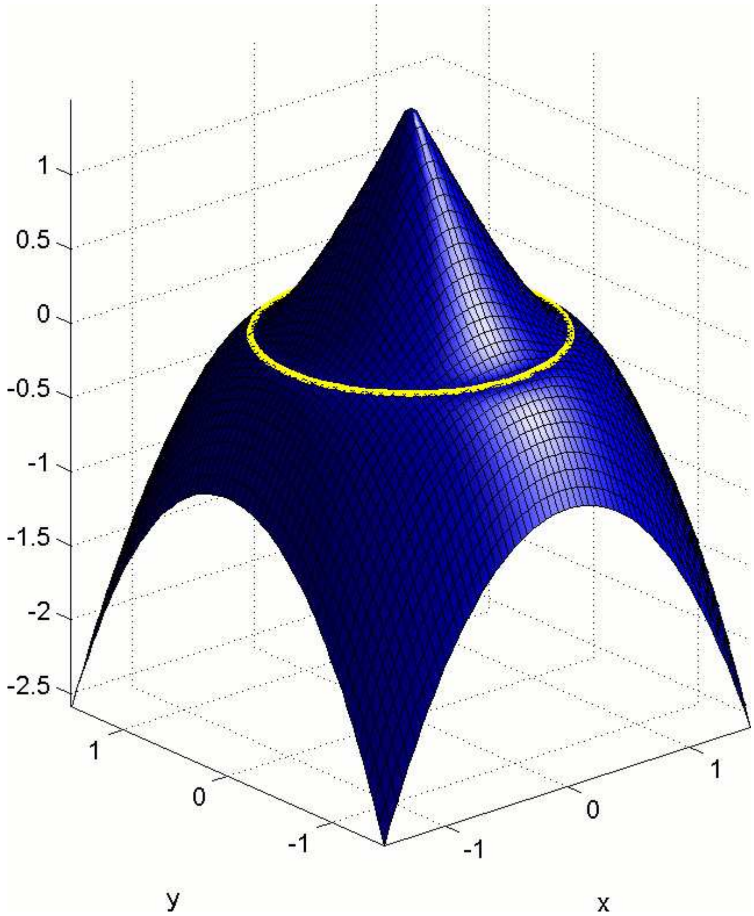

Figure 6. $H = -4$ and $H = 4$ together. The circle Γ is also ploted.

It's interesting to note that $h(r) = -i \cdot g(r)$ – not surprisingly, since (in)formally

$$\det\left(D^2 h\right) = \det\left(D^2(-i\,g)\right) = (i)^2 D\,e\,t\left(D^2 g\right) = -(-4) = +4$$

It would be interesting to investigate the possibility of putting together both these formulations under the same umbrella – for example, one could try to define complex-valued solutions for the equation $\det\left(D^2 f\right) = 4$ and use only their absolute values, say, as distance functions.

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