The iterated Aluthge transforms of a matrix converge

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Abstract

Given an $r \times r$ complex matrix T, if $T = U|T|$ is the polar decomposition of T, then, the Aluthge transform is defined by

$$\Delta(T) = |T|^{1/2} U |T|^{1/2}.$$

Let $\Delta^n(T)$ denote the n-times iterated Aluthge transform of T, i.e. $\Delta^0(T) = T$ and $\Delta^n(T) = \Delta(\Delta^{n-1}(T))$, $n \in \mathbb{N}$. We prove that the sequence $\{\Delta^n(T)\}_{n\in\mathbb{N}}$ converges for every $r \times r$ matrix T. This result was conjecturated by Jung, Ko and Pearcy in 2003. We also analyze the regularity of the limit function.

Keywords: Aluthge transform, stable manifold theorem, similarity orbit, polar decomposition.

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1 Introduction

Let $\mathcal{H}$ be a Hilbert space and T a bounded operator defined on $\mathcal{H}$ whose polar decomposition is $T = U|T|$. The *Aluthge transform* of T is the operator $\Delta(T) = |T|^{1/2}U\ |T|^{1/2}$. This transform was introduced in [1] to study p-hyponormal and log-hyponormal operators. Roughly speaking, the idea behind the Aluthge transform is to convert an operator into other operator which shares with the first one some spectral properties but it is closer to being a normal operator.

The Aluthge transform has received much attention in recent years. One reason is its connection with the invariant subspace problem. Jung, Ko and Pearcy proved in [13]

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that T has a nontrivial invariant subspace if an only if $\Delta(T)$ does. On the other hand, Dykema and Schultz proved in [9] that the Brown measure is preserved by the Aluthge transform.

Another reason is related with the iterated Aluthge transform. Let $\Delta^0(T) = T$ and $\Delta^n(T) = \Delta(\Delta^{n-1}(T))$ for every $n \in \mathbb{N}$. In [14] Jung, Ko and Peacy raised the following conjecture:

Conjecture 1. The sequence of iterates $\{\Delta^n(T)\}_{n\in\mathbb{N}}$ converges, for every matrix T.

Although many results supported this conjecture (see for instance [2] and [18]), there were only partial solutions. For instance, Ando and Yamazaki proved in [3] that Conjecture 1 is true for 2×2 matrices, Dykema and Schultz in [9] proved that the conjecture is true for an operator T in a finite factor such that the unitary part of its polar decomposition normalizes an abelian subalgebra that contains $|T|$, and Huang and Tam proved in [12] that the conjecture is true for matrices whose eigenvalues have different moduli.

In our previous work [6], we introduced a new approach to studying this problem, which was based on techniques from dynamical systems. This approach allowed to show that Conjecture 1 is true for every diagonalizable matrix.

In this paper, using again dynamical techniques, combined with some geometrical arguments, we completely solve Conjecture 1. There are many fruitful points of contact between the theories of dynamical systems and operator algebras. The combination of dynamical and geometrical techniques used to study Conjecture 1 suggests a new possible interaction between both theories. On one hand, it provides another field of applications of the stability theory of hyperbolic systems and invariant manifolds. In this sense, the work by Shub and Vasquez on the QR algorithm is an important precedent (see [17]). On the other hand, it provides to the operator theorists a new set of powerful tools to deal with problems where the usual techniques fail. In our case, besides the solution, it also provides a better understanding of the problem. The dynamical perspective not only allows to prove Conjecture 1, but it also provides further information related to the regularity of the limit function and the rate of convergence of the iterated sequence.

By a result proved in [5], Conjecture 1 reduces to the invertible case. On the other hand, according to a result independently proved by Jung, Ko and Pearcy in [14], and by Ando in [2], for every invertible $r \times r$ matrix T, the sequence of iterates $\{\Delta^n(T)\}$ goes toward the set of normal operators which have the same characteristic polynomial as T. This set can be characterized as the unitary orbit of some matrix D that also shares the characteristic polynomial with T. Let $\mathcal{U}(D)$ denote this unitary orbit.

If T is diagonalizable, then T and all the iterates $\Delta^n(T)$ belong to the similarity orbit of D, denoted by $\mathcal{S}(D)$, which is a riemannian manifold that contains $\mathcal{U}(D)$ as a compact submanifold. Note that all the points of $\mathcal{U}(D)$ are normal matrices, and therefore they are fixed points for the Aluthge transform. These facts suggest the possibility of pursuing a dynamic approach in $\mathcal{S}(D)$ based in the stable manifold theorem. This approach was carried out in [6] to prove the convergence of the iterated Aluthge transform sequence for diagonalizable matrices.

However, the non-diagonalizable case is different, since the geometry context of the problem is more complicated. Indeed, if T is not diagonalizable, $\mathcal{U}(D)$ is contained in the boundary of $\mathcal{S}(T)$, which also contains the orbits of matrices with smaller Jordan forms than the Jordan form of T. The boundary of $\mathcal{S}(T)$ can be thought as a sort of lattice of boundaries. Therefore, in order to prove Conjecture 1, we put the problem in a different setting so that both cases can be analyzed together. The Aluthge transform is viewed as an endomorphism on the space of invertible matrices, and we consider all the orbits mentioned before not as a manifold, but as the basin of attraction $B_\Delta(\mathcal{U}(D))$ i.e., those matrices T such that the sequence $\{\Delta^n(T)\}_{n\in\mathbb{N}}$ goes to $\mathcal{U}(D)$ as $n \xrightarrow[n\to\infty]{} \infty$. The basin $B_\Delta(\mathcal{U}(D))$ can also be characterized as the set of those matrices that have the same characteristic polynomial as D.

The stable manifold theorem can be extended to $B_\Delta(\mathcal{U}(D))$ (see Thm. 3.2.2), and no differential structure is required in the basin. Using this theorem we construct, through each T in the basin close enough to $\mathcal{U}(D)$, a Δ-invariant manifold $\mathcal{W}_T^{ss}$ which satisfies that

$$\mathcal{W}_T^{ss} \subseteq \{S : \|\Delta^n(T) - \Delta^n(S)\| < C\gamma^n \quad \text{for every} \quad n \in \mathbb{N}\},$$

where C and $\gamma < 1$ are constants that only depend on the distance among different eigenvalues of D. Hence, if the sequence $\Delta^\infty(S)$ converges for some $S \in \mathcal{W}_T^{ss}$, then the same must happen for T. For the diagonalizable case, in [6] we have considered only the stable manifolds $\mathcal{W}_N^{ss}$ for points $N \in \mathcal{U}(D)$. Then, using an argument which involves the inverse mapping theorem, we deduced that the union of these manifolds contains an open neighborhood of $\mathcal{U}(D)$ in $\mathcal{S}(D)$.

That approach fails in the general case, because the basin in not a manifold. In this case we prove that, for every T in the basin near to $\mathcal{U}(D)$, the stable manifolds $\mathcal{W}_T^{ss}$ intersect the set $\mathcal{O}_D$ of those matrices in the basin with orthogonal spectral projections. The set $\mathcal{O}_D$ can be studied with the usual properties of the Aluthge transform, showing that Conjecture 1 holds for its elements. However, $\mathcal{O}_D$ does not have a differential structure. To avoid this problem, in order to see that $\mathcal{W}_T^{ss} \cap \mathcal{O}_D \neq \varnothing$, we project the stable manifold $\mathcal{W}_T^{ss}$ and $\mathcal{O}_D$ to the orbit $\mathcal{S}(D)$, using spectral projections, in such a way that $\mathcal{O}_D$ projects onto the manifold $\mathcal{U}(D)$. Then, inside the manifold $\mathcal{S}(D)$, the desired result follows by a geometric argument based in known results about transversal intersections.

Another problem arises proving the continuity of the limit map Δ^∞ on $\mathcal{G}l_r(\mathbb{C})$: If $N_0 \in \mathcal{G}l_r(\mathbb{C})$ is normal, but has eigenvectors with multiplicity greater that 1, then in every open neighborhood of N_0 there exist matrices with very close but different eigenvalues. This fact reduces drastically the rate of convergence for such matrices, even in the diagonalizable case (see section 6.1). To solve this problem we separate the spectrum of these matrices in blocks which are near to each eigenvalue of N_0, even if in these blocks the eigenvalues are different. Then, we repeat the strategy of the proof of the convergence, but with respect to spectral projections relative the blocks indexed by the spectrum of N_0 (instead of using the spectrum of each matrix near N_0). As before, we show that these projections converge at an uniform velocity to an orthogonal system of projections, near the system of N_0. Then, one can see easily that, whatever were the limit and the rate of convergence to this limit, it must remain close to N_0, because the (block) spectral projections and the global spectra are close.

Using the results of [7], all the results mentioned before can be extended to the so called λ-Alutghe transform $\Delta_\lambda(T) = |T|^\lambda U \ |T|^{1-\lambda}$, for every $\lambda \in (0,1)$.

This paper is organized as follows: in section 2, we collect several preliminary definitions and results about the Aluthge transform, the geometry of similarity and unitary orbits, the stable manifold theorem in local basins, and the known properties of the spectral projections. In section 3, we compute the derivative of Δ in the whole space $\mathcal{M}_r(\mathbb{C})$, and we state the Dynamical Systems aspects of Δ. Particularly the stable manifold theorem on the local basin of a compact set of fixed points. In section 4 we prove Conjecture 1 about the convergence $\Delta^n(T) \xrightarrow[n\to\infty]{} \Delta^\infty(T)$ for every $T \in \mathcal{M}_r(\mathbb{C})$. In section 5 we study the regularity of the limit map $T \mapsto \Delta^\infty(T)$, mainly for T invertible. Section 6 contains concluding remarks about the rate of convergence, and the extension of the main results to the λ-Aluthge transforms, for every $\lambda \in (0,1)$. In the Appendix, we write the proof of two technical but escential results of sections 3 and 4.

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2 Preliminaries.

In this paper $\mathcal{M}_r(\mathbb{C})$ denotes the algebra of complex $r \times r$ matrices, $\mathcal{G}l_r(\mathbb{C})$ the group of all invertible elements of $\mathcal{M}_r(\mathbb{C})$, $\mathcal{U}(r)$ the group of unitary operators, and $\mathcal{M}_r^h(\mathbb{C})$ (resp. $\mathcal{M}_r^{ah}(\mathbb{C})$) denotes the real subspace of hermitian (resp. antihermitian) matrices. We denote $\mathcal{N}(r) = \{N \in \mathcal{M}_r(\mathbb{C}) : N \text{ is normal}\}$. If $v \in \mathbb{C}^r$, we denote by $\operatorname{diag}(v) \in \mathcal{M}_r(\mathbb{C})$ the diagonal matrix with v in its diagonal.

Given $T \in \mathcal{M}_r(\mathbb{C})$, $R(T)$ denotes the range or image of T, $\ker(T)$ the null space of T, $\operatorname{rk}(T) = \dim R(T)$ the rank of T, $\sigma(T)$ the spectrum of T, $\lambda(T) \in \mathbb{C}^r$ the vector of eigenvalues of T (counted with multiplicity), $\rho(T)$ the spectral radius of T, $\operatorname{tr}(T)$ the trace of T, and T^* the adjoint of T. We shall consider the space of matrices $\mathcal{M}_r(\mathbb{C})$ as a real Hilbert space with the inner product defined by

$$\langle A,\ B\rangle = \mathbb{R}\mathrm{e}\left(\operatorname{tr}(B^*A)\right).$$

The norm induced by this inner product is the Frobenius norm, that is denoted by $\|\cdot\|_2$. For $T \in \mathcal{M}_r(\mathbb{C})$ and $\mathcal{A} \subseteq \mathcal{M}_r(\mathbb{C})$, by means of $\operatorname{dist}(T,\mathcal{A})$ we denote the distance between them, with respect to the Frobenius norm.

On the other hand, let M be a manifold. By means of TM we denote the tangent bundle of M and by means of T_xM we denote the tangent space at the point $x \in M$. Given a function $f \in C^k(M)$ $(k \geq 1)$, we denote by $T_xf\,(V)$ the derivative of f at the point x applied to the tangent vector $V \in T_xM$. Given an open subset $\mathcal{U} \subseteq \mathbb{R}^m$, in

$C^k(\mathcal{U}, M)$ we shall consider the C^k-topology. In this topology, two functions are close if the functions and all their derivatives until order k are close uniformly on compact subsets. We denote by $\mathrm{Emb}^k(\mathcal{U}, M)$ the subset of $C^k(\mathcal{U}, M)$ consisting of the embeddings from $\mathcal{U}$ into M, endowed with the relative C^k-topology.

2.1 Basic facts about the Aluthge transform

Definition 2.1.1. Let $T \in \mathcal{M}_r(\mathbb{C})$, and suppose that $T = U|T|$ is the polar decomposition of T. Then, the Aluthge transform of T is defined by

$$\Delta(T) = |T|^{1/2}\, U\, |T|^{1/2}$$

Throughout this paper, $\Delta^n(T)$ denotes the n-times iterated Aluthge transform of T, i.e.

$$\Delta^0(T) = T; \qquad \text{and} \qquad \Delta^n(T) = \Delta\left(\Delta^{n-1}(T)\right) \quad n \in \mathbb{N}.$$

The following proposition contains some properties of the Aluthge transform which follows easily from its definition.

Proposition 2.1.2. Let $T \in \mathcal{M}_r(\mathbb{C})$. Then:

1. $\Delta(T) = T$ if and only if T is normal.
2. $\Delta(\lambda T) = \lambda \Delta(T)$ for every $\lambda \in \mathbb{C}$.
3. $\Delta(VTV^*) = V\Delta(T)V^*$ for every $V \in \mathcal{U}(r)$.
4. $\|\Delta(T)\|_2 \leqslant \|T\|_2$. In [5] it is proved that equality only holds if $T \in \mathcal{N}(r)$.
5. T and $\Delta(T)$ have the same characteristic polynomial, in particular, $\sigma(\Delta(T)) = \sigma(T)$.

The following result is easy to see, and it will be very useful in the sequel.

Proposition 2.1.3. If $T = T_1 \oplus T_2 \in \mathcal{M}_r(\mathbb{C})$ with respect to a reducing subspace $\mathcal{S} \subseteq \mathbb{C}^r$, then $\Delta(T) = \Delta(T_1) \oplus \Delta(T_2)$.

Next theorem states the regularity properties of Aluthge transforms (see [9] or [6]).

Theorem 2.1.4. The map Δ is continuous in $\mathcal{M}_r(\mathbb{C})$ and it is of class C^∞ in $\mathcal{G}l_r(\mathbb{C})$.

Remark 2.1.5. The map Δ fails to be differentiable at several matrices $T \in \mathcal{M}_r(\mathbb{C}) \setminus \mathcal{G}l_r(\mathbb{C})$. Indeed, supose that Δ were differentiable at $T = 0$. In this case, given $X \in \mathcal{M}_r(\mathbb{C})$,

$$T_0\Delta(X) = \frac{d}{dt}\,\Delta(tX)\Big|_{t=0} = \frac{d}{dt}\,t\,\Delta(X)\Big|_{t=0} = \Delta(X)\ .$$

But this is impossible, because the map $X \mapsto \Delta(X)$ is not linear. Using Proposition 2.1.3, this fact can be easily extended to any $T \in \mathcal{M}_r(\mathbb{C}) \setminus \mathcal{G}l_r(\mathbb{C})$ such that $\ker T$ is orthogonal to $R(T)$ (for example, every non invertible normal matrix). ▲

Now, we recall a result proved by Jung, Ko and Pearcy in [14], and by Ando in [2].

Proposition 2.1.6. If $T \in \mathcal{M}_r(\mathbb{C})$, the limit points of the sequence $\{\Delta^n(T)\}_{n\in\mathbb{N}}$ are normal. Moreover, if L is a limit point, then $\sigma(L) = \sigma(T)$ with the same algebraic multiplicity. ■

Remark 2.1.7. Proposition 2.1.6 has the next easy consequences:

1. Let $T \in \mathcal{M}_r(\mathbb{C})$. Denote by $\lambda(T) \in \mathbb{C}^r$ the vector of eigenvalues of T (counted with multiplicity). Then $\|\Delta^n(T)\|_2^2 \searrow_{n\to\infty} \sum_{i=1}^r |\lambda_i(T)|^2$.
2. If $\sigma(T) = \{\mu\}$, then $\Delta^n(T) \xrightarrow[n\to\infty]{} \mu I$, because μI is the unique normal matrix with spectrum $\{\mu\}$. ▲

2.2 Similarity orbits

Definition 2.2.1. Let $D \in \mathcal{M}_r(\mathbb{C})$. By means of $\mathcal{S}(D)$ we denote the similarity orbit of D:

$$\mathcal{S}(D) = \{\ SDS^{-1}\ :\ S \in \mathcal{G}l_r(\mathbb{C})\ \} .$$

On the other hand, $\mathcal{U}(D) = \{\ UDU^*\ :\ U \in \mathcal{U}(r)\ \}$ denotes the unitary orbit of D. We donote by $\pi_D : \mathcal{G}l_r(\mathbb{C}) \to \mathcal{S}(D) \subseteq \mathcal{M}_r(\mathbb{C})$ the C^∞ map defined by $\pi_D(S) = SDS^{-1}$, for every $S \in \mathcal{G}l_r(\mathbb{C})$. With the same name we note its restriction to the unitary group: $\pi_D : \mathcal{U}(r) \to \mathcal{U}(D)$. ▲

Remark 2.2.2. Let $T \in \mathcal{M}_r(\mathbb{C})$ and $N \in \mathcal{N}(r)$ with $\lambda(N) = \lambda(T)$. Then

$$\mathcal{U}(N) = \{M \in \mathcal{N}(r) : \lambda(M) = \lambda(T)\} .$$

On the other hand, by Schur's triangulation theorem, there exists $N_0 \in \mathcal{U}(N)$ such that $\|T\|_2^2 - \sum_{i=1}^r |\lambda_i(T)|^2 = \|T - N_0\|_2^2 \geq \operatorname{dist}(T, \mathcal{U}(N))^2$. ▲

The following two results are well known (see, for example, [8] or [4]):

Proposition 2.2.3. The similarity orbit $\mathcal{S}(D)$ is a C^∞ submanifold of $\mathcal{M}_r(\mathbb{C})$, and the projection $\pi_D : \mathcal{G}l_r(\mathbb{C}) \to \mathcal{S}(D)$ becomes a submersion. Moreover, $\mathcal{U}(D)$ is a compact submanifold of $\mathcal{S}(D)$, which consists of the normal elements of $\mathcal{S}(D)$, and $\pi_D : \mathcal{U}(r) \to \mathcal{U}(D)$ is a submersion. ■

Remark 2.2.4. For every $N \in \mathcal{S}(D)$, it is well known that

$$T_N\,\mathcal{S}(D) = T_I(\pi_N)(\mathcal{M}_r(\mathbb{C})) = \{[A, N] = AN - NA :\ A \in \mathcal{M}_r(\mathbb{C})\}.$$

If $\sigma(D) = \{\mu_1, \dots, \mu_k\}$, and $E_i(N)$ are the spectral projections of $N \in \mathcal{S}(D)$ associated to disjoint open neighborhoods of each μ_i, then $N = \sum_{i=1}^k \mu_i E_i(N)$. Therefore

$$
\begin{aligned}
T_N \mathcal{S}(D) &= \{AN - NA : \ A \in \mathcal{M}_r(\mathbb{C})\} \\
&= \left\{ \sum_{i,j=1}^{k} (\mu_j - \mu_i) E_i(N) A E_j(N) : \ A \in \mathcal{M}_r(\mathbb{C}) \right\} \\
&= \{X \in \mathcal{M}_r(\mathbb{C}) : E_i(N) X E_i(N) = 0 \,, \ 1 \le i \le k\} \ . \qquad (1)
\end{aligned}
$$

Throughout this paper we shall consider on $\mathcal{S}(D)$ (and in $\mathcal{U}(D)$) the Riemannian structure inherited from $\mathcal{M}_r(\mathbb{C})$ (using the usual inner product on their tangent spaces).

Observe that, for every $U \in \mathcal{U}(r)$, it holds that $U\mathcal{S}(D)U^* = \mathcal{S}(D)$ and the map $T \mapsto UTU^*$ is isometric, on $\mathcal{S}(D)$, with respect to the Riemannian metric as well as with respect to the $\|\cdot\|_2$ metric of $\mathcal{M}_r(\mathbb{C})$. ▲

In the previous work [6], we have proved the following result:

Theorem 2.2.5. Let $D = \operatorname{diag}(d_1, \dots, d_r) \in \mathcal{M}_r(\mathbb{C})$ be an invertible diagonal matrix. For every $N \in \mathcal{U}(D)$, there exists a subspace $\mathcal{E}^s_N$ of the tangent space $T_N\mathcal{S}(D)$ such that

1. $T_N\mathcal{S}(D) = \mathcal{E}^s_N \oplus T_N\mathcal{U}(D)$;
2. Both, $\mathcal{E}^s_N$ and $T_N\mathcal{U}(D)$, are $T\,\Delta$-invariant;
3. $T\,\Delta_N\big|_{T_N\mathcal{U}(D)} = I_{T_N\mathcal{U}(D)}$ and $\left\|T\,\Delta_N\big|_{\mathcal{E}^s_N}\right\| \le k_D < 1$, where
$$k_D = \max_{i,\,j:\ d_i \ne d_j} \frac{|1 + e^{i(\arg(d_j) - \arg(d_i))}|\,|d_i|^{1/2}|d_j|^{1/2}}{|d_i| + |d_j|} \ ;$$
4. If $U \in \mathcal{U}(r)$ satisfies $N = UDU^*$, then $\mathcal{E}^s_N = U(\mathcal{E}^s_D)U^*$.

In particular, the map $\mathcal{U}(D) \ni N \mapsto \mathcal{E}^s_N$ is smooth. This fact can be formulated in terms of the projections P_N onto $\mathcal{E}^s_N$ parallel to $T_N\mathcal{U}(D)$, $N \in \mathcal{U}(D)$. ■

2.3 Stable manifold theorem for the Basin of attraction

Let M be a smooth Riemann manifold, f a smooth endomorphism of M, and $N \subseteq M$ a compact set such that $f(N) = N$. The basin of attraction of N is the set

$$B_f(N) = \{y \in M : \operatorname{dist}(f^n(y), N) \xrightarrow[n\to\infty]{} 0\} \ .$$

Given $\varepsilon > 0$, the local basin of N is the set

$$B_f(N)_\varepsilon = \{y \in B_f(N) : \operatorname{dist}(f^n(y), N) < \varepsilon \ , \quad \text{for every } n \in \mathbb{N}\} \ .$$

The following result is standard when stated on a compact f-invariant subset N (see the Appendix of [6], [11] or [16]). The following version, which extend the prelamination $\mathcal{W}^s$ to its local basin $B_f(N)_\varepsilon$ is also well known. In Remark 2.3.2, we shall expose briefly the principal steps of the proof of the version for N, and then explain how that proof can be extended to "its basin of attraction."

Theorem 2.3.1 (Stable manifold theorem). Let f be a C^k endomorphism of M and let N be a compact f-invariant subset of M. Let us assume that for some $\varepsilon > 0$ there exist two continuous subbundles of $T_{B_f(N)_\varepsilon}M$, denoted by $\mathcal{E}^s$ and $\mathcal{F}$, such that, for every $x \in B_f(N)_\varepsilon$,

1. $T_{B_f(N)_\varepsilon}M = \mathcal{E}^s \oplus \mathcal{F}$.
2. $\mathcal{E}^s_x$ is T_xf-invariant in the sense that $T_xf(\mathcal{E}^s_x) \subseteq \mathcal{E}^s_{f(x)}$.
3. $\mathcal{F}_z$ is T_zf-invariant, for every $z \in N$.
4. There exists $\rho \in (0,1)$ such that $T_x f$ restricted to $\mathcal{F}_x$ expand it by a factor greater than ρ, and $T_xf : \mathcal{E}^s_x \to \mathcal{E}^s_{f(x)}$ has norm lower than ρ.

Then, there is a continuous, f-invariant and self coherent C^k-pre-lamination $\mathcal{W}^s : B_f(N)_\varepsilon \to \mathrm{Emb}^k((-1,1)^m, M)$ (endowed with the C^k-topology) such that, for every $x \in B_f(N)_\varepsilon$,

1. $\mathcal{W}^s(x)(0) = x$,
2. $\mathcal{W}^s_x = \mathcal{W}^s(x)((-1,1)^m)$ is tangent to $\mathcal{E}^s_x$,
3. $\mathcal{W}^s_x \subseteq \Big\{ y \in M \;:\; \mathrm{dist}(f^n(x), f^n(y)) < \mathrm{dist}(x,y)\rho^n \Big\}$. ■

Remark 2.3.2. The proof of the existence of a map $\mathcal{W}^s : N \to \mathrm{Emb}^k((-1,1)^m, M)$ which satisfies all the the mentioned conditions, consists in using the graph transform operator. We shall see that it is well defined if we only consider forward iterates. Therefore, since the basin of attraction of N is properly mapped inside by f, the graph transform operator is well defined on $B_f(N)_\varepsilon$, allowing to extend the proof of stable manifolds to the whole local basin. Recall that to define the graph transform operator, first we consider $C^k(\hat{\mathcal{E}^s}_x, \hat{\mathcal{F}}_x)$, the set of C^k maps from $\hat{\mathcal{E}^s}_x$ to $\hat{\mathcal{F}}_x$, where

$$\hat{\mathcal{E}^s}_x(\mu) = \exp(\mathcal{E}^s_x \cap (T_xM)_\mu), \quad \hat{\mathcal{F}}_x(\mu) = \exp(\mathcal{F}_x \cap (T_xM)_\mu)$$

and $\exp_x : (T_xM)_\mu \to M$ is the exponential map acting on $(T_xM)_\mu$, the ball of radius μ in T_xM. Later we consider the space

$$C^{k,0}(\hat{\mathcal{E}}^s, \hat{\mathcal{F}}) = \{\sigma : N \to C^k(\hat{\mathcal{E}^s}_x, \hat{\mathcal{F}}_x)\}$$

i.e.: for each $x \in N$ we take $\sigma_x \in C^k(\hat{\mathcal{E}^s}_x, \hat{\mathcal{F}}_x)$ and we assume that $x \mapsto \sigma_x$ moves continuously with x. We can represent $C^{k,0}(\hat{\mathcal{E}^s}, \hat{\mathcal{F}})$ as a vector bundle over N given by $N \times \{C^k(\hat{\mathcal{E}^s}_x, \hat{\mathcal{F}})\}_{x\in X}$. Then, we take the maps

$$f_x^1 = p_x^1 \circ f : M \to \hat{\mathcal{E}^s}_x \quad \text{and} \quad f_x^2 = p_x^2 \circ f : M \to \hat{\mathcal{F}}_x \ ,$$

where p_x^1 is the projection on $\hat{\mathcal{E}^s}_x$ and p_x^2 is the projection on $\hat{\mathcal{F}}_x$. Now we take the *graph transform operator*. If f is a diffeomorphism, then we can obtain an explicit formula for the graph transform:

$$\Gamma_f(\sigma_x) = \big(f_x^2 \circ (id, \sigma_{f(x)})\big)^{-1} \circ \big(f_x^1 \circ (id, \sigma_x)\big)\big|_{\hat{\mathcal{E}^s}_x} \ . \tag{2}$$

On the other hand, if f is an endomorphism, the graph transform can be defined implicitly. In both cases, this map is well defined in $B_f(N)_\varepsilon$ and therefore the whole proof can be carried out, in the sense of proving that the graph transform operator is a contractive map and therefore it has a fixed point. ▲

2.4 Spectral projections

In this section we state the basic properties of the spectral projections of matrices, which are constructed by using the Riesz functional calculus. A complete exposition on this theory can be found in Kato's book [15, Ch. 2]. Given $T \in \mathcal{M}_r(\mathbb{C})$ we call $\lambda = \lambda(T) \in \mathbb{C}^r$ its vector of eigenvalues. Let $\sigma(T) = \{\mu_1, \dots, \mu_k\}$, taking one μ_i for each group of repeated $\lambda_j(T) = \mu_i$ in $\lambda(T)$ (i.e., $k \le r$). Fix $D = \operatorname{diag}(\lambda) \in \mathcal{M}_r(\mathbb{C})$.

Definition 2.4.1. Given $\lambda = \lambda(D) \in \mathbb{C}^r$ and $\mu = (\mu_1, \dots, \mu_k) \in \mathbb{C}^k$ as before ($\mu_i \ne \mu_j$), let

1. $\varepsilon_\mu = \dfrac{1}{3} \min\limits_{i\ne j} |\mu_i - \mu_j|$ and $\Omega_\mu = \bigcup\limits_{1\le i\le k} B(\mu_i\,,\, \varepsilon_\mu)$.

2. $\tilde{\mathcal{M}}_\mu = \{T \in \mathcal{M}_r(\mathbb{C}) : \sigma(T) \subseteq \Omega_\mu\}$, which is open in $\mathcal{M}_r(\mathbb{C})$.

3. Let $E : \tilde{\mathcal{M}}_\mu \to \mathcal{M}_r(\mathbb{C})^k$ given by

$$\tilde{\mathcal{M}}_\mu \ni T \mapsto E(T) = (E_1(T), \cdots, E_k(T)\,)\,,$$

   where $E_i(T) = \aleph_{B(\mu_i\,,\,\varepsilon_\mu)}(T)$ is the spectral projection of $T \in \tilde{\mathcal{M}}_\mu$, associated to $B(\mu_i\,,\, \varepsilon_\mu)$.

4. Denote $Q = (Q_1, \dots, Q_k) = E(D)$ and consider the open set

$$\mathcal{M}_\lambda = \{T \in \tilde{\mathcal{M}}_\mu : \ \operatorname{rk}(E_i(T)) = \ \operatorname{rk}(Q_i)\ , \quad 1 \le i \le k\}\ , \tag{3}$$

   which is the connected component of D in $\tilde{\mathcal{M}}_\mu$.

5. Let $\Pi_E : \mathcal{M}_\lambda \to \mathcal{M}_r(\mathbb{C})$ given by $\Pi_E(T) = \sum_{i=1}^{k} \mu_i E_i(T)$, for every $T \in \mathcal{M}_\lambda$. ▲

Remark 2.4.2. Given $\lambda = \lambda(D) \in \mathbb{C}^r$ and $\mu = (\mu_1, \dots, \mu_k) \in \mathbb{C}^k$ as before, the following properties hold:

1. For every $1 \le i \le k$, $Q_i = Q_i^*$. Also $Q_i Q_j = 0$ (if $i \ne j$) and $\sum_i Q_i = I$. The entries of $E(T)$ for other $T \in \mathcal{M}_\lambda$ satisfy the same properties, but they can be not selfadjoint.
2. Each map E_i (so that the map E) is of class C^∞ in $\mathcal{M}_\lambda$.
3. $E(\mathcal{M}_\lambda) = \mathcal{S}(Q) := \{(SQ_1S^{-1}, \dots, SQ_kS^{-1}) : S \in \mathcal{G}l_r(\mathbb{C})\}$.
4. Moreover, if $T \in \mathcal{M}_\lambda$ y $S \in \mathcal{G}l_r(\mathbb{C})$, then $E(STS^{-1}) = SE(T)S^{-1}$.

Then, the map $\Pi_E : \mathcal{M}_\lambda \to \mathcal{M}_r(\mathbb{C})$ satisfies the following properties:

1. It is of class C^∞ on $\mathcal{M}_\lambda$.
2. For every $T \in \mathcal{S}(D)$, $\Pi_E(T) = T$.
3. $\Pi_E(\mathcal{M}_\lambda) = \mathcal{S}(D)$, and $\rho(T - \Pi_E(T)) < \varepsilon_\mu$ for every $T \in \mathcal{M}_\lambda$.
4. If $T \in \mathcal{M}_\lambda$ and $S \in \mathcal{G}l_r(\mathbb{C})$, then $\Pi_E(STS^{-1}) = S\Pi_E(T)S^{-1}$. ▲

Remark 2.4.3. With the previous notations, for every $M \in \mathcal{M}_\lambda$, we consider the subspace

$$\mathcal{A}_M = \{B \in \mathcal{M}_r(\mathbb{C}) : BE_i(M) = E_i(M)B \quad 1 \le i \le k\}\ , \tag{4}$$

of *block diagonal* matrices, with respect to $E(M)$. It is easy to see that $\mathcal{A}_M = \ker T_M\Pi_E$ and $R(T_M\Pi_E) = T_N\mathcal{S}(D)$. By Eq. (1), if $N = \Pi_E(M) \in \mathcal{S}(D)$, then $\mathcal{M}_r(\mathbb{C}) = \mathcal{A}_M \oplus T_N\mathcal{S}(D)$, and the sum becomes orthogonal if $M \in \mathcal{U}(D)$.

Since ${\Pi_E}^2 = \Pi_E$, if $M \in \mathcal{S}(D)$, then $T_M\Pi_E$ is the projector with kernel $\mathcal{A}_M$ and and image $T_M\mathcal{S}(D)$. Observe that

$$\mathcal{A}_M = \{M\}' := \{B \in \mathcal{M}_r(\mathbb{C}) : MB = BM\} \tag{5}$$

for every $M \in \mathcal{S}(D)$, since in this case $M = \Pi_E(M)$. ▲

3 Some dynamical aspects of the Aluthge transform

The main aim of this section is to introduce the dynamical setting as well as some results that will be used in the next section to prove the convergence of the iterated Aluthge transform sequence.

3.1 The derivative of Δ in $\mathcal{M}_r(\mathbb{C})$

Let $N \in \mathcal{M}_r(\mathbb{C})$ be an invertible normal matrix. Theorem 2.2.5 gives a description of the action of $T_N\Delta$ on $T_N\mathcal{S}(N)$. By Remark 2.2.4, in order to obtain a complete characterization of the action of $T_N\Delta$ on $\mathcal{M}_r(\mathbb{C})$, it is enough to describe the action of $T_N\Delta$ on its orthogonal complement, i.e. the subspace $\mathcal{A}_N$ described in Remark 2.4.3.

Proposition 3.1.1. *Let $D = \operatorname{diag}(d_1, \ldots, d_r) \in \mathcal{G}l_r(\mathbb{C})$ be a diagonal matrix with k different eigenvectors. Fix $N \in \mathcal{U}(D)$ and consider the subspace $\mathcal{A}_N \subseteq \mathcal{M}_r(\mathbb{C})$ defined in Eq. (4). Then $T_N\Delta|_{\mathcal{A}_N} = I_{\mathcal{A}_N}$.*

Proof. If $N \in \mathcal{U}(D)$, then N is normal, and $\mathcal{A}_N = \{N\}'$. Hence, if $Y \in \mathcal{A}_N$ is normal, then $N + tY$ is also normal for every $t \in \mathbb{R}$. This implies that $T_N\Delta(Y) = \frac{d}{dt}\Delta(N+tY)\Big|_{t=0} = Y$. On the other hand, since $\mathcal{A}_N$ is closed by taking adjoints, then $\mathbb{R}e(X) \in \mathcal{A}_N$ and $\mathbb{I}m(X) \in \mathcal{A}_N$ for every $X \in \mathcal{A}_N$. Therefore $T_N\Delta(X) = X$. ■

Corollary 3.1.2. *Let $D = \operatorname{diag}(d_1, \ldots, d_r) \in \mathcal{M}_r(\mathbb{C})$ be an invertible diagonal matrix. For $N \in \mathcal{U}(D)$, denote by $F_N = \mathcal{A}_N \oplus T_N\mathcal{U}(D)$. Then, the subspaces F_N and $\mathcal{E}^s_N$ (defined in Theorem 2.2.5) satisfy that, for every $N \in \mathcal{U}(D)$,*

1. *$\mathcal{M}_r(\mathbb{C}) = F_N \oplus \mathcal{E}^s_N$.*
2. *Both subspaces are $T_N\Delta$ invariant.*
3. *$T_N\Delta\big|_{F_N} = I_{F_N}$ and $\left\|T_N\Delta|_{\mathcal{E}^s_N}\right\| \leq k_D < 1$, where k_D is defined as in Theorem 2.2.5.*
4. *The distributions $N \mapsto F_N$ and $N \mapsto \mathcal{E}^s_N$ are smooth.*

Proof. By Theorem 2.2.5 and Remark 2.4.3,

$$\mathcal{M}_r(\mathbb{C}) = \mathcal{A}_N \oplus T_N\mathcal{S}(D) = \mathcal{A}_N \oplus T_N\mathcal{U}(D) \oplus \mathcal{E}^s_N = F_N \oplus \mathcal{E}^s_N .$$

By Proposition 3.1.1, one deduces that $T_N\Delta\big|_{F_N} = I_{F_N}$. The remainder conditions follow easily from Theorem 2.2.5. ■

Remark 3.1.3. With the notations of Corollary 3.1.2, the subspaces F_N and $\mathcal{E}^s_N$ can be characterized by mean of the functional calculus applied to the linear maps $T_N\Delta$, for every $N \in \mathcal{U}(D)$. Indeed,

$$F_N = R\big(\aleph_{B(1,\varepsilon)}(T_N\Delta)\big) \quad \text{and} \quad \mathcal{E}^s_N = R\big(\aleph_{B(0,k_D+\varepsilon)}(T_N\Delta)\big),$$

for every $\varepsilon > 0$ sufficiently small. In particular, this implies that the distribution of subspaces F_N and $\mathcal{E}^s_N$ can be extended smoothly to an open neighborhood of $\mathcal{U}(D)$. ▲

3.2 Stable manifolds

Let $\mathbb{P} \subseteq \mathcal{M}_r(\mathbb{C})$ be a compact set of fixed points for Δ, i.e. a compact set of normal matrices. Recall that its basin of attraction is the set

$$B_\Delta(\mathbb{P}) = \{T \in \mathcal{M}_r(\mathbb{C}) : \operatorname{dist}(\Delta^n(T), \mathbb{P}) \xrightarrow[n\to\infty]{} 0\}$$

and, for every $\varepsilon > 0$, the local basin is the set

$$B_\Delta(\mathbb{P})_\varepsilon = \{T \in B_\Delta(\mathbb{P}) : \operatorname{dist}(\Delta^n(T), \mathbb{P}) < \varepsilon\,,\ n \in \mathbb{N}\}\ .$$

In this subsection, using the stable manifold theorem 2.3.1, we shall prove that, if $\mathbb{P}$ has a distribution of subspaces with good properties (like the distribution of Corollary 3.1.2 for $\mathbb{P} = \mathcal{U}(D)$), through each $T \in B_\Delta(\mathbb{P})$ closed enough to $\mathbb{P}$ there is a stable manifold $\mathcal{W}_T^{ss}$ with the property

$$\mathcal{W}_T^{ss} \subseteq \{B \in \mathcal{M}_r(\mathbb{C}) :\ \|\Delta^n(T) - \Delta^n(B)\| < C\gamma^n\},$$

where $\gamma < 1$ and C is a positive constant. With this aim, firstly we need to extend to some local basin the distribution of subspaces given on $\mathbb{P}$. This extension is a quite standard procedure in dynamical systems. For completeness, we include a sketch of its proof (adapted to our case) in the Appendix A.

Proposition 3.2.1. *Let $\mathbb{P}$ be a compact set consisting of fixed points of Δ. Suppose that, for every $N \in \mathbb{P}$, there are subspaces $\mathcal{E}_N^s$ and $\mathcal{F}_N$ of $\mathcal{M}_r(\mathbb{C})$ with the following properties:*

1. *$\mathcal{M}_r(\mathbb{C}) = \mathcal{E}_N^s \oplus \mathcal{F}_N$.*
2. *The distributions $N \mapsto \mathcal{E}_N^s$ and $N \mapsto \mathcal{F}_N$ are continuous.*
3. *There exist $\rho \in (0,1)$ which does not depend on N such that*

$$\left\| T_N\,\Delta|_{\mathcal{E}_N^s} \right\| < 1 - \rho \quad \text{and} \quad \|(I - T_N\,\Delta)\,|_{\mathcal{F}_N}\| < \frac{\rho}{2}\ , \tag{6}$$

4. *Both subspaces $\mathcal{E}_N^s$ and $\mathcal{F}_N$ are $T_N\Delta$ invariant.*

Then, there exists $\varepsilon > 0$ such that the distributions $N \mapsto \mathcal{E}_N^s$ and $N \mapsto \mathcal{F}_N$ can be extended to the local basin $B_\Delta(\mathbb{P})_\varepsilon$, verifying conditions 1, 2, 3 and the following new condition:

4' For every $T \in B_\Delta(\mathbb{P})_\varepsilon$, the subspace $\mathcal{E}_T^s$ is $T_T\,\Delta$-invariant, i.e., $T_T\Delta(\mathcal{E}_T^s) \subseteq \mathcal{E}_{\Delta(T)}^s$.

Proof. See section A.1 of the appendix.

Now we are ready to state and prove the announced result on stable manifolds.

Proposition 3.2.2. *Let $\mathbb{P}$ be a compact set consisting of fixed points of Δ, with two distributions $N \mapsto \mathcal{E}^s_N$ and $N \mapsto \mathcal{F}_N$ which satisfy the hypothesis of Proposition 3.2.1. Then, there exist $\varepsilon > 0$ and a C^2-pre-lamination $\mathcal{W}^s : B_\Delta(\mathbb{P})_\varepsilon \to \mathrm{Emb}^2((-1,1)^m, B_\Delta(\mathbb{P}))$ (endowed with the C^2-topology) of class C^0 such that, for every $T \in B_\Delta(\mathbb{P})_\varepsilon$,*

1. *$\mathcal{W}^s(T)(0) = T$.*
2. *If $\mathcal{W}^{ss}_T$ is the submanifold $\mathcal{W}^s(T)\big((-1,1)^m\big)$, then $T_T\mathcal{W}^{ss}_T = \mathcal{E}^s_T$.*
3. *There are constants $\gamma < 1$ and $C > 0$ such that*

$$\mathcal{W}^{ss}_T \subseteq \{B \in \mathcal{M}_r(\mathbb{C}) : \ \|\Delta^n(T) - \Delta^n(B)\| < C\gamma^n\}. \tag{7}$$

Proof. By Proposition 3.2.1, the distributions $\mathbb{P} \ni N \mapsto \mathcal{E}^s_N$, $\mathcal{F}_N$ can be extended to a local basin $B_\Delta(\mathbb{P})_\varepsilon$, satisfying the hypothesis the Stable Manifold Theorem 2.3.1. Observe that the condition $\|(I - T_T\Delta)|_{\mathcal{F}_T}\| < \frac{\varrho}{2}$ implies that $\|T_T\Delta(Y)\| > (1-\frac{\varrho}{2})\|Y\|$ for every $Y \in \mathcal{F}_T$. ■

4 Convergence of the sequence $\Delta^n(T)$

This section is entirely devoted to the proof of Jung, Ko and Pearcy's conjecture. The basic tools are the results of the previous section.

4.1 The case $\mathbb{P} = \mathcal{U}(D)$

Let $D \in \mathcal{M}_r(\mathbb{C})$ is an invertible diagonal matrix. In this section we shall consider the compact invariant set $\mathbb{P} = \mathcal{U}(D)$. Observe that the distributions $N \mapsto F_N$ and $N \mapsto \mathcal{E}^s_N$ ($N \in \mathbb{P}$) given by Corollary 3.1.2 clearly verify the hypothesis of Propositions 3.2.1 and 3.2.2. Thus, by Proposition 3.2.2, there exist a continuous prelamination $\mathcal{W}^s : B_\Delta(\mathbb{P})_\varepsilon \to \mathrm{Emb}^2((-1,1)^k, B_\Delta(\mathbb{P}))$ and submanifolds $\mathcal{W}^{ss}_T$ (for every $T \in B_\Delta(\mathbb{P})_\varepsilon$). which will be also used throughout this section.

In this setting we can give simple characterizations of the basins of $\mathbb{P}$. Indeed, by Proposition 2.1.6 and Remark 2.2.2,

$$B_\Delta(\mathcal{U}(D)) = \{T \in \mathcal{M}_r(\mathbb{C}) : \lambda(T) = \lambda(D)\}. \tag{8}$$

Given $T \in B_\Delta(\mathcal{U}(D))$, let $d_n(T) = \|\Delta^n(T)\|_2^2 - \sum_{i=1}^r |\lambda_i(T)|^2$. By Remarks 2.1.7 and 2.2.2, $\mathrm{dist}(\Delta^n(T), \mathcal{U}(N)) \le d_n(T) \underset{n\to\infty}{\searrow} 0$. Then,

$$\{T \in B_\Delta(\mathcal{U}(D)) : d_1(T) < \varepsilon\} \subseteq B_\Delta(\mathcal{U}(D))_\varepsilon, \tag{9}$$

and it is also an open neighborhood of $\mathcal{U}(D)$ in $B_\Delta(\mathcal{U}(D))$. Therefore, if $T \in B_\Delta(\mathcal{U}(D)$ is close enough to $\mathcal{U}(D)$, then $T \in B_\Delta(\mathcal{U}(D))_\varepsilon$ (and we do not need to check the $\mathrm{dist}(\Delta^n(T), \mathcal{U}(N))$ for $n > 1$).

Observe that if $T \in B_\Delta(\mathcal{U}(D))$, despite the equality $\lambda(T) = \lambda(D)$, T can have any Jordan form.

4.2 The sets $\mathcal{O}_D$

In this subsection we identify some convenient sets of matrices where the iterated Aluthge transform sequence converges (possibly slowly). By their properties, these sets will play a key role in the proof of the convergence of the iterated Aluthge transform sequence. Let D be an invertible diagonal matrix, $\lambda = \lambda(D)$, and $\Pi_E : \mathcal{M}_\lambda \to \mathcal{S}(D)$, the map defined in Section 2.4. If $\mathbb{P} = \mathcal{U}(D)$, consider the following subset of $B_\Delta(\mathbb{P})$:

$$\mathcal{O}_D = \{T \in B_\Delta(\mathbb{P}) : \Pi_E(T) \in \mathbb{P}\} = \Pi_E^{-1}(\mathbb{P}) \cap B_\Delta(\mathbb{P}) \ . \tag{10}$$

Note that, if $T \in \mathcal{O}_D$, then the system of projectors $E(T)$ is orthogonal. Hence we get the next simple consequence.

Proposition 4.2.1. *If $T \in \mathcal{O}_D$, then $\Delta^n(T) \xrightarrow[n\to\infty]{} \Pi_E(T) \in \mathcal{U}(D)$.*

Proof. If $T \in \mathcal{O}_D \subseteq B_\Delta(\mathcal{U}(D))$ then $\lambda(T) = \lambda(D)$, by Eq. (8). On the other hand, if $N = \Pi_E(T)$, then $E(T) = E(N)$ is an **orthogonal** system of projectors, and $T \in \mathcal{A}_N$, the subspace defined in Eq. (4). Write $T = T_1 \oplus \cdots \oplus T_k$, where each $T_i = T|_{R(E_i(T))}$. By Proposition 2.1.3, $\Delta^n(T) = \Delta^n(T_1) \oplus \cdots \oplus \Delta^n(T_k)$, for every $n \in \mathbb{N}$. Since $\lambda(T) = \lambda(D)$, then $\sigma(T_i) = \{\mu_i\}$ and, by Remark 2.1.7,

$$\Delta^n(T_i) \xrightarrow[n\to\infty]{} \mu_i \, I_{R(E_i(T))} \ , \quad 1 \le i \le k \ .$$

Therefore $\Delta^n(T) \xrightarrow[n\to\infty]{} \sum_{i=1}^{k} \mu_i \, E_i(T) = \Pi_E(T)$. ■

Another important characteristic of the sets $\mathcal{O}_D$ is that each element of $B_\Delta(\mathbb{P})$ "close enough" to $\mathbb{P}$ is exponentially attracted toward $\mathcal{O}_D$. This property is precisely described in the following statement.

Proposition 4.2.2. Let $D \in \mathcal{M}_r(\mathbb{C})$ be an invertible diagonal matrix, $\mathbb{P} = \mathcal{U}(D)$ and $\mathcal{W}^s : B_\Delta(\mathbb{P})_\varepsilon \to \mathrm{Emb}^2((-1,1)^k, B_\Delta(\mathbb{P}))$ the prelamination given by Proposition 3.2.2. Then, there exists $\eta < \varepsilon$ such that $\mathcal{W}^{ss}_T \cap \mathcal{O}_D \neq \varnothing$ for every $T \in B_\Delta(\mathbb{P})_\eta$.

The proof is rather technical. Since this result is the key part of the proof of the conjecture, we give a brief description of the proof here, and we leave a detailed proof to Appendix A.

If $T \in B_\Delta(\mathbb{P})_\varepsilon$ is near $N \in \mathcal{U}(D)$, then the set $\mathcal{V}^{ss}_T = \Pi_E(\mathcal{W}^{ss}_T)$ is a smooth submanifold of $\mathcal{S}(D)$ with the same dimension as $\mathcal{W}^{ss}_T$, and it remains C^2-close to $\mathcal{W}^{ss}_N$. This facts can be deduced from the properties of the projection Π_E stated in Subsection 2.4.

In order to show that $\mathcal{W}^{ss}_T \cap \mathcal{O}_D \neq \varnothing$, it suffices to prove that $\mathcal{V}^{ss}_T \cap \mathcal{U}(D) \neq \varnothing$. Finally, this fact follows from a well known argument of transversal intersection (inside the manifold $\mathcal{S}(D)$), by using the dimension of the tangent spaces, and the fact that $\mathcal{W}^{ss}_N$ intersects $\mathcal{U}(D)$ transversally.

4.3 The proof of Jung, Ko and Pearcy's conjecture

Now, we are in conditions to prove the main result of this paper:

Theorem 4.3.1. *For every $T \in \mathcal{M}_r(\mathbb{C})$, the sequence $\Delta^n(T)$ converges.*

Proof. By Corollary 4.16 of [5], we can assume that $T \in \mathcal{G}l_r(\mathbb{C})$. Let $D \in \mathcal{G}l_r(\mathbb{C})$ be a diagonal matrix such that $\lambda(T) = \lambda(D)$, and let $\mathbb{P} = \mathcal{U}(D)$. By Eq. (8), $T \in B_\Delta(\mathbb{P})$. By Eq. (9), replacing T by $\Delta^n(T)$ for some n large enough, we can assume that $T \in B_\Delta(\mathbb{P})_\rho$, for any fixed $\rho > 0$.

Consider now the stable manifold $\mathcal{W}^{ss}_T$, constructed in Proposition 3.2.2, for every $T \in B_\Delta(\mathbb{P})_\varepsilon$. By Proposition 4.2.2, there exists $0 < \eta < \varepsilon$ such that $\mathcal{W}^{ss}_T \cap \mathcal{O}_D \neq \varnothing$, for every $T \in B_\Delta(\mathbb{P})_\eta$. If $M \in \mathcal{W}^{ss}_T \cap \mathcal{O}_D$ then, by Proposition 4.2.1 and Eq. (7) of Proposition 3.2.2, we deduce that $\Pi_E(M) = \lim_{n\to\infty} \Delta^n(M) = \lim_{n\to\infty} \Delta^n(T)$. ■

5 Regularity of the map Δ^∞

Given $T \in \mathcal{M}_r(\mathbb{C})$ we denote $\Delta^\infty(T) = \lim_{n\to\infty} \Delta^n(T)$, which is a normal matrix. Observe that the map $\Delta^\infty : \mathcal{M}_r(\mathbb{C}) \to \mathcal{N}(r)$ is a retraction. In this section we study the regularity of this retraction.

5.1 Differentiability vs. continuity

In [6] we proved that the map Δ^∞ is of class C^∞, when it is restricted to the open dense set of those matrices in $\mathcal{M}_r(\mathbb{C})$ with r different eigenvalues. The following proposition shows that this can not be extended globally to the set of all matrices.

Proposition 5.1.1. *The map Δ^∞ can not be C^1 in a neighborhood of the identity.*

Proof. Suppose that Δ^∞ is C^1 in a neighborhood of the identity. By the same argument used in the proof of Proposition 3.1.1, it follows that $T_I\Delta^\infty$ is the identity map (in this case, $\mathcal{A}_I = \mathcal{M}_r(\mathbb{C})$). This implies that Δ^∞ is a local diffeomorphism. However, this is impossible because it takes values in the set of normal operators. ■

5.2 Continuity of Δ^∞ on $\mathcal{G}l_r(\mathbb{C})$

For a sake of convenience, throughout this subsection we shall use the spectral norm, instead of the Frobenius norm, to measure distances in $\mathcal{M}_r(\mathbb{C})$.

Remark 5.2.1. Since Δ^∞ is the limit of continuous maps and it is a retraction, in order to show that it is continuous on $\mathcal{G}l_r(\mathbb{C})$, it is enough to prove the continuity at the normal matrices of $\mathcal{G}l_r(\mathbb{C})$. Indeed, observe that Δ^n is continuous for every $n \in \mathbb{N}$. Then, for every $T \in \mathcal{G}l_r(\mathbb{C})$, and every neighborhood $\mathcal{W}$ of $\Delta^\infty(T)$, there exists $n \in \mathbb{N}$ and a neighborhood $\mathcal{U}$ of T such that $\Delta^n(\mathcal{U}) \subseteq \mathcal{W}$. Then, note that $\Delta^\infty \circ \Delta^n = \Delta^\infty$. ▲

From now on, let $N_0 \in \mathcal{N}(r)$ be a fixed normal invertible matrix such that $\lambda(N_0) = \lambda$ and $\sigma(N_0) = (\mu_1, \dots, \mu_k)$. Let $\varepsilon_\mu = \frac{1}{3}\min_{i\neq j}|\mu_i - \mu_j|$. Consider the open set $\mathcal{M}_\lambda$ defined in equation (3). Recall that, for $T \in \mathcal{M}_\lambda$, we call $\Pi_E(T) = \sum_{i=1}^{k} \mu_i E_i(T) \in \mathcal{S}(N_0)$.

Definition 5.2.2. With the previous notations, we denote

1. $\mathcal{M}_{\lambda,\eta}$ the open subset of $\mathcal{M}_\lambda$ obtained in the same way, but by replacing ε_μ by $\eta \in (0, \varepsilon_\mu)$. Observe that $\rho(T - \Pi_E(T)) < \eta$ for every $T \in \mathcal{M}_{\lambda,\eta}$.

2. Given $\beta > 0$, we denote $\mathbb{P}_\beta = \{N \in \mathcal{N}(r) : \operatorname{dist}(N, \mathcal{U}(N_0)) \leq \beta\}$. Observe that $\mathbb{P}_\beta$ is compact. ▲

Lemma 5.2.3. *With the previous notations, let $\beta > 0$ such that the closed ball $\overline{B(N_0, \beta)}$ is contained in $\mathcal{M}_\lambda$. Then*

1. *$\mathbb{P}_\beta \subseteq \mathcal{M}_\lambda$.*

2. *For every $\eta < \min\{\beta, \varepsilon_\mu\}$, it holds that $\mathcal{M}_{\lambda,\eta} \subseteq B_\Delta(\mathbb{P}_\beta)$.*

3. *Moreover, if $T \in \mathcal{M}_{\lambda,\eta}$ and $N = \Delta^\infty(T)$, then $N \in \mathcal{M}_{\lambda,\eta}$ and $\|N - \Pi_E(N)\| < \eta$.*

Proof. If $N \in \mathbb{P}_\beta$, let $U \in \mathcal{U}(r)$ such that $\|N - UN_0U^*\| \leq \beta$ (recall that $\mathcal{U}(N_0)$ is compact). Then $U^*NU \in \overline{B(N_0, \beta)} \subseteq \mathcal{M}_\lambda$, so that also $N \in \mathcal{M}_\lambda$.

Let $T \in \mathcal{M}_{\lambda,\eta}$ and denote $N = \Delta^\infty(T)$. Since $\lambda(N) = \lambda(T)$, then also $N \in \mathcal{M}_{\lambda,\eta}$. Since N is normal, then $\Pi_E(N) \in \mathcal{U}(N_0)$ and

$$\eta > \rho(N - \Pi_E(N)) = \|N - \Pi_E(N)\| ,$$

because N commutes with $\Pi_E(N)$, so that $N - \Pi_E(N)$ is normal. Therefore, $N \in \mathbb{P}_\eta \subseteq \mathbb{P}_\beta$ and $T \in B_\Delta(\mathbb{P}_\beta)$. ■

Theorem 5.2.4. *The map Δ^∞ is continuous on $\mathcal{G}l_r(\mathbb{C})$.*

Proof. By Remark 5.2.1, it is enough to prove the continuity at the normal matrices of $\mathcal{G}l_r(\mathbb{C})$. Fix $N_0 \in \mathcal{G}l_r(\mathbb{C})$ a normal matrix. Let $\varepsilon > 0$, such that $B(N_0, \varepsilon) \subseteq \mathcal{G}l_r(\mathbb{C})$. We shall use the notatios of the previous statements relative to N_0. For $\beta < \min\{\frac{\varepsilon}{2}, \varepsilon_\mu\}$ small enough, we can extend to the compact set $\mathbb{P}_\beta$ the distribution of subspaces $N \mapsto F_N$ and $\mathcal{E}^s_N$ given by Corolary 3.1.2 (for $\mathbb{P} = \mathcal{U}(N_0)$), by using the functional calculus on the derivatives $T_M\Delta$, for $M \in \mathbb{P}_\beta$ (see Remark 3.1.3). In this case, the subspaces F_M and $\mathcal{E}^s_M$ are $T_M\Delta$-invariant, $T_M\Delta\big|_{F_M}$ is near I_{F_M} and $\left\| T_M\,\Delta|_{\mathcal{E}^s_M} \right\| \leq k'_{N_0} < 1$, for some $k_{N_0} < k'_{N_0} < 1$.

Observe that the set $\mathbb{P}_\beta$ consists of fixed points for Δ. Hence this distribution are in the hypothesis of Proposition 3.2.2. Let $\rho > 0$ such that $B(N_0, \rho) \subseteq \mathcal{M}_{\lambda,\beta} \subseteq B_\Delta(\mathbb{P}_\beta)$. Following the same steps of the proof of Proposition 4.2.2, but using Lemma A.2.4

instead of Lemma A.2.2, we obtain that, if ρ is small enough then, for every $T \in B(N_0\,,\rho)$, there exists

$$N_1 \in \Pi_E(\mathcal{W}_T^{ss}) \cap \mathcal{U}\,(N_0) \cap B\big(N_0\,,\frac{\varepsilon}{2}\big)\ .$$

Let $S \in \mathcal{W}_T^{ss}$ such that $\Pi_E(S) = N_1$. Since $E(S) = E(N_1)$ is an orthogonal system of projectors, Proposition 2.1.3 and Eq. (7) of Proposition 3.2.2 assure that

$$\Delta^\infty(T) = \Delta^\infty(S) = N_2 \quad \text{ and } \quad \Pi_E(N_2) = \Pi_E(S) = N_1\ .$$

Since $T \in \mathcal{M}_{\lambda,\beta}$, Lemma 5.2.3 assures that

$$\|N_2 - N_1\| = \|N_2 - \Pi_E(N_2)\| < \beta < \frac{\varepsilon}{2}\ .$$

This shows that $\Delta^\infty(B(N_0\,,\rho)) \subseteq B(N_0\,,\varepsilon)$, i.e., that Δ^∞ is continuous at N_0. ■

Remark 5.2.5. By Remark 2.1.5, the Aluthge transform fails to be differentiable at every non invertible normal matrix. By this fact, we can not use the previous techniques for proving continuity of Δ^∞ on $\mathcal{M}_r(\mathbb{C}) \setminus \mathcal{G}l_r(\mathbb{C})$. We conjecture that it is, indeed, continuous on $\mathcal{M}_r(\mathbb{C})$, but we have no proof for non invertible matrices. ▲

6 Concluding remarks

6.1 Rate of convergence

In [6] we proved that, if $T \in \mathcal{M}_r(\mathbb{C})$ is diagonalizable, then after some iterations the rate of convergence of the sequence $\Delta^n(T)$ becomes exponential. More precisely, for some $n_0 \in \mathbb{N}$ and every $n \ge n_0$, there exist $C > 0$ and $0 < \gamma < 1$ such that $\|\Delta^n(T) - \Delta^\infty(T)\| < C\gamma^n$. This exponential rate depends on the spectrum of T. Actually, if $\lambda(T) = \lambda(D)$ for some diagonal matrix D, then $\gamma = k_D$, the constant which appears in Theorem 2.2.5. Using the formula for k_D, one can see that it is closer to 1 (so that the rate of convergence becomes slower) if the different eigenvalues are closer one to each other.

These facts are not longer true if T is not diagonalizable, since the rate of convergence for such a T depends on the rate of convergence for some $M \in \mathcal{W}_T^{ss} \cap \mathcal{O}_D$ (with the notation of Proposition 4.2.2), which can be much slower (and not exponential). Observe that the proof of the convergence of the sequence $\{\Delta^n(M)\}$, given in Proposition 4.2.1, does not study the rate of convergence. It only shows that there exists an unique possible limit point for the sequence.

Nevertheless, using Proposition 4.2.2 and Eq. (7), it is easy to see that the system of projections $E(\Delta^n(T))$ converges to $E(\Delta^\infty(T))$ exponentially, because $E(M) = E(\Delta^\infty(T))$. As in the case of diagonalizable matrices the rate of convergence of the spectral projections depends on the spectrum of T, which agree with the spectrum of M.

Note that the spectrum of T and the spectral projections of M completely characterize the limit $\Delta^\infty(T)$. Indeed, if $\sigma(T) = \{\mu_1, \ldots, \mu_k\}$, then

$$\Delta^\infty(T) = \Delta^\infty(M) = \Pi_E(M) = \sum_{j=1}^{k} \mu_j\, E_j(M) \ .$$

6.2 λ-Aluthge transform

Given $\lambda \in (0,1)$ and a matrix $T \in \mathcal{M}_r(\mathbb{C})$ whose polar decomposition is $T = U|T|$, the λ-Aluthge transform of T is defined by

$$\Delta_\lambda(T) = |T|^\lambda U |T|^{1-\lambda} \ .$$

All the results obtained in this paper are also true for the λ-Aluthge transform for every $\lambda \in (0,1)$, with almost the same proofs. Indeed, note that the basic results about Aluthge transform used throughout sections 3 and 4 are Theorem 2.2.5 and those stated in subsection 2.1. All these results were extended to every λ-Aluthge transform (see [5] and [7]). The unique difference is that the constant k_D of Theorem 2.2.5 now depends on λ (see Theorem 3.2.1 of [7]). Anyway, the new constants are still lower than one for every $\lambda \in (0,1)$. Moreover, they are uniformly lower than one on compact subsets of $(0,1)$.

Another result which depends particularly on the Aluthge transform is Proposition 3.1.1, which is used to prove Corollary 3.1.2. Nevertheless, it is easy to see that both results are still true for every $\lambda \in (0,1)$. On the other hand, the proof of Theorem 5.2.4 uses the same facts about the Aluthge transform. So that, it also remains true for Δ_λ, for every $\lambda \in (0,1)$. We resume all these remarks in the following statement:

Theorem 6.2.1. For every $T \in \mathcal{M}_r(\mathbb{C})$ and $\lambda \in (0,1)$, the sequence $\Delta_\lambda^n(T)$ converges to a normal matrix $\Delta_\lambda^\infty(T)$. The map $T \mapsto \Delta_\lambda^\infty(T)$ is continuous on $\mathcal{G}l_r(\mathbb{C})$. ■

We extend the conjecture given in Remark 5.2.5 to the following:

Conjecture 2. The map $(0,1) \times \mathcal{M}_r(\mathbb{C}) \ni (\lambda, T) \mapsto \Delta_\lambda^\infty(T)$ is continuous .

Using the same ideas as in section 4 of [7], it can be proved that the above map is continuous if it is restricted to $(0,1) \times \mathcal{G}l_r(\mathbb{C})$. ▲

A Appendix

A.1 Proof of Proposition 3.2.1

Recall that $\mathbb{P}$ is a compact set consisting of fixed points of Δ with two complementary, continuous and $T_N\Delta$ invariant distributions $N \mapsto \mathcal{E}_N^s$ and $N \mapsto \mathcal{F}_N$ such that

$$\left\| T_N\, \Delta|_{\mathcal{E}_N^s} \right\| < 1 - \rho \quad \text{and} \quad \|(I - T_N\,\Delta)\,|_{\mathcal{F}_N}\| < \frac{\rho}{2}\ , \quad N \in \mathbb{P}\ , \tag{11}$$

for some $\rho \in (0,1)$ which does not depend on N. The aim of the Proposition is to extend them to distributions defined in some local basin of $\mathbb{P}$ with almost the same properties.

The first step is to extend these distributions using the functional calculus: Fix $\varepsilon > 0$ such that $\sigma(T_T\Delta) \subseteq B(1, \frac{\rho}{2}) \cup B(0, 1-\rho)$, for every $T \in B_\Delta(\mathbb{P})_\varepsilon$. As in Remark 3.1.3, consider the spectral subspaces

$$F_T = R\big(\aleph_{B(1,\frac{\rho}{2})}(T_T\Delta)\big) \quad \text{and} \quad E_T = R\big(\aleph_{B(0,1-\rho)}(T_T\Delta)\big) .$$

Observe that Eq. (11) assures that $E_N = \mathcal{E}^s_N$ and $F_N = \mathcal{F}_N$ for every $N \in \mathbb{P}$. Since the functional calculus is smooth and $\mathbb{P}$ is compact, we can assume that, for every $T \in B_\Delta(\mathbb{P})_\varepsilon$, the angle between F_T and E_T is uniformly bounded from below, and $T_T\Delta$ satisfies inequalities as in Eq. (11), when it is restricted to E_T and F_T. Let us take the cones $C_T = C(\alpha, E_T)$ of size α in the direction E_T. For every small α, we can assume that

a) There exists $\gamma > 0$ such that $C_T \cap C(\gamma, F_T) = \{0\}$ for every $T \in B_\Delta(\mathbb{P})_\varepsilon$.

b) Every subspace $E'_T \subseteq C_T$ with $\dim E'_T = \dim E_T$ satisfies inequalities as in Eq (11).

Claim A.1.1. *There exist positive constants $\lambda_0 < 1$ and $\alpha > 0$ such that, if ε is a small enough, then for every $T \in B_\Delta(\mathbb{P})_\varepsilon$ it holds that $[T_T\Delta]^{-1}(C_{\Delta(T)})$ is a cone of size not greater than $\lambda_0\,\alpha$ inside C_T.*

Proof of the claim: First observe that, by the properties of the subspaces E_T and F_T, there exist $\lambda_1 < 1$ and $\alpha > 0$ such that, for every $T \in B_\Delta(\mathbb{P})_\varepsilon$ it holds that $[T_T\Delta]^{-1}(C_T) \subseteq C(\lambda_1\,\alpha, E_T)$ which is a cone of size $\lambda_1\,\alpha$ inside C_T.

Take $T \in B_\Delta(\mathbb{P})_\varepsilon$ and its image $\Delta(T) \in B_\Delta(\mathbb{P})_\varepsilon$. Observe that Δ commutes with unitary conjugations, and Δ is uniformly continuous on compact sets. Hence, if ε is taken small enough, then T is arbitrarily (and uniformly) close to $\Delta(T)$ for every $T \in B_\Delta(\mathbb{P})_\varepsilon$. Therefore, E_T is arbitrarily and uniformly close to $E_{\Delta(T)}$, and the same occurs between $C_{\Delta(T)}$ and C_T. Putting all together, it follows that

$$[T_T\Delta]^{-1}(C_{\Delta(T)}) \sim [T_T\Delta]^{-1}(C_T) \subseteq C(\lambda_1\,\alpha, E_T) .$$

Therefore, there exists $\lambda_1 < \lambda_0 < 1$ such that $[T\Delta]^{-1}(C_{\Delta(T)})$ is a cone of size not greater than $\lambda_0\,\alpha$ inside C_T. This completes the proof of the Claim.

It is easy to see that the Claim implies that, if C is a cone of size $\beta < \alpha$ inside $C_{\Delta(T)}$ and of the same dimension, then $[T_{\Delta(T)}\Delta]^{-1}(C)$ is a cone of size not greater than $\lambda_0\beta$ inside C_T. For each $T \in B_\Delta(\mathbb{P})_\varepsilon$, consider the sequence $\{\Delta^n(T)\}_{n\in\mathbb{N}}$ and the sequence of cones

$$C_1 = [T_T\Delta]^{-1}(C_{\Delta(T)}) \quad \text{and} \quad \{C_n\}_{n\in\mathbb{N}} = \{[T_T\Delta^n]^{-1}(C_{\Delta^n(T)})\}_{n\in\mathbb{N}}$$

in $T_T\mathcal{M}_r(\mathbb{C})$. The following facts hold: For every $n \in \mathbb{N}$,

$$\begin{aligned} C_{n+1} &= [T_T\Delta^n]^{-1}\big([T_{\Delta^n(T)}\Delta]^{-1}C_{\Delta^{n+1}(T)}\big) \\ &\subseteq [T_T\Delta^n]^{-1}(C_{\Delta^n(T)}) = C_n \ . \end{aligned}$$

Therefore $C_{n+1} \subseteq C_n \subseteq C_1 \subseteq C_T$ and every C_n is a cone of size not greater than $\lambda_0^n \alpha$. An easy argument of dimensions shows that every set C_n contains a subspace of dimension equal to $\dim E_T$ (even if the derivarives $T_{\Delta^n(T)}\Delta$ are not bijective). Therefore,

$$\mathcal{E}_T^s := \bigcap_{n\in\mathbb{N}} C_n = \bigcap_{n\in\mathbb{N}} [T_{\Delta^n(T)}\Delta]^{-n}(C_{\Delta^n(T)})$$

is a a well defined unique direction, and $\dim \mathcal{E}_T^s = \dim E_T$. Observe that the direction is invariant and $\mathcal{E}_T^s \subseteq C_T$, and so it is contracted by $T\Delta$. Take $\mathcal{F}_T = F_T$, $T \in B_\Delta(\mathbb{P})_\varepsilon$, which is continuous by construction. The continuity of $\mathcal{E}_T^s$ follows from the fact that this subbundle is invariant and uniformly contracted for any forward iterate and from the uniqueness of a subbundle (with maximal dimension) exhibiting these properties. Finally, the subspaces $\mathcal{E}_T^s$ and $\mathcal{F}_T$ satisfy Eq (11) by construction. ■

A.2 Proof of Proposition 4.2.2

In this section we shall use the following notations: If $D \in \mathcal{M}_r(\mathbb{C})$ is an invertible diagonal matrix then $\mathbb{P} = \mathcal{U}(D)$ and, for every $T \in B_\Delta(\mathbb{P})_\varepsilon$, by means of $\mathcal{W}_T^{ss} = \mathcal{W}^{ss}(T) : (-1,1)^m \to B_\Delta(\mathbb{P})$ we denote the maps given by Proposition 3.2.2. The invariant manifolds will be denoted by $\mathcal{W}_T^{ss}\big((-1,1)^m\big)$. Finally, for a sake of simplicity, for every $t > 0$, $\mathcal{Q}_t$ denotes the m-dimensional cube $(-t,t)^m$.

Observe that $\mathcal{W}_T^{ss}(\mathcal{Q}_1)$ intersects $\mathcal{O}_D$ if and only if $\Pi_E(\mathcal{W}_T^{ss}(\mathcal{Q}_1))$ intersects $\mathbb{P}$. The proof of Proposition 4.2.2 uses this remark and it is based on some well known results about transversal intersections, using that $\Pi_E(\mathcal{W}_T^{ss}(\mathcal{Q}_1))$ is "C^2-close" to another manifold ($\mathcal{W}_N^{ss}(\mathcal{Q}_1)$ for some $N \in \mathbb{P}$ near T) which intersecs transversally $\mathbb{P}$, both contained in $\mathcal{S}(D)$. We give a proof adapted to our case, divided into three lemmas: We begin with the following classical result (see for example [10, pg.36]).

Lemma A.2.1. Let $U \subseteq \mathbb{R}^m$ be an open set and $W \subseteq U$ an open set with compact closure $\overline{W} \subseteq U$. Let $M \subseteq \mathbb{R}^n$ be a smooth submanifold and $f : U \to M$ a C^1 embedding. There exists $\varepsilon > 0$ such that, if

$$g : U \to M \text{ is } C^1, \quad \|T_x g - T_x f\| < \varepsilon \quad \text{and} \quad \|g(x) - f(x)\| < \varepsilon$$

for every $x \in W$, then $g|_W$ is an embedding. ■

Lemma A.2.2. *Let D and $\mathbb{P}$ be as in Proposition 4.2.2. Then there is $\eta < \varepsilon$ such that the map*

$$\mathcal{V} : B_\Delta(\mathbb{P})_\eta \to \mathrm{Emb}^2\big(\mathcal{Q}_{\frac{1}{2}}, \mathcal{S}(D)\big) \text{ given by } \mathcal{V}_T = \Pi_E \circ \mathcal{W}_T^{ss}|_{\mathcal{Q}_{\frac{1}{2}}},$$

is well defined and continuous with respect to the the C^2 topology of $\mathrm{Emb}^2\big(\mathcal{Q}_{\frac{1}{2}}, \mathcal{S}(D)\big)$.

Proof. Consider the map $\widetilde{\mathcal{V}} : B_\Delta(\mathbb{P})_\varepsilon \to C^2(\mathcal{Q}_1, \mathcal{S}(D))$ given by $\widetilde{\mathcal{V}}_T = \Pi_E \circ \mathcal{W}_T^{ss}$. By Proposition 3.2.2 and Remark 2.4.2, $\widetilde{\mathcal{V}}$ is well defined and continuous, if $C^2(\mathcal{Q}_1, \mathcal{S}(D))$

is endowed with the C^2 topology. Observe that $\mathcal{W}_N^{ss}$ takes values in $\mathcal{S}(D)$ for every $N \in \mathbb{P}$. Indeed, this follows by Cor. 3.1.2 of [6], or by rewriting the proof of Proposition 3.2.2 inside $\mathcal{S}(D)$ in this case. Therefore, $\widetilde{\mathcal{V}}_N = \mathcal{W}_N^{ss}$ for every $N \in \mathbb{P}$, because Π_E is the identity on $\mathcal{S}(D)$.

Given $N \in \mathbb{P}$, Lemma A.2.1 assures that there exists ε_N such that, if $\mathcal{T} : \mathcal{Q}_1 \to \mathcal{M}_r(\mathbb{C})$ is a C^1 map which satisfies that

$$\|T_x \mathcal{W}_N^s - T_x \mathcal{T}\| < \varepsilon_N \quad \text{and} \quad \|\mathcal{W}_N^s(x) - \mathcal{T}(x)\| < \varepsilon_N \ , \tag{12}$$

for every $x \in \mathcal{Q}_{\frac{1}{2}}$, then $\mathcal{T}\big|_{\mathcal{Q}_{\frac{1}{2}}}$ is an embedding. By the continuity of $\widetilde{\mathcal{V}}$, there is a neighborhood $\mathcal{U}_N$ of N in $B_\Delta(\mathbb{P})_\varepsilon$ such that, for every $T \in \mathcal{U}_N$, the map $\widetilde{\mathcal{V}}_T$ satisfies (12). Take $\eta > 0$ such that $B_\Delta(\mathbb{P})_\eta \subseteq \bigcup_{N\in\mathbb{P}} \mathcal{U}_N$. Then, $\mathcal{V}(T) = \widetilde{\mathcal{V}}_T\big|_{\mathcal{Q}_{\frac{1}{2}}} \in \operatorname{Emb}^2(\mathcal{Q}_{\frac{1}{2}}, \mathcal{S}(D))$, for every $T \in B_\Delta(\mathbb{P})_\eta$ i.e., $\mathcal{V}$ is well defined. The continuity of $\mathcal{V}$ follows from the fact that both $\widetilde{\mathcal{V}}$ and the restriction map $\mathcal{T} \mapsto \mathcal{T}\big|_{\mathcal{Q}_{\frac{1}{2}}}$ are continuous with respect to the the C^2 topology. ■

Lemma A.2.3. *Let D and $\mathbb{P}$ be as in Proposition 4.2.2. Given $N_0 \in \mathbb{P}$ and $\varepsilon > 0$, there exists a C^2-neighborhood Ω of $\mathcal{W}_{N_0}^{ss}\big|_{\mathcal{Q}_{\frac{1}{2}}}$ in the space $\operatorname{Emb}^2(\mathcal{Q}_{\frac{1}{2}}, \mathcal{S}(D))$, such that $\mathcal{T}(\mathcal{Q}_{\frac{1}{2}})$ intersects the submanifold $\mathbb{P}$ at a point $N \in B(N_0, \varepsilon)$, for every $\mathcal{T} \in \Omega$.*

Proof. Let $(\mathcal{U}_{N_0}, \varphi)$ be a chart in $\mathcal{S}(D)$ such that $N_0 \in \mathcal{U}_{N_0} \subseteq B(\varepsilon, N_0)$, $\varphi(N_0) = 0$, and $\varphi(\mathbb{P} \cap \mathcal{U}_{N_0}) = \varphi(\mathcal{U}_{N_0}) \cap (\{0\} \oplus \mathbb{R}^{n-m})$, where $\mathbb{R}^n \simeq \mathbb{R}^m \oplus \mathbb{R}^{n-m}$. Let P denote the orthogonal projection from $\mathbb{R}^n$ onto $\mathbb{R}^m \oplus \{0\}$.

By Proposition 3.2.2, the intersection $\mathcal{W}_{N_0}^{ss}(\mathcal{Q}_{\frac{1}{2}}) \cap \mathcal{U}(D) = \{N_0\}$ is transversal. Then, there exist $\delta \in (0, 1/2)$ and a C^2-neighborhood Ω_0 of $\mathcal{W}_{N_0}^{ss}\big|_{\mathcal{Q}_{\frac{1}{2}}}$ in $\operatorname{Emb}^2(\mathcal{Q}_{\frac{1}{2}}, \mathcal{S}(D))$ such that, for every $\mathcal{T} \in \Omega_0$,

1. $\mathcal{T}(\overline{\mathcal{Q}_\delta}) \subseteq \mathcal{U}_{N_0}$;
2. $\ker P \oplus T_M \widetilde{\mathcal{T}}(\mathcal{Q}_\delta) = \mathbb{R}^n$, where $\widetilde{\mathcal{T}} = \varphi \circ \mathcal{T}\big|_{\mathcal{Q}_\delta}$ and $M \in \widetilde{\mathcal{T}}(\mathcal{Q}_\delta)$.
3. The angle between $\ker P$ and $T_M \widetilde{\mathcal{T}}(\mathcal{Q}_\delta)$ is uniformly bounded from below.

Note that items (2) and (3) imply that, $\mathbb{R}^k \oplus \{0\}$. On the other hand, item (2) also implies that for every $\mathcal{T} \in \Omega_0$ and every $M \in \widetilde{\mathcal{T}}(\mathcal{Q}_\delta)$, the linear map P acting on $T_M \widetilde{\mathcal{T}}(\mathcal{Q}_\delta)$ is uniformly bounded from below. On the other hand, the norm of the second derivative of $P \circ \widetilde{\mathcal{T}}$ is bounded on $\mathcal{Q}_{\frac{\delta}{2}}$. Hence there exists $\mu > 0$ so that, for every $M \in \widetilde{\mathcal{T}}\left(\mathcal{Q}_{\frac{\delta}{2}}\right)$,

$$B(P(M), \mu) \subseteq P\big(\widetilde{\mathcal{T}}(\mathcal{Q}_\delta)\big) \ . \tag{13}$$

Take $\Omega \subseteq \Omega_0$ such that $\|\widetilde{\mathcal{W}}_{N_0}^{ss}(x) - \widetilde{\mathcal{T}}(x)\| < \mu/2$ for every $\mathcal{T} \in \Omega$ and every $x \in \mathcal{Q}_\delta$, where $\widetilde{\mathcal{W}}_{N_0}^{ss} = \varphi \circ \mathcal{W}_{N_0}^{ss}\big|_{\mathcal{Q}_\delta}$. As $\widetilde{\mathcal{W}}_{N_0}^{ss}(0) = 0$, Eq. (13) implies that $0 \in P\big(\widetilde{\mathcal{T}}(\mathcal{Q}_\delta)\big)$, for every $\mathcal{T} \in \Omega$.

Thus $\mathcal{T} \cap \mathcal{U}(D) \cap \mathcal{U}_{N_0} \neq \varnothing$, because $\mathcal{T}(\overline{\mathcal{Q}_\delta}) \subseteq \mathcal{U}_{N_0}$. In particular, $\mathcal{T}(\mathcal{Q}_{\frac{1}{2}})$ intersects the submanifold $\mathbb{P}$ transversally at a point $N \in \mathcal{U}_{N_0} \subseteq B(N_0, \varepsilon)$. ■

Proof of Proposition 4.2.2: Given $N \in \mathbb{P}$, Lemma A.2.3 assures that there is a C^2-neighborhood Ω_N of $\mathcal{W}^{ss}_N\big|_{\mathcal{Q}_{\frac{1}{2}}}$ in $\mathrm{Emb}^2(\mathcal{Q}_{\frac{1}{2}}, \mathcal{S}(D))$ such that $\mathcal{T}(\mathcal{Q}_{\frac{1}{2}}) \cap \mathbb{P} \neq \varnothing$ for every $\mathcal{T} \in \Omega_N$. Let $\mathcal{V}$ be the function defined in Lemma A.2.2, and let $\mathcal{U}_N = \mathcal{V}^{-1}(\Omega_N)$. Define $\mathcal{U}_{\mathbb{P}} = \bigcup_{N \in \mathbb{P}} \mathcal{U}_N$. Therefore, $\mathcal{U}_{\mathbb{P}}$ is an open neighborhood of $\mathbb{P}$ contained in $B_\Delta(\mathbb{P})$. Since $\mathbb{P}$ is compact, there exists $0 < \eta < \varepsilon$ such that $B_\Delta(\mathbb{P})_\eta \subseteq \mathcal{U}_{\mathbb{P}}$. Then, for every $T \in B_\Delta(\mathbb{P})_\eta$, $\Pi_E\big(\mathcal{W}^{ss}_T(\mathcal{Q}_{\frac{1}{2}})\big)$ itersects $\mathbb{P}$. By Proposition 3.2.2, $\mathcal{W}^{ss}_T(\mathcal{Q}_{\frac{1}{2}}) \subseteq B_\Delta(\mathbb{P})$. So that $\mathcal{W}^{ss}_T(\mathcal{Q}_{\frac{1}{2}}) \cap \mathcal{O}_D \neq \varnothing$. ■

The proof of the next result, which is used in the proof of the continuity of the limit function Δ^∞, follows the same lines as the proof of Lemma A.2.2.

Lemma A.2.4. Let $N_0 \in \mathcal{M}_r(\mathbb{C})$ be a normal matrix, $\mathbb{P}_\beta$ as in Definition 5.2.2 and $\mathcal{W}^{ss} : B_\Delta(\mathbb{P}_\beta)_\varepsilon \to \mathrm{Emb}^2(\mathcal{Q}_1, B_\Delta(\mathbb{P}))$ the prelamination given by Proposition 3.2.2. If Π_E is defined with respect to the spectrum of N_0, then there exists $\eta < \beta$ so that the map

$$\mathcal{V} : B_\Delta(\mathbb{P}_\eta)_\eta \to \mathrm{Emb}^2\big(\, \mathcal{Q}_{\frac{1}{2}}, \mathcal{S}(N_0)\, \big)$$

given by $\mathcal{V}_T = \Pi_E \circ \mathcal{W}^{ss}_T\big|_{\mathcal{Q}_{\frac{1}{2}}}$ is well defined and continuous with the C^2 topology of $\mathrm{Emb}^2\big(\, \mathcal{Q}_{\frac{1}{2}}, \mathcal{S}(N_0)\, \big)$. ■

References

[1] A. Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integral Equations Operator Theory 13 (1990), 307-315.

[2] T. Ando, *Aluthge Transforms and the Convex Hull of the Eigenvalues of a Matrix*, Linear Multilinear Algebra 52 (2004), 281-292.

[3] T. Ando and T. Yamazaki, *The iterated Aluthge transforms of a 2-by-2 matrix converge*, Linear Algebra Appl. 375 (2003), 299-309.

[4] E. Andruchow and D. Stojanoff, *Differentiable structure of similarity orbits*, J. Operator Theory 21 (1989), 349-366.

[5] J. Antezana, P. Massey and D. Stojanoff, *λ-Aluthge transforms and Schatten ideals*, Linear Algebra Appl. 405 (2005), 177-199.

[6] J. Antezana, E. Pujals and D. Stojanoff, *Convergence of iterated Aluthge transform sequence for diagonalizable matrices*, Advances in Mathematics 216 (2007) 255-278.

[7] J. Antezana, E. Pujals and D. Stojanoff, *Convergence of iterated Aluthge transform sequence for diagonalizable matrices II - λ-Aluthge transform,* preprint available in arXiv.

[8] G. Corach, H. Porta and L. Recht, *The geometry of spaces of projections in C^*-algebras*, Adv. Math. 101 (1993), 59-77.

[9] K. Dykema and H. Schultz, *On Aluthge Transforms: continuity properties and Brown measure*, preprint available in arXiv.

[10] M. W. Hirsch, *Differential topology*, Graduate Texts in Mathematics 33, Springer-Verlag, New York 1994.

[11] M. W. Hirsch, C. C. Pugh, and M. Shub, *Invariant manifolds*, Lecture Notes in Mathematics, Vol. 583. Springer-Verlag, Berlin-New York, 1977.

[12] Huajun Huang and Tin-Yau Tam, *On the Convergence of the Aluthge sequence*, Oper. Matrices 1 (2007), no. 1, 121-141.

[13] I. Jung, E. Ko, and C. Pearcy, *Aluthge transform of operators*, Integral Equations Operator Theory 37 (2000), 437-448.

[14] I. Jung, E. Ko, and C. Pearcy, *The Iterated Aluthge Transform of an operator*, Integral Equations Operator Theory 45 (2003), 375-387.

[15] T. Kato, Perturbation theory for linear operators. Reprint of the 1980 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995.

[16] M. Shub, *Global Stability of Dynamical Systems, Springer*, 1986 (this book is a translation of the French edition with added chapters and corrections).

[17] M. Shub and A. T. Vasquez, *Some linearly induced Morse-Smale systems, the QR algorithm and the Toda lattice* The legacy of Sonya Kovalevskaya, Proc. Symp., Radcliffe Coll. 1985, Contemp. Math. 64, 181-194 (1987).

[18] T. Yamazaki, *An expression of the spectral radius via Aluthge tranformation*, Proc. Amer. Math. Soc. 130 (2002), 1131-1137.

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