LYAPUNOV STABLE CHAIN RECURRENT CLASS

JIAGANG YANG

ABSTRACT. We show that there exists a generic subset $R \subset C^1(M) \setminus \overline{HT}$, such that for $f \in R$, the union of stable manifolds for periodic points of f is dense, it gives an answer to Bonatti's conjecture in the set $(\overline{HT})^c$. In fact, we prove that when $f \in R$ and C is any Lyapunov stable chain recurrent class of f, it should be a homoclinic class, what's more, suppose i_0 is the minimal index for the periodic points in C, then C has an index i_0 dominated splitting $E_{i_0}^{cs} \oplus E_{i_0+1}^{cu}$, and either $E_{i_0}^{cs}|_C$ is contracting or C is an index $i_0 - 1$ fundamental limit and the bundle $E_{i_0}^{cs}|_C$ has a codimension-1 sub-dominated splitting $E_{i_0}^{cs}|_C = E_{i_0-1}^s \oplus E_1^{cs}$ where $E_{i_0-1}^s$ is contracting and $dim(E_1^{cs})=1$

1. INTRODUCTION

This paper is about generic dynamics, a subject that has been very active in the last years. The theory of generic dynamics is trying to give a description of a large class of differential dynamics, especially it can help us understanding the non-hyperbolic diffeomorphisms which is one of the most important aim of modern dynamical theory.

The stable manifold for hyperbolic periodic point is one of the most basic and important object in differential dynamic, such submanifold has a special converging property, and the complicated phenomena: homoclinic intersection just comes from the transverse intersection between the stable manifold and unstable manifold. When a diffeomorphism f is hyperbolic, it's well known that the union of stable manifolds of f's periodic points of is dense, but people discovered that the set of hyperbolic diffeomorphisms are not dense among differential dynamics, so we want to know that if the results on the hyperbolic systems can indicate the same property will be hold for generic non-hyperbolic systems. Here we proved that:

Theorem 1: *There exists a generic subset $R \subset (\overline{HT^1})^c$ such that for any $f \in R$,* $\bigcup_{p \in Per(f)} W^s(p)$ *is dense in M.*

These result gives a partial answer to the following Bonatti's conjecture:

Conjecture 1 (Bonatti): *There exists a generic subset $R \subset C^1(M)$ such that for any $f \in R$,* $\bigcup_{p \in Per(f)} W^s(p)$ *is dense in M.*

The Bonatti's conjecture is one step towards the following famous conjecture.

C^r **Palis conjecture:** *Diffeomorphisms of M exhibiting either a homoclinic tangency or heterdimensional cycle are C^r dense in the complement of the C^1 closure of hyperbolic systems.*

Since until now, almost all the perturbation tools just work in C^1 topology, so in this paper we just consider C^1 diffeomorphisms and talk about C^1 typical phenomena.

In fact, I believe something even stronger than Palis conjecture should be live:

Conjecture 3 (Tameness conjecture): *There exists a generic subset $R \subset (\overline{HT^1})^c$ such that any $f \in R$ is tame.*

It's not difficult to get C^1 Palis conjecture from tameness conjecture, but until now we can't prove the tameness conjecture even in the simplest open set: the small open neighborhood of the map: *linear Anosov map$|_{T^2} \times Id_{S^1}$*. In the flow case, it looks like true in the set $\mathcal{F}^1(M)$.

In the direction of proving the Tameness conjecture, I propose the following two intermediate problems:

Conjecture 4: *There exists a generic subset $R \subset (\overline{HT^1})^c$ such that for any $f \in R$, its chain recurrent classes are all homoclinic classes.*

Conjecture 5: *There exists a generic subset $R \subset (\overline{HT^1})^c$ such that for any $f \in R$, suppose C is a homoclinic class of f, and $i_0 = \min_{i}\{i : C \bigcap Per_i(f) \neq \phi\}$, then C has an index i_0 dominated splitting $T_C M = E_{i_0}^s \oplus E_{i_0+1}^{cu}$ where $E_{i_0}^s$ is contracting.*

Here I want to point out that the above two weaker conjectures are still enough to prove Palis conjecture, now let's show some simple idea of how to induce C^1 Palis conjecture from the above two conjectures: suppose $f \in R$ and it's far away from heterdimensional cycle ($f \in (\overline{HC} \bigcup \overline{HT})^c$), let C be any chain recurrent class of f, then by conjecture 4, C is a homoclinic class, and by $f \in (\overline{HC})^c$, all the periodic points in C have the same index i_0, then by conjecture 5, C is hyperbolic and has an index i_0 dominated splitting $T_C M = E_{i_0}^s \oplus E_{i_0+1}^u$, then it's easy to know f has just finite chain recurrent classes so it satisfies Axiom A, f satisfies the non-cycle condition is just a well known C^1 generic result from [5]'s connecting lemma.

The above two conjectures have been proved by [37] when M is a boundless surface (in fact, they proved tameness conjecture in this case). In higher dimensional manifold they are still far away to be proved. The following conjectures are weaker more and look like easier to prove:

Conjecture 6: *There exists a generic subset $R \subset (\overline{HT^1})^c$ such that for any $f \in R$, suppose C is any aperiodic class of f, then C has a partial hyperbolic splitting $T_C M = E^s \oplus E^c \oplus E^u$ where $E^s, E^u \neq \phi$ and $dim(E^c) = 1$.*

Conjecture 7: *There exists a generic subset $R \subset (\overline{HT^1})^c$ such that for any $f \in R$, suppose C is a homoclinic class of f and $i_0 = \min_{i}\{i : C \bigcap Per_i(f) \neq \phi\}$, then C has an index i_0 dominated splitting*

$T_C M = E_{i_0}^{cs} \oplus E_{i_0+1}^{cu}$, and either $E_{i_0}^{cs}$ is contracting or $E_{i_0}^{cs}$ has a codimension-1 sub-dominated splitting $E_{i_0}^{cs} = E_{i_0-1}^{s} \oplus E_1^c$ where $E_{i_0-1}^{s}$ is hyperbolic and $dim(E_1^c|_C) = 1$.

All the conjectures above just talk about general chain recurrent classes, before we prove them, we should check them in some special situation. In this paper we'll use a special chain recurrent class: Lyapunov stable chain recurrent class to check these conjectures, and we can show that for this special kind of chain recurrent class Conjecture 4 and half of Conjecture 7 are right, they give some evidence that the above conjectures may be right. The precisely statements are following:

Theorem 2: *There exists a generic subset $R \subset (\overline{HT^1})^c$ such that for any $f \in R$, its Lyapunov stable chain recurrent classes should be homoclinic classes.*

Theorem 3: *There exists a generic subset $R \subset (\overline{HT^1})^c$ such that for any $f \in R$, suppose C is any Lyapunov stable homoclinic class of f, let $i_0 = \min_i\{i : C \bigcap Per_i(f) \neq \phi\}$, then C has an index i_0 dominated splitting $T_C M = E_{i_0}^{cs} \oplus E_{i_0+1}^{cu}$, and*

- *either $E_{i_0}^{cs}$ is contracting and C is an index i_0 fundamental limit*
- *or $E_{i_0}^{cs}$ has a codimension-1 sub-dominated splitting $E_{i_0}^{cs} = E_{i_0-1}^{s} \oplus E_1^c$ where $E_{i_0-1}^{s}$ is contracting and $dim(E_1^c|_C) = 1$, and C is an index $i_0 - 1$ and index i_0 fundamental limit.*

In §3 we'll state some generic properties and give an important technique lemma, its proof will be given in §7. In §4 I'll introduce some properties for fundamental limit and Crovisier's central model, in §5 I'll state the main lemma and use it to prove theorem 1,2,3. The proof of the main lemma is given in §6.

2. DEFINITIONS AND NOTATIONS

Let M be a compact boundless Riemannian manifold, since when M is a surface [37] has proved that hyperbolic diffeomorphisms are open and dense in $C^1(M) \setminus \overline{HT}$, we suppose $dim(M) = d > 2$ in this paper. Let $Per(f)$ denote the set of periodic points of f and $\Omega(f)$ the non-wondering set of f, for $p \in Per(f)$, $\pi(p)$ means the period of p. If p is a hyperbolic periodic point, the index of p is the dimension of the stable bundle. We denote $Per_i(f)$ the set of the index i periodic periodic points of f, and we call a point x is an index i preperiodic point of f if there exists a family of diffeomorphisms $g_n \xrightarrow{C^1} f$, where g_n has an index i periodic point p_n and $p_n \longrightarrow x$. $P_i^*(f)$ is the set of index i preperiodic points of f.

Remark 2.1. *It's easy to know $\overline{P_i(f)} \subset P_i^*(f)$.*

Let Λ be an invariant compact set of f, we call Λ is an index i fundamental limit if there exists a family of diffeomorphisms g_n C^1 converging to f, p_n is an index i periodic point of g_n and $Orb(p_n)$ converge to Λ in *Hausdorff* topology. So if $\Lambda(f)$ is an index i fundamental limit, we have $\Lambda(f) \subset P_i^*(f)$. Λ is minimal index i fundamental limit if $\Lambda(f)$ is an index i fundamental limit and any invariant compact subset $\Lambda_0 \subsetneq \Lambda$ is not an index i fundamental limit. In [51] we have showed the following result:

Lemma 2.2. *Any index i fundamental limit contains a minimal index i fundamental limit.*

For two points $x, y \in M$ and some $\delta > 0$, we say there exists a δ-pseudo orbit connects x and y if there exist points $x = x_0, x_1, \cdots, x_n = y$ such that $d(f(x_i), x_{i+1}) < \delta$ for $i = 0, 1, \cdots, n-1$, and we denote it $x \underset{\delta}{\dashv} y$. We say $x \dashv y$ if for any $\delta > 0$ we have $x \underset{\delta}{\dashv} y$ and denote $x \vdash\!\dashv y$ if $x \dashv y$ and $y \dashv x$. A point x is called a chain recurrent point if $x \vdash\!\dashv x$. $CR(f)$ denotes the set of chain recurrent points of f, it's easy to know that $\vdash\!\dashv$ is a closed equivalent relation on $CR(f)$, and every equivalent class of such relation should be compact and called chain recurrent class. A chain recurrent class C of f is called Lyapunov stable if there exists a family of neighborhoods $\{U_n\}$ of C satisfying:

a) $\overline{U_{n+1}} \subset U_n$,
b) $\bigcap U_n = C$,
c) $f(\overline{U_n}) \subset U_n$.

Remark 2.3. *Conley proved that any homeomorphism f has at least one Lyapunov stable chain recurrent class.*

Lemma 2.4. *Let C be a Lyapunov stable chain recurrent class of f, then if $y \in W^u(C)$ (that means $\lim_{i \to \infty} \min_{z \in C}\{d(f^{-i}(y), z)\} \longrightarrow 0$), we have $y \in C$.*

Proof : For any U_n the neighborhood of C given in the definition of Lyapunov stable chain recurrent class, there exists an $i > 0$ such that $f^{-i}(y) \in U_n$, then $y \in f^i(U_n) \subset U_n$, so $y \in \bigcap_n U_n = C$. □

Let K be a compact invariant set of f, and x, y are two points in K, we denote $x \underset{K}{\dashv} y$ if for any $\delta > 0$, we have a δ-pseudo orbit in K connects x and y. If for any two points $x, y \in K$ we have $x \underset{K}{\dashv} y$, we call K a chain recurrent set. Let C be a chain recurrent class of f, we say C is an aperiodic class if C does not contain periodic point.

Let Λ be an invariant compact set of f, for $l \in \mathbb{N}$, $0 < \lambda < 1$ and $1 \leq i < d$, we say Λ has an index $i - (l, \lambda)$ dominated splitting if we have a continuous invariant splitting $T_\Lambda M = E \oplus F$ where $dim(E_x) = i$ for any $x \in \Lambda$ and $\| Df^l|_E(x) \| \cdot \| Df^{-l}|_F(f^l x) \| < \lambda$ for all $x \in \Lambda$. For simplicity, sometimes we just say $\Lambda(f)$ has an index i dominated splitting. A compact invariant set can have many dominated splittings, but for fixed i, the index i dominated splitting is unique.

We say a diffeomorphism f has C^r tangency if $f \in C^r(M)$, f has hyperbolic periodic point p and there exists a non-transverse intersection between $W^s(p)$ and $W^u(p)$. HT^r denote the set of the diffeomorphisms which have C^r tangency, usually we just use HT denote HT^1. We call a diffeomorphism f is far away from tangency if $f \in C^1(M) \setminus \overline{HT}$. The following proposition shows the relation between dominated splitting and far away from tangency.

Proposition 2.5. ([43]) *f is C^1 far away from tangency if and only if there exists (l, λ) such that $P_i^*(f)$ has index $i - (l, \lambda)$ dominated splitting for $0 < i < d$.*

Usually dominated splitting is not a hyperbolic splitting, Mañé showed that in some special case, one bundle of the dominated splitting is hyperbolic.

Proposition 2.6. ([29]) *Suppose $\Lambda(f)$ has an index i dominated splitting $E \oplus F$ ($i \neq 0$), if $\Lambda(f) \bigcap P_j^*(f) = \phi$ for $0 \leq j < i$, then E is a contracting bundle.*

3. GENERIC PROPERTIES

Here we'll introduce some C^1 generic properties.

For a topology space X, we call a set $R \subset X$ is a generic subset of X if R is countable intersection of open and dense subsets of X, and we call a property is a generic property of X if there exists some generic subset R of X holds such property. Especially, when $X = C^1(M)$ and R is a generic subset of $C^1(M)$, we just call R is C^1 generic, and we call any generic property of $C^1(M)$ 'a C^1 generic property' or 'the property is C^1 generic'.

It's easy to know that if R is C^1 generic and R_1 is a generic subset of R, then R_1 is also C^1 generic.

At first let's state some well known C^1 generic properties.

Proposition 3.1. *There is a C^1 generic subset R_0 such that for any $f \in R_0$, one has*

1) *f is Kupka-Smale (every periodic point p in $Per(f)$ is hyperbolic and the invariant manifolds of periodic points are everywhere transverse).*
2) *$CR(f) = \Omega = \overline{Per(f)}$.*
3) *$P_i^*(f) = \overline{P_i(f)}$*
4) *any chain recurrent set is the Hausdorff limit of periodic orbits.*
5) *any index i fundamental limit is the Hausdorff limit of index i periodic orbits of f.*
6) *any chain recurrent class containing a periodic point p is the homoclinic class $H(p, f)$.*
7) *suppose C is a homoclinic class of f, and $i_0 = min\{i : C \bigcap Per_i(f) \neq \phi\}$, $i_1 = max\{i : C \bigcap Per_i(f) \neq \phi\}$, then for any $i_0 \leq i \leq i_1$, we have $C \bigcap Per_i(f) \neq \phi$ and C is index i fundamental limit.*
8) *if all the Lyapunov stable chain recurrent classes of f are homoclinic classes, then $\bigcup\limits_{p \in Per(f)} W^s(p)$ is dense in M.*

Proof 1) comes from Kupka-Smale theorem, 2) is proved in [5], 3),4),5),6) are all well known, 7) is proved in [2], 8) is proved in [31]. □

By proposition 3.1, for any f in R_0, every chain recurrent class C of f is either an aperiodic class or a homoclinic class. If $\#(C) = \infty$, we say C is non-trivial.

The following technique lemma gives a new C^1 generic property whose proof would be given in §7.

Lemma 3.2. *(Technique lemma). There exists a generic subset R_0' of R_0 such that for $f \in R_0'$, suppose C is a non-trivial chain recurrent class of f, $\Lambda \subsetneq C$ is a compact chain recurrent set without periodic point, then for $0 < s < 1$ and any point $y \in (C \setminus \Lambda) \bigcap W^{s(u)}(\Lambda)$, for any small neighborhood O of y*

and any small neighborhood V of Λ, there exists a periodic point q of f satisfying $Orb(q) \bigcap O \neq \phi$, and $\frac{\#\{Orb(q) \bigcap V\}}{\pi(q)} > s$.

Since R'_0 is a generic subset of R_0 and R_0 is C^1 generic, R'_0 is a C^1 generic subset also.

Corollary 3.3. *There exist a generic subset $R \subset R'_0 \setminus \overline{HT}$ such that for $f \in R$, if C is a chain recurrent class of f, $\Lambda \subsetneq C$ is a non-trivial minimal set with partial hyperbolic splitting $E^s_i \oplus E^c_1 \oplus E^u_{i+1}$ where $dim(E^c_1(\Lambda)) = 1$ and $E^c_1(\Lambda)$ is not hyperbolic, then $W^u(\Lambda) \bigcap C \subset P^*_i \bigcap P^*_{i+1}$ and*

- *either C contains index $i+1$ or index i periodic point and it's an index i fundamental limit,*
- *or for any $y \in (C \bigcap W^u(\Lambda)) \setminus \Lambda$, and $\{V_n\}$ is a family of neighborhoods of Λ satisfying $\overline{V_{n+1}} \subset V_n$ and $\bigcap V_n = \Lambda$, there exists $\{q_n\}$ a family of index i (or $i+1$) periodic points of f such that $(y \bigcup \Lambda) \subset \lim_{n \to \infty} Orb(q_n)$ and $\lim_{n \to \infty} \frac{\#\{Orb(q_n) \bigcap V_n\}}{\pi(q_n)} \longrightarrow 1^-$.*

Proof : At first let's suppose $f \in R'_0 \setminus \overline{HT}$. When $i = 0$ (or $i+1 = d$), theorem 1 of [51] has shown C contains index 1 (d-1) periodic point and C is an index 0 and index 1 (index d and index $d-1$) fundamental limit, so from now we suppose $E^s_i|_\Lambda, E^u_{i+2}|_\Lambda \neq \phi$, and here we just prove the above result for case i, the proof of the case $i+1$ is similar.

Fix any $y \in C \bigcap W^u(\Lambda) \setminus \Lambda$ and V_n is a family of neighborhood of Λ such that $\overline{V_{n+1}} \subset V_n$ and $\bigcap_{n \geq 1} V_n = \Lambda$, choose $\varepsilon_n > 0$ and $0 < s_n < 1$ satisfying $\varepsilon_n \longrightarrow 0^+$ and $s_n \longrightarrow 1^-$. By the technique lemma, there exists a family of periodic points $\{q_n(f)\}$ such that $y \bigcup \Lambda \subset \lim_{n \to \infty} Orb(q_n)$ and $\{q_n\}$ satisfies $\frac{\#\{Orb(q_n) \bigcap V_n\}}{\pi(q_n)} > s_n$. We can let all the $q_n(f)$ have the same index j, we suppose $j \geq i$, since the proof of the other case is the same.

Let $j_1 = \min_{j \geq i}\{j :$ there exists a family of C^1 diffeomorphism g_n such that $\lim_{n \to \infty} g_n \longrightarrow f$ and g_n has an index j periodic point $p_n(g_n)$ such that $\lim_{n \to \infty} Orb_{g_n}(q_n(g_n)) \supset y \bigcup \Lambda$ and $\frac{\#\{Orb_{g_n}(q_n) \bigcap V_n\}}{\pi(q_n)} > s_n\}$.

We claim that

(a) either C contains index $i+1$ or index i periodic point and it's an index i fundamental limit,

(b) or $j_1 = i$.

Proof of the claim

• If $j_1 = i$, we get (b).

• If $j_1 > i$, we'll show (a) is true.

Suppose g_n is the family of diffeomorphisms and $q_n(g_n)$ is the index j_1 periodic point of g_n given in the definition of j_1. Let $\lim_{n \to \infty} Orb_{g_n}(q_n) = C_0$, then $C_0 \subset P^*_{j_1}$, by proposition 2.5, C_0 has an index j_1 dominated splitting $E^{cs}_{j_1} \oplus E^{cu}_{j_1+1}|_{C_0}$.

By the definition of j_1 and Franks lemma, we know that $\{Dg_n|_{E^{cs}_{j_1}(Orb_{g_n}(q_n))}\}^\infty_{n=1}$ is stable contracting. By lemma 4.9, lemma 4.10 and remark 4.11 of [51], there exist N_0, l, $0 < \lambda < 1$ such that for $\pi_{g_n}(q_n) > N_0$, we have $q'_n \in Orb_{g_n}(q_n)$ satisfying $\prod_{j=0}^{s-1} \|Dg_n^l|_{E^{cs}_{j_1}(g_n^{jl}(q'_n))}\| \leq \lambda^s$ for $s \geq 1$. Since Λ is minimal and non-trivial, from $\Lambda \subset \lim_{n \to \infty} Orb_{g_n}(q_n(g_n))$, we know $\lim_{n \to \infty} \pi_{g_n}(q_n(g_n)) \longrightarrow \infty$, so we can suppose $\pi_{g_n}(q_n(g_n)) > N_0$ always. The above point q'_n is called hyperbolic time for bundle $E^{cs}_{j_1}$, its existence comes from Pliss lemma, since $Orb_{g_n}(q_n)$ stays a lot of time in V_n, so in fact from the Pliss lemma we can always choose

$q'_n \in V_n$, then we can suppose $\lim_{n \to \infty} q'_n = x_0 \in \Lambda$, by $\lim_{n \to \infty} g_n \xrightarrow{C^1} f$, we have

$$(1) \qquad \prod_{j=0}^{s-1} \|Df^l|_{E^{cs}_{j_1}(f^{jl}(x_0))}\| \le \lambda^s \text{ for } s \ge 1.$$

Since Λ has two dominated splitting $(E^{cs}_i \oplus E^c_1) \oplus E^{cu}_{i+2}$ and $E^{cs}_{j_1} \oplus E^{cu}_{j_1+1}$ with $j_1 \ge i+1$, by lemma 4.30 of [51], we have that $E^{cs}_i \oplus E^c_1 \subset E^{cs}_{j_1}$, so by (1), we get $\prod_{j=0}^{s-1} \|Df^l|_{(E^{cs}_{i_0} \oplus E^c_1)(f^{jl}(x_0))}\| \le \lambda^s$ for $s \ge 1$. By Λ is minimal and $E^c_1|_{\Lambda}$ is not hyperbolic, the splitting $(E^{cs}_i \oplus E^c_1) \oplus E^{cu}_{i+2}|_{\Lambda}$ satisfies all the assumptions of weakly selecting lemma, by weakly selecting lemma given in [51] and corollary 4.26 there, C contains index $i+1$ periodic point and C is an index i fundamental limit, so C satisfies (a). □

Now with a generic argument like we'll do in §7.1, in the proof of above claim we can replace $R'_0 \setminus \overline{HT}$ by a generic subset $R \subset R'_0 \setminus \overline{HT}$ such that if $f \in R$ and (a) is false, f itself will have a family of index i periodic points $\{q_n\}$ such that $(y \bigcup \Lambda) \subset \lim_{n \to \infty} Orb(q_n)$ and $\lim_{n \to \infty} \frac{\#\{Orb(q_n) \bigcap V_n\}}{\pi(q_n)} \longrightarrow 1^-$. □

We'll show the generic set R satisfies theorem 1, 2 and 3.

4. Fundamental limit and Crovisier's central model

4.1. **The minimal index j_0 fundamental limit.** Let $f \in R$, C is any non-trivial chain recurrent class of f, suppose $j_0 = \min_j\{j : C \bigcap P^*_j \ne \phi\}$ and Λ be a minimal index j_0 fundamental limit, by lemma 2.2, such set always exists. Now we'll recall some results about j_0 and the set Λ, they are all given in [51]

Lemma 4.1. *Suppose $f \in R$, C is a chain recurrent class of f, $j_0 = \min_j\{j : C \bigcap P^*_j \ne \phi\}$, Λ is a minimal index j_0 fundamental limit in C, then*

- *either Λ is a non-trivial minimal set with partial hyperbolic splitting $E^s_{j_0} \oplus E^c_1 \oplus E^u_{j_0+2}$*
- *or C contains a periodic point with index j_0 or $j_0 + 1$ and C is an index j_0 fundamental limit.*

Lemma 4.2. *Suppose $f \in R$, C is a non-trivial chain recurrent class of f, if $C \bigcap P^*_0 \ne \phi$, then C should be a homoclinic class containing index 1 periodic points and C is an index 0 fundamental limit.*

4.2. **Partial hyperbolic splitting and Crovisier's central model.** $f \in C^1(M)$, Λ is a minimal set of f with partial hyperbolic splitting $E^s_{j_0} \oplus E^c_1 \oplus E^u_{j_0+2}$ where $E^s_{j_0}, E^u_{j_0+2} \ne \phi$, $dim(E^c_1|_{\Lambda}) = 1$ and $E^c_1(\Lambda)$ is not hyperbolic, let C be the chain recurrent class containing Λ and V_0 be a small neighborhood of Λ, then the maximal invariant set of $\overline{V_0}$: $\Lambda_0 = \bigcap_j f^j(\overline{V_0})$ will have a partial hyperbolic splitting $E^s_{j_0} \oplus E^c_1 \oplus E^u_{j_0+2}$ also. In fact, we can extend such splitting to $\overline{V_0}$ (it's not invariant anymore). For every point $x \in \overline{V_0}$, we define some cones on its tangent space $C^i_a(x) = \{v | v \in T_x M$, there exists $v' \in E^i(x)$ such that $d(\frac{v}{|v|}, \frac{v'}{|v'|}) < a\}_{i=s,c,u,cs,cu}$. When a is small enough, $C^i_a(x) \bigcap C^j_a(x) = \phi$ $_{(i \ne j=s,c,u)}$, $C^{cs}_a(x) \bigcap C^u_a(x) = \phi$, $C^{cu}_a(x) \bigcap C^s_a(x) = \phi$ and $Df(C^i_a(x)) \subset C^i_a(f(x))$ $_{i=u,cu}$, $Df^{-1}(C^i_a(x)) \subset C^i_a(f^{-1}(x))$ $_{i=s,cs}$ for $x \in \Lambda_0$.

We say a submanifold D^i $(i = s, c, u, cs, cu)$ tangents with cone C^i_a when $dim D^i = dim(E^i)$ (we denote $E^{cs} = E^c_1 \oplus E^s$, $E^{cu} = E^c_1 \oplus E^u$) and for $x \in D^i$, $T_x D^i \subset C^i_a(x)$. For simplicity, sometimes we just call it i-disk, especially when $i = c$, we call D^c a central curve. We say an i-disk D^i has center x with size δ if $x \in D^i$, and respecting the Riemannian metric restricting on D^i, the ball centered on x with radius δ is

in D^i. We say an i-disk D^i has center x with radius δ if $x \in D^i$, and respecting the Riemannian metric restricting on D^i, the distance between any point $y \in D^i$ and x is smaller than δ.

We say a smooth central curve γ is a central segment if $f^i(\gamma) \subset V_0$ and $f^i(\gamma)$ is a central curve for any $i \in \mathbb{Z}$, so if γ is a central segment, $\gamma \subset \Lambda_0$, and it's easy to know $T_x\gamma = E_1^c(x)$ for any $x \in \gamma$. We say a smooth central curve γ is a positive(negative) central segment if $f^i(\gamma) \subset V_0$ and $f^i(\gamma)$ is a central curve for any $i \geq (\leq)0$, so if γ is a positive (negative) central segment, $\gamma \subset \bigcap\limits_{-\infty}^{0} f^i(\overline{V_1})$ $(\bigcap\limits_{0}^{\infty} f^i(\overline{V_1}))$,

Now let's consider the orientation of the central bundle $E_1^c(\Lambda)$.

Definition 4.3. *We say $E_1^c(\Lambda)$ has an f-orientation if $E_1^c(\Lambda)$ is orientable and Df preserves its orientation.*

Lemma 4.4. *For a compact neighborhood V_1 of Λ satisfying $V_1 \subset V_0$ and let $\Lambda_1 = \bigcap\limits_{i=-\infty}^{\infty} f^i(V_1)$, $\Lambda_1^+ = \bigcap\limits_{i=-\infty}^{0} f^i(V_1)$, $\Lambda_1^- = \bigcap\limits_{i=0}^{\infty} f^i(V_1)$, then there exist $\delta_0 > 0$, $\delta_0/2 > \delta_1 > \delta_2 > 0$ such that they satisfy the following properties:*

  a) *If $E_1^c(\Lambda)$ has an f orientation, $E_1^c(\Lambda_1)$ has an f orientation also.*
  b) *for any $x \in V_1$, $B_{\delta_0}(x) \subset V_0$ and $E_1^c(B_{\delta_0}(x))$ is orientable, so it gives orientation for any central curve in $B_{\delta_0}(x)$, and we suppose δ_0 is small enough such that any central curve in $B_{\delta_0}(x)$ never intersects with itself.*
  c) *for any $x \in \Lambda_1^+$, x has δ_1 uniform size of strong stable manifold $W_{\delta_1}^{ss}(x)$ and $W_{\delta_1}^{ss}(x)$ is an s disk; for any $x \in \Lambda_1^-$, x has δ_1 uniform size of strong unstable manifold $W_{\delta_1}^{uu}(x)$ and $W_{\delta_1}^{uu}(x)$ is an u disk.*
  d) *for any $x \in \Lambda_1$, there exists a central curve $l_{\delta_1}(x)$ with center x and radius δ_1, such that there exists a continuous function $\Phi^c : \Lambda_1 \longrightarrow Emb^1(I, M)$ satisfying $\Phi^c(x) = l_{\delta_1}(x)$ where $x \in \Lambda_1$, and if let $l_{\delta_2}(x) \subset l_{\delta_1}(x)$ be the central curve with center x and radius δ_1, then $f(l_{\delta_2}(x)) \subset l_{\delta_1}(f(x))$ and $f^{-1}(l_{\delta_2}(x)) \subset l_{\delta_1}(f^{-1}(x))$.*
  e) *For any $0 < \varepsilon < \delta_1$, there exists $\delta > 0$ such that for any positive central segment $\gamma \subset \Lambda_1^+$ with $\varepsilon < length(\gamma) < \delta_1$, $W_{loc}^s(\gamma) = \bigcup\limits_{x \in \gamma} W_{\delta_1}^{ss}(x)$ is a cs disk with uniform size δ, and for any $x \in Int(\gamma)$, there exists $\delta_x > 0$ such that for any $y \in B_{\delta_x}(x) \bigcap \Lambda_1$, we have $W_{\delta_1}^{uu}(y) \pitchfork W_{loc}^s(\gamma) \neq \phi$. And if C has the following dominated splitting $E_{j_0}^s \oplus E_1^c \oplus W_{j_0+2}^u$, then there exists U_n a small neighborhood of C such that any $z \in C' = \bigcap\limits_i f^i(U_n)$ will have uniform size of strong unstable manifold $W_{\delta_1}^{uu}(z)$ and if $z \in B_{\delta_x}(x) \bigcap C'$, we still have $W_{\delta_1}^{uu}(z) \pitchfork W_{loc}^s(\gamma) \neq \phi$.*

Proof a), b) are obviously, c) is [21]'s result about strong stable manifold theorem, d) is [21]'s result about cental manifolds, the first part of e) is the stable manifold theorem for normally hyperbolic submanifold; about the second part, when U_n is small enough, C' will have the dominated splitting $E_{j_0}^s \oplus E_1^c \oplus W_{j_0+2}^u$ also, and we can even extend such splitting to U_n and get two cones $C_{a_0}^s|_{C'}$ and $C_{a_0}^{cu}|_{C'}$ which match the respectively cones in V_1, then when δ_1 is small enough, any $z \in C'$ will have uniform size of strong stable manifold $W_{\delta_1}^{ss}(z)$ and it's an s-disk (tangents the cone $C_{a_0}^s|_{U_n}$), so when $z \in C'$ near x enough, we'll have $W_{\delta_1}^{ss}(z) \pitchfork W_{loc}^u(\gamma) \neq \phi$. $\square$

Now let's introduce Crovisier's result, we divide the statement to two cases: $E_1^c(\Lambda)$ has an f orientation or not. At first, suppose $E_1^c(\Lambda)$ has an f orientation, and we call the direction right.

Lemma 4.5. *($E_1^c(\Lambda)$ has an f orientation): $0 < \delta_2 < \delta_1 < \delta_0/2$ are given by lemma 4.4, and $l_{\delta_1}(x)$ $_{(x \in \Lambda_1)}$ are given there also, let $l_{\delta_1}^+(x) \subset l_{\delta_1}(x)$ be the central curve in the right of x, then*

   a) *either for some $x_0 \in \Lambda$, there exists a central segment $\gamma_0 \subset l_{\delta_1}^+(x_0)$ where γ_0 contains x_0 and $\gamma_0 \subset \Lambda_0$, in fact, γ_0 is in the same chain recurrent class with Λ respect the map $f|_{V_0}$.*

   b) *or for any $x \in \Lambda_1$ there exists a central curve $\gamma_x^+ \subset l_{\delta_1}^+(x)$ such that γ_x^+ containing x, γ_x^+ $_{(x \in \Lambda_1)}$ is a family of smooth curve and they are C^0 continuously depend on $x \in \Lambda_1$, and either $f(\overline{\gamma_x^+}) \subset \gamma_{f(x)}^+$ for all $x \in \Lambda_1$ or $f^{-1}(\overline{\gamma_x^+}) \subset \gamma_{f^{-1}(x)}^+$ for all $x \in \Lambda_1$.*

In the case b) of lemma 4.5, if we have $f(\overline{\gamma_x^+}) \subset \gamma_{f(x)}^+$, we call the right central curve is 1-step contracting, if $f^{-1}(\overline{\gamma_x^+}) \subset \gamma_{f^{-1}(x)}^+$, we call it's one step expanding.

Lemma 4.6. *([13],[51]) When $f \in R$, and a) of lemma 4.5 happens, then C is a homoclinic class containing index i_0 or $i_0 + 1$ periodic point and C is an index i_0 fundamental limit.*

Lemma 4.7. *($E_1^c(\Lambda)$ has non f-orientation) $0 < \delta_2 < \delta_1 < \delta_0/2$ are given by lemma 4.4, and $l_{\delta_1}(x)$ $_{(x \in \Lambda_1)}$ are given there also, then*

   a) *either for some $x_0 \in \Lambda$, there exists a central segment $\gamma_0 \subset l_{\delta_1}(x_0)$ such that $x_0 \in \gamma_0$ and $\gamma_0 \subset \Lambda_1$, and if $f \in R$, then C is a homoclinic class containing index i_0 or $i_0 + 1$ periodic point and C is an index i_0 fundamental limit.*

   b) *or for every $x \in \Lambda_1$ there exists a central curve $\gamma_x \subset l_{\delta_1}(x)$ containing x and γ_x $_{(x \in \Lambda_1)}$ is a family of smooth curve C^0 continuously depend on $x \in \Lambda_1$, and either $f(\overline{\gamma_x}) \subset \gamma_{f(x)}$ for all $x \in \Lambda_1$ or $f^{-1}(\overline{\gamma_x}) \subset \gamma_{f^{-1}(x)}$ for all $x \in \Lambda_1$.*

5. Proof of theorem 1, 2 and 3

At first, let's state the main lemma, its proof would be given in §6.

Lemma 5.1. *(The main lemma) Suppose $f \in R$, C is a non-trivial Lyapunov stable chain recurrent class of f, let $j_0 = \min_j \{j : C \bigcap P_j^* \neq \phi\}$, then C contains index j_0 or $j_0 + 1$ periodic point and C is an index j_0 fundamental limit.*

It's easy to see that theorem 2 is a simply corollary of the main lemma.

Now I'll show the proof of theorem 1:

Proof of Theorem 1: It's just a corollary of generic property 8) of proposition 3.1 and theorem 2. □

Proof of Theorem 3: Recall $j_0 = \min_j \{j : C \bigcap P_j^* \neq \phi\}$ and $i_0 = \min_i \{i : C \bigcap Per_i(f) \neq \phi\}$, so $j_0 \leq i_0$, by lemma 5.1, we have $j_0 \geq i_0 - 1$.

So either $j_0 = i_0$ or $j_0 = i_0 - 1$.

When $j_0 = i_0$, then by generic property 6) of proposition 3.1, $C \subset \overline{Per_{i_0}(f)} \subset P_{i_0}^*(f)$. By proposition 2.5 and $f \in R \subset (\overline{HT})^c$, C has an index i_0 partial hyperbolic splitting $T_C M = E_{i_0}^{cs} \oplus E_{i_0+1}^{cu}$. By the definition of j_0 and the assumption $i_0 = j_0$, we know $C \bigcap P_j^* = \phi$ for $j < i_0$, so from proposition

2.6, $E_{j_0}^{cs}|_C$ is hyperbolic, we denote it by $E_{i_0}^s|_C$, then on C we have the following dominated splitting $T_C M = E_{i_0}^s \oplus E_{i_0+1}^{cu}$. And since C contains index i_0 periodic point, C is an index i_0 fundamental limit.

When $j_0 = i_0 - 1$, by lemma 5.1, C is an index $i_0 - 1$ fundamental limit, so $C \subset P_{i_0-1}^*$ and we've known that C contains index i_0 periodic point, so $C \subset P_{i_0}^*$, then $C \subset P_{i_0-1}^* \bigcap P_{i_0}^*$, from $f \in R \subset (\overline{HT})^c$ and proposition 2.5, C has an index $i_0 - 1$ dominated splitting $E_{i_o-1}^{cs} \oplus E_{i_0}^{cu}|_C$ and an index i_0 dominated splitting $E_{i_0}^{cs} \oplus E_{i_0+1}^{cu}|_C$. Let $E_1^c|_C = E_{i_0}^{cs} \bigcap E_{i_0}^{cu}|_C$, then C will have the following dominated splitting $E_{i_0-1}^s \oplus E_1^c \oplus E_{i_0+1}^{cu}|_C$. By the definition of j_0, $C \bigcap P_j^* = \phi$ for $j < i_0 - 1$, so from proposition 2.6, $E_{i_0-1}^{cs}|_C$ is hyperbolic, we denote it $E_{i_0-1}^s(C)$. $\hfill\square$

6. PROOF OF THE MAIN LEMMA

Proof At first, we can suppose $j_0 \neq 0$, since if $j_0 = 0$, by lemma 4.2, C is a homoclinic class containing index 1 periodic points and C is an index 0 fundamental limit, then we proved the main lemma.

From lemma 2.2, there always exists a minimal index j_0 fundamental limit in C, we denote one of them Λ, by lemma 4.1, we can suppose Λ is a non-trivial minimal set with a partial hyperbolic splitting $E_{j_0}^s \oplus E_1^c \oplus E_{j_0+2}^u|_\Lambda$.

At first, let's prove C contains an index j_0 or $j_0 + 1$ periodic point.

Now we divide the proof into two cases: $E_1^c(\Lambda)$ has an f orientation or not.

Case A: $E_1^c(\Lambda)$ has an f orientation.

At first, like in §4, choose V_0 a small neighborhood of Λ such that $\Lambda_0 = \bigcap f^i(\overline{V_0})$ will have also a partial hyperbolic splitting $E_{j_0}^s \oplus E_1^c \oplus E_{j_0+2}^u|_{\Lambda_0}$ and $E_1^c|_{\Lambda_0}$ also has an f orientation, and when V_0 is small enough, we can always suppose the splitting can be extended to V_0 (of course, it's not invariant any more). Choose a_0 small enough such that for $x \in V_0$, we have $C_{a_0}^s(x) \bigcap C_{a_0}^{cu}(x) = \phi$, $C_{a_0}^{cs}(x) \bigcap C_{a_0}^u(x) = \phi$, and $C_{a_0}^i \bigcap C_{a_0}^j = \phi$ for $(i \neq j \in s, c, u)$.

Choose another small neighborhood V_1 of Λ satisfying $\overline{V_1} \subset V_0$, let $\Lambda_1 = \bigcap_{i=-\infty}^{\infty} f^i(\overline{V_1})$, then by lemma 4.3, Λ_1 has a family of cental curves with uniform size and locally invariant. Since $E_1^c(\Lambda_1)$ has an f orientation, we choose one orientation and call the direction right, at first let's consider the right central curves. By lemma 4.5, we can suppose the family of right cental curves have 1-step contracting or expanding property (if it's not, by (a) of lemma 4.5 and lemma 4.6, C is a homoclinic class containing index j_0 or $j_0 + 1$ periodic point and C is an index j_0 fundamental limit.) that means for every point $x \in \Lambda_1$, there exists a smooth central curve $\gamma_x^+ \subset l_{\delta_1}^+(x)$ on the right of x such that

- γ_x^+ continuously depends on $x \in \Lambda_1$,
- $f(\overline{\gamma_x^+}) \subset \gamma_{f(x)}^+$ for all $x \in \Lambda_1$ or $f^{-1}(\overline{\gamma_x^+}) \subset \gamma_{f^{-1}(x)}^+$ for all $x \in \Lambda_1$,
- there exist ε_0 such that $length(\gamma_x^+) > \varepsilon_0$.

Since Λ is minimal, by 4) of proposition 3.1, there exists a family of periodic points $\{p_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} Orb(p_n) \longrightarrow \Lambda$, so we can suppose $Orb(p_n) \subset V_1$ for $n \geq 1$, that means $Orb(p_n) \subset \Lambda_1$, then on the right of p_n, we have a central curve $\gamma_{p_n}^+$ such that either $f(\overline{\gamma_{p_n}^+}) \subset \gamma_{f(p_n)}^+$ (if the right central curves of Λ_1 is 1-step contracting) or $f^{-1}(\overline{\gamma_{p_n}^+}) \subset \gamma_{f^{-1}(p_n)}^+$ (when the right central curves of Λ_1 is 1-step expanding). For simplicity, we denote the central curve $\gamma_{p_n}^+$ by γ_n^+, so we have $f^{\pi(p_n)}(\overline{\gamma_n^+}) \subset \gamma_n^+$ or $f^{-\pi(p_n)}(\overline{\gamma_n^+}) \subset \gamma_n^+$. Let

$\Gamma_n^+ = \bigcap_{i=-\infty}^{\infty} f^{i\pi(p_n)}(\gamma_n^+)$, then Γ_n^+ is a periodic segment with periodic $\pi(p_n)$, let q_n^+ be one of the extreme point of Γ_n^+ different with p_n when Γ_n^+ is not trivial and $q_n^+ = p_n$ when Γ_n^+ is trivial, let $h_n^+ = \gamma_n^+ \setminus \Gamma_n^+$, then there exists ε_1 doesn't depend on n such that $length(h_n^+) > \varepsilon_1$. It's easy to know that when the right central curves are 1-step contracting, $h_n^+ \subset W^s(q_n^+)$ and if the right central curves are 1-step expanding, we have $h_n^+ \subset W^u(q_n^+)$. With the same argument on the left central curves, we can get γ_n^-, q_n^-, Γ_n^-, h_n^- also.

Remark 6.1. $\Gamma_n = \Gamma_n^+ \bigcup \Gamma_n^-$ *is a periodic central segment with period $\pi(p_n)$ and $f|_{\Gamma_n}$ is Kupka-Smale diffeomorphism, that means Γ_n just has finite fixed points and they are sinks or sources.*

Now considering the contracting or expanding properties of the two half parts of cental curves, we divide the proof into three subcases:

A.1 Two sides of central curves are 1-step contracting.

A.2 Right central curves are 1-step expanding and the left central curves are 1-step contracting.

A.3 Two sides of central curves are 1-step expanding.

Subcase A.1: Two sides of central curves are 1-step contracting.

In this subcase, we can show there exists periodic point $p \in C$ with index j_0 or $j_0 + 1$ and $Orb(p) \subset V_0$.

We have known that $\gamma_{f^i(p_n)}^+ \subset l_{\delta_1}^+(f^i(p_n)) \subset B_{\delta_1}(f^i(p_n)) \subset V_0$, and by 1-step contracting property , we have $f^i(\gamma_n) \subset \gamma_{f^i(p_n)}$ for $i \geq 0$, so $\gamma_n \subset \Lambda^+$, that means any $x \in \gamma_n$ has uniform size of δ_1 strong stable manifold $W_{\delta_1}^{ss}(x)$. Since γ_n is a positive central segment, by the property of normally hyperbolic manifold and $length(\gamma_n) > \varepsilon_0$ for all n, there exists δ such that $W^s(\gamma_n) = \bigcup_{x \in \gamma_n} W^{ss}(x)$ is a cs disk with uniform size δ, it's easy to know $W^s(\gamma_n) = \bigcup_{p \in Per(\gamma_n)} W^s(p)$.

Let's suppose $\lim_{n \to \infty} p_n = x_0 \in \Lambda$, then there exists n big enough, such that $W^{uu}(x_0) \pitchfork W^{cs}(\gamma_n) \neq \phi$, suppose $a \in W^{uu}(x_0) \pitchfork W^{cs}(\gamma_n)$, by lemma 2.4 we have $a \in W^{uu}(x_0) \subset C$, it's easy to know $a \in W^s(p)$ for some $p \in Per(\gamma_n)$, so $p \in \omega(a) \subset C$, recall that all the central curves are in V_0, so $Orb(p) \subset V_0$ and p has index j_0 or $j_0 + 1$.

Subcase A.2: Right central curves are 1-step expanding and the left central curves are 1-step contracting.

In this subcase, we can show that

(a) either there exists periodic point $p \in C$ with index j_0 or $j_0 + 1$ and $Orb(p) \subset V_0$

(b) or there exists periodic point $p \in C$ with index j_0.

From now we suppose that (a) is false, we claim that we can always suppose $\lim_{n \to \infty} length(\Gamma_n^+) \longrightarrow 0$.
Proof of the claim: Suppose there exist δ' and $\{\Gamma_{n_i}^+\}_{i=0}^{\infty}$ such that $length(\Gamma_{n_i}^+) > \delta'$ for $i \geq 0$, then $\gamma_{n_i}^- \bigcup \Gamma_{n_i}^+$ has uniform size and is a positive central curves and $f^j(\gamma_{n_i}^- \bigcup \Gamma_{n_i}^+) \subset V$ for any $j \geq 0$, so like the argument in Case A.1, $W^s(\gamma_{n_i}^- \bigcup \Gamma_{n_i}^+)$ with center p_{n_i} has uniform size and when i big enough, we have $W^{uu}(x_0) \pitchfork W^s(\gamma_{n_i}^- \bigcup \Gamma_{n_i}^+) \neq \phi$, then C contains an index j_0 or $j_0 + 1$ periodic point p with $Orb(p) \subset V_0$, that's a contradiction with our assumption that a) is false. $\square$

Recall that central curves are a family of C^1 curves continuous depend on $x \in \Lambda_1$, so we know $\lim_{n\to\infty} \gamma_n^+ \longrightarrow \gamma_{x_0}^+$, with $length(\Gamma_n^+) \longrightarrow 0$ we can know $\lim_{n\to\infty} h_n^+ \longrightarrow \gamma_{x_0}^+$.

Since the right central curves are 1-step expanding we can know $f^{-i}(h_n^+) \subset V_1$ for all $i \geq 0$, so $f^{-i}(\gamma_{x_0}^+) \subset V_0$ for all $i \geq 0$, that means $\gamma_{x_0}^+$ is a negative central segment. With $length(\gamma_{x_0}^+) > \varepsilon_0$, that means $W^u(\gamma_{x_0}^+) = \bigcup_{x \in \gamma_{x_0}^+} W_{\delta_1}^{uu}(x)$ is a cu disk.

We claim that $\gamma_{x_0}^+ \subset C$.

Proof of the claim: Since C is Lyapunov stable, that means that there exists a family of open neighborhood $\{U_n\}_{n=1}^{\infty}$ of C such that

a) $\overline{U_{n+1}} \subset U_n$

b) $f(\overline{U_n}) \subset U_n$

c) $\bigcap_n U_n = C$.

By the property of $\lim_{n\to\infty} Orb(p_n) = \Lambda$, we can suppose $Orb(p_n) \subset U_n$ always, since $\lim_{n\to\infty} \Gamma_n^+ \longrightarrow 0$, we can suppose $q_n^+ \in U_n$ also. By the property of b), we can know that $W^u(q_n^+) \subset U_n$, since we've known that $h_n^+ \subset W^u(q_n^+)$, so $h_n^+ \subset U_n$, then $\gamma_{x_0}^+ = \lim_{n\to\infty} \gamma_n^+ \subset \bigcap_{n\geq 1} U_n = C$. $\qquad\square$

Remark 6.2. *By the above argument, we can know that for any $x \in \Lambda$, $\gamma_x^+ \subset C$, by $f^{-i}(\gamma_x^+) \subset \gamma_{f^{-i}(x)}^+$ for $i \in \mathbb{N}$ we know γ_x^+ is a negative central segment, so by e) of lemma 4.3, γ_x^+ has unstable manifold $W^u(\gamma_x^+)$. By lemma 2.4, $W^u(\gamma_x^+) \subset C$.*

Choose $y \in \gamma_{x_0}^+ \setminus x_0$, then $y \in C$ also. Now we claim that we can always suppose $y \in W^u(\Lambda)$.

Proof of the claim: At first let's note that $y \in \gamma_{x_0}^+ \setminus x_0$ for some $x_0 \in \Lambda$ and $f^{-i}(\gamma_{x_0}^+) \subset \gamma_{f^{-i}(x_0)}^+$ for $i \in \mathbb{N}$ because the right central model is 1-step expanding. So $f^{-i}(y) \in \gamma_{f^{-i}(x_0)}^+$ for $i \in \mathbb{N}$, that also means $\alpha(y) \subset V_0$ and it has partial hyperbolic splitting $E_{j_0}^s \oplus E_1^c \oplus E_{j_0+2}^u$. Now we just need show the length of the curve in $\gamma_{f^{-i}(x_0)}^+$ which connecting $f^{-i}(y)$ and $f^{-i}(x_0)$ converges to 0.

Now we suppose the length doesn't converge to 0, that means there exists $i_n \longrightarrow \infty$ such that the length of the curve in $\gamma_{f^{-i_n}(x_0)}^+$ which connecting $f^{-i_n}(y)$ and $f^{-i_n}(x_0)$ doesn't converge to 0, suppose $\lim_{n\to\infty} f^{-i_n}(x_0) \longrightarrow x_1$, then $\lim_{n\to\infty} f^{-i_n}(y) \longrightarrow y_1 \in \gamma_{(x_1)}^+ \setminus x_1$. Since $y_1 \in \alpha(y)$ and $\alpha(y)$ is a chain recurrent set in V_1, by generic property 4) of proposition 3.1, there exists a family of periodic points $\{p_n\} \subset V_0$ such that $p_n \longrightarrow y_1$, it's easy to know that $Orb(p_n)$ has index j_0 or $j_0 + 1$ and $Orb(p_n)$ has uniform size of strong stable manifold $W_\delta^{ss}(p_n)$, by e) of lemma 4.4, we know $W_\delta^{ss}(p_n) \bigcap W^u(\gamma_{x_1}^+) \ni a \neq \phi$.

Remark 6.2 has shown that $a \in C$, so $Orb(p_n) \subset \omega(a) \subset C$, recall that $\{p_n\} \subset V_1$, then we proved (a), it's a contradiction with the assumption that a) is false. $\qquad\square$

By the technique lemma, there exists a family of periodic points $\{q_n\}$ such that $\lim_{n\to\infty} q_n = y$ and $y \bigcup \Lambda \subset \lim_{n\to\infty} Orb(q_n)$. By the corollary 3.3, we can suppose $\{q_n\}$ all have index j_0 or index $j_0 + 1$. Denote $C_0 = \lim_{n\to\infty} Orb(p_n)$, then $C_0 \subset P_{j_0}^* \bigcap P_{j_0+1}^*$, hence C_0 has a dominated splitting $E_{j_0}^s \oplus E_1^c \oplus E_{j_0+2}^{cu}$, then by e) of lemma 4.4, q_n has uniform size of strong stable manifold $W_{\delta_1}^{ss}(q_n)$ tangent at q_n with $E_{j_0}^s(q_n)$, and when n big enough, we have $W_{\delta_1}^{ss}(q_n) \pitchfork W^u(\gamma_{x_0}^+) \neq \phi$. Let $a \in W_{\delta_1}^{ss}(q_n) \pitchfork W^u(l_{x_0}^+)$, then $a \in W^u(\gamma_{x_0}^+) = \bigcup_{x \in \gamma_{x_0}^+} W_{\delta_1}^{uu}(x) \subset C$ and $q_n \in \omega(a) \subset C$, so C contains an index j_0 and index $j_0 + 1$

periodic point.

Subcase A.3: Two sides of central curves are 1-step expanding.

In this subcase, we can show that

(a) either there exists periodic point $p \in C$ with index j_0 or $j_0 + 1$ and $Orb(p) \subset V_0$

(b) or there exists periodic point $p \in C$ with index j_0.

We claim that if (a) is false, we can suppose there exists a subsequence $\{n_j\}$ such that $\lim\limits_{j \to \infty} \Gamma^+_{n_j} \longrightarrow 0$ or $\lim\limits_{j \to \infty} \Gamma^-_{n_j} \longrightarrow 0$.

Proof of the claim: Suppose it's wrong, then there exists $\varepsilon > 0$ such that $length(\Gamma^+_n) > \varepsilon$ and $length(\Gamma^-_n) > \varepsilon$, then Γ_n is a central segment with uniform size , with the same argument in case A.1, we can show C contains index j_0 or index $j_0 + 1$ periodic point, and its orbit is contained in V_0.   □

Now change by a subsequence, we can suppose $\lim\limits_{n \to \infty} \Gamma^+_n \longrightarrow 0$, now the rest argument is the same with case A 2.

Case B: $E^c_1(\Lambda)$ has no f orientation:

In this case, we can locally define orientation, and in this case locally the two sides of central curves are either 1-step expanding or 1-step contracting, the rest argument is almost the same with Case A.1 and Case A.3.

Now let's prove that C is an index j_0 fundamental limit, here we choose a family of neighborhoods $\{V_n\}$ of Λ such that $V_{n+1} \subset V_n$ and $\bigcap V_n = \Lambda$, then by above argument, we can show that

(a) either C contains index j_0 periodic point,

(b) or C contains periodic point $p_n \in C$ with index $j_0 + 1$ and $Orb(p_n) \subset V_n$.

In the case (a), of course C is an index j_0 fundamental limit; in the case (b) we just need the following lemma given in [51]:

Lemma 6.3. *Suppose $f \in R$, C is a non-trivial chain recurrent class of f, and $\Lambda \subsetneq C$ is a minimal set with partial hyperbolic splitting $E^s_{j_0} \oplus E^c_1 \oplus E^u_{j_0+2}$ where $dim(E^c_1(\Lambda)) = 1$ and $E^c_1(\Lambda)$ is not hyperbolic, if there exists a family of periodic points $\{p_n\}$ in C satisfying $\lim\limits_{n \to \infty} Orb(p_n) = \Lambda$, then C is index j_0 and $j_0 + 1$ fundamental limit.*

Remark 6.4. *The proof of the above lemma is divided into two cases:*

(A) *there exists $\delta > 0$ such that for any p_n, we have $Df^{\pi(p_n)}|_{E^c_1(p_n)} < e^{-\delta \pi(p_n)}$,*

(B) *for any $\frac{1}{m}$, there exists p_{n_m} such that $Df^{\pi(p_{n_m})}|_{E^c_1(p_n)} > e^{-\frac{1}{m}\pi(p_{n_m})}$.*

In the first case we use weakly selecting lemma, and in case (B) we use lemma 4.25 of [51] which basically is a transition property.

□

7. Proof of technique lemma

The proof of the technique lemma depends on generic assumption heavily, with many generic assumptions, we can find some segment of orbit with 'good' position, then after using connecting lemma and another generic property, we can get the periodic points which we need.

In §7.1, we'll introduce some new C^1 generic properties in order to define the generic set given in technique lemma. In §7.2, we'll recall the proof of connecting lemma, especially about the 'cutting tool', because we need an important fact which just appears in the proof of connecting lemma. In §7.3, we'll prove the technique lemma.

7.1. Some new C^1 generic properties.

. Suppose $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ is a topological basis of M satisfying for any $\varepsilon > 0$, there exists a subsequence $\{U_{\alpha_i}\}_{i=1}^\infty$ such that $diam(U_{\alpha_i}) < \varepsilon$ and $\bigcup_i (U_{\alpha_i})$ is a cover of M. Fix this topological basis, we'll get some new C^1 generic properties.

At first, let's recall some definitions, suppose K is a compact set of M, $f \in C^1(M)$ has been given, $x, y \in K$, $x \underset{K}{\dashv} y$ means that for any $\varepsilon > 0$, there exists an ε-pseudo orbit in K beginning from x and ending at y. If $K = M$, we just denote $x \dashv y$.

The following result has been proved in [Cr2]:

Lemma 7.1. *There exists a generic subset $R_{1,0}^*$ such that any $f \in R_{1,0}^*$ will satisfy the following property: suppose K is a compact set, W is any neighborhood of K, $x_1, x_2 \in K$ satisfy $x_1 \underset{K}{\dashv} x_2$, $U_1, U_2 \subset W$ are neighborhoods of x_1, x_2 respectively, then there exists a segment of orbit of f in W beginning from U_0 and ending in U_2. More precisely, there exists $a \in U_0$ and $i_1 > 0$ such that $f^{i_1}(a) \in U_1$ and $f^i(a) \in W$ for $0 \le i \le i_1$.*

Lemma 7.2. *There exists a generic subset $R_{1,1}^*$ such that any $f \in R_{1,1}^*$ will satisfy the following property: suppose Λ is a invariant compact subset of f, $y \notin \Lambda$, $0 < s < 1$, $\{\Phi_i\}_{i=1}^K \subset \{U_\alpha\}_{\alpha\in\mathcal{A}}$ is an open cover for Λ and $O \in \{U_\alpha\}_{\alpha\in\mathcal{A}}$ is a small neighborhood of y, if there exist $g_n \xrightarrow{c^1} f$ and g_n has periodic point p_n satisfying $\dfrac{\#\{Orb_{g_n}(p_n)\bigcap(\bigcup\limits_{i=1}^K \Phi_i)\}}{\pi_{g_n}(p_n)} > s$ and $Orb_{g_n}(p_n)\bigcap O \neq \phi$, then f itself has a periodic point p satisfying $\dfrac{\#\{Orb(p)\bigcap(\bigcup\limits_{i=1}^K \Phi_i)\}}{\pi(p)} > s$ and $Orb(p_n)\bigcap O \neq \phi$.*

Proof : Consider the set $\{(\Phi_{\beta_1}, \cdots, \Phi_{\beta_{N(\beta)}})\}_{\beta\in\mathcal{B}_0}$ where $\Phi_{\beta_i} \in \{U_\alpha\}_{\alpha\in\mathcal{A}}$, it's easy to know $\mathcal{B}_0$ is countable.

For any $\beta \in \mathcal{B}_0$, denote

- $H_\beta = \{f|\ f \in C(M), f$ has a C^1 neighborhood $\mathcal{U}$ such that for any $g \in \mathcal{U}$, g has a periodic orbit p_g satisfying $\dfrac{\#\{Orb_g(p_g)\bigcap(\bigcup\limits_{i=1}^K \Phi_i)\}}{\pi_g(p_g)} > s$ and $Orb_g(p_g)\bigcap O \neq \phi\}$,
- $N_\beta = \{f|\ f \in C^1(M), f$ has a C^1 neighborhood $\mathcal{U}$ such that for any $g \in \mathcal{U}$, g has no any periodic orbit p_g satisfying $\dfrac{\#\{Orb_g(p_g)\bigcap(\bigcup\limits_{i=1}^K \Phi_i)\}}{\pi_g(p_g)} > s$ and $Orb_g(p_g)\bigcap O \neq \phi\}$.

It's easy to know $H_\beta\bigcup N_\beta$ is open and dense in $C^1(M)$. Let $R_{1,0}^* = \bigcap\limits_{\beta\in\mathcal{B}_0} (H_\beta\bigcup N_\beta)$, we'll show $R_{1,0}^*$ satisfies the property we need.

For any $f \in R_{1,0}^*$ and any $\beta^* \in \mathcal{B}_0$, suppose there exists a family of C^1 diffeomorphisms $\{g_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} g_n = f$ and any g_n has a periodic orbit p_n satisfying $\dfrac{\#\{Orb_{g_n}(p_n) \bigcap (\bigcup_{i=1}^K \Phi_i)\}}{\pi_{g_n}(p_n)} > s$ and $Orb_{g_n}(p_n) \bigcap O \neq \phi$, then $f \notin N_{\beta^*}$. That means $f \in H_{\beta^*}$, so we proved this lemma. $\qquad\square$

With the same argument like above, we can get the following result :

Lemma 7.3. *There exists a generic subset $R_{1,2}^*$ such that any $f \in R_{1,2}^*$ will satisfy the following property: for finite number of open set $\{\Phi_i\}_{i=1}^N \subset \{U_\alpha\}_{\alpha \in \mathcal{A}}$ and $U_0, U_1, U_2, U_3 \in \{U_\alpha\}_{\alpha \in \mathcal{A}}$ such that $U_0, U_1, U_2, U_3 \subset \bigcup_{i=1}^N \Phi_i$, if there exist $a_n \in U_0$, $g_n \xrightarrow{C^1} f$ and $0 < i_{1,n} < i_{2,n}$ such that $g_n^i(a_n) \in \bigcup_{i=1}^N \Phi_i$ for $0 \leq i \leq i_{2,n}$, $g_n^{i_{1,n}}(a_n) \in U_1$, $g_n^{i_{2,n}}(a_n) \in U_3$ and $g_n^i(a_n) \notin \overline{U_2}$ for $0 \leq i \leq i_{1,n}$, then there exist $a \in U_0$ and $0 < i_1 < i_2$ such that $f^i(a) \in \bigcup_{i=1}^N \Phi_i$ for $0 \leq i \leq i_2$, $f^{i_1}(a) \in U_1$, $f^{i_2}(a) \in U_3$ and $f^i(a_n) \notin \overline{U_2}$ for $0 \leq i \leq i_1$.*

Now let $R_0^* = R_0 \bigcap R_{1,0}^* \bigcap R_{1,1}^* \bigcap R_{1,2}^*$, and in §7.3 we'll show the set will satisfy the technique lemma.

7.2. **Introduction of connecting lemma.** Connecting lemma was proved by Hayashi [20] at first, and then was extended to the conservative setting by Xia, Wen [48]. the following statement of connecting lemma was given by Lan Wen as an uniform version of connecting lemma.

Lemma 7.4. *(connecting lemma [44]) For any C^1 neighborhood $\mathcal{U}$ of f, there exist $\rho > 1$, a positive integer L and $\delta_0 > 0$ such that for any z and $\delta < \delta_0$ satisfying $\overline{f^i(B_\delta(z))} \bigcap \overline{f^j(B_\delta(z))} = \phi$ for $0 \leq i \neq j \leq L$, then for any two points p and q outside the cube $\Delta = \bigcup_{i=1}^L f^i(B_\delta(z))$, if the positive f-orbit of p hits the ball $B_{\delta/\rho}(z)$ after p and if the negative f-orbit of q hits the small ball $B_{\delta/\rho}(z)$, then there is $g \in \mathcal{U}$ such that $g = f$ off Δ and q is on the positive g-orbit of p.*

Remark 7.5. *Suppose we have another point $z_1 \in M$ satisfying $\Delta_1 \bigcap \Delta = \phi$ where $\Delta_1 = \bigcup_{i=1}^L f^i(B_\delta(z_1))$, then if we use twice connecting lemma in Δ and Δ_1, we can still get a diffeomorphism g in $\mathcal{U}$.*

Now we'll show the idea of the proof of connecting lemma, because we need some special property which just appears in the proof.

In the proof, the main idea is Hayashi's 'cutting' tool, by it we can cut some orbits from p's original f-orbit and q's original f-orbit, and then connect the rest part in Δ. More precisely description is following. Suppose $f^{s_m}(p) \in B_{\delta/\rho}(z)$ and there exists $0 < s_1 < s_2 < \cdots < s_m$ such that $f^{s_i} \in B_\delta(z)$ for $1 \leq i \leq m$ and $f^s(p) \notin B_\delta(z)$ for $s \in \{0, 1, \cdots, s_m\} \setminus \{s_1, s_2, \cdots, s_m\}$. For q, there exists $0 < t_1 < t_2 < \cdots < t_n$ such that $f^{-t_i}(q) \in B_\delta(z)$ for $1 \leq i \leq n$, $f^{t_n}(q) \in B_{\delta/\rho}(z)$ and $f^{-t}(q) \notin B_\delta(z)$ for $t \in \{0, 1, \cdots, t_n\} \setminus \{t_1, t_2, \cdots, t_n\}$. By some rule, we can cut some f-orbits in p's orbit like $\{f^{s_i+1}(p), f^{s_i+2}(p), \cdots, f^{s_j}(p)\}_{j>i}$ and cut some f-orbits in q's orbit like $\{f^{-t_i}(q), \cdots, f^{-t_j+2}(q), f^{-t_j+1}(q)\}_{j>i}$, the rest segment is like:

$$P' = (p, f(p), \cdots, f^{s_{i_1}}(p); f^{s_{i_2}+1}, \cdots, f^{s_{i_3}}(p); \cdots; f^{s_{i_{(k(p)-1)}}+1}(p), \cdots, f^{s_{i_{k(p)}}}(p)),$$

$$Q' = (f^{-t_{j_{k(q)}}+1}(q), \cdots, f^{-t_{j_{k(q)-1}}}(q); \cdots; f^{-t_{j_3}+1}, \cdots, f^{-t_{j_2}}(q); f^{-t_{j_1}+1}(q) \cdots, f^{-1}(q), q,).$$

Denote $X = P' \bigcup Q'$, and $\pi(X)$ is the length of X, it's easy to know X is a 2δ-pseudo orbits. Then we can do several perturbations called 'push' in Δ and get a diffeomorphism g such that q is on the positive g-orbit of p, in fact, we have $g^{\pi(X)}(p) = q$. It's because after the push, we can connect $f^{s_{i_1}}(p)$ and $f^{s_{i_2}+L}(p)$, $\cdots$; $f^{s_{i_{k(p)-2}}}(p)$ and $f^{s_{i_{k(p)-1}}+L}(p)$; $f^{s_{i_{k(p)}}}(p)$ and $f^{-t_{j_{k(q)}}+L}(q)$; $f^{-t_{j_{k(q)-1}}}(q)$ and $f^{-t_{j_{k(q)-2}}+L}(q)$; $\cdots$;

$f^{-t_{j_2}}(q)$ and $f^{-t_{j_1}+L}(q)$ by L times pushes in Δ, we don't cut orbits anymore, and it's important to note that the supports of different pushes don't intersect with each other, so we don't change the length of X, we just push the points of X in Δ and get a connected orbit. By the above argument, it's easy to know $g|_{M\backslash\Delta} = f|_{M\backslash\Delta}$ and $g(\Delta) = f(\Delta)$.

Remark 7.6. *In the above argument, suppose there exists an open set V such that $f^i(p) \in V$ for $0 \leq i \leq s_m$ and $\Delta \subset V$, then after cutting and pushes, we can know $\{p, g(p), \cdots, g^{\pi(P')}(p)\} \subset V$. What's more, we can show that $\#\{\{g^i(p)\}_{i=0}^{\pi(P')+\pi(Q')} \bigcap (V)^c\} < t_n$.*

7.3. **Proof of technique lemma. Proof** : Here we just prove the technique lemma for $y \in C \bigcap W^s(\Lambda) \backslash C$, the proof for the other case is similar.

Fix V_0 a small neighborhood of Λ, $U_0 \in \{U_\alpha\}_{\alpha \in \mathcal{A}}$ a small neighborhood of y such that $\overline{V_0} \bigcap \overline{U_0} = \phi$ and $\overline{U_0} \subset O$, $\overline{V_0} \subset V$. Let $\{\Phi_i\}_{i=1}^N \subset \{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of Λ such that $\bigcup_{i=1}^N \Phi_i \subset V_0$, choose V_1 another small compact neighborhood of Λ such that $V_1 \subset \bigcup_{i=1}^N \Phi_i$, and choose $O(y) \in \{U_\alpha\}_{\alpha \in \mathcal{A}}$ is a neighborhood of y such that $\overline{O(y)} \subset U_0$.

Choose $x_0 \in \omega(y) \subset \Lambda$, denote $z_0 = f^{i_0}(y)$ is the last time the positive orbit of y enters V_1, then we have $z_0 \in V_1$, it's easy to know that z_0 is not a periodic point.
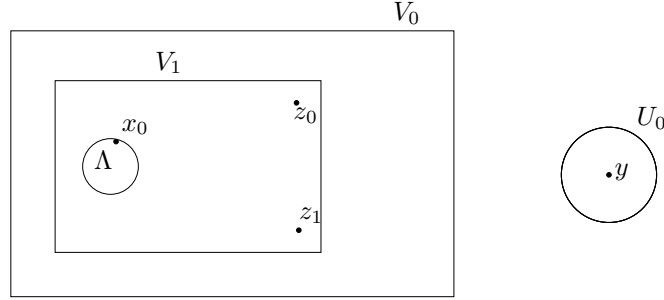
Now choose $\{\delta_n\}_{n=1}^\infty$ satisfying $\delta_n \longrightarrow 0^+$, for every δ_n, there exists a δ_n-pseudo orbit from x_0 to y, denote z_n^- the first time the pseudo orbit leaves V_1, suppose $\lim_{n\to\infty} f^{-1}(z_n^-) = z_1$, then $Orb^-(z_1) \in \overline{V_1}$ and $x_0 \dashv_{V_1} z_1$. We can always suppose z_1 is not a periodic point, since if z_1 is a periodic point, by f is a Kupka-Smale diffeomorphism, z_1 should be a hyperbolic periodic point, then there exists a point $z_1' \in W_{loc}^s(z)$ such that $Orb^-(z_1') \in \overline{V_1}$ and $z_1' \dashv_{V_1} x_0$, then we can replace z_1 by z_1'.

Before we enter the details of the proof, we'll show some ideas of the proof. In the beginning we show that there exists a orbit beginning from a neighborhood of z_1 to a neighborhood of z_0. Then we show there exists another segment of orbit in $\overline{V_1}$ beginning from a neighborhood of z_0 passing a very small neighborhood of x_0 and ending in a neighborhood of z_1, and most important, the orbit between the neighborhood of z_0 and the neighborhood of x_0 will never pass z_1's neighborhood. Here we should note that until now we just use generic property, and we don't do any perturbation yet. Now we'll use connecting lemma twice to connect the above two orbits and get a periodic orbit, more precisely, at first we use connecting lemma at z_1's neighborhood and then we use connecting lemma near z_0's neighborhood, and we can show after the perturbations, the periodic orbit we get will spend a long time in V_1, then with generic assumption again, we can know that f itself has such kind of periodic orbit.

At first, we need the following lemma:

Lemma 7.7. *There exists $\delta_0 > 0$ such that for any $\delta_1^*, \delta_2^*, \delta_3^* < \delta_0$, there exist $a \in B_{\delta_1^*}(z_0)$ and $0 < i_1 < i_2$ such that $f^{i_1}(a) \in B_{\delta_2^*}(x_0)$, $f^{i_2}(a) \in B_{\delta_3^*}(z_1)$, $f^i(a) \in \bigcup_{i=1}^{N} \Phi_i$ for $0 \leq i \leq i_2$ and $f^i(a) \notin \overline{B_{\delta_3^*}(z_1)}$ for $0 \leq i \leq i_1$.*

Proof : Since $Orb^-(z_1) \subset V_1$ and $f^{-i_0}(z_0) = y \notin V_1$, we get $z_1 \notin Orb^+(z_0)$, with the fact $z_1 \notin \omega(z_0)$, we can choose $Y^+(z_1) \in \{U_\alpha\}_{\alpha \in \mathcal{A}}$ a small neighborhood of z_1 such that $\overline{Y^+(z_1)} \bigcap Orb^+(z_0) = \phi$, $Y^+(z_1) \subset \bigcup_{i=1}^{N} \Phi_i$ and $\overline{Y^+(z_1)} \bigcap \Lambda = \phi$. Choose $\delta_0 > 0$ small enough such that

- $B_{\delta_0}(z_1) \subset Y^+(z_1)$,
- $B_{\delta_0}(z_0) \subset \bigcup_{i=1}^{N} \Phi_i$, $\overline{B_{\delta_0}(z_0)} \bigcap \Lambda = \phi$, and $B_{\delta_0}(z_0) \bigcap Y(z_1) = \phi$,
- $B_{\delta_0}(x_0) \subset \bigcup_{i=1}^{N} \Phi_i$, $B_{\delta_0}(x_0) \bigcap Y(z_1) = \phi$, $B_{\delta_0}(x_0) \bigcap B_{\delta_0}(z_0) = \phi$.

Now suppose $\delta_1^*, \delta_2^*, \delta_3^* < \delta_0$ are fixed, we can choose $X(z_0) \in \{U_\alpha\}_{\alpha \in \mathcal{A}}$ a small neighborhood of z_0 satisfying $X(z_0) \subset B_{\delta_1^*}(z_0)$ and choose $Y^-(z_1) \in \{U_\alpha\}_{\alpha \in \mathcal{U}}$ a small neighborhood of z_1 such that $Y^-(z_1) \subset B_{\delta_0}(z_1) \subset Y^+(z_1)$. For any small $\varepsilon_n > 0$, by connecting lemma, $B_{\varepsilon_n}(f)$ gives us parameters L_n, δ_n and ρ_n, we choose $W_n^+, W_n^- \in \{U_\alpha\}_{\alpha \in \mathcal{A}}$ neighborhoods of x_0 small enough such that

- $W_n^+, W_n^- \subset B_{\delta_2^*}(x_0)$,
- there exists $0 < \delta < \delta_n$ such that $W_n^- \subset B_{\delta/\rho_n}(x_0) \subset B_\delta(x_0) \subset W_n^+ \subset B_{\delta_2^*}(x_0)$ and we have $f^i(W_n^+) \bigcap f^j(W_n^-)) = \phi$ for $0 \leq i \neq j \leq L_n$.
- denote $\Delta_n = \bigcup_{i=0}^{L_n} W_n^+(x_0)$, then $\Delta_n \subset \bigcup_{i=1}^{N} \Phi_i$ and $\Delta_n \bigcap X(z_0) = \phi$, $\Delta_n \bigcap Y^+(z_1) = \phi$.

Since Λ is an invariant compact subset and $z_0, z_1 \notin \Lambda$ and $x_0 \in \Lambda$ is not a periodic point, we can always choose such kind of neighborhoods.

Since $x_0 \in \omega(z_0) \subset \Lambda$, then there exists $i_{1,n}^*$ such that $f^{i_{1,n}^*}(z_0) \in W_n^-$; because $x_0 \underset{V_1}{\dashv} z_1$, by lemma 7.1, there exist $b_n \in Y^-(z_1)$ and j_n such that $f^{-j_n}(b_n) \in W_n^-$ and $f^{-j}(b_n) \in \bigcup_{i=1}^{N} \Phi_i$ for $0 \leq j \leq j_n$.

Recall $W_n^- \subset B_{\delta/\rho_n}(x_0)$, use connecting lemma to connect z_0 and b_n in Δ_n, we can get a new diffeomorphism g_n and $i_{0,n}, i_{1,n}$ such that $g_n^{i_{0,n}}(a_n) \in W_n^+$, $g_n^{i_{1,n}}(a_n) = b_n \in Y^-(z_1)$; since the original pseudo orbits are both in $\bigcup_{i=1}^{N} \Phi_i$ and $\Delta_n \subset \bigcup_{i=1}^{N} \Phi_i$, we can know that $g_n^j(a_n) \in \bigcup_{i=1}^{N} \Phi_i$ for all $0 \leq j \leq i_{1,n}$.

From $\{f^j(z_0)\}_{j=0}^{i_{1,n}^*} \subset (\overline{Y^+(z_1)})^c$, by remark 7.6, we can choose $i_{0,n}$ such that $(g_n)^j(a_n) \notin \overline{Y^+(z_1)}$ for $0 \leq j \leq i_{0,n}$.
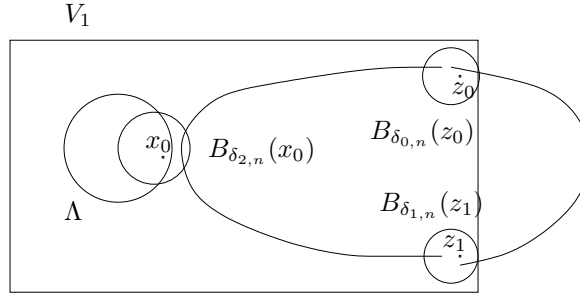
Now fix $n_0 \in \mathbb{N}$ and consider the neighborhood $W_{n_0}^+$ of x_0, with generic property lemma 7.3, there exists $a \in X(z_0)$ and $0 < i_1 < i_2$ such that $f^i(a) \in \bigcup_{i=1}^{N} \Phi_i$ for $0 \leq i \leq i_2$, $f^{i_1}(a) \in W_{n_0}^+$, $f^{i_2}(a) \in Y^-(z_1)$ and $f^i(a) \notin \overline{Y^+(z_1)}$ for $0 \leq i \leq i_1$. With the facts $X(z_0) \subset B_{\delta_1^*}(z_0)$, $W_n^+ \subset B_{\delta_2^*}(x_0)$ and $Y^-(z_1) \subset B_{\delta_3^*}(z_1) \subset Y^+(z_1)$, we finish the proof. $\qquad\square$

Now for any sequence $\varepsilon_n \longrightarrow 0^+$ and $s_n \longrightarrow 1^-$, consider $B_{\varepsilon_n}(f)$ the ε_n-neighborhood of f in $C^1(M)$, by connecting lemma $B_{\varepsilon_n}(f)$ gives us a family of parameters $\rho_n \longrightarrow \infty$, $\delta_n \longrightarrow 0$ and L_n. Then there exist $\delta_{0,n} \longrightarrow 0^+$ such that

A1 $\delta_{0,n} < \delta_n$, $\delta_{0,n} < \delta_0$

A2 $\delta_{0,n+1} < \delta_{0,n}/\rho_n$,

A3 $f^i(B_{\delta_{0,n}}(z_0)) \bigcap f^j(B_{\delta_{0,n}}(z_0)) = \phi$ for $0 \le i \ne j \le L_n$ and $\bigcup_{i=1}^{L_n} f^i(B_{\delta_{0,n}}(z_0)) \subset \bigcup_{i=1}^{N} \Phi_i$,

A4 $\overline{B_{\delta_{0,n}}(z_0)} \bigcap \Lambda = \phi$ and $\bigcup_{i=0}^{L_n} f^i(B_{\delta_{0,n}}(z_0)) \bigcap \bigcup_{i=0}^{L_n} f^{-i}(z_1) = \phi$.

Since z_0 is not periodic point, $Orb^+(z_0) \subset \overline{V_1} \subset \bigcup_{i=1}^{N} \Phi_i$ and $\omega(z_0) \subset \Lambda$, we can always choose the above sequence $\{\delta_{0,n}\}$ for z_0. For z_1 we can also choose a sequence $\{\delta_{1,n}\}$ such that

B1 $\delta_{1,n} < \delta_n$, $\delta_{1,n} < \delta_0$

B2 $\delta_{1,n+1} < \delta_{1,n}/\rho_n$,

B3 $f^{-i}(B_{\delta_{1,n}}(z_1)) \bigcap f^{-j}(B_{\delta_{1,n}}(z_1)) = \phi$ for $0 \le i \ne j \le L_n$ and $\bigcup_{i=0}^{L_n} f^{-i}(B_{\delta_{1,n}}(z_1)) \subset \bigcup_{i=1}^{N} \Phi_i$,

B4 ~~PStrag(replacements~~ and $\bigcup_{i=0}^{L_n} f^i(B_{\delta_{0,n}}(z_0)) \bigcap \bigcup_{i=0}^{L_n} f^{-i}(B_{\delta_{1,n}}(z_1)) = \phi$.

Then by lemma 7.1, there exists a family of points $\{a_n\}$ in $B_{\delta_{1,n}/\rho_n}(z_1)$ and $i_{0,n}$ such that $f^{i_{0,n}}(a_n) \in B_{\delta_{0,n}/\rho_n}(z_0)$. We define $\Delta_{0,n} = \bigcup_{i=1}^{L_n} f^i(B_{\delta_{0,n}}(z_0))$ and $\Delta_{1,n} = \bigcup_{i=1}^{L_n} f^{-i}(B_{\delta_{1,n}}(z_1))$.

Now we'll choose a sequence of number $\delta_{2,n} \longrightarrow 0^+$ such that:

C1 $\delta_{2,n+1} < \delta_{2,n}$, $\delta_{2,n} < \delta_0$,

C2 $B_{\delta_{2,n}}(x_0) \subset V_1$, $B_{\delta_{2,n}}(x_0) \bigcap \Delta_{0,n} = \phi$ and $B_{\delta_{2,n}}(x_0) \bigcap \Delta_{1,n} = \phi$.

C3 For any j_0 satisfying $f^{j_0}(B_{\delta_{0,n}}(z_0)) \bigcap B_{\delta_{2,n}}(x_0) \ne \phi$, we have $\frac{i_{0,n}}{j_0} < 1 - s$.

Since Λ is an invariant compact subset in V_1, we can always choose such neighborhoods.

Now by lemma 7.7, for $B_{\delta_{0,n}/\rho_n}(z_0)$, $B_{\delta_{1,n}/\rho_n}(z_1)$ and $B_{\delta_{2,n}}(x_0)$ there exists an orbit in $\bigcup_{i=1}^{N} \Phi_i$ beginning in $B_{\delta_{0,n}/\rho_n}(z_0)$ passing $B_{\delta_{2,n}}(x_0)$ and ending in $B_{\delta_{1,n}/\rho_n}(z_1)$. More precisely, it means that there exist $b_n \in B_{\delta_{1,n}/\rho_n}(z_0)$ and $0 < j_{0,n}^* \le j_{1,n}^* < j_{2,n}^*$ such that:

D1 $f^j(b_n) \in \bigcup_{i=1}^{N} \Phi_i$ for $0 \le j \le j_{2,n}^*$,

D2 $f^{j_{0,n}^*}(b_n) \in B_{\delta_{0,n}}(z_0)$, $f^{j_{1,n}^*}(b_n) \in B_{\delta_{2,n}}(x_0)$, $f^{j_{2,n}^*}(b_n) \in B_{\delta_{1,n}/\rho_n}(z_1)$,

D3 $f^j(b_n) \notin B_{\delta_{0,n}}(z_0)$ for $j_{0,n}^* < j \le j_{1,n}^*$, and $f^j(b_n) \notin \overline{B_{\delta_{2,n}}(z_1)}$ for $0 \le j \le j_{1,n}^*$.

Remark 7.8. *In fact, we can know that $\{f^j(b_n)\}_{j=0}^{j_{1,n}^*} \bigcap \Delta_{1,n} = \phi$ and $\{f^j(b_n)\}_{j=j_{0,n}^*+L_n}^{j_{1,n}^*} \bigcap \Delta_{0,n} = \phi$, so in the following proof, when we use connecting lemma in $\Delta_{0,n}, \Delta_{1,n}$ twice, we can get a new diffeomorphism g_n and a periodic orbit $Orb_{g_n}(p_n)$ of g_n such that the segment $\{f^j(b_n)\}_{j=j_{0,n}^*+L_n}^{j_{1,n}^*} \subset Orb_{g_n}(p_n)$ and $\#\{Orb_{g_n}(p_n) \bigcap (\bigcup_{i=1}^{N} \Phi_i)^c\} < i_{0,n}$, then by C3, we can know that $\frac{\#\{Orb_{g_n}(p_n) \bigcap \bigcup_{i=1}^{N} \Phi_i\}}{\pi_{g_n}(p_n)} > 1 - \frac{i_{0,n}}{j_{1,n}^* - j_{0,n}^*} > s$.*

Now fix an n, let's consider the two points $f^{i_{0,n}}(a_n)$ and b_n, we know the positive f-orbit of b_n hits $B_{\delta_{1,n}/\rho_n}(z_1)$ after b_n and the negative f-orbit of $f^{i_{0,n}}(a_n)$ hits $B_{\delta_{1,n}/\rho_n}(z_1)$ also, by connecting lemma, the fact $\Delta_{1,n} \subset \bigcup_{i=1}^{N} \Phi_i$ and remark 7.6, 7.8, there exists $g_n^* \in B_{\varepsilon_n}(f)$ such that $g_n^* \equiv f$ off $\Delta_{1,n} = \bigcup_{i=0}^{L_n-1} f^{-i}(B_{\delta_{1,n}}(z_1))$ and there exists $j_{2,n}, j_{3,n}$ such that

    E1 $(g_n^*)^j(b_n) = f^j(b_n)$ for $0 \leq j \leq j_{1,n}^*$,

    E2 $(g_n^*)^{j_{2,n}}(b_n) \in B_{\delta_{1,n}}(z_1)$, $(g_n^*)^{j_{3,n}}(b_n) \in B_{\delta_{0,n}/\rho_n}(z_0)$,

    E3 $(g_n^*)^j(b_n) \in \bigcup_{i=1}^{N} \Phi_i$ for $0 \leq j \leq j_{2,n}$ and $j_{3,n} - j_{2,n} < i_{0,n}$.

Remark 7.9. *Above argument shows that* $\#\{\{(g_n^*)^j(b_n)\}_{j=0}^{j_{3,n}} \bigcap (\bigcup_{i=1}^{N} \Phi_i)\} < i_{0,n}$.

Now we'll use connecting lemma in the neighborhood of z_0, let's consider $f^{j_{1,n}^*}(b_n)$, it's near x_0, we know that the positive g_n^*-orbit of $f^{j_{1,n}^*}(b_n)$ hits $B_{\delta_{0,n}/\rho_n}(z_0)$ after $f^{j_{1,n}^*}(b_n)$ and the negative g_n^*-orbit of $f^{j_{1,n}^*}(b_n)$ hits $B_{\delta_{0,n}/\rho_n}(z_0)$ also, by connecting lemma, the fact $\Delta_{0,n} = \bigcup f^j(B_{\delta_n'}(z_0)) \subset \bigcup_{i=1}^{N} \Phi_i$ and remark 7.6, there exists $g_n \in B_{\varepsilon_n}(f)$ such that $g_n \equiv f$ off $\Delta_{0,n}^*$ and there exists j_0, j_1 such that

    F1 $g_n^{j_{1,n}^*+j_1}(b_n) = g_n^{j_{1,n}^*-j_0}(b_n) \in B_{\delta_n'}(z_0)$,

    F2 $f^{j_{1,n}^*-j}(b_n) = (g_n^*)^{j_{1,n}^*-j}(b_n) = (g_n)^{j_{1,n}^*-j}(b_n)$ for $0 \leq j \leq j_{1,n}^* - j_{0,n}^*$, it means that
$$\#\{Orb_{g_n}(f^{j_{1,n}^*+j_1}(b_n)) \bigcap \bigcup_{i=1}^{N} \Phi_i\} \geq j_{1,n}^* - j_{0,n}^*,$$

    F3 $\#\{Orb(f^{j_{1,n}^*+j_1}(b_n)) \bigcap (\bigcup_{i=1}^{N} \Phi_i)^c\} \leq j_{3,n} - j_{2,n} \leq i_{0,n}$.

We denote the above periodic orbits for g_n by $Orb(p_n)$ where $p_n = f^{j_{1,n}^*+j_1}(b_n)$, so we know that
$$\frac{\#\{Orb_{g_n}(p_n) \bigcap \bigcup_{i=1}^{N} \Phi_i\}}{Orb_{g_n}(p_n)} = 1 - \frac{\#\{Orb_{g_n}(p_n) \bigcap (\bigcup_{i=1}^{N} \Phi_i)^c\}}{Orb_{g_n}(p_n)} \geq 1 - \frac{i_{0,n}}{j_{1,n}^* - j_{0,n}^*} > 1 - (1 - s) = s$$ and $\lim_{n \to \infty} p_n \longrightarrow z_0$.

So we show that there exists a family of diffeomorphisms $\{g_n\}$ such that $g_n \xrightarrow{C^1} f$ and g_n has periodic point p_n such that $\frac{\#\{Orb_{g_n}(p_n) \bigcap \bigcup_{i=1}^{N} \Phi_i\}}{Orb_{g_n}(p_n)} > s$ and $p_n = f^{j_{1,n}^*+j_1}(b_n) \longrightarrow z_0$, recall that $z_0 = f^{i_0}(y)$, we know that when n is big enough, $Orb_{g_n}(p_n)$ will pass through U_0 the neighborhood of y, so by generic property lemma 7.2, f itself has periodic point p such that $\frac{\#\{Orb(p) \bigcap \bigcup_{i=1}^{N} \Phi_i\}}{Orb(p)} > s$ and $Orb(p) \bigcap U_0 \neq \phi$. $\square$

REFERENCES

[1] F. Abdenur, C. Bonatti and S. Crovisier, Global dominated splittings and the C^1 Newhouse phenomenon, *Proceedings of the American Mathematical Society* **134**, (2006), 2229-2237.

[2] F. Abdenur, C. Bonatti and S. Crovisier, L.J. Diaz and L. Wen, *Periodic points and homoclinic classes, preprint* (2006).

[3] R. Abraham and S. Smale, Nongenericity of Ω-stability, Global analysis I, *Proc. Symp. Pure Math. AMS* **14** (1970), 5-8.

[4] M-C. Arnaud, Creation de connexions en topologie C^1. *Ergodic Theory and Dynamical System* **31** (2001), 339-381.

[5] C. Bonatti and S. Crovisier, Recurrence et genericite(French), *Invent. math.,* **158** (2004), 33-104

[6] C. Bonatti, L. J. Díaz and E. Pujals, A C^1-generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many class or sources, *Ann. of Math.* **158** (2003), 355-418

[7] C. Bonatti, L. J. Díaz, Connexions hétérocliniques et généricite d'une infinité de puits ou de sources, *Annales Scientifiques de l'cole Normal Suprieure de Paris*, **32 (4)**, (1999) 135-150,

[8] C. Bonatti, L. J. Díaz and M. Viana, Dyanamics beyond uniform hyperbolic, **Volume 102** of *Encyclopaedia of Mathematical Sciences.* Springer- Verlag, Berlin, 2005. A global geometric and probabilistic perspective, Mathematical Physics, III.

[9] C. Bonatti, S. Gan and L. Wen, On the existence of non-trivial homoclinic class, *preprint* (2005)

[10] C. Bonatti, N. Gourmelon and T. Vivier, Perturbation of the derivative along periodic orbits, *preprint* (2004)

[11] C. Conley, Isolated invariant sets and Morse index, CBM Regional Conference Series in Mathematics, **38**, AMS Providence, R.I.,(1978).

[12] S. Crovisier, Periodic orbits and chain transitive sets of C^1 diffeomorphisms, *preprint* (2004).

[13] S. Crovisier, Birth of homoclinic intersections: a model for the central dynamics of partial hyperbolic systems, *preprint* (2006).

[14] J. Franks, Necessary conditions for stability of diffeomorphisms, *Trans. Amer. Math. Soc.*, **158** (1977), 301-308.

[15] S. Gan, Private talk.

[16] S. Gan, Another proof for C^1 stability conjecture for flows, *SCIENCE IN CHINA (Series A)* **41 No. 10** (October 1998) 1076-1082

[17] S. Gan, The Star Systems X^* and a Proof of the $C^1 Omega$-stability Conjecture for Flows, *Journal of Differential Equations*, **163** (2000) 1–17

[18] S. Gan and L. Wen, Heteroclinic cycles and homoclinic closures for generic diffeomorphisms, *Journal of Dynamics and Differential Equations,*, **15** (2003), 451-471.

[19] S. Gan and L. Wen, Nonsingular star flow satisfy Axion M and the nocycle condition, *Ivent. Math.,* **164** (2006), 279-315.

[20] S. Hayashi, Connecting invariant manifolds and the solution of the C^1 stability and Ω-stable conjecture for flows, *Ann. math.*, **145** (1997), 81-137.

[21] M. Hirsch, C. Pugh and M. Shub, Invariant manifolds, volume 583 of *Lect. Notes in Math.* Springer Verlag, New york, 1977

[22] Shantao Liao, Obstruction sets I, *Acta Math. Sinica*, **23** (1980), 411- 453.

[23] Shantao Liao, Obstruction sets II, *Acta Sci. Natur. Univ. Pekinensis*, **2** (1981), 1-36.

[24] Shantao Liao, On the stability conjecture, *Chinese Annals of Math.*, **1** (1980), 9-30.(in English)

[25] Shantao Liao, An existence theorem for periodic orbits, *Acta Sci. Natur. Univ. Pekinensis*, **1** (1979), 1-20.

[26] Shantao Liao, Qualitative Theory of Differentiable Dynamical Systems, *China Science Press*, (1996).(in English)

[27] R. Mañé, Quasi-Anosov diffeomorphisms and hyperbolic manifolds, *Trans. Amer. Math. Soc.*, **229** (1977), 351-370.

[28] R. Mañé, Contributions to the stability conjecture, *Topology*, **17** (1978), 383-396.

[29] R. Mañé, An ergodic closing lemma, *Ann. Math.*, **116** (1982), 503-540.

[30] R. Mañé, A proof of the C^1 stability conjecture. *Inst. Hautes Etudes Sci. Publ. Math.* **66** (1988), 161-210.

[31] C. Morales and M. J. Pacifico, Lyapunov stability of ω-limit sets, *Discrete Contin. Dyn. Syst.* **8** (2002), no. 3, 671-674.

[32] S. Newhouse, Non-density of Axiom A(a) on S 2. *Proc. A. M. S. Symp pure math*, **14** (1970), 191-202, 335-347.

[33] S. Newhouse, Diffeomorphisms with infinitely many sinks, *Topology*, **13**, 9-18, (1974).

[34] J. Palis and M. Viana, High dimension diffeomorphisms displaying infinitely sinks, *Ann. Math.*, **140** (1994), 1-71.

[35] V. Pliss, On a conjecture due to Smale, *Diff. Uravnenija.*, **8** (1972), 268-282.

[36] C. Pugh, The closing lemma, *Amer. J. Math.*, **89** (1967), 956-1009.

[37] E. Pujals and M. Sambarino, Homoclinic tangencies and hyperbolicity for surface diffeomorphisms, *Ann. Math.*, **151** (2000), 961-1023.

[38] E. Pujals and M. Sambarino, Density of hyperbolicity and tangencies in sectional dissipative regions, *preprint* (2005).

[39] J. Selgrade, Isolated invariant sets for flows on vector bundles, *Trans. Amer. Math. Soc.*, **203** (1975), 259-390.

[40] R. Sacker and G. Sell, Existence of dichotomies and invariant splittings for linear differential systems, *J. Diff. Eq.*, **22** (1976), 478-496.

[41] M. Shub, Topological transitive diffeomorphisms in T^4, Lecture Notes in Math. Vol. **206**, Springer Verlag, 1971.

[42] L. Wen, On the C^1-stability conjecture for flows. *Journal of Differential Equations,* **129**(1995) 334-357.

[43] L. Wen, Homoclinic tangencies and dominated splittings, *Nonlinearity*, **15** (2002), 1445-1469.

[44] L. Wen, A uniform C^1 connecting lemma, *Discrete and continuous dynamical systems*, **8** (2002), 257-265.

[45] L. Wen, Generic diffeomorphisms away from homoclinic tangencies and heterodimensional cycles, *Bull. Braz. Math. Soc. (N.S.)*, **35** (2004), 419- 452.

[46] L. Wen, Selection of Quasi-hyperbolic strings, *preprint* (2006).

[47] L. Wen and S. Gan, Obstruction sets, Obstruction sets, quasihyperbolicity and linear transversality.(Chinese) *Beijing Daxue Xuebao Ziran Kexue Ban*, **42** (2006), 1-10.

[48] Z. Xia and L. Wen, C^1 connecting lemmas, *Trans. Amer. Math. Soc.* **352** (2000), 5213-5230.

[49] D. Yang, S. Gan, L, Wen, Minimal Non-hyperbolicity and Index-Completeness, *preprine* (2007).

[50] J. Yang, Ergodic measure far away from tangency, *preprint* (2007).

[51] J. Yang, Newhouse phenomena and homoclinic class, *preprint* (2007).

[52] J. Yang, Index complement conjecture, *preprint* (2007).

[53] Y. Zhang and S. Gan, On Mañé's Proof of the C^1 Stability Conjecture, *Acta mathematica Sinica, English Series* Vol. **21**, No. **3**, June, 2005, 533-540.

IMPA, Est. D. Castorina 110, 22460-320 Rio de Janeiro, Brazil

E-mail address: yangjg@impa.br