$\mathbf{L^1}$ STABILITY OF SPATIALLY PERIODIC SOLUTIONS IN RELATIVISTIC GAS DYNAMICS

DANIELA CALVO, RINALDO M. COLOMBO, AND HERMANO FRID

ABSTRACT. This paper proves the well posedness of spatially periodic solutions of the relativistic isentropic gas dynamics equations. The pressure is given by a γ-law with initial data of large amplitude, provided $\gamma - 1$ is sufficiently small. As a byproduct of our techniques, we obtain the same results for the classical case. At the limit $c \to +\infty$, the solutions of the relativistic system converge to the solutions of the classical one, the convergence rate being $1/c^2$. We also construct the semigroup of solutions of the Cauchy problem for initial data with bounded total variation, which can be large, as long as $\gamma - 1$ is small.

1. INTRODUCTION

We consider the 2×2 hyperbolic system of conservation laws describing the one-dimensional motion of an isentropic relativistic gas in Euler coordinates, which reads

$$\left\{\begin{array}{l} \partial_t \left(\rho \dfrac{1 + \left(\frac{v}{c}\right)^2 \frac{p(\rho)}{c^2 \rho}}{1 - \left(\frac{v}{c}\right)^2} \right) + \partial_x \left(\rho\, v \dfrac{1 + \frac{p(\rho)}{c^2\rho}}{1 - \left(\frac{v}{c}\right)^2} \right) = 0 \\ \partial_t \left(\rho\, v \dfrac{1 + \frac{p(\rho)}{c^2\rho}}{1 - \left(\frac{v}{c}\right)^2} \right) + \partial_x \left(\dfrac{\rho\, v^2 + p(\rho)}{1 - \left(\frac{v}{c}\right)^2} \right) = 0 \end{array}\right. \tag{1.1}$$

Here, ρ is the gas density, v its velocity and p the pressure. We consider the case of a polytropic gas, in which the pressure is given by the so called γ-law, $p(\rho) = \zeta^2 \rho^\gamma$, with $1 \le \gamma < 2$.

The main result of this paper states the existence of a Standard Riemann Semigroup (SRS, cf. [4]) of periodic solutions to (1.1), which may have large amplitude, provided $\gamma - 1$ is sufficiently small. In particular, this means that the initial value problem with periodic initial data is well posed in $\mathbf{L^1}$ globally in time, as long as $\gamma - 1$ is sufficiently small. In this case, the total variation per period of the initial data may be taken arbitrarily large, according to the smallness of $\gamma - 1$.

While proving the $\mathbf{L^1}$-stability of periodic solutions, we also construct the SRS for the Cauchy problem. So, our results contain, in particular, the existence of a SRS for the Cauchy problem, with initial data with arbitrarily large amplitude and total variation, as long as $\gamma - 1$ is sufficiently small.

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The above system has been considered in the literature by many authors, such as [9, 10, 15] and, in the case $\gamma = 1$, [3, 11, 18].

It is immediate to see that in the classical limit $c \to +\infty$, system (1.1) formally converges to the classical Euler equations of isentropic gas dynamics

$$\left\{\begin{array}{l} \partial_t \rho + \partial_x (\rho\, v) = 0 \\ \partial_t (\rho\, v) + \partial_x \left(\rho\, v^2 + p(\rho)\right) = 0 \end{array}\right. \qquad p(\rho) = \zeta^2\, \rho^\gamma. \tag{1.2}$$

The present analytical techniques apply (more easily) also to the non-relativistic case (1.2) yielding, in particular, the well posedness of the solutions constructed in [16, 18] and, more importantly, the well posedness of the periodic solutions constructed in [15]. Furthermore, we prove that in the classical limit $c \to +\infty$, the SRS generated by (1.1) converges to that of (1.2), the rate of convergence being $1/c^2$, recovering, in particular, the results in [9].

In the next section we state the main results of the paper, and at the end we describe the sections along which the main results, as well as the additional results concerning the non-relativistic case and the limit as $c \to +\infty$, are presented.

2. Statements of the Main Results

Bakhvalov introduced in [2] a class of 2×2 strictly hyperbolic and genuinely non-linear systems, characterized by the particular geometry of the shock curves in the plane of Riemann invariants, for which a global existence result can be proved for initial data with large oscillation and only locally bounded total variation.

Namely, consider a strictly hyperbolic, genuinely nonlinear 2×2 system

$$\partial_t \mathbf{u} + \partial_x f(\mathbf{u}) = 0, \tag{2.1}$$

where $\mathbf{u} = (u_1,\, u_2)$ and $f(\mathbf{u}) = (f_1(\mathbf{u}),\, f_2(\mathbf{u}))$. Let z, w be a pair of Riemann invariants for (2.1) such that the map $(u_1, u_2) \mapsto (z, w)$ is one-to-one in its domain. Parametrize the shock curves of the first and second family by

$$\begin{array}{ll} z = R_1(w; z_0, w_0),\ w \le w_0; & z = L_1(w; z_0, w_0),\ w \ge w_0; \\ z = R_2(w; z_0, w_0),\ w \le w_0; & z = L_2(w; z_0, w_0),\ w \ge w_0. \end{array} \tag{2.2}$$

In (2.2), the state (z, w) can be connected on the left by L_i and on the right by R_i to (z_0, w_0) by a shock of the i-th family. Finally, for fixed $W, Z \in \mathbb{R}$, let

$$\Omega = \left\{(z, w) \colon z \ge Z \text{ and } w \le W\right\}. \tag{2.3}$$

The next hypotheses impose conditions on the shock curves under which the solvability of the Cauchy problem with locally bounded variation is obtained.

A_1: $\max_{i=1,2} \sup_{(z,w)\in\Omega} \left|\lambda_i(z,w)\right| < \infty$.

A_2: $\forall (z,w) \in \Omega$ with $w \neq w_0$, $1 < \dfrac{\partial R_1}{\partial w}, \dfrac{\partial L_1}{\partial w} < +\infty$, $0 < \dfrac{\partial R_2}{\partial w}, \dfrac{\partial L_2}{\partial w} < 1$.

A_3: For $i = 1, 2$, let $z_r = R_i(w_r; z_l, w_l)$. Then the shock curves $z = R_i(w; z_l, w_l)$, for $w \leq w_l$, and $z = L_i(w; z_r, w_r)$, for $w \geq w_r$, intersect only in the points (z_l, w_l), (z_r, w_r).

A_4: If four points (z_l, w_l), (z_r, w_r), (z_m, w_m) and $(\hat{z}_m, \hat{w}_m)$ satisfy $z_m = R_2(w_m; z_l, w_l)$, $z_r = R_1(w_r; z_m, w_m)$, $\hat{z}_m = R_1(\hat{w}_m; z_l, w_l)$ and $z_r = R_2(w_r; \hat{z}_m, \hat{w}_m)$, then $(z_l - \hat{z}_m) + (\hat{w}_m - w_r) \leq (w_l - w_m) + (z_m - z_r)$.

System (2.1) belongs to Bakhvalov's class over Ω if it satisfies A_1 - A_4.

Theorem 2.1 ([2, Theorem 1]). *Fix Ω as in* (2.3) *with $Z, W \in \mathbb{R}^2$. Let system* (2.1) *be strictly hyperbolic and genuinely nonlinear in Ω. If* (2.1) *belongs to Bakhvalov's class over Ω, then for all $\mathbf{u}_o \in \mathbf{BV}_{loc}(\mathbb{R}; \Omega)$, the Cauchy problem for* (2.1) *with datum $\mathbf{u}_o$ admits a global weak entropy solution.*

Remark 2.1. The region considered by Bakhvalov is a subset of Ω, so his theorem is a little stronger than the above statement.

The proof of Theorem 2.1 involves the construction of a functional F, non-increasing in time, defined on approximate solutions obtained by the Glimm scheme, see [2]. This functional is constant along solutions to Riemann problems and, hence, may be seen as a function of the two initial states. Let $\mathbf{u}_l$ and $\mathbf{u}_r$ be the left and right constant states of a given Riemann problem whose solution consists of a shock (or rarefaction) wave of first family, say σ_1, followed be a shock (or rarefaction) wave of second family, say σ_2.

Definition 2.1. Define $F(\mathbf{u}_l, \mathbf{u}_r) = [\![\, \Delta z(\sigma_1) \,]\!]_- + [\![\, \Delta w(\sigma_2) \,]\!]_-$, with $[\![\, s \,]\!]_- = \max\{-s, 0\}$ denoting the negative part of s.

As usual, the Riemann coordinates are assumed to have a *positive* increment across rarefactions and a negative one across a shock.

Lemma 2.1 ([2, Lemma 1]). *Under the same assumptions of Theorem 2.1, for any three states $\mathbf{u}_l$, $\mathbf{u}_m$ and $\mathbf{u}_r$ in Ω,*

$$F(\mathbf{u}_l, \mathbf{u}_r) \leq F(\mathbf{u}_l, \mathbf{u}_m) + F(\mathbf{u}_m, \mathbf{u}_r). \tag{2.4}$$

The equality holds in (2.4) *if and only if $\mathbf{u}_m$ is a value attained by the solution corresponding to the Riemann data $\mathbf{u}_l$, for $x < 0$, and $\mathbf{u}_r$, for $x > 0$.*

As in [15], it is sufficient to use a local version of the above lemma. For any set B in the $(z,\, w)$ plane, define $R[B]$ to be the set of all values attained by the solution to any Riemann problem with initial data in B. The following is [15, Lemma 3.2].

Lemma 2.2. *Let B_0 and B_1 be rectangles in the $(z,\, w)$ plane with the property that $R[R[B_0]] \subset B_1$ and system* (2.1) *verifies Bakhvalov conditions A_i, $i = 1, .., 4$, when restricted to B_1. Then for any three states $\mathbf{u}_l$, $\mathbf{u}_m$ and $\mathbf{u}_r$ in B_0 we have*

$$F(\mathbf{u}_l, \mathbf{u}_r) \leq F(\mathbf{u}_l, \mathbf{u}_m) + F(\mathbf{u}_m, \mathbf{u}_r), \tag{2.5}$$

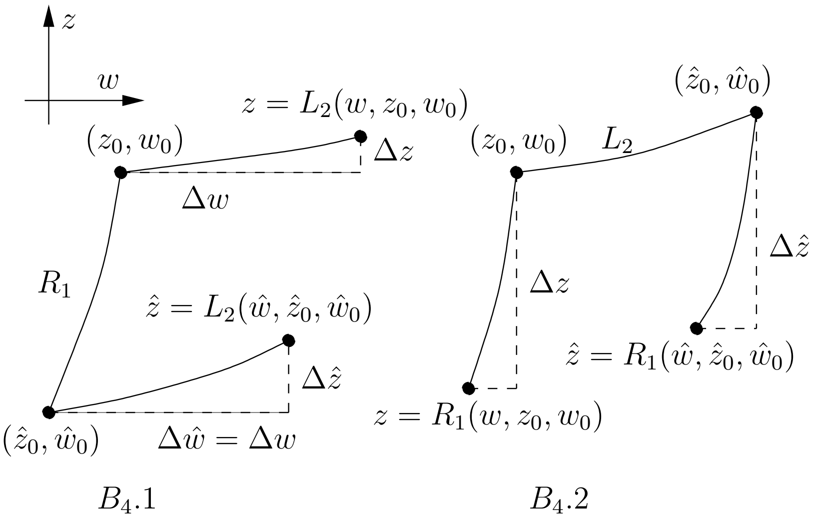

FIGURE 1. Hypothesis B_4.

and equality holds in (2.4) *if* $\mathbf{u}_m$ *is a value assumed by the Riemann solution corresponding to the Riemann data* $\mathbf{u}_l$*, for* $x < 0$*, and* $\mathbf{u}_r$*, for* $x > 0$.

It is convenient to substitute condition A_4 by the following stronger condition introduced by DiPerna in [12]. Define

$$\begin{aligned} R_1(z_0, w_0) &= \left\{(z,w) : z = R_1(w; z_0, w_0),\ w \le w_0\right\}, \\ L_2(z_0, w_0) &= \left\{(z,w) : z = L_2(w; z_0, w_0),\ w \ge w_0\right\}, \end{aligned}$$

and

$$\Delta w = w - w_0,\ \Delta \hat w = \hat w - \hat w_0,\ \Delta z = z - z_0,\ \Delta \hat z = \hat z - \hat z_0.$$

Condition B_4 consists of the following

$B_4.1$: Let $(\hat z_0, \hat w_0) \in R_1(z_0, w_0)$. If $z = L_2(w; z_0, w_0)$, $\hat z = L_2(\hat w; \hat z_0, \hat w_0)$ and $\Delta \hat w = \Delta w$, then $\Delta \hat z \ge \Delta z$.

$B_4.2$: Let $(\hat z_0, \hat w_0) \in L_2(z_0, w_0)$. If $z = R_1(w; z_0, w_0)$, $\hat z = R_1(\hat w; \hat z_0, \hat w_0)$ and $\Delta \hat z = \Delta z$, then $\Delta \hat w \ge \Delta w$.

The above conditions depend on the choice of the pair of Riemann invariants.

System (1.1) falls within (2.1) by setting

$$\begin{aligned} u_1 &= \rho \frac{1 + \left(\frac{v}{c}\right)^2 \frac{p(\rho)}{c^2 \rho}}{1 - \left(\frac{v}{c}\right)^2} \\ u_2 &= \rho\, v \frac{1 + \frac{p(\rho)}{c^2\rho}}{1 - \left(\frac{v}{c}\right)^2} \end{aligned} \qquad f(u_1, u_2) = \left[\begin{array}{c} \rho\, v \dfrac{1 + \frac{p(\rho)}{c^2\rho}}{1 - \left(\frac{v}{c}\right)^2} \\ \dfrac{\rho\, v^2 + p(\rho)}{1 - \left(\frac{v}{c}\right)^2} \end{array}\right]. \tag{2.6}$$

The characteristic speeds of (1.1) are:

$$\lambda_1 = \frac{v - \sqrt{p'(\rho)}}{1 - \frac{v\sqrt{p'(\rho)}}{c^2}} \quad \text{and} \quad \lambda_2 = \frac{v + \sqrt{p'(\rho)}}{1 + \frac{v\sqrt{p'(\rho)}}{c^2}}.$$

For later use, introduce the physically relevant set

$$\mathcal{U} = \left\{\mathbf{u} \in \mathring{\mathbb{R}}^+ \times \mathbb{R} \colon \rho > 0,\ p' < c^2,\ v^2 < c^2\right\}. \tag{2.7}$$

It is immediate to verify that if $\mathbf{u} \in \mathcal{U}$, then $-c < \lambda_1(\mathbf{u}) < \lambda_2(\mathbf{u}) < c$. Throughout the paper, $\mathbf{u} \in \mathcal{U}$ denotes the conserved variables (2.6), while $\mathbf{v} \in \mathcal{V}$, with $\mathcal{V} = \mathbf{v}(\mathcal{U})$, denotes the Riemann coordinates $\mathbf{v} = (v_1, v_2)$, see [10, formulæ (2.24)-(2.25)]:

$$v_1 = \frac{c}{2}\log\frac{c+v}{c-v} - \int_0^\rho \frac{\sqrt{p'(r)}}{r + \frac{p(r)}{c^2}}\,dr, \qquad v_2 = \frac{c}{2}\log\frac{c+v}{c-v} + \int_0^\rho \frac{\sqrt{p'(r)}}{r + \frac{p(r)}{c^2}}\,dr. \tag{2.8}$$

We remark that $\mathbf{v}$ is defined only for $|v| < c$ and that the vacuum state corresponds to the line $v_1 = v_2$, while $v_2 > v_1$ corresponds to $\rho > 0$. In the case of the γ-law $p = \zeta^2\rho^\gamma$, the above integrals can be explicitly computed:

$$\begin{aligned}
\int_0^\rho \frac{\sqrt{p'(r)}}{r + \frac{p(r)}{c^2}}\,dr &= c\,\frac{2\sqrt{\gamma}}{\gamma - 1}\arctan\left(\frac{\zeta}{c}\,\rho^{(\gamma-1)/2}\right) && \text{if } \gamma \in\,]1,2]\,,\\
\int_{\rho_*}^\rho \frac{\sqrt{p'(r)}}{r + \frac{p(r)}{c^2}}\,dr &= \frac{\zeta}{1 + \left(\frac{\zeta}{c}\right)^2}\ln(\rho/\rho_*) && \text{if } \gamma = 1.
\end{aligned}$$

In [15, Theorem 4.1], it is established the existence of domains $\mathcal{U}_\gamma$, $1 < \gamma < 2$, satisfying:

(i) For any compact $K \subset \mathcal{U}$, we have $K \subset \mathcal{U}_\gamma$, for $\gamma - 1$ sufficiently small;

(ii) In $\mathcal{V}_\gamma := \mathbf{v}(\mathcal{U}_\gamma)$ it is possible to define new Riemann invariants
$$z = z(v_1, \gamma), \qquad w = w(v_2, \gamma)$$
with respect to which the corresponding shock curves satisfy the Bakhvalov's conditions, recalled in what follows.

The referred result in [15] extends to the relativistic system (1.1) a previous result of DiPerna [12] for the corresponding non-relativistic system (1.2).

With the classical Riemann coordinates (v_1, v_2), (see (2.8)), system (1.1) satisfies conditions A_1 - A_3. This follows from the lemmas in [15, Section 2]. However, neither B_4 nor A_4 hold for the classical Riemann invariants v_1 and v_2 of system (1.1). The situation is parallel to that of the system of non-relativistic isentropic gas dynamics, in which DiPerna showed in [12] that it is still possible to find a pair of Riemann invariants $z = z(v_1, \gamma)$, $w = w(v_2, \gamma)$ for which the system satisfies A_1 - A_3 and B_4, at least locally.

Using these new Riemann invariants $z = z(v_1, \gamma)$ and $w = w(v_2, \gamma)$ we next define a functional on periodic piecewise constant functions

$$\bar{\mathbf{u}}(x) = \sum_{\alpha=0}^{N} \mathbf{u}^\alpha \chi_{]x_{\alpha-1}, x_\alpha]}(x), \qquad \begin{array}{l} x \in \Pi \quad \mathbf{u}^0 = \mathbf{u}^N \\ \mathbf{u}^\alpha \in \mathcal{U},\ \alpha = 0, \ldots, N, \end{array} \tag{2.9}$$

where Π denotes the interval of periodicity. We set

$$\mathcal{L}[\bar{\mathbf{u}}] := \sum_{\alpha=1}^{N} F(\mathbf{u}^{\alpha-1}, \mathbf{u}^\alpha), \tag{2.10}$$

where F is as in Definition 2.1.

The construction of wave front tracking approximate solutions does not use the exact Riemann solution, but an approximate solution, which depends on the approximation parameter $\varepsilon > 0$ (see [5]). Accordingly, we use a function $F^\varepsilon(\mathbf{u}_l, \mathbf{u}_r)$ whose definition is similar to that of $F(\mathbf{u}_l, \mathbf{u}_r)$ with the only difference that instead of the exact Riemann solution it uses the approximate one. Coherently, we define

$$\mathcal{L}^\varepsilon[\bar{\mathbf{u}}] := \sum_{\alpha=1}^{N} F^\varepsilon(\mathbf{u}^{\alpha-1}, \mathbf{u}^\alpha). \tag{2.11}$$

For the construction of periodic wave front-tracking approximate solutions of (1.1), $\mathbf{u}^\varepsilon(t)$, through an interval $[0,T]$, for any $T > 0$, if $\varepsilon > 0$ is sufficiently small, a key point is the fact that $\mathcal{L}^\varepsilon[\mathbf{u}^\varepsilon(t)]$ is non-increasing for $t \in [0,T]$.

Let $\mathbf{BV}_\Pi(\mathbb{R},\mathcal{U})$ be the space of periodic functions on $\mathbb{R}$, with periodic interval Π, assuming values in $\mathcal{U}$, of bounded total variation *per* period.

Given $\mathbf{u}_\Pi \in \mathcal{U}$, our approximate SRS, S_t^ε, “almost” preserves the domains $\mathcal{D}'_\gamma \subset \mathbf{BV}_\Pi(\mathbb{R},\mathcal{U})$, consisting of Π-periodic piecewise constant functions $\mathbf{u}(x)$ satisfying $\mathcal{L}[\mathbf{u}] < M'_\gamma$, for some $M'_\gamma > 0$, with $M'_\gamma \to \infty$, as $\gamma \to 1$, and

$$\frac{1}{\Pi}\int_\Pi \mathbf{u}(x)\,dx = \mathbf{u}_\Pi, \tag{2.12}$$

in the sense that $\mathcal{L}^\varepsilon[S_t^\varepsilon\mathbf{u}] < M'_\gamma$, for $t \in [0,T]$, and

$$\left\|\frac{1}{\Pi}\int_\Pi S_t^\varepsilon\mathbf{u}(x)\,dx - \mathbf{u}_\Pi\right\| < \delta, \qquad t \in [0,T],$$

for any $\delta > 0$, if $\varepsilon > 0$ is sufficiently small.

We measure the total variation *per* period of a periodic function $\mathbf{u} : \mathbb{R} \to \mathcal{U}$, denoted $\mathrm{TV}(\mathbf{u}|\Pi)$, by means of the sum of the total variations in one period of each of the Riemann coordinates:

$$\begin{aligned}\mathrm{TV}(\mathbf{u}|\Pi) = \sup\Big\{\sum_{i=1}^{2}\sum_{\alpha=1}^{N}\Big|v_i\big(\mathbf{u}(x_\alpha)\big) - v_i\big(\mathbf{u}(x_{\alpha-1})\big)\Big|,\\ N \in \mathbb{N},\ x_{\alpha-1} < x_\alpha,\ x_0,\ldots,x_N \in \Pi\Big\}.\end{aligned} \tag{2.13}$$

We can now state our main theorem establishing the existence of a Standard Riemann Semigroup of periodic solutions with large oscillation and total variation *per* period. The definition of SRS in the periodic case is the obvious adaptation from [4, Definition 9.1].

Theorem 2.2. *Let $\mathbf{u}_\Pi \in \mathcal{U}$ and $1 < \gamma < 2$. Then, there exist domains $\mathcal{U}_\gamma \subset \mathcal{U}$, and constants $M_\gamma > 0$ satisfying $M_\gamma \to \infty$ as $\gamma \to 1$, $p'(\rho) < c^2$ for $\mathbf{u} \in \mathcal{U}_\gamma$ and, for any given compact $K \subset \mathcal{U}$, $K \subset \mathcal{U}_\gamma$, provided $\gamma - 1$ is sufficiently small. Moreover, there exists a Standard Riemann Semigroup $S\colon [0,+\infty[\times \mathcal{D}_\gamma \to \mathcal{D}_\gamma$, in the sense that*

(a): *For any $\mathbf{u} \in \mathcal{D}_\gamma$,*

$$\|S_t\mathbf{u} - S_{t'}\mathbf{u}\|_{\mathbf{L^1}(\Pi)} \le C\big|t - t'\big|, \qquad \text{for } t, t' \in [0,\infty[, \tag{2.14}$$

for a constant $C > 0$ depending only on the functions in $\mathcal{D}_\gamma$;

(b): *For $\mathbf{u}, \mathbf{u}' \in \mathcal{D}_\gamma$,*

$$\big\|S_t\mathbf{u} - S_t\mathbf{u}'\big\|_{\mathbf{L^1}(\Pi)} \le e^{Ct}\big\|\mathbf{u} - \mathbf{u}'\big\|_{\mathbf{L^1}(\Pi)}, \tag{2.15}$$

also for some constant $C > 0$ depending only on bounds on the functions in $\mathcal{D}_\gamma$;

(c): *If* $\mathbf{u} \in \mathcal{D}_\gamma$ *is piecewise constant, then for* $T > 0$ *sufficiently small,* $S_t\mathbf{u}$ *(*$t \in [0,T]$*) coincides with the function obtained by piecing together the Riemann solutions corresponding to each of the jump discontinuities in* $\mathbf{u}$*.*

Further, we have also the following properties:

(d): $\mathcal{D}_\gamma \supseteq \Big\{\mathbf{u} \in \mathbf{BV}_\Pi(\mathbb{R},\mathcal{U}_\gamma)\colon \frac{1}{\Pi}\int_\Pi \mathbf{u}(x)\,dx = \mathbf{u}_\Pi\,,\ \mathrm{TV}\,(\mathbf{u}|\Pi) \le M_\gamma\Big\}$.

(e): *for all* $\mathbf{u} \in \mathcal{D}_\gamma$*, the trajectory* $t \mapsto S_t\mathbf{u}$ *coincides with the Glimm solution constructed in* [15]*.*

Theorem 2.2 follows immediately from two major results which are stated subsequently. The first one, Theorem 2.3, establishes the existence of periodic wave front tracking approximate solutions, $\bar{\mathbf{u}}^\varepsilon(t,\cdot) = S_t^\varepsilon\bar{\mathbf{u}}$, defined for $t \in [0,T]$, for any $T > 0$, and any periodic piecewise constant function $\bar{u}$ assuming values in $\mathcal{U}$, provided that $\varepsilon > 0$ and $\gamma - 1 > 0$ are sufficiently small. The second one, Theorem 2.4, establishes the stability in $\mathbf{L^1}(\Pi)$ of the periodic wave front tracking approximate solutions with respect to their initial data.

Theorem 2.3. *Given any periodic piecewise constant function* $\mathbf{u}\colon \mathbb{R} \to \mathcal{U}$ *of the form* (2.9) *and any* $T > 0$*, it is possible to construct periodic wave front tracking approximate solutions,* $\mathbf{u}^\varepsilon(t,\cdot) = S_t^\varepsilon\mathbf{u}$*, defined for* $t \in [0,T]$*, for any* $T > 0$*, provided that* $\varepsilon > 0$ *and* $\gamma - 1 > 0$ *are sufficiently small. The approximate solutions satisfy*

$$\mathcal{L}^\varepsilon[S_{t'}^\varepsilon\mathbf{u}] \le \mathcal{L}^\varepsilon[S_t^\varepsilon\mathbf{u}], \qquad \text{for } 0 \le t < t' \le T, \tag{2.16}$$

$$\int_\Pi \big|S_t^\varepsilon\mathbf{u} - S_{t'}^\varepsilon\mathbf{u}\big|\,dx \le C\big|t - t'\big|\,, \qquad \text{for } t,t' \in [0,T], \tag{2.17}$$

where $C > 0$ *is a constant depending only on* $\mathrm{TV}\,(\mathbf{u}|\Pi)$*, and* $\mathbf{u}_\Pi$ *is given by* (2.12)*. Moreover, there exists a subsequence* $\varepsilon_i \to 0$ *for which* $S_t^{\varepsilon_i}\mathbf{u} \to \mathbf{u}(t,\cdot) =: S_t\mathbf{u}$ *as* $\varepsilon_i \to 0$*, where* $\mathbf{u}(t,\cdot)$ *is an entropy solution of* (1.1) *with initial data* $\mathbf{u}$*.*

Theorem 2.4. *For* $\mathbf{u}_\Pi \in \mathcal{U}$ *and* $\gamma \in\,]1,2[$*, there exist constants* $M_\gamma > 0$*, with* $M_\gamma \to \infty$*, as* $\gamma \to 1+$*, and domains* $\mathcal{U}_\gamma \subset \mathcal{U}$*, with* $\mathcal{U}_\gamma \supset K$*, for any compact* $K \subset \mathcal{U}$*, for* γ *sufficiently close to* 1*, with the following property. If* $\mathbf{u}', \mathbf{u}''$ *are two periodic piecewise constant functions, of the form* (2.9)*, with mean values*

$$\mathbf{u}'_\Pi := \frac{1}{|\Pi|}\int_\Pi \mathbf{u}'(x)\,dx, \quad \mathbf{u}''_\Pi := \frac{1}{|\Pi|}\int_\Pi \mathbf{u}''(x)\,dx, \qquad \mathbf{u}'_\Pi, \mathbf{u}''_\Pi \in \mathcal{U}_\gamma,$$

respectively, assuming values in $\mathcal{U}_\gamma$*, with*

$$\big|\mathbf{u}'_\Pi - \mathbf{u}_\Pi\big|, \big|\mathbf{u}''_\Pi - \mathbf{u}_\Pi\big|, \mathrm{TV}\,(\mathbf{u}|\Pi), \mathrm{TV}\,(\mathbf{u}'|\Pi) < M_\gamma,$$

then $S_t^\varepsilon\mathbf{u}$ *and* $S_t^\varepsilon\mathbf{u}'$ *are defined for* $t \in [0,T]$*, for any* $T > 0$*, provided that* $\varepsilon > 0$ *is sufficiently small. Moreover, we have*

$$\int_\Pi \big|S_t^\varepsilon\mathbf{u}(x) - S_t^\varepsilon\mathbf{u}'(x)\big|\,dx \le e^{Ct}\int_\Pi \big|\mathbf{u}(x) - \mathbf{u}'(x)\big|\,dx, \tag{2.18}$$

for some constant $C > 0$ *depending only on* M_γ*.*

The fact that $S_t\mathbf{u}$ coincides with the Glimm solution constructed in [15] follows from well known uniqueness theorems, cf. [4, 6, 7, 8, 14].

The rest of this paper is organized as follows. In Section 3, we construct the wave front tracking approximate solutions and prove Theorem 2.3.

Section 4 is devoted to the proof of Theorem 2.4. The latter involves the construction of wave front tracking approximate solutions for the usual Cauchy problem with initial data of bounded total variation, and the proof of the $\mathbf{L^1}$-stability of such approximate solutions. In Section 5, we briefly show how our results immediately apply also to the non-relativistic system (1.2). Finally, in Section 6, we show the convergence of the semigroup solutions of the relativistic system (1.1) to the semigroup solution of (1.2) when $c \to \infty$.

3. Periodic Wave Front Tracking Approximate Solutions

This section is devoted to the proof of Theorem 2.3. The latter is based on the fact that, with respect to the new Riemann invariants $z(v_1, \gamma)$ and $w(v_2, \gamma)$, system (1.1) satisfies Bakhvalov's conditions A_1 - A_4. We recall that conditions A_1 - A_3 are satisfied also by v_1, v_2 (see [10, 15]).

The precise definition of $z(v_1, \gamma)$ and $w(v_2, \gamma)$ is given in [15, formula (4.2)] and is immaterial for our purposes here. Only those relevant properties of the functions $z(v_1, \gamma)$ and $w(v_2, \gamma)$ stated in the following lemma are sufficient, see [15, Section 6] for more details. Remember the definition (2.7) of $\mathcal{U}$.

Lemma 3.1. *The new Riemann invariants $z(v_1, \gamma)$, $w(v_2, \gamma)$, with respect to which system* (1.1) *satisfies Bakhvalov's conditions A_1 - A_4, may be defined for $\mathbf{v}$ belonging to a domain $\mathcal{V}_\gamma$ and $\mathbf{u} \in \mathcal{U}_\gamma := \mathbf{v}^{-1}(\mathcal{V}_\gamma)$, $\mathcal{U}_\gamma \subset \mathcal{U}$ and $\mathcal{U}_\gamma \supset K$, for any compact $K \subset \mathcal{U}$, provided that $\gamma - 1 > 0$ is sufficiently small. Moreover, after a suitable normalization, $z(v_1, \gamma)$ and $w(v_2, \gamma)$ satisfy*

$$
\begin{array}{lll}
 & \lim_{\gamma\to 1} z(v_1,\gamma) = v_1 & \lim_{\gamma\to 1} w(v_2,\gamma) = v_2 \\
(3.1) & \lim_{\gamma\to 1} \dfrac{\partial z}{\partial v_1}(v_1,\gamma) = 1 & \lim_{\gamma\to 1} \dfrac{\partial w}{\partial v_2}(v_2,\gamma) = 1 \\
 & \lim_{\gamma\to 1} \dfrac{\partial^k z}{\partial v_1^k}(v_1,\gamma) = 0 \text{ for } k \geq 2 & \lim_{\gamma\to 1} \dfrac{\partial^k w}{\partial v_2^k}(v_2,\gamma) = 0 \text{ for } k \geq 2
\end{array}
$$

locally uniformly in v_1 and v_2, respectively.

Although the conserved variable $\mathbf{u}$ depends on γ through the pressure $p = p_\gamma(\rho) = \zeta^2\rho^\gamma$, we may consider $\mathbf{u}$ as independent of γ, because of the nice behavior of p_γ as $\gamma \to 1+$, on compact subsets of $]0, +\infty[$, as stated in the following lemma, whose elementary proof is left to the reader.

Lemma 3.2. *As $\gamma \to 1+$, the pressure law p_γ converges to p_1 uniformly in $\mathbf{C}^k$, for any $k \in \mathbb{N}$, on any compact subset of $]0, +\infty[$.*

In the following, we will always use the Riemann coordinates $z(v_1, \gamma)$, $w(v_2, \gamma)$, whose relevant properties are described in Lemma 3.1, but henceforth we denote them simply by v_1 and v_2, respectively. Except for the fact that now we assume that the pair (v_1, v_2) also satisfies A_4, for any other property that will be needed in the following, we can mix up these two pairs of Riemann coordinates without any problem, due to (3.1).

Lemma 3.3. *In the Riemann coordinates, the Lax curves of* (1.1) *departing from* $\mathbf{v}$ *can be parametrized as*

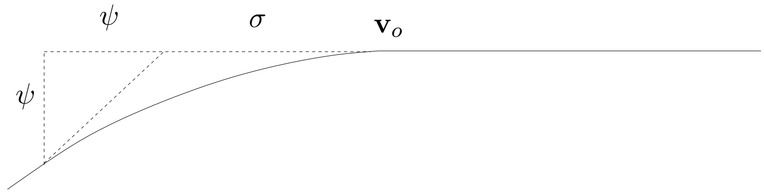

FIGURE 2. The construction of the parametrization (3.2).

$$\begin{array}{rcl} \mathcal{L}_1(\mathbf{v},\sigma) & = & \big(v_1+\sigma+\psi(\mathbf{v},\gamma,\sigma),\, v_2+\psi(\mathbf{v},\gamma,\sigma)\big) \\ \mathcal{L}_2(\mathbf{v},\sigma) & = & \big(v_1+\psi(\mathbf{v},\gamma,\sigma),\, v_2+\sigma+\psi(\mathbf{v},\gamma,\sigma)\big) \end{array} \tag{3.2}$$

with a suitable function ψ *of class* $\mathbf{C}^{2,1}$ *such that* $\psi(\mathbf{v},\gamma,\sigma)=0$ *for all* $\sigma\ge 0$ *and* $\psi(\mathbf{v},\gamma,\sigma)\to\varphi(\sigma)$ *in* $\mathbf{C}^2$ *uniformly on compact sets as* $\gamma\to 1$*, where*

$$\varphi(\sigma)=\begin{cases} 0 & \sigma\ge 0, \\ -\frac{\sigma}{2}+c\,\mathrm{arcsinh}\left(\frac{2\zeta_c}{c}\sinh\frac{\sigma}{4\zeta_c}\right) & \sigma\le 0, \end{cases} \qquad \zeta_c=\frac{\zeta}{1+(\zeta/c)^2}\,. \tag{3.3}$$

Moreover, ψ *is locally Lipschitz, for all* $\sigma\le 0$*,* $\psi\le 0$*,* $\partial_\sigma\psi\ge 0$ *and setting*

$$H(\gamma,M,\mathbf{u}^-,\mathbf{u}^+)=\sup\left\{\partial_\sigma\psi(\mathbf{v},\gamma,\sigma)\colon \begin{array}{c}\mathbf{v}\in\mathcal{V},\ \gamma\in[1,\bar\gamma] \\ \text{and } \psi(\mathbf{v},\gamma,\sigma)\in\mathcal{K}\end{array}\right\}, \tag{3.4}$$

where $\mathcal{K}$ *is any compact subset of* $\mathcal{V}$*, we have* $H(\gamma,M,\mathbf{u}^-,\mathbf{u}^+)<+\infty$ *and*

$$\lim_{\gamma\to 1}H(\gamma,M,\mathbf{u}^-,\mathbf{u}^+)=H(1,M,\mathbf{u}^-,\mathbf{u}^+).$$

Proof. **1.** We observe that when $\gamma>1$, the shock curves are not translation invariant as in the case $\gamma=1$ (see [11, 18]) and therefore the function ψ depends also on $\mathbf{v}$.

2. The parametrization (3.3) is in [11, Section 4], see Figure 2 for its geometric construction. The regularity of Ψ follows from that of p_γ and, hence, of the flux function defining (1.1), see also [18, Proposition 1].

3. The inequalities $\psi\le 0$ and $\partial_\sigma\psi\ge 0$ are consequences of the following two facts. First, a tedious but straightforward computation shows that

$$\varphi(0)=\varphi'(0)=\varphi''(0)=0, \qquad \lim_{\sigma\to 0-}\varphi'''(\sigma)>0.$$

It is also easy to see that, for each fixed $\mathbf{v}\in\mathcal{V}$, $\lim_{\gamma\to 1+}\partial_\sigma^k\psi(\mathbf{v},\gamma,\sigma)=\varphi^k(\sigma)$, uniformly in $[-M,0[$, for any $M>0$, $k\in\mathbb{N}$. Hence, given $\delta>0$ such that $\varphi'''(\sigma)>0$ for $\sigma\in[-\delta,0[$, we conclude that

$$\partial_\sigma^3\psi(\mathbf{v},\gamma,\sigma)>0, \qquad \text{for } \sigma\in[-\delta,0[\,,$$

for and all $\gamma>1$ sufficiently close to 1. We thus obtain, in particular,

$$\partial_\sigma^2\psi(\mathbf{v},\gamma,\sigma)<0, \quad \partial_\sigma\psi(\mathbf{v},\gamma,\sigma)>0, \qquad \text{for } \sigma\in[-\delta,0[\,.$$

4. Second, note that for $\sigma\le-\delta$ we have

$$\frac{2\zeta_c}{c}\cdot\sinh\frac{\sigma}{4\zeta_c}<\sinh\left(\frac{2\zeta_c}{c}\cdot\frac{\sigma}{4\zeta_c}\right) \quad \text{where} \quad \frac{2\zeta_c}{c}\in\,]0,1[\ \text{ and } \frac{\sigma}{2c}<0\,.$$

The latter inequality follows from the strict concavity of $s \to \sinh s$ for $s \leq -\delta$. Moreover, we have $\partial_\sigma \varphi < 0$ since

$$\cosh \frac{\sigma}{4\zeta_c} > \sqrt{1 + \left(\frac{2\zeta_c}{c}\right)^2 \sinh^2 \frac{\sigma}{4\zeta_c}}$$

which holds thanks to $2\zeta_c/c \in \,]0,1[$. Hence, we obtain the corresponding inequalities for $\psi(\mathbf{v},\gamma,\sigma)$ and $\partial_\sigma\psi(\mathbf{v},\gamma,\sigma)$, for each fixed $\mathbf{v} \in \mathcal{V}$, for $\sigma \in [-\delta, -M]$, for any $M > 0$, if $\gamma > 1$ is sufficiently close to 1.

5. The boundedness of $H(\gamma, M, \mathbf{u}^-, \mathbf{u}^+)$ follows from the regularity of ψ and the compactness of $\mathcal{K}$. The final limit is a consequence of the locally uniform convergence $\gamma \to 1$, see Lemma 3.2. □

Following [5], for a fixed $\varepsilon > 0$, we are lead to consider the interpolation between the i-shock and the i-rarefaction wave ($i = 1, 2$) (approximate Lax curves):

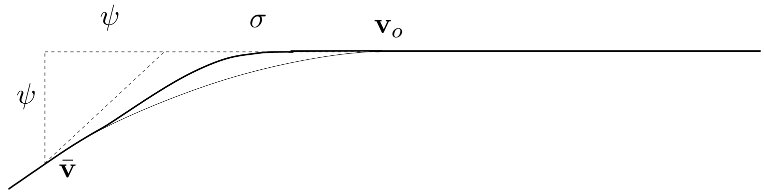

FIGURE 3. The parametrization of the approximate Lax curves (3.5).

$$\begin{array}{rcl} \mathcal{L}_1^\varepsilon(\mathbf{v},\sigma) & = & \big(v_1 + \sigma + \psi_\varepsilon(\mathbf{v},\gamma,\sigma),\, v_2 + \psi_\varepsilon(\mathbf{v},\gamma,\sigma)\big) \\ \mathcal{L}_2^\varepsilon(\mathbf{v},\sigma) & = & \big(v_1 + \psi_\varepsilon(\mathbf{v},\gamma,\sigma),\, v_2 + \sigma + \psi_\varepsilon(\mathbf{v},\gamma,\sigma)\big) \end{array} \tag{3.5}$$

where

$$\psi_\varepsilon(\mathbf{v},\gamma,\sigma) = \Phi(\sigma/\sqrt{\varepsilon})\,\psi(\mathbf{v},\gamma,\sigma) \tag{3.6}$$

and Φ is any $\mathbf{C}^\infty$ function satisfying

$$\begin{array}{lll} \Phi(s) = 1 & \text{for } s & \leq \; -2 \\ \Phi(s) = 0 & \text{for } s & \geq \; -1 \\ \Phi(s) \in [0,1],\ \Phi'(s) \in [-2,0] & \text{for } s & \in \; [-2,-1]\,. \end{array}$$

We note that the interpolated Lax curves admit the parametrization (3.5) for ε sufficiently small. Indeed, $\partial_\sigma \mathcal{L}_1^\varepsilon(\mathbf{v},0) = [1 \quad 0]^T$, hence the half-line exiting $\bar{\mathbf{v}}$ and parallel to $v_1 = v_2$ does not intersect the approximate Lax curve between $\bar{\mathbf{v}}$ and $\mathbf{v}_o$, see Figure 3, provided ε is sufficiently small. We thus have the following analog of Lemma 3.3.

Lemma 3.4. *There exist $\gamma_o \in \,]1,\bar\gamma]$, $\varepsilon_o > 0$ such that for $\varepsilon \in \,]0,\varepsilon_o]$ and $\gamma \in [1,\gamma_o]$, the function ψ_ε in (3.6) satisfies $\psi_\varepsilon \leq 0$, $\partial_\sigma\psi^\varepsilon \geq 0$ and, for any compact $\mathcal{K} \subset \mathcal{V}$,*

$$H^\varepsilon(\gamma, M, \mathbf{u}^-, \mathbf{u}^+) = \sup \left\{ \partial_\sigma \psi_\varepsilon(\mathbf{v},\gamma,\sigma) \colon \begin{array}{c} \mathbf{v} \in \mathcal{K},\ \psi_\varepsilon(\mathbf{v},\gamma,\sigma) \in \mathcal{K} \\ \text{and } \gamma \in [1,\gamma_o] \end{array} \right\} < +\infty\,.$$

Moreover, $\psi_\varepsilon(\cdot,\gamma,\cdot) \to \psi$ and $H^\varepsilon(\gamma, M, \mathbf{u}^l, \mathbf{u}^r) \to H(\gamma, M, \mathbf{u}^l, \mathbf{u}^r)$ as $\varepsilon \to 0$, uniformly on compact sets.

Given $\varepsilon > 0$, a left and right state $\mathbf{u}^l$, $\mathbf{u}^r$, with Riemann coordinates $\mathbf{v}^l = (v_1^l, v_2^l)$ and $\mathbf{v}^r = (v_1^r, v_2^r)$, as in [11, Section 2] or [10, Theorem 4.1], we construct an approximate solution of the Riemann problem associated to (1.1). First, determine the unique values σ_1 and σ_2 and a middle state $\mathbf{v}^m$ such that $\mathbf{v}^r = \mathcal{L}_2^\varepsilon(\mathbf{v}^m, \sigma_2)$ and $\mathbf{v}^m = \mathcal{L}_1^\varepsilon(\mathbf{v}^l, \sigma_1)$. If $\sigma_1 \geq 0$, then the states $\mathbf{v}^l$ and $\mathbf{v}^m$ are connected by a 1-rarefaction wave. We approximate this rarefaction wave introducing a fixed ε grid in the (v_1, v_2) plane. Let the integers h, k be such that $h\varepsilon \leq v_1^l < (h+1)\varepsilon$ and $k\varepsilon \leq v_1^m < (k+1)\varepsilon$. Introducing the states $\omega_1^j = (j\varepsilon, v_2^l)$ and $\widehat{\omega}_1^j = \left((j+\frac{1}{2})\varepsilon, v_2^l\right)$ for $j = h, \ldots, k$, we construct the ε-approximate solution through the following rarefaction fan:

$$\mathbf{v}^\varepsilon(t,x) = \begin{cases} \mathbf{v}^l & \text{if } x/t \in \left]-\infty, \lambda_1(\widehat{\omega}_1^h)\right] \\ \omega_1^j & \text{if } x/t \in \left]\lambda_1(\widehat{\omega}_1^{j-1}), \lambda_1(\widehat{\omega}_1^j)\right[\,, \quad j = h+1, \ldots, k \\ \mathbf{v}^m & \text{if } x/t \in \left[\lambda(\widehat{\omega}_1^k), +\infty\right[\,. \end{cases}$$

On the other hand, if $\sigma_1 < 0$, the states $\mathbf{v}^l$ and $\mathbf{v}^m$ are connected by a shock:

$$\mathbf{v}^\varepsilon(t,x) = \begin{cases} \mathbf{v}^l & \text{if } x < \lambda_1^\Phi(\mathbf{v}^l, \sigma_1)t \\ \mathbf{v}^m & \text{if } x > \lambda_1^\Phi(\mathbf{v}^l, \sigma_1)t\,. \end{cases}$$

Let $\lambda_1(\mathbf{v}, \sigma)$ denote the Rankine-Hugoniot speed of the (exact) shock between the states $\mathbf{v}$ and $\mathcal{L}_1(\mathbf{v}, \sigma)$. Then, the shock speed λ_1^Φ is defined as

$$\begin{aligned} \lambda_1^\Phi(\mathbf{v}^l, \sigma_1) &= \Phi\left(\frac{\sigma_1}{\sqrt{\varepsilon}}\right)\lambda_1^s(\mathbf{v}^l, \sigma_1) + \left(1 - \Phi\left(\frac{\sigma_1}{\sqrt{\varepsilon}}\right)\right)\lambda_1^r(\mathbf{v}^l, \sigma_1), \\ \lambda_1^s(\mathbf{v}^l, \sigma_1) &= \lambda_1\left(\mathbf{v}^l, \mathcal{L}_1(\mathbf{v}^l, \sigma_1)\right) \\ \lambda_1^r(\mathbf{v}^l, \sigma_1) &= \sum_j \frac{\text{meas}\left([j\varepsilon, (j+1)\varepsilon] \cap [v_1^m, v_1^l]\right)}{|\sigma_1|}\lambda_1(\hat{\omega}_1^j). \end{aligned}$$

The construction of the ε-approximate solution for waves of the second family is analogous to the previous case, we refer to [5, 11] for details.

Let now $\mathbf{u}(x)$ be a periodic piecewise constant initial condition as in (2.9). A piecewise constant ε-approximate solution to the Cauchy problem for (1.1) is constructed as follows. At time $\tau_0 = 0$ solve the Riemann problems defined by the jumps in $\mathbf{u}(x)$ using the above algorithm. This yields a piecewise constant approximate solution $(t,x) \mapsto \mathbf{u}^\varepsilon(t,x)$ defined up to the time $\tau_1 > \tau_0$ where the first set of interactions takes place. The Riemann problems arising at time τ_1 are again approximately solved using the algorithm above. Then, $\mathbf{u}^\varepsilon$ can be defined up to the time τ_2 when the next interaction takes place, etc. As usual, we denote $S_t^\varepsilon \mathbf{u} := \mathbf{u}^\varepsilon(t, \cdot)$.

Of great importance in the wave front tracking technique is the control of the interactions. Aiming at continuous dependence, the usual slight modifications of the wave speeds to avoid multiple interactions cannot be adopted. On the other hand, we treat below in details the case of simple interactions, leaving to the inductive procedures developed in [5, 11] in the case of many waves interacting simultaneously.

Let $\bar{\mathcal{D}}$ be the set of periodic piecewise constant functions with values in $\mathcal{U}$ as in (2.9). We denote by $\sigma_{i,\alpha}$ $(i = 1, 2)$ the total size of the waves of the i-th family in the ε-approximate solution of the Riemann problem for (1.1) at x_α with states $\mathbf{u}_\alpha$ and $\mathbf{u}_{\alpha+1}$. In the Riemann coordinates, this means

$$\mathbf{v}^\alpha = \mathcal{L}_2^\varepsilon\left(\mathcal{L}_1^\varepsilon(\mathbf{v}^{\alpha-1}, \sigma_{1,\alpha}), \sigma_{2,\alpha}\right) \qquad \text{for } \alpha = 0, \ldots, N\,, \tag{3.7}$$

where $\mathbf{u}^0 = \mathbf{u}^N$. Given $\mathbf{u}_-, \mathbf{u}_+ \in \mathcal{U}$, let $\mathbf{v}_-, \mathbf{v}_+$ be their respective images in the plane of Riemann coordinates. We solve the corresponding ε-approximate Riemann problem obtaining $\mathbf{v}_+ = \mathcal{L}_2^\varepsilon\left(\mathcal{L}_1^\varepsilon(\mathbf{v}_-, \sigma_1), \sigma_2\right)$. Define, similarly to Definition 2.1,

$$F^\varepsilon(\mathbf{u}_-, \mathbf{u}_+) := [\![\, \Delta v_1(\sigma_1) \,]\!]_- + [\![\, \Delta v_2(\sigma_2) \,]\!]_-\,, \tag{3.8}$$

with the same notation as in Definition 2.1. Now if $\mathbf{u}(\cdot) \in \bar{\mathcal{D}}$, we define

$$\mathcal{L}^\varepsilon[\mathbf{u}] := \sum_{\alpha=1}^{N} F^\varepsilon(\mathbf{u}_{\alpha-1}, \mathbf{u}_\alpha).$$

Proof of Theorem 2.3. **1.** Given any $T > 0$, to prove that we can construct the approximate solution $S_t^\varepsilon\mathbf{u}$ throughout the whole interval $[0, T]$ we need to show that after all possible interactions between wave fronts in $S_t^\varepsilon\mathbf{u}$, as t increases, $S_t^\varepsilon\mathbf{u}$ keeps assuming values in $\mathcal{U}_\gamma$, where the special Riemann coordinates v_1, v_2 are defined. We achieve this by showing that $\mathrm{TV}\,(S_t^\varepsilon\mathbf{u}|\Pi)$ keeps being always uniformly bounded and that the mean value

$$(S_t^\varepsilon\mathbf{u})_\Pi := \frac{1}{\Pi}\int_\Pi S_t^\varepsilon\mathbf{u}(x)\,dx$$

can be made arbitrarily close to $\mathbf{u}_\Pi := (S_0^\varepsilon\mathbf{u})_\Pi = (\mathbf{u}(\cdot))_\Pi$ if $\varepsilon > 0$ is sufficiently small.

2. As in [13], we construct the approximate solutions in a number of time-steps of fixed length T_0, independent of ε, using (2.16)–(2.17) and the convergence of the approximate solutions at each time-step as $\varepsilon \to 0$ to an entropy solution $S_t\mathbf{u}$ of (1.1) with initial data $\mathbf{u}(\cdot)$, in order to pass from one time-step to the next one.

3. The control of $\mathrm{TV}\,(S_t^\varepsilon\mathbf{u}|\Pi)$ is achieved once we show (2.16). Observe that, by the geometry of the approximate wave curves, both v_1 and v_2 decrease across approximate shocks of both families, and increase across approximate rarefactions of both families. Observe also that, because of property A_2, the absolute value of the change in v_1 across approximate shocks of the first family dominates that of v_2 across the same waves, while the absolute value of the change in v_2 across approximate shocks of the second family dominates that of v_1 across the same waves. Clearly, then, $\mathcal{L}^\varepsilon[S_t^\varepsilon\mathbf{u}]$ is equivalent to the negative variation *per* period of $S_t^\varepsilon\mathbf{u}$. By periodicity, the total variation *per* period equals twice the negative variation *per* period, so $\mathcal{L}^\varepsilon[S_t^\varepsilon\mathbf{u}]$ is equivalent to $\mathrm{TV}\,(S_t^\varepsilon\mathbf{u}|\Pi)$, that is,

$$\frac{1}{C}\mathcal{L}^\varepsilon[S_t^\varepsilon\mathbf{u}] \le \mathrm{TV}\,(S_t^\varepsilon\mathbf{u}|\Pi) \le C\mathcal{L}^\varepsilon[S_t^\varepsilon\mathbf{u}]\,, \tag{3.9}$$

for some constant $C > 0$ depending only on (1.1) and $\mathcal{U}_\gamma$.

4. As usual, we say that a wave on the left approaches a wave to its right, if the former belongs to a family of order greater than that of the latter, or if they both belong to the same family and at least one of them is a shock. Now, suppose that a wave connecting a state $\mathbf{u}_l$ to a state $\mathbf{u}_m$ interacts with a wave connecting $\mathbf{u}_m$ to $\mathbf{u}_r$ (see Figure 4). Assume also that the interaction

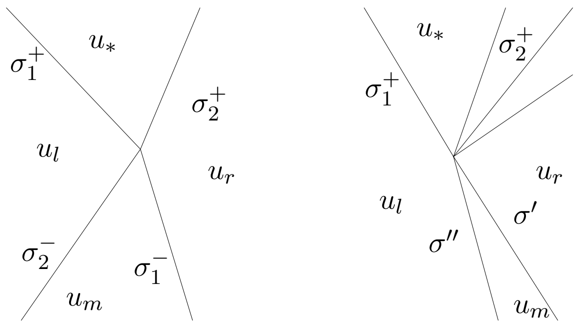

FIGURE 4. Notation for the interaction estimates.

produces two wave fronts of total size σ_1^+ and σ_2^+, connecting the states $\mathbf{u}_l$ to $\mathbf{u}'_m$ and $\mathbf{u}'_m$ to $\mathbf{u}_r$, respectively. We are going to show that

$$F^\varepsilon(\mathbf{u}_l, \mathbf{u}_r) \le F^\varepsilon(\mathbf{u}_l, \mathbf{u}_m) + F^\varepsilon(\mathbf{u}_m, \mathbf{u}_r). \tag{3.10}$$

5. We must analyze all possibilities according whether σ' is a shock or rarefaction of the first or second family and σ'' is a shock or rarefaction of the first or second family. There are in total 10 cases of approaching waves. Of all these cases, the most delicate is the one in which σ_2^- is a shock of the second family, and σ_1^- is a shock of the first family, see Figure 4, left. Bakhvalov's condition A_4 refers exactly to this type of interaction (see Figure 5).

6. In the case of the γ-law systems of gas dynamics, as pointed out by DiPerna [12], Bakhvalov's condition A_4 is satisfied in the plane of the special Riemann coordinates introduced in [12], as a consequence of the validity of DiPerna's conditions $B_4.1, B_4.2$. The latter can be viewed also as follows. Let $\mathbf{v}_0$ be a given reference state, let $R_1 := \{(v_1, v_2) : v_2 = R_1(v_1; \mathbf{v}_0),\ v_1 \le v_{01}\}$ be the right shock curve of the first family departing from $\mathbf{v}_0$ (i.e., curve whose points can be connected on the right of $\mathbf{v}_0$ by a 1-shock). Let $L_2 := \{(v_1, v_2) : v_1 = L_2(v_2; \mathbf{v}_0),\ v_2 \ge v_{02}\}$ be the left shock curve of the second family departing from $\mathbf{v}_0$ (i.e., curve whose points can be connected on the left of $\mathbf{v}_0$ by a 2-shock). If $\mathbf{v}_*$ is any point in R_1, the left shock curve of the second family departing from $\mathbf{v}_*$ is the graph of a function $v_1 = g(v_2) := L_2(v_2; \mathbf{v}_*)$. If $T_{\mathbf{v}_*} : \mathbb{R}^2 \to \mathbb{R}^2$ is the translation in the Riemann coordinates plane taking $\mathbf{v}_0$ to $\mathbf{v}_*$, the $T_{\mathbf{v}_*}$-translate of L_2 is the graph of the function $v_1 = \bar{g}(v_2) := v_{*1} + L_2(v_2 - v_{*2}; \mathbf{v}_0)$. DiPerna's condition $B_4.1$ is equivalent to the fact that the inequality $\bar{g}(v_2) \le g(v_2)$ holds. Similarly, from any point $\mathbf{v}_{**} \in L_2$, the right shock curve of the first family is the graph of the function $v_2 = h(v_1) := R_1(v_1; \mathbf{v}_{**})$ and the $T_{\mathbf{v}_{**}}$-translate of R_1 is the graph of the function $v_2 = \bar{h}(v_1) = v_{**2} + R_1(v_1 - v_{**1}; \mathbf{v}_0)$. Again, DiPerna's condition $B_4.2$ means exactly that we must have $\bar{h}_1(v_1) \ge h(v_1)$. See Fig. 5 where $\mathbf{v}_0 = \mathbf{u}_m$, $\mathbf{v}_* = \mathbf{u}_r$, $\mathbf{v}_{**} = \mathbf{u}_l$, $R_1 = R_1^-$, $L_2 = L_2^-$, $L_2(\,\cdot\,; \mathbf{v}_*) = L_2^+$,

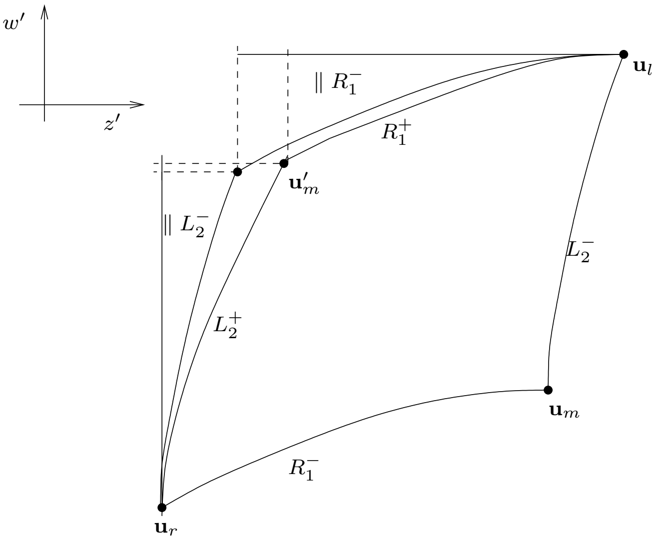

FIGURE 5. Bakhvalov's condition A_4 through DiPerna's conditions $B_4.1, B_4.2$.

$R_1(\,\cdot\,;\mathbf{v}_{**}) = R_1^+$, and the translates of R_1 and L_2 are $\| R_1^-$ and $\| L_2^-$, respectively. As we see in Fig. 5, the polygon formed by the curves L_2^-, R_1^-, L_2^+, R_1^+ is contained in the polygon formed by the curves L_2^-, R_1^-, $\| L_2^-$ and $\| R_1^-$ and this implies A_4 due to the validity of A_2, which impose constraints on the inclinations of the curves R_i and L_i.

7. The analysis in [12], for the classical case, and [15], for the relativistic case, shows that DiPerna's transformation is C^2-stable in the sense that if $R_1^\varepsilon := \{(v_1, v_2) := v_2 = R_1^\varepsilon(v_1; v_0),\ v_1 \le v_{01}\}$ and $L_2^\varepsilon := \{(v_1, v_2) := v_1 = L_2^\varepsilon(v_1; v_0),\ v_2 \ge v_{02}\}$ are curves sufficiently close (in a compact interval) to the curves R_1 and L_2, defined in the last step, in the C^2-norm, we still have the inequalities between the corresponding functions g, $\bar g$, h and $\bar h$, where now $\mathbf{v}_*$ runs along R_1^ε and $\mathbf{v}_{**}$ runs along L_2^ε.

8. Now, observe that we may define a parametrization similar to (3.2) for the *left* wave curves departing from a given point $\mathbf{v}$, that is,

$$\begin{aligned} \tilde{\mathcal{L}}_1(\mathbf{v},\sigma) &= \Big(v_1 + \sigma + \tilde\psi(\mathbf{v},\gamma,\sigma),\ v_2 + \tilde\psi(\mathbf{v},\gamma,\sigma)\Big) \\ \tilde{\mathcal{L}}_2(\mathbf{v},\sigma) &= \Big(v_1 + \tilde\psi(\mathbf{v},\gamma,\sigma),\ v_2 + \sigma + \tilde\psi(\mathbf{v},\gamma,\sigma)\Big) \end{aligned} \tag{3.11}$$

with a suitable function $\tilde\psi$ of class $\mathbf{C^{2,1}}$ such that $\tilde\psi(\mathbf{v},\gamma,\sigma) = 0$ for all $\sigma \le 0$ which converges in C^2 as $\gamma \to 1$ to the function corresponding to $\gamma = 1$. We can also define the aproximate left wave curves similarly to (3.5), that is,

$$\begin{aligned} \tilde{\mathcal{L}}_1^\varepsilon(\mathbf{v},\sigma) &= \Big(v_1 + \sigma + \tilde\psi_\varepsilon(\mathbf{v},\gamma,\sigma),\ v_2 + \tilde\psi_\varepsilon(\mathbf{v},\gamma,\sigma)\Big) \\ \tilde{\mathcal{L}}_2^\varepsilon(\mathbf{v},\sigma) &= \Big(v_1 + \tilde\psi_\varepsilon(\mathbf{v},\gamma,\sigma),\ v_2 + \sigma + \tilde\psi_\varepsilon(\mathbf{v},\gamma,\sigma)\Big) \end{aligned} \tag{3.12}$$

where

$$\tilde\psi_\varepsilon(\mathbf{v},\gamma,\sigma) = \Phi(-\sigma/\sqrt{\varepsilon})\ \tilde\psi(\mathbf{v},\gamma,\sigma), \tag{3.13}$$

and Φ is defined as before.

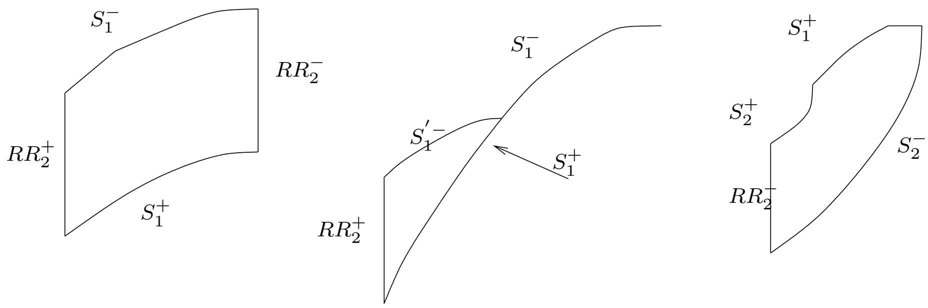

FIGURE 6. Three examples of possible interactions.

9. To simplify the reasoning we assume here that the our wave front tracking approximate solution is constructed exactly as described before with the only difference that the approximate Riemann problems are solved using the approximate *right* 1-wave curve $\mathcal{L}_1^\varepsilon$ and the approximate *left* 2-wave curve $\tilde{\mathcal{L}}_2^\varepsilon$. This choice would not cause any change in the development of the theory of [5]. So given $\mathbf{v}^l$, $\mathbf{v}^r$ we find $\mathbf{v}^m$ as the intersection of $\mathcal{L}_1^\varepsilon(\mathbf{v}^l,\sigma)$ with $\tilde{\mathcal{L}_2}^\varepsilon(\mathbf{v}^r,\sigma)$, and construct the approximate Riemann solution as before. Hence, from the considerations in steps **6.** and **7.** and the form in which the approximate wave curves are defined (as convex combinations of the coordinate lines and the exact wave curves, see (3.5) and (3.12)) we conclude the following. If g^ε, $\bar{g}^\varepsilon$, h^ε and $\bar{h}^\varepsilon$ are the functions defined as above replacing the exact right 1-shock and left 2-shock curves by the approximate ones, with $\mathbf{v}_*$ running along R_1^ε and $\mathbf{v}_{**}$ running along L_2^ε, we still have $\bar{g}^\varepsilon(v_2) \le g^\varepsilon(v_2)$ and $\bar{h}^\varepsilon(v_1) \ge h^\varepsilon(v_1)$. This implies that Fig. 5 also describes the interaction between two approximate shock waves. Therefore, we conclude that (3.10) holds for this type of interaction.

10. For the other possible types of interactions, the fact that (3.10) holds is immediate and we only need to draw pictures to see that clearly. For instance, Figure 6 describes three examples of possible interactions: (i) a 2-rarefaction RR_2^- with a 1-shock S_1^- giving a 1-shock S_1^+ and a 2-rarefaction RR_2^+; (ii) a 1-shock S_1^- with a 1-shock $S_1^{'-}$ giving a 1-shock S_1^+ and a 2-rarefaction RR_2^+; (iii) a 2-shock S_2^- with a 2-rarefaction RR_2^- giving a 1-shock S_1^+ and a 2-shock S_2^+.

11. From the validity of (3.10) at each possible interaction, we conclude that $\mathcal{L}^\varepsilon[S_t^\varepsilon \mathbf{u}]$ decreases at each interaction time, being constant in time intervals that do not contain any interaction. Therefore, (2.16) holds.

12. Since $\mathcal{L}^\varepsilon[S_t^\varepsilon \mathbf{u}]$ is non-increasing with time and, by construction, $S_t^\varepsilon \mathbf{u}$ is spatially Π-periodic, it follows from (3.9) that the total variation *per* period of $S_t^\varepsilon \bar{\mathbf{u}}$ is uniformly bounded.

13. Now, the proof of (2.17) follows similarly to the one of the corresponding property of the wave front tracking approximate solution for the usual Cauchy problem (see [5, 11]), by using the periodicity of $S_t^\varepsilon \mathbf{u}$ and the uniform boundedness of TV $(S_t^\varepsilon \mathbf{u}|\Pi)$.

14. The properties (2.16) and (2.17) satisfied by the Π-periodic wave front tracking approximate solution $S_t^\varepsilon \mathbf{u}$ allow us to repeat the reasoning in [13, 15]

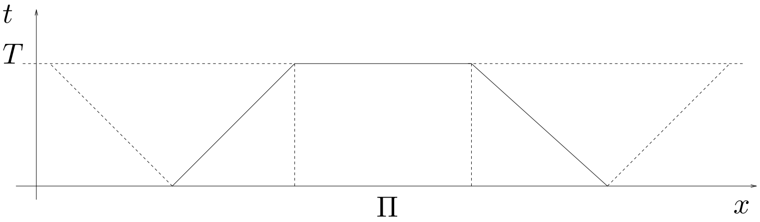

FIGURE 7. $\mathbf{L^1}$-stability of the periodic solution in theorem 2.2.

and construct the approximate solutions in an arbitrary time-interval $[0,T]$, as long as $\varepsilon > 0$ is sufficiently small.

15. Indeed, from (2.16) and (2.17) we first obtain a $T_0 > 0$ such that the approximate solutions may be constructed in the time-interval $[0,T_0]$. This $T_0 > 0$ is such that the image in the Riemann coordinates plane of the mean-values $(S_t^\varepsilon \bar{\mathbf{u}})_\Pi$ do not leave a square box $Q(\mathbf{v}_0, R)$ whose side length equals a certain $R > 0$ and is centered at a certain point $\mathbf{v}_0$, during the time-interval $[0,T_0]$, as long as $\mathbf{u}_\Pi$ lies in the concentric box $Q(\mathbf{v}_0, R/2)$ of side length $R/2$. Such $T_0 > 0$ always exists due to (2.16) and (2.17) (*cf.* [13, 15]).

16. Now, since the approximate solutions converge to an entropy solution of (1.1), which follows in a standard way (see [5]), we have that, for $\varepsilon > 0$ sufficiently small, the mean-values $(S_t^\varepsilon \bar{\mathbf{u}})_\Pi$, for $t \in [0,T_0]$, will belong to the box $Q(\mathbf{v}_0, R/2)$ since they converge to $\mathbf{u}_\Pi$, uniformly in $t \in [0,T_0]$. Therefore, we may construct the approximate solutions also in the time-interval $[T_0, 2T_0]$ if $\varepsilon > 0$ is sufficiently small, and so on. In this way, we can cover the given time-interval $[0,T]$ with a finite number of intervals of the form $[(k-1)T_0, kT_0]$, $k \in \mathbb{N}$, and obtain that the approximate solutions can be constructed in any time-interval $[0,T]$, as long as $\varepsilon > 0$ is sufficiently small.

17. As already mentioned, the convergence of the approximate solutions to an entropy solution of (1.1) follows in a standard way. This concludes the proof. □

4. The $\mathbf{L^1}$-Stability of the Periodic Wave Front Tracking Approximate Solutions

This section is devoted to the proof of Theorem 2.4.

Since the $\mathbf{L^1}$-stability is a local property, in the periodic case we can reduce its proof to the usual case of the Cauchy problem as follows.

We define approximate wave-front tracking solutions, $S_t^{C,\varepsilon}\mathbf{u}_*, S_t^{C,\varepsilon}\mathbf{u}'_*$ as in [5, 11], for initial data $\mathbf{u}_*, \mathbf{u}'_*$ which coincides with the Π-periodic piecewise constant initial data $\mathbf{u}, \mathbf{u}'$ on three period intervals, and is constant, equal to $\mathbf{u}_\Pi, \mathbf{u}'_\Pi$ outside these intervals. So, if $\Pi = [-L, L]$, these initial data coincide on $[-3L, 3L]$ (see Figure 7). The corresponding approximate wave front tracking solutions, $S_t^{C,\varepsilon}\mathbf{u}_*$ and $S_t^{P,\varepsilon}\mathbf{u}$, coincide over Π, on a time-interval $[0,T_*]$, where T_* depends only on an upper bound of the characteristic speeds, because of the finite speed of propagation property. The same is true for the approximate solutions $S_t^{C,\varepsilon}\mathbf{u}'_*$ and $S_t^{P,\varepsilon}\mathbf{u}'$.

If we can prove that any two approximate solutions, such as $S_t^{C,\varepsilon}\mathbf{u}_*$ and $S_t^{C,\varepsilon}\mathbf{u}'_*$, satisfy

$$\left\| S_t^{C,\varepsilon}\mathbf{u}_* - S_t^{C,\varepsilon}\mathbf{u}'_* \right\|_{\mathbf{L^1}(\mathbb{R})} \le C_0 \left\| \mathbf{u}_* - \mathbf{u}'_* \right\|_{\mathbf{L^1}(\mathbb{R})}, \tag{4.1}$$

for some constant C_0 not depending on $\varepsilon, \mathbf{u}_*, \mathbf{u}'_*$, but only on the bounds for the data of the problem, we then obtain

$$\left\| S_t^{P,\varepsilon}\mathbf{u} - S_t^{P,\varepsilon}\mathbf{u}' \right\|_{\mathbf{L^1}(\Pi)} \le 3C_0 \left\| \mathbf{u} - \mathbf{u}' \right\|_{\mathbf{L^1}(\Pi)}, \qquad t \in [0, T_*]. \tag{4.2}$$

This reasoning can be repeated for the intervals $[(k-1)T_*, kT_*]$, $k \in \mathbb{N}$, as long as they are contained in the intervals $[0, T_\varepsilon]$, $[0, T'_\varepsilon]$, where $S_t^{P,\varepsilon}\mathbf{u}$ and $S_t^{P,\varepsilon}\mathbf{u}'$ are defined, for which we have shown that $T_\varepsilon, T'_\varepsilon \to +\infty$ as $\varepsilon \to 0$. Indeed, the possibility of repeating the procedure follows from the fact that $\mathcal{L}^\varepsilon[S_t^{P,\varepsilon}\mathbf{u}]$ and $\mathcal{L}^\varepsilon[S_t^{P,\varepsilon}\mathbf{u}']$ do not increase with time, which guarantees the uniform boundedness of $\mathrm{TV}\,(S_t^{P,\varepsilon}\mathbf{u}|\Pi)$ for $t \in [0, T_\varepsilon]$ and $\mathrm{TV}\,(S_t^{P,\varepsilon}\mathbf{u}'|\Pi)$ for $t \in [0, T'_\varepsilon]$.

We thus get

$$\left\| S_t^{P,\varepsilon}\mathbf{u} - S_t^{P,\varepsilon}\mathbf{u}' \right\|_{\mathbf{L^1}(\Pi)} \le (3C_0)^k \left\| \mathbf{u} - \mathbf{u}' \right\|_{\mathbf{L^1}(\Pi)}, \qquad t \in \big[(k-1)T_*, kT_*\big],\ k \in \mathbb{N}, \tag{4.3}$$

which then gives the desired stability property (2.18).

By the above arguments, in this section we consider only the stability property for the Cauchy problem as in [5, 11].

From now on, we follow closely the notation in [11].

Before we begin, we state the following simple proposition which shows that we may prevent vacuum by means of a suitable bound on the total variation of the function which we measure in the Riemann coordinates by

$$\mathrm{TV}\,(\mathbf{u}) = \sup \left\{ \sum_{i=1}^{2} \sum_{\alpha} \Big| v_i\big(\mathbf{u}(x_\alpha)\big) - v_i\big(\mathbf{u}(x_{\alpha-1})\big) \Big| : x_{\alpha-1} < x_\alpha \right\}. \tag{4.4}$$

Proposition 4.1. *Fix a positive M and two states $\mathbf{u}^-, \mathbf{u}^+ \in\,]0, +\infty[\, \times \mathbb{R}$ such that*

$$\int_0^{\rho^-} \frac{\sqrt{p'(r)}}{r + \frac{p(r)}{c^2}}\, dr > \frac{M}{4} \quad \text{and} \quad \int_0^{\rho^+} \frac{\sqrt{p'(r)}}{r + \frac{p(r)}{c^2}}\, dr > \frac{M}{4}. \tag{4.5}$$

Then, any function $\mathbf{u}\colon \mathbb{R} \mapsto\,]0, +\infty[\, \times \mathbb{R}$ satisfying

$$\lim_{x\to-\infty} \mathbf{u}(x) = \mathbf{u}^-, \qquad \lim_{x\to+\infty} \mathbf{u}(x) = \mathbf{u}^+ \qquad \text{and} \qquad \mathrm{TV}\,(\mathbf{u}) < M$$

does not attain as value the vacuum state.

Proof. Assume that $\mathbf{u}^*\colon \mathbb{R} \mapsto\,]0, +\infty[\, \times \mathbb{R}$ satisfies (4.5), $\lim_{x\to-\infty} \mathbf{u}^*(x) = \mathbf{u}^-$, $\lim_{x\to+\infty} \mathbf{u}^*(x) = \mathbf{u}^+$ and moreover $\rho^*(x_*) = 0$ for some $x_* \in \mathbb{R}$. Then, by (4.4)

$$\mathrm{TV}\,(\mathbf{u}^*) \;\ge\; \sum_{i=1}^{2} \left(\left| v_i^- - v_i^*(x_*) \right| + \left| v_i^*(x_*) - v_i^+ \right| \right)$$

$$
\begin{aligned}
&\geq \left|v_1^- - v_2^-\right| + \left|v_1^+ - v_2^+\right| \\
&= 2\left(\int_0^{\rho^-} \frac{\sqrt{p'(r)}}{r + \frac{p(r)}{c^2}}\, dr + \int_0^{\rho^+} \frac{\sqrt{p'(r)}}{r + \frac{p(r)}{c^2}}\, dr\right) \\
&> M
\end{aligned}
$$

completing the proof. □

We are going to prove the following theorem.

Theorem 4.1. *Choose a positive M and states $\mathbf{u}^-, \mathbf{u}^+$ satisfying* (4.5) *for some $\bar\gamma \in \left]1, 2\right]$. Then, there exists $\gamma_o \in \left]1, \bar\gamma\right]$ such that for all $\gamma \in \left[1, \gamma_o\right[$, system* (1.1) *generates a Standard Riemann Semigroup $S\colon \left[0, +\infty\right[\times \mathcal{D} \mapsto \mathcal{D}$. Moreover, for a suitable $\kappa_\gamma \in \left]0, 1\right[$,*

(1) $\mathcal{D} \supseteq \mathrm{cl}_{\mathbf{L}^1} \left\{ \mathbf{u} \in \mathbf{BV}(\mathbb{R}; \mathcal{U})\colon \begin{array}{c} \lim_{x \to -\infty} \mathbf{u}(x) = \mathbf{u}^- \\ \lim_{x \to +\infty} \mathbf{u}(x) = \mathbf{u}^+ \\ \mathrm{TV}\,(\mathbf{u}) \leq \kappa_\gamma\, M \end{array} \right\}$;

(2) if $\mathrm{TV}\,(\mathbf{u}_o) \leq \kappa_\gamma M$, *then* $\mathrm{TV}\,(S_t \mathbf{u}_o) \leq M$ *for all* $t \geq 0$*;*

(3) $\lim_{\gamma \to 1} \kappa_\gamma = 1/\big(1 + H(1, M, \mathbf{u}^-, \mathbf{u}^+)\big)$.

Let $\bar{\mathcal{D}}$ denote the set of piecewise constant functions with values in $\mathcal{U}$. For any $\mathbf{u} \in \bar{\mathcal{D}}$,

$$
\mathbf{u} = \mathbf{u}^-\, \chi_{\left]-\infty, x_0\right]} + \sum_\alpha \mathbf{u}_\alpha\, \chi_{\left]x_{\alpha-1}, x_\alpha\right]} + \mathbf{u}^+\, \chi_{\left]x_N, +\infty\right[}\,, \tag{4.6}
$$

we denote by $\sigma_{i,\alpha}$ ($i = 1, 2$) the total size of the waves of the i-th family in the ε-approximate solution of the Riemann Problem for (1.1) at x_α with states $\mathbf{u}_\alpha$ and $\mathbf{u}_{\alpha+1}$, see (3.7).

Let $\mathcal{A}$ denote the set of all couples of approaching waves. We say that a pair of waves $(\sigma_{i,\alpha}, \sigma_{j,\beta})$ is approaching if either, $\alpha < \beta$ and $j < i$, or if $j = i$, $\min\{\sigma_{i,\alpha}, \sigma_{i,\beta}\} < 0$, see [4, Chapter 7] or [17]. Following [11, (2.11)], we introduce the functionals

$$
\begin{aligned}
V^\varepsilon(\mathbf{u}) &= \sum_\alpha \sum_i \big(1 - \eta\, \mathrm{sgn}\, \sigma_{i,\alpha}\big) \left|\sigma_{i,\alpha}\right| \\
Q^\varepsilon(\mathbf{u}) &= \sum_{\sigma_{i,\alpha}, \sigma_{j,\beta} \in \mathcal{A}} \left|\sigma_{i,\alpha} \sigma_{j,\beta}\right| \\
\Upsilon^\varepsilon(\mathbf{u}) &= V^\varepsilon(\mathbf{u}) + \frac{1}{K} \cdot Q^\varepsilon(\mathbf{u})
\end{aligned}
\tag{4.7}
$$

where $\eta \in \left]0, 1\right[$ and $K > 0$ are constants depending only on $\mathbf{u}^-, \mathbf{u}^+, M$ and their values will be defined below. The dependence of Υ^ε on ε is due to the dependence on ε of the wave sizes in the ε–solution to Riemann problems. Clearly, the functional $\Upsilon^\varepsilon(\mathbf{u})$ is equivalent to the total variation of $\mathbf{u}$.

Our first goal will be to show that $\Upsilon^\varepsilon(S_t^\varepsilon \mathbf{u})$ decreases with time. The decrease of $\Upsilon^\varepsilon(S_t^\varepsilon \mathbf{u})$ with time will then be used to prove (4.1).

To simplify the notation, in the remainder of this paper, C denotes a generic *"large"* constant dependent only on the domain $\mathcal{U}$ where the conserved quantities may vary. The actual value of C is unimportant for the results obtained here.

Throughout this section, we refer to Figure 4 for the interaction estimates.

Lemma 4.1. *Let $n \in \mathbb{N}$ with $n \geq 1$ and $A \subset \mathbb{R}^3$ be a compact set with the origin in its interior. If $g \in \mathbf{C}^{2,1}(\Omega;\mathbb{R}^n)$ satisfies $g(0,y,z) = g(x,0,z) = g(x,y,1) = 0$, then*

$$\left\| g(x,y,z) \right\| \leq \mathbf{Lip}(D^2 g) \cdot |x| \cdot |y| \cdot |z-1| \,.$$

The proof is a straightforward extension of [4, Lemma 2.5].

Lemma 4.2. *Let $\mathbf{u}^l, \mathbf{u}^m, \mathbf{u}^r$ belong to $\mathcal{U}$ and the waves σ_1^- of the first family and σ_2^- of the second family interact. Call σ_1^+, σ_2^+ the total sizes of the waves exiting the interaction, see Figure 4, left. Then, there exists a constant C dependent only on $\mathcal{U}$ such that*

$$\left|\sigma_1^+ - \sigma_1^-\right| + \left|\sigma_2^+ - \sigma_2^-\right| \leq C \cdot (\gamma - 1) \cdot \left|\sigma_1^- \, \sigma_2^-\right| . \tag{4.8}$$

provided $\gamma - 1$ is sufficiently small. Moreover

$$\operatorname{sgn} \sigma_1^+ = \operatorname{sgn} \sigma_1^- \qquad \text{and} \qquad \operatorname{sgn} \sigma_2^+ = \operatorname{sgn} \sigma_2^- \tag{4.9}$$

Proof. Consider first (4.8) and introduce for a fixed $\mathbf{v}^l$ the functions

$$g(\sigma_1^-, \sigma_2^-, \gamma) = \begin{bmatrix} \sigma_1^+ - \sigma_1^- \\ \sigma_2^+ - \sigma_2^- \end{bmatrix} .$$

Clearly, $g(0, \sigma_2^-, \gamma) = g(\sigma_1^-, 0, \gamma) = 0$. Moreover, if $\gamma = 1$, the equality $g(\sigma_1^-, \sigma_2^-, 1) = 0$ holds if and only if

$$\mathcal{L}_1^\varepsilon \left(\mathcal{L}_2^\varepsilon(\mathbf{v}^l, \sigma_2^+), \sigma_1^+ \right) = \mathcal{L}_2^\varepsilon \left(\mathcal{L}_1^\varepsilon(\mathbf{v}^l, \sigma_1^-), \sigma_2^- \right) .$$

Clearly, $\sigma_1^- = \sigma_1^+$ and $\sigma_2^- = \sigma_2^+$ is a solution to the latter equality. It is the unique solution, since the map $(\sigma_1^-, \sigma_2^-) \to \left(\sigma_1^+(\sigma_1^-, \sigma_2^-), \sigma_2^+(\sigma_1^-, \sigma_2^-) \right)$ is globally invertible by Hadamard global inverse function theorem, see [1, Theorem 1.8] and [11, Lemma 3.2]. The estimate (4.8) now follows from Lemma 4.1.

To prove (4.9), choose γ so that $C(\gamma - 1)M \leq 1/2$ and apply (4.8). □

Lemma 4.3. *Fix $\mathbf{u}^l, \mathbf{u}^m, \mathbf{u}^r \in \mathcal{U}$ and let the waves σ', σ'' both of the first family interact and call σ_1^+, σ_2^+ the total sizes of the waves exiting the interaction, see Figure 4, right. Then, if $\gamma - 1$ is sufficiently small,*

(1) if $\sigma'' < 0$ and $\sigma' < 0$, then $\sigma_1^+ - \sigma_2^+ = \sigma' + \sigma''$;

(2) if $\sigma'' > 0$, $\sigma' < 0$ and $\sigma_1^+ < 0$, then $\sigma_1^+ - \sigma_2^+ = \sigma' + \sigma''$. Moreover

$$\begin{aligned} \left|\sigma_1^+\right| - \left|\sigma'\right| &< 0 \\ \left|\sigma_2^+\right| &\leq C(\gamma - 1)\left|\sigma'\sigma''\right| + C\left(\left|\sigma'\right| - \left|\sigma_1^+\right| \right) ; \end{aligned}$$

(3) if $\sigma' < 0$, $\sigma'' < 0$ and $\sigma_1^+ > 0$, then $\sigma_2^+ = 0$ and $\sigma_1^+ = \sigma' + \sigma''$.

Moreover, whenever $\sigma'\sigma'' < 0$, there exists a $C > 0$, depending only on $\mathcal{U}$, such that

$$\left|\sigma_1^+ - (\sigma' + \sigma'')\right| + \left|\sigma_2^+\right| \leq C \cdot \left|\sigma'\sigma''\right| \cdot \left(\left|\sigma'\right| + \left|\sigma''\right| \right) . \tag{4.10}$$

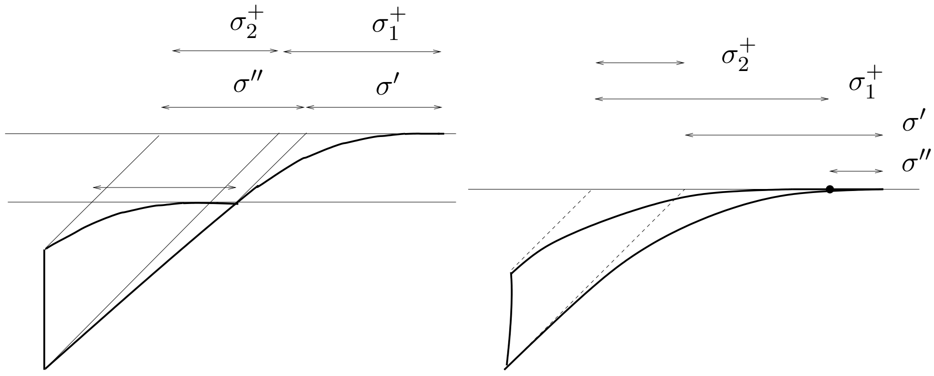

FIGURE 8. Left, proof of 1. and, right, proof of 2. in Lemma 4.3.

Proof. The proof of 1. follows directly from the parametrization (3.5), see Figure 8, left. Similarly, to prove the first equality in 2. see Figure 8, right.

Consider the function $F(\mathbf{v}^l, \gamma, \sigma', \sigma'') = \sigma_2^+(\mathbf{v}^l, \sigma', \sigma'') - \sigma''$. It is known that $F(\mathbf{v}^l, 1, \sigma', \sigma'') < 0$ for all $\mathbf{v}^l, \sigma', \sigma''$. Therefore, on the compact set $\mathcal{U}$, if γ is sufficiently small, also $F(\mathbf{v}^l, \gamma, \sigma', \sigma'') < 0$. Hence, $\left|\sigma_2^+\right| < \sigma''$ and $\left|\sigma_1^+\right| - \left|\sigma'\right| < 0$. Moreover,

$$\begin{aligned}
\psi_\varepsilon(\mathbf{v}^m, \gamma, \sigma') &= \psi_\varepsilon(\mathbf{v}^l, \gamma, \sigma_1^+) + \sigma_2^+ + \psi_\varepsilon(\mathbf{v}_*, \gamma, \sigma_2^+) \\
\psi_\varepsilon(\mathbf{v}^m, \gamma, \sigma') &\leq \psi_\varepsilon(\mathbf{v}^l, \gamma, \sigma_1^+) + \sigma_2^+ \\
\sigma_2^+ &\geq \psi_\varepsilon(\mathbf{v}^m, \gamma, \sigma') - \psi_\varepsilon(\mathbf{v}^l, \gamma, \sigma_1^+) \\
\left|\sigma_2^+\right| &\leq \left|\psi_\varepsilon(\mathbf{v}^m, \gamma, \sigma') - \psi_\varepsilon(\mathbf{v}^l, \gamma, \sigma')\right| \\
&\quad + \left|\psi_\varepsilon(\mathbf{v}^l, \gamma, \sigma') - \psi_\varepsilon(\mathbf{v}^l, \gamma, \sigma_1^+)\right| \\
&\leq C(\gamma - 1)\left|\sigma'\sigma''\right| + C\left|\sigma' - \sigma_1^+\right| \\
&\leq C(\gamma - 1)\left|\sigma'\sigma''\right| + C\left(\left|\sigma'\right| - \left|\sigma_1^+\right|\right)
\end{aligned}$$

where we applied Lemma 4.1 to the function

$$(\sigma', \sigma'', \gamma) \to \psi_\varepsilon\left(\mathbf{v}^m(\sigma''), \gamma, \sigma'\right) - \psi_\varepsilon(\mathbf{v}^l, \gamma, \sigma').$$

The bound (4.10) is exactly as [11, Lemma 3.1].

The latter case 3. is immediate. □

Entirely analogous estimates hold for waves of the second family.

Finally, introduce the set

$$\mathcal{D}_M^\varepsilon = \left\{\mathbf{u} \in \bar{\mathcal{D}} \colon \Upsilon^\varepsilon(\mathbf{u}) \leq 2M\right\}. \tag{4.11}$$

We observe that $\mathcal{D}_M^\varepsilon$ depends only on ε, $\mathbf{u}^-$, $\mathbf{u}^+$ and M.

Lemma 4.4. *If* $\mathbf{u} \in \mathcal{D}_M^\varepsilon$, H_γ *is as in* (3.4), Υ^ε *is as in* (4.7) *with* $K \geq M$ *and the total variation is measured as in* (4.4), *then*

$$\frac{1-\eta}{1+H_\gamma} \cdot \mathrm{TV}\,(\mathbf{u}) \leq \Upsilon^\varepsilon(\mathbf{u}) \leq 2(1+\eta)\,\mathrm{TV}\,(\mathbf{u}).$$

Proof. Write $\mathbf{u}$ as in (4.6). By (4.7), thanks to $K \geq M$ and $\mathbf{u} \in \mathcal{D}_M^\varepsilon$

$$\begin{aligned}
\Upsilon^\varepsilon(\mathbf{u}) &\leq (1+\eta)\sum_{i,\alpha}\left|\sigma_{i,\alpha}\right| + \frac{1}{K}\left(\sum_{i,\alpha}\left|\sigma_{i,\alpha}\right|\right)^2 \\
&\leq (1+\eta)\sum_{i,\alpha}\left|\sigma_{i,\alpha}\right| + \frac{\mathrm{TV}\,(\mathbf{u})}{K}\sum_{i,\alpha}\left|\sigma_{i,\alpha}\right| \\
&\leq 2(1+\eta)\sum_{i,\alpha}\left|\sigma_{i,\alpha}\right| \leq 2(1+\eta)\,\mathrm{TV}\,(\mathbf{u})\,.
\end{aligned}$$

Using the definition (4.4) of the total variation, the form (3.5) of the interpolated Lax curves and the bound on $\partial_\sigma\psi^\varepsilon$ provided by Lemma 3.4,

$$\begin{aligned}
\mathrm{TV}\,(\mathbf{u}) &= \sum_{i,\alpha}\Big(\left|\sigma_{i,\alpha}\right| + \left|\psi^\varepsilon(\mathbf{v}_\alpha,\gamma,\sigma_{i,\alpha})\right|\Big) \\
&\leq (1+H_\gamma)\cdot\sum_{i,\alpha}\left|\sigma_{i,\alpha}\right| \leq \frac{1+H_\gamma}{1-\eta}\cdot\Upsilon^\varepsilon(\mathbf{u})
\end{aligned}$$

completing the proof. □

This allows to choose M so that all functions $\mathbf{u}$ of the form (4.6) satisfying (3.7) with $K\ \mathrm{TV}\,(\mathbf{u}) + \big(\mathrm{TV}\,(\mathbf{u})\big)^2 \leq M$ are also in $\mathcal{D}_M^\varepsilon$ for all ε.

As usual, below we use $\Delta V(t)$ to denote the variation of the functional V at the interaction time t, similarly for Q and Υ.

Lemma 4.5. *There exist constants $\eta \in \,]0,1[$, $K \in [M,+\infty[$, $\gamma_o > 1$ and $\varepsilon_o > 0$ such that for all $\gamma \in [1,\gamma_o[$, for all $\varepsilon \in \,]0,\varepsilon_o]$ and at any time $\bar t > 0$ at which two waves σ_1^- and σ_2^- of different families interact (see Figure 4, left), the following estimates hold:*

$$\Delta\Upsilon^\varepsilon(\bar t) \leq -\frac{1}{2K}\cdot\left|\sigma_1^-\sigma_2^-\right|.$$

Moreover, at any time $\bar t > 0$ at which two waves σ' and σ'' of the same family interact (see Figure 4, right),

$$\Delta\Upsilon^\varepsilon(\bar t) \leq -\frac{1}{2K}\cdot\left|\sigma'\sigma''\right|.$$

Proof. Consider the different possible interactions separately.

1. First the interaction between waves of different families. Remember (4.8) in Lemma 4.2. Therefore,

$$\begin{aligned}
\Delta V^\varepsilon(\bar t) &= (1-\eta\,\mathrm{sgn}\,\sigma_1^+)\left|\sigma_1^+\right| + (1-\eta\,\mathrm{sgn}\,\sigma_2^+)\left|\sigma_2^+\right| \\
&\quad -(1-\eta\,\mathrm{sgn}\,\sigma_1^-)\left|\sigma_1^-\right| - (1-\eta\,\mathrm{sgn}\,\sigma_2^-)\left|\sigma_2^-\right| \\
&\leq \left|\sigma_1^+ - \sigma_1^-\right| + \left|\sigma_2^+ - \sigma_2^-\right| \\
&\leq C\cdot(\gamma-1)\cdot\left|\sigma_1^-\sigma_2^-\right|. \\
\Delta Q^\varepsilon(\bar t) &= \sum_{\sigma_{j,\beta}\in\mathcal{A}(\sigma_1^+)}\left|\sigma_1^+\sigma_{j,\beta}\right| + \sum_{\sigma_{j,\beta}\in\mathcal{A}(\sigma_2^+)}\left|\sigma_2^+\sigma_{j,\beta}\right|
\end{aligned}$$

$$
\begin{aligned}
& - \sum_{\sigma_{j,\beta}\in\mathcal{A}(\sigma_1^-)\setminus\{\sigma_2^-\}} \left|\sigma_1^- \,\sigma_{j,\beta}\right| - \sum_{\sigma_{j,\beta}\in\mathcal{A}(\sigma_2^-)\setminus\{\sigma_1^-\}} \left|\sigma_2^- \sigma_{j,\beta}\right| - \left|\sigma_1^- \,\sigma_2^-\right| \\
\leq\; & \left|\sigma_1^+ - \sigma_1^-\right| \sum_{\sigma_{j,\beta}\in\mathcal{A}(\sigma_1^+)} \left|\sigma_{j,\beta}\right| + \left|\sigma_2^+ - \sigma_2^-\right| \sum_{\sigma_{j,\beta}\in\mathcal{A}(\sigma_2^+)} \left|\sigma_{j,\beta}\right| \\
& -\left|\sigma_1^-\sigma_2^-\right| \\
\leq\; & \big(C\,M\,(\gamma-1)-1\big)\left|\sigma_1^-\sigma_2^-\right| .
\end{aligned}
$$

$$
\Delta\Upsilon^\varepsilon(\bar t) \;\leq\; \frac{1}{K}\big(C\,(K+M)(\gamma-1)-1\big)\left|\sigma_1^-\sigma_2^-\right|
$$

so that the condition $\gamma < 1 + \big(2(K+M)C\big)^{-1}$ ensures that the desired estimate holds.

Consider an interaction between waves of the same family. Following the same lines of [11, Lemma 3.1], we consider the different cases.

2. $\sigma' < 0$, $\sigma'' < 0$ and $\sigma_1^+ < 0$. Note that $\sigma_2^+ \geq 0$. Moreover by (4.7)

$$
\begin{aligned}
\Delta V^\varepsilon(\mathbf{u}) &\leq -2\eta\left|\sigma_2^+\right| \\
\Delta Q^\varepsilon(\mathbf{u}) &\leq -\left|\sigma'\sigma''\right| + 2M\left|\sigma_2^+\right| \\
\Delta\Upsilon^\varepsilon(\mathbf{u}) &\leq 2\left(\frac{M}{K}-\eta\right)\left|\sigma_2^+\right| - \frac{1}{K}\left|\sigma'\sigma''\right| \leq -\frac{1}{K}\left|\sigma'\,\sigma''\right|
\end{aligned}
$$

as soon as $K \geq M/\eta$.

3. $\sigma' < 0$, $\sigma'' > 0$ and $\sigma_1^+ < 0$. Moreover, following 2. in Lemma 4.3, see also [11, Lemma 3.1],

$$
\begin{aligned}
\Delta V^\varepsilon(\mathbf{u}) &= (1+\eta)\left(\left|\sigma_1^+\right| - \left|\sigma'\right|\right) + (1+\eta)\left|\sigma_2^+\right| - (1-\eta)\left|\sigma''\right| \\
&= 2\left(\left|\sigma_1^+\right| - \left|\sigma'\right| + \eta\left|\sigma_2^+\right|\right) \\
&\leq 2(C\eta-1)\left(\left|\sigma'\right| - \left|\sigma_1^+\right|\right) + C\eta(\gamma-1)\left|\sigma'\sigma''\right| \\
\Delta Q^\varepsilon(\mathbf{u}) &\leq \left(\left|\sigma_1^+\right| - \left|\sigma'\right|\right)\sum_{\sigma_\alpha\in\mathcal{A}(\sigma')}\left|\sigma_\alpha\right| + \left|\sigma_2^+\right|M - \left|\sigma'\sigma''\right| \\
&\leq M\left(\left|\sigma'\right| - \left|\sigma_1^+\right| + \left|\sigma_2^+\right|\right) - \left|\sigma'\sigma''\right| \\
&\leq M(C+1)\left(\left|\sigma'\right| - \left|\sigma_1^+\right|\right) + \big(CM\eta(\gamma-1)-1\big)\left|\sigma'\sigma''\right| \\
\Delta\Upsilon^\varepsilon(\mathbf{u}) &\leq \left(2C\eta + \frac{M(1+C)}{K} - 1\right)\left(\left|\sigma'\right| - \left|\sigma_1^+\right|\right) \\
&\qquad + \frac{1}{K}\big((K+M)C\eta(\gamma-1)-1\big)\left|\sigma'\sigma''\right| \\
&\leq -\frac{1}{2K}\left|\sigma'\sigma''\right|
\end{aligned}
$$

as soon as $\eta < 1/(4C)$, $K > 2M(1+C)$ and $\gamma < 1 + 1/\big(2C\eta(K+M)\big)$.

4. $\sigma' < 0$, $\sigma'' > 0$ and $\sigma_1^+ > 0$. By 3. in Lemma 4.3,

$$\begin{aligned} \Delta V^\varepsilon(\mathbf{u}) &= -2\left|\sigma'\right| \le 0 \\ \Delta Q^\varepsilon(\mathbf{u}) &\le \left(\left|\sigma_1^+\right| - \left|\sigma''\right|\right) \sum_{\sigma\in\mathcal{A}(\sigma_1^+)} |\sigma| - \left|\sigma'\sigma''\right| \le -\left|\sigma'\sigma''\right| \\ \Delta\Upsilon^\varepsilon(\mathbf{u}) &\le -\frac{1}{K}\left|\sigma'\sigma''\right|. \end{aligned}$$

To complete the proof, we only choose the parameters K, η and γ as follows

$$\begin{array}{rcll} \eta & < & 1/(4C) & \text{Case 1} \\ K & > & \max\left\{M/\eta, 2M(1+C)\right\} & \text{Cases 2 and 3} \\ \gamma & < & 1+\min\Big\{1/\left(2(K+M)C\right), 1/\left(2C\eta(K+M)\right)\Big\} & \text{Cases 1 and 3} \end{array}$$

to satisfy all the above requirements. □

We thus proved the following proposition.

Proposition 4.2. *Fix $M > 0$. Then there exists a constant $\eta > 0$, independent of ε, such that, for any $\bar{\mathbf{u}} \in \mathcal{D}_M^\varepsilon$, the wave-front tracking algorithm constructs an approximate weak solution $\mathbf{u}^\varepsilon : [0, +\infty[\times\mathbb{R} \to \Omega$ of* (1.1)*, with the following properties:*

(i) *$\mathbf{u}^\varepsilon(t,\cdot) \in \mathcal{D}_M^\varepsilon$ for all $t \ge 0$;*
(ii) *the function $t \to \Upsilon^\varepsilon(\mathbf{u}^\varepsilon(t,\cdot))$ is non-increasing;*
(iii) *any strip of the form $[0,T]\times\mathbb{R}$ contains finitely many interaction points of $\mathbf{u}^\varepsilon$;*
(iv) *$\mathrm{TV}\left(\mathbf{u}^\varepsilon(t,\cdot)\right)$ is uniformly bounded.*

To denote the globally defined, ε-approximate solution, we use the notation

$$\mathbf{u}^\varepsilon(t,\cdot) = S_t^\varepsilon \bar{\mathbf{u}}. \tag{4.12}$$

The proof then works towards an estimate independent of ε of the Lipschitz constant for the semigroup S^ε in the $\mathbf{L^1}$ norm. The basic technique is to shift the locations x_α of the jumps in the initial condition $\bar{\mathbf{u}}$ at constant rates ξ^α, and estimate the rates at which the jumps in the corresponding solution $\mathbf{u}^\varepsilon(t,\cdot)$ are shifted, for any fixed $t > 0$.

Definition 4.1. Let $]a,b[$ be an open interval. An elementary path in $\mathcal{D}_M^\varepsilon$ is a map $\gamma :]a,b[\mapsto \mathcal{D}_M^\varepsilon$ of the form

$$\gamma(\theta) = \mathbf{u}^-\,\chi_{]-\infty,x_0^\theta[} + \sum_{\alpha=1}^{n-1} \mathbf{u}^\alpha\,\chi_{]x_{\alpha-1}^\theta, x_\alpha^\theta]} + \mathbf{u}^+\,\chi_{]x_n^\theta,+\infty]}\,,$$

with $x_\alpha^\theta = \bar{x}_\alpha + \xi_\alpha\theta$ and $x_{\alpha-1}^\theta < x_\alpha^\theta$ for all $\theta \in]a,b[$ and $\alpha = 0,\ldots,n$.

Definition 4.2. A continuous map $\gamma : [a,b] \to \mathbf{L^1}_{\mathrm{loc}}$ is a pseudopolygonal if there exist countably many disjoint open intervals $J_h \subset [a,b]$ such that:

(i) the restriction of γ to each J_h is an elementary path;
(ii) the set $[a,b]\setminus \cup_{h\ge1} J_h$ is countable.

Moreover, as in [5, 11], we can prove the following result.

Proposition 4.3. *Let γ_0 be a pseudopolygonal in $\mathcal{D}_M^\varepsilon$. Then, for all $\tau > 0$, the path $\gamma_\tau = S_\tau^\varepsilon\gamma_0$ is also a pseudopolygonal.*

For $\varepsilon > 0$, the weighted length of the elementary path $\gamma :]a, b[\to \mathcal{D}^\varepsilon_M$ is

$$\|\gamma\| = (b - a) \cdot \Upsilon^\varepsilon_\xi(\mathbf{u}) \tag{4.13}$$

where the functional Υ^ε_ξ will be defined below. In the above definition, $\|\gamma\|$ does not depend on the particular choice of θ such that $\gamma(\theta) = \mathbf{u}$, since the map $\theta \mapsto \Upsilon^\varepsilon_\xi\left(\gamma(\theta)\right)$ is constant along elementary paths.

Definition 4.3. The weighted length of a pseudopolygonal is the sum of the weighted lengths of its elementary paths. For any two piecewise constant functions $u, w \in \mathcal{D}^\varepsilon_M$, their weighted distance is

$$d_\varepsilon(u, w) = \inf \left\{ \begin{array}{c} \|\gamma\| \ \text{ such that } \gamma \colon [0,1] \mapsto \mathcal{D}^\varepsilon_M \\ \text{is a pseudopolygonal joining } u \ \text{ with } w \end{array} \right\} .$$

We introduce the functional Υ^ε_ξ used in the definition of the length of pseudopolygonals:

$$\begin{array}{rcl} S^\varepsilon_{i,\alpha} & = & 2\left(\displaystyle\sum_\beta \sum_{j=1}^2 [\![\, \sigma_{j,\beta} \,]\!]_- \right) - [\![\, \sigma_{i,\alpha} \,]\!]_- \\ R^\varepsilon_\alpha & = & \displaystyle\sum_{\beta<\alpha} \left|\sigma_{2,\beta}\right| + \sum_{\beta>\alpha} \left|\sigma_{1,\beta}\right| \\ \Upsilon^\varepsilon_\xi & = & \displaystyle\sum_\alpha \sum_{i=1}^2 \left|\sigma_{i,\alpha}\, \xi_\alpha\right| \exp\left(K_1\, S^\varepsilon_{i,\alpha} + K_2\, R^\varepsilon_{i,\alpha} + K_3\, \Upsilon^\varepsilon\right) \end{array} \tag{4.14}$$

where $[\![\, s \,]\!]_- = \max\{-s, 0\}$ denotes the negative part of s and V^ε is defined in (4.7). The constants K_1, K_2 and K_3 are determined below.

The basic interaction estimates on shifting interactions are the following.

Lemma 4.6. *Consider an interaction as in Figure 4, left. Then*

$$\sum_{i=1}^2 \left|\sigma_i^+ \xi_i^+\right| - \left|\sigma_1^- \xi_1^-\right| < C \cdot \left|\sigma_1^- \sigma_2^-\right| \cdot \left(\left|\xi_1^-\right| + \left|\xi_2^-\right|\right)$$

while in case of Figure 4, right

$$\left|\sigma_1^+ \xi_1^+\right| - \left|\sigma' \xi'\right| - \left|\sigma'' \xi''\right| + \sum_\alpha \left|\sigma^+_{2,\alpha} \xi^+_{2,\alpha}\right| < C \cdot \left|\sigma' \sigma''\right| \left(\left|\xi'\right| + \left|\xi''\right|\right) .$$

An elementary estimate that will be used throughout the forthcoming proofs is the following:

$$\forall\, a, b \in \mathbb{R} \qquad e^a - e^b \le (a - b)\, e^a .$$

Lemma 4.7. *There exist constants K_1, K_2 and K_3 such that at any interaction Υ^ε_ξ does not increase.*

Proof. Following the lines of the proof of [11, Lemma 3.4], we consider several different cases. For the sake of notational simplicity, we omit the dependence on ε in the functionals below and we keep it fixed throughout this proof.

1. Interaction between two waves of different families. Using the notation in Figure 4, left, the estimate (4.8) and Lemma 4.5, we have:

$$\begin{array}{rclrcl} \Delta S_i & \leq & C(\gamma-1)\left|\sigma_1^-\sigma_2^-\right| & \Delta R_1 & = & -\left|\sigma_2^-\right| \\ \Delta R_2 & = & -\left|\sigma_1^-\right| & \Delta \Upsilon & \leq & -\frac{1}{2K}\left|\sigma_1^-\sigma_2^-\right|. \end{array}$$

Therefore

$$\begin{aligned}
&\Delta\Upsilon_\xi \\
\leq\ & \sum_{i=1}^{2}\left(\left|\sigma_i^+\xi_i^+\right| - \left|\sigma_i^-\xi_i^-\right|\right)\exp\left(K_1S_1^+ + K_2R_1^+ + K_3\Upsilon^+\right) \\
&+\sum_{i=1}^{2}\left|\sigma_i^-\xi_i^-\right|\left(K_1\Delta S_i + K_2\Delta R_i + K_3\Delta\Upsilon\right)\times \\
&\quad\times\exp\left(K_1S_i^+ + K_2R_i^+ + K_3\Upsilon^+\right) \\
\leq\ & C\left|\sigma_1^-\sigma_2^-\right|\left(\left|\xi_1^-\right| + \left|\xi_2^-\right|\right)\exp\left(K_1S_1^+ + K_2R_1^+ + K_3\Upsilon^+\right) \\
&+\left|\sigma_1^-\xi_1^-\right|\left(K_1\Delta S_1 + K_2\Delta R_1 + K_3\Delta\Upsilon\right)\times \\
&\quad\times\exp\left(K_1S_1^+ + K_2R_1^+ + K_3\Upsilon^+\right) \\
&+C\left|\sigma_1^-\sigma_2^-\right|\left(\left|\xi_1^-\right| + \left|\xi_2^-\right|\right)\exp\left(K_1S_2^+ + K_2R_2^+ + K_3\Upsilon^+\right) \\
&+\left|\sigma_2^-\xi_2^-\right|\left(K_1\Delta S_2 + K_2\Delta R_2 + K_3\Delta\Upsilon\right)\times \\
&\quad\times\exp\left(K_1S_2^+ + K_2R_2^+ + K_3\Upsilon^+\right) \\
\leq\ & \left(\left|\xi_1^-\right| + \left|\xi_2^-\right|\right)\left|\sigma_1^-\right|\left(C\left|\sigma_2^-\right| + K_1\Delta S_1 + K_2\Delta R_1 + K_3\Delta\Upsilon\right)\times \\
&\quad\times\exp\left(K_1S_1^+ + K_2R_1^+ + K_3\Upsilon^+\right) \\
&+\left(\left|\xi_1^-\right| + \left|\xi_2^-\right|\right)\left|\sigma_2^-\right|\left(C\left|\sigma_1^-\right| + K_1\Delta S_2 + K_2\Delta R_2 + K_3\Delta\Upsilon\right)\times \\
&\quad\times\exp\left(K_1S_2^+ + K_2R_2^+ + K_3\Upsilon^+\right) \\
\leq\ & \left(\left|\xi_1^-\right| + \left|\xi_2^-\right|\right)\left|\sigma_1^-\right|\times \\
&\quad\times\left((C-K_2)\left|\sigma_2^-\right| + \left(CK_1(\gamma-1) - \frac{K_3}{2K}\right)\left|\sigma_1^-\sigma_2^-\right|\right) \\
&\quad\times\exp\left(K_1S_1^+ + K_2R_1^+ + K_3\Upsilon^+\right) \\
&+\left(\left|\xi_1^-\right| + \left|\xi_2^-\right|\right)\left|\sigma_2^-\right|\times \\
&\quad\times\left((C-K_2)\left|\sigma_1^-\right| + \left(CK_1(\gamma-1) - \frac{K_3}{2K}\right)\left|\sigma_1^-\sigma_2^-\right|\right)
\end{aligned}$$

$$\begin{aligned}
&\times \exp\left(K_1 S_2^+ + K_2 R_2^+ + K_3 \Upsilon^+\right) \\
\leq\;& 0
\end{aligned}$$

provided $K_2 \geq C$ and $\gamma \leq 1 + K_3/(2CKK_1)$.

In the next cases, it is useful to separate the waves taking part to the interaction from those on the left or on the right of the interaction point:

$$\Delta\Upsilon_\xi = \Delta\Upsilon_\xi^{\text{left}} + \Delta\Upsilon_\xi^{\text{right}} \Delta\Upsilon_\xi^{\text{int}}\,.$$

2. Interaction between shocks of the same family. Concerning the waves on the left of the interaction point, by (4.7), (4.14) and Lemma 4.5, we have

$$\Delta S_{i,\alpha} = -2\left|\sigma_2^+\right|, \quad \Delta R_\alpha = -\left|\sigma_2^+\right|, \quad \Delta\Upsilon^\varepsilon \leq -\frac{1}{2K}\left|\sigma'\sigma''\right|,$$

therefore $\Delta\Upsilon_\xi^{\text{left}} \leq 0$.

Concerning the waves on the right of the interaction point, analogously we get

$$\Delta S_{i,\alpha} = -2\left|\sigma_2^+\right|, \quad \Delta R_\alpha = \left|\sigma_2^+\right|, \quad \Delta\Upsilon^\varepsilon \leq -\frac{1}{2K}\left|\sigma'\sigma''\right|,$$

therefore, if $K_1 \geq K_2/2$, then $\Delta\Upsilon_\xi^{\text{right}} \leq 0$.

Now let us consider the waves entering and exiting the interaction point. By (4.7), (4.14), 1. in Lemma 4.3, (4.10) and Lemma 4.5, we get

$$\begin{array}{rclcrcl}
S_1^+ - S' &=& -\left|\sigma''\right| - \left|\sigma_2^+\right| & \qquad & S_1^+ - S'' &=& -\left|\sigma'\right| - \left|\sigma_2^+\right| \\
S_{2,\alpha}^+ - S_1^+ &\leq& \left|\sigma_2^+\right| & & R_1^+ - R'' &=& -\left|\sigma'\right| \\
R_1^+ &=& R' & & R_{2,\alpha}^+ - R' &\leq& \left|\sigma_2^+\right|.
\end{array}$$

Then, thanks to Lemma 4.6, 1. in Lemma 4.3 and (4.10), we get

$$\begin{aligned}
&\Delta\Upsilon_\xi^{\text{int}} \\
=\;& \left|\sigma_1^+\xi_1^+\right| \exp(K_1 S_1^+ + K_2 R_1^+ + K_3\Upsilon^+) \\
&+\sum_\alpha \left|\sigma_{2,\alpha}^+\xi_{2,\alpha}^+\right| \exp(K_1 S_{2,\alpha}^+ + K_2 R_{2,\alpha}^+ + K_3\Upsilon^+) \\
&-\left|\sigma'\xi'\right| \exp(K_1 S' + K_2 R' + K_3\Upsilon^-) \\
&-\left|\sigma''\xi''\right| \exp(K_1 S'' + K_2 R'' + K_3\Upsilon^-) \\
=\;& \left(\left|\sigma_1^+\xi_1^+\right| - \left|\sigma'\xi'\right| - \left|\sigma''\xi''\right|\right) \exp(K_1 S_1^+ + K_2 R' + K_3\Upsilon^-) \\
&+\left|\sigma_1^+\xi_1^+\right| \exp(K_1 S_1^+ + K_2 R') \left(\exp(K_3\Upsilon^+) - \exp(K_3\Upsilon^-)\right) \\
&+\left|\sigma'\xi'\right| \left(\exp(K_1 S_1^+) - \exp(K_1 S')\right) \exp(K_2 R' + K_3\Upsilon^-) \\
&+\left|\sigma''\xi''\right| \left(\exp(K_1 S_1^+ + K_2 R') - \exp(K_1 S'' + K_2 R'')\right) \exp(K_3\Upsilon^-) \\
&+\sum_\alpha \left|\sigma_{2,\alpha}^+\xi_{2,\alpha}^+\right| \exp(K_1 S_1^+ + K_2 R' + K_3\Upsilon^-) \\
&+\sum_\alpha \left|\sigma_{2,\alpha}^+\xi_{2,\alpha}^+\right| \times
\end{aligned}$$

$$
\begin{aligned}
& \times \Bigl(\exp(K_1 S^+_{2,\alpha} + K_2 R^+_{2,\alpha} + K_3 \Upsilon^+) - \exp(K_1 S^+_1 + K_2 R' + K_3 \Upsilon^-)\Bigr) \\
\le\ & C\left|\sigma'\sigma''\right| \Bigl(\left|\xi'\right| + \left|\xi''\right|\Bigr) \exp\Bigl(K_1 S^+_1 + K_2 R' + K_3 \Upsilon^-\Bigr) \\
& -K_1 \left|\sigma'\xi'\right| \left(\left|\sigma''\right| + \left|\sigma^+_2\right|\right) \exp\Bigl(K_1 S^+_1 + K_2 R' + K_3 \Upsilon^-\Bigr) \\
& -K_1 \left|\sigma''\xi''\right| \left(\left|\sigma'\right| + \left|\sigma^+_2\right|\right) \exp\Bigl(K_1 S^+_1 + K_2 R' + K_3 \Upsilon^-\Bigr) \\
& + \sum_\alpha \left|\sigma^+_{2,\alpha}\xi^+_{2,\alpha}\right| \Bigl((K_1 + K_2)\left|\sigma^+_2\right| - K_3/(2K)\left|\sigma'\sigma''\right|\Bigr) \times \\
& \quad \times \exp(K_1 S^+_1 + K_2 R' + K_3 \Upsilon^-) \\
\le\ & (C - K_1)\left|\sigma'\sigma''\right| \Bigl(\left|\xi'\right| + \left|\xi''\right|\Bigr) \exp(K_1 S^+_1 + K_2 R' + K_3 \Upsilon^-) \\
& + \sum_\alpha \left|\sigma^+_{2,\alpha}\xi^+_{2,\alpha}\right| \bigl(C(K_1 + K_2)M - K_3/(2K)\bigr) \left|\sigma'\sigma''\right| \times \\
& \quad \times \exp(K_1 S^+_1 + K_2 R' + K_3 \Upsilon^-) \\
\le\ & 0
\end{aligned}
$$

which is satisfied if $K_1 \ge C$ and $K_3 \ge 2CKM(K_1 + K_2)$.

3. Interaction between a 1-shock σ' and a 1-rarefaction σ'' resulting in a 1-shock. In this case, σ^+_2 is a 2-shock. Let us consider the waves on the left of the interaction point. Thanks to Lemma 4.5, 2. in Lemma 4.3 and (4.10)

$$
\Delta S^{\mathrm{left}}_{i,\alpha} \le 2\left|\sigma^+_2\right|, \quad \Delta R^{\mathrm{left}}_{\alpha} \le \left|\sigma^+_2\right|, \quad \Delta\Upsilon^{\mathrm{left}} \le -\frac{1}{2K}\left|\sigma'\sigma''\right| .
$$

And therefore

$$
\begin{aligned}
K_1 \Delta S^{\mathrm{left}}_{i,\alpha} + K_2 \Delta R^{\mathrm{left}}_{\alpha} + K_3 \Delta\Upsilon^{\mathrm{left}} &\le (2K_1 + K_2)\left|\sigma^+_2\right| - \frac{K_3}{2K}\left|\sigma'\sigma''\right| \\
&\le \left((2K_1 + K_2)CM - \frac{K_3}{2K}\right)\left|\sigma'\sigma''\right|
\end{aligned}
$$

so that $\Delta\Upsilon^{\mathrm{left}}_\xi \le 0$ provided $K_3 > 2K(2K_1 + K_2)CM$.

Considering the waves on the right of the interaction point, we also get

$$
\Delta S^{\mathrm{right}}_{i,\alpha} \le 2\left|\sigma^+_2\right|, \quad \Delta R^{\mathrm{right}}_{\alpha} = \left|\sigma^+_2\right|, \quad \Delta\Upsilon^{\mathrm{right}} \le -\frac{1}{2K}\left|\sigma'\sigma''\right|
$$

so that $\Delta\Upsilon_\xi \le 0$ under the same conditions as above.

Concerning the interacting waves, using 3 in Lemma 4.3, we compute

$$
\begin{aligned}
S^+_1 - S' &= \left|\sigma^+_1\right| - \left|\sigma'\right| + 2\left|\sigma^+_2\right| = 3\left|\sigma^+_2\right| - \left|\sigma''\right| \\
S^+_1 - S'' &= \left|\sigma^+_1\right| - 2\left|\sigma'\right| + 2\left|\sigma^+_2\right| \le \left|\sigma^+_2\right| - \left|\sigma'\right| \\
S^+_{2,} - S^+_1 &= \left|\sigma^+_1\right| - \left|\sigma^+_2\right| = \left|\sigma'\right| - \left|\sigma''\right| \le \left|\sigma'\right| \\
R^+_2 - R'' &= -\left|\sigma'\right| \\
R^+_1 - R'' &= -\left|\sigma'\right| \\
R' - R'' &= -\left|\sigma'\right| \le 0
\end{aligned}
$$

Therefore, by Lemma 4.5 and 2. in Lemma 4.3, we get

$$
\begin{aligned}
& \Delta\Upsilon^{\mathrm{int}}_{\xi} \\
= \; & \left(\left|\sigma_1^+\xi_1^+\right| - \left|\sigma'\xi'\right| - \left|\sigma''\xi''\right| + \left|\sigma_2^+\xi_2^+\right|\right) \exp(K_1 S_1^+ + K_2 R'' + K_3 \Upsilon^-) \\
& + \left|\sigma'\xi'\right| \left(e^{K_1 S_1^+ + K_2 R'' + K_3\Upsilon^+} - e^{K_1 S' + K_2 R' + K_3 \Upsilon^-}\right) \\
& + \left|\sigma''\xi''\right| \left(e^{K_1 S_1^+ + K_2 R'' + K_3\Upsilon^+} - e^{K_1 S'' + K_2 R'' + K_3 \Upsilon^-}\right) \\
& + \left|\sigma_2^+\xi_2^+\right| \left(e^{K_1 S_2^+ + K_2 R_2^+ + K_3\Upsilon^+} - e^{K_1 S_1^+ + K_2 R'' + K_3 \Upsilon^+}\right) \\
\leq \; & C\left|\sigma'\sigma''\right| \left(\left|\xi'\right| + \left|\xi''\right|\right) \exp(K_1 S_1^+ + K_2 R'' + K_3 \Upsilon^+) \\
& + \left|\sigma'\xi'\right| \left(3K_1\left|\sigma_2^+\right| - K_1\left|\sigma''\right| - \frac{K_3}{2K}\left|\sigma'\sigma''\right|\right) e^{K_1 S_1^+ + K_2 R'' + K_3\Upsilon^+} \\
& + \left|\sigma''\xi''\right| \left(K_1\left|\sigma_2^+\right| - K_1\left|\sigma'\right| - \frac{K_3}{2K}\left|\sigma'\sigma''\right|\right) \times \\
& \qquad \times \exp(K_1 S_1^+ + K_2 R'' + K_3 \Upsilon^+) \\
& + \left|\sigma_2^+\xi_2^+\right| \left(K_1\left|\sigma'\right| - K_2\left|\sigma'\right|\right) \exp(K_1 S_1^+ + K_2 R'' + K_3 \Upsilon^+) \\
\leq \; & \left|\sigma'\sigma''\right| \left(\left|\xi'\right| + \left|\xi''\right|\right) (C - K_1) \exp(K_1 S_1^+ + K_2 R'' + K_3 \Upsilon^+) \\
& + \left|\sigma'\xi'\right| \left(3CK_1 - \frac{K_3}{2K}\right) \left|\sigma'\sigma''\right| \exp(K_1 S_1^+ + K_2 R'' + K_3 \Upsilon^+) \\
& + \left|\sigma''\xi''\right| \left(K_1 C - \frac{K_3}{2K}\right) \left|\sigma'\sigma''\right| \exp(K_1 S_1^+ + K_2 R'' + K_3 \Upsilon^+) \\
& + \left|\sigma_2^+\xi_2^+\right| (K_1 - K_2)\left|\sigma'\right| \exp(K_1 S_1^+ + K_2 R'' + K_3 \Upsilon^+) \\
\leq \; & 0
\end{aligned}
$$

provided $K_1 > C$, $K_2 > K_1$ and $K_3 > 6CKK_1$.

4. Interaction between a 1-shock σ' and a 1-rarefaction σ'' resulting in a 1-rarefaction. As observed before, $\sigma_2^+ = 0$. For the waves not taking part in the interaction we have $\Delta S_{i,\alpha} = -2\left|\sigma'\right|$. For the waves on the left of the interaction point: $\Delta R_\alpha \leq -\left|\sigma'\right|$ while for waves on the right $\Delta R_\alpha = 0$. Therefore $\Delta\Upsilon^{\mathrm{left}} \leq 0$ and $\Delta\Upsilon^{\mathrm{right}} \leq 0$.

Regarding the waves entering or exiting the interaction point, we compute

$$
\begin{aligned}
S_1^+ - S' &\leq -\left|\sigma'\right| & \qquad S_1^+ - S'' &\leq -2\left|\sigma'\right| \\
R_1^+ - R' &\leq -\left|\sigma'\right| & \qquad R_1^+ - R'' &\leq -\left|\sigma_1^+\right|.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
& \Delta\Upsilon^{\mathrm{int}}_{\xi} \\
= \; & \left(\left|\sigma_1^+\xi_1^+\right| - \left|\sigma'\xi'\right| - \left|\sigma''\xi''\right|\right) \exp(K_1 S_1^+ + K_2 R_1^+ + K_3 \Upsilon^+) \\
& + \left|\sigma'\xi'\right| \left(e^{K_1 S_1^+ + K_2 R_1^+ + K_3\Upsilon^+} - e^{K_1 S' + K_2 R' + K_3 \Upsilon^-}\right)
\end{aligned}
$$

$$
\begin{aligned}
& +|\sigma''\xi''| \left(e^{K_1S_1^+ + K_2R_1^+ + K_3\Upsilon^+} - e^{K_1S'' + K_2R'' + K_3\Upsilon^-} \right) \\
\leq\ & C|\sigma'\sigma''| \left(|\xi'| + |\xi''| \right) \exp(K_1S_1^+ + K_2R_1^+ + K_3\Upsilon^+) \\
& +|\sigma'\xi'| \left(-(K_1 + 2K_3)|\sigma'| - K_2|\sigma''| \right) e^{K_1S_1^+ + K_2R_1^+ + K_3\Upsilon^+} \\
& +|\sigma''\xi''| \left(-2(K_1 + K_3)|\sigma'| \right) \exp(K_1S_1^+ + K_2R_1^+ + K_3\Upsilon^+) \\
\leq\ & |\sigma'\sigma''| \left((C - K_2)|\xi'| + (C - 2(K_1 + K_3)|\xi''| \right) \\
& \quad \exp(K_1S_1^+ + K_2R_1^+ + K_3\Upsilon^+) \\
\leq\ & 0
\end{aligned}
$$

provided $K_2 > C$ and $2(K_1 + K_3) > C$. □

Proposition 4.4. *Let us consider the system* (1.1) *and let us take M as in Theorem 4.1. Then there exist positive constants K_1, K_2 and K_3, independent of ε in* (4.14)*, such that the following holds: if γ_0 is a pseudopolygonal, then the weighted length $\|\gamma_\tau\|$ of the pseudopolygonal $\gamma_\tau = S^\varepsilon_\tau\gamma_0$ is a non-increasing function of time, i.e. the map $t \to \Upsilon^\varepsilon_\xi(S^\varepsilon_t\gamma)$ is non-increasing.*

The proof is the same as in [5].

Proposition 4.5. *Any two functions u, u' in $\mathcal{D}^\varepsilon_M$ can be joined by a pseudopolygonal entirely contained in $\mathcal{D}^\varepsilon_M$. Moreover, the weighted length of this pseudopolygonal is uniformly equivalent to the usual $\mathbf{L^1}$-distance, i.e.,*

$$
\frac{1}{C} \cdot \|\gamma\|_{\mathbf{L^1}} \leq (b-a) \cdot \Upsilon^\varepsilon_\xi(\gamma) \leq C \cdot \|\gamma\|_{\mathbf{L^1}} .
$$

Proposition 4.6. *Let M be as in Theorem 4.1. Then, the semigroup $S^\varepsilon\colon [0, +\infty[\times\mathcal{D}^\varepsilon_M \to \mathcal{D}^\varepsilon_M$ defined by* (4.12) *is uniformly Lipschitz continuous with respect to the $\mathbf{L^1}$ distance, with a constant independent of ε.*

As in [5], to complete the proof of Theorem 4.1, we consider a sequence of semigroups S^{ε_n} with $\lim_{n\to+\infty} \varepsilon_n = 0$, and construct the limit semigroup as $S = \lim_{n\to+\infty} S^{\varepsilon_n}$. More precisely, for $\bar{\mathbf{u}} \in \mathcal{D}$ and $t \geq 0$, we define

$$
S_t\bar{\mathbf{u}} = \lim_{n\to+\infty} S^{\varepsilon_n},
$$

where $\bar{\mathbf{u}} \in \mathcal{D}^{\varepsilon_n}_M$ is any sequence converging to $\bar{\mathbf{u}}$ in $\mathbf{L^1}$. The limit is unique and depends continuously on the initial data.

With easy computations, we can verify that if $\mathrm{TV}\,(\mathbf{u}_o) \leq \kappa_\gamma M$, then $\mathrm{TV}\,(S_t\mathbf{u}_o) \leq M$ for all $t \geq 0$, and $\lim_{\gamma\to1} \kappa_\gamma = \frac{1}{1+H(1,M,\mathbf{u}^-,\mathbf{u}^+)}$; therefore the conclusion of the proof of Theorem 4.1 follows as in [5].

5. The Classical Case

This section describes the modifications necessary to show that also the classical case (1.2) fits in Theorem 4.1. Again, $\mathbf{u}$ denotes the conserved quantities and f the flow:

$$
\begin{array}{rcl} u_1 & = & \rho \\ u_2 & = & \rho v \end{array} \qquad\qquad f(u_1, u_2) = \left[\begin{array}{c} \rho v \\ \rho v^2 + p(\rho) \end{array} \right] .
$$

Conditions (4.5) reduce to the known inequalities [17, (XX)]

$$\int_0^{\rho^-} \frac{\sqrt{p'(r)}}{r}\, dr > \frac{M}{4} \quad \text{and} \quad \int_0^{\rho^+} \frac{\sqrt{p'(r)}}{r}\, dr > \frac{M}{4}\,. \tag{5.1}$$

All the results obtained for the relativistic system (1.1) have immediate analogue for the classical case (1.2). For instance, Theorem 4.1 can be restated as follows. (Recall the definition (2.7) of $\mathcal{U}$).

Theorem 5.1. *Choose a positive M and states $\mathbf{u}^-, \mathbf{u}^+$ satisfying* (5.1) *for some $\bar\gamma \in \,]1,2]$. Then, there exists $\gamma_o \in \,]1,\bar\gamma]$ such that for all $\gamma \in [1,\gamma_o[$, system* (1.1) *generates a Standard Riemann Semigroup $S\colon\ [0,+\infty[\times\mathcal{D} \mapsto \mathcal{D}$. Moreover, for a suitable $\kappa_\gamma \in \,]0,1[$,*

(1) $\mathcal{D} \supseteq \mathrm{cl}_{\mathbf{L}^1}\left\{ \mathbf{u} \in \mathbf{BV}(\mathbb{R};\mathcal{U})\colon \begin{array}{l} \lim_{x\to-\infty}\mathbf{u}(x) = \mathbf{u}^- \\ \lim_{x\to+\infty}\mathbf{u}(x) = \mathbf{u}^+ \\ \mathrm{TV}\,(\mathbf{u}) \leq \kappa_\gamma\, M \end{array} \right\}$;

(2) if $\mathrm{TV}\,(\mathbf{u}_o) \leq \kappa_\gamma M$, *then* $\mathrm{TV}\,(S_t\mathbf{u}_o) \leq M$ *for all* $t \geq 0$*;*

(3) $\lim_{\gamma\to1}\kappa_\gamma = \frac{1}{1+H(1,M,\mathbf{u}^-,\mathbf{u}^+)}$ *with* $H(1,M,\mathbf{u}^-,\mathbf{u}^+)$ *as in* (3.4).

The proof is entirely similar to that of Theorem 4.1, so we only sketch it.

Coherently with the limit $c \to +\infty$ in (2.8), the Riemann coordinates are

$$v_1 = v - \int_0^{\rho} \frac{\sqrt{p'(r)}}{r}\, dr \quad \text{and} \quad v_2 = v + \int_0^{\rho} \frac{\sqrt{p'(r)}}{r}\, dr \tag{5.2}$$

and, using the γ-law,

$$\begin{array}{rclll} \displaystyle\int_0^{\rho^-} \frac{\sqrt{p'(r)}}{r}\, dr & = & \displaystyle\frac{3-\gamma}{\gamma-1}\sqrt{\gamma}\,\zeta\,\rho^{(\gamma-1)/2} & \text{if } \gamma > 1 \\ \displaystyle\int_{\rho_*}^{\rho} \frac{\sqrt{p'(r)}}{r}\, dr & = & \zeta\,\ln(\rho/\rho_*) & \text{if } \gamma = 1\,. \end{array}$$

The function φ in (3.3) becomes (see [11, § 4]),

$$\varphi_\infty(\sigma) = \left\{ \begin{array}{lll} 0 & \sigma \geq 0 \\ -\frac{1}{2}\sigma + 2\zeta\sinh\frac{\sigma}{4\zeta} & \sigma < 0\,. \end{array} \right.$$

Note that, in both cases, the relations above coincide with the formal limits for $c \to +\infty$ of the analogous relativistic conditions. In the classical case, Theorem 4.1 can thus be entirely rephrased, providing Lipschitz continuous dependence to the solutions constructed in [16].

6. The Limit $c \to +\infty$

Now, we can extend to the case "γ near to 1" the rigorous classical limit $c \to +\infty$ obtained in [3, Theorem 4.1] for $\gamma = 1$, see also [9]. We prove that as $c \to +\infty$ any solution of (1.1) converges to the corresponding solution of the classical p-system (1.2) with $1/c^2$ as rate of convergence.

Below, we denote by $S^c\colon\ [0,+\infty[\,\times\,\mathcal{D}^{c_o} \mapsto \mathcal{D}^c$ the semigroup constructed in Theorem 4.1 and by $S\colon\ [0,+\infty[\,\times\,\mathcal{D} \mapsto \mathcal{D}$ the one defined in Theorem 5.1. Throughout this section, $\gamma \in \,]1,\gamma_o]$.

Proposition 6.1. *Fix* $\gamma \in [1,3]$. *Let* $c_o \in\,]0,+\infty[$ *be fixed. Choose a positive* M *and states* $\mathbf{u}^-$, $\mathbf{u}^+$ *satisfying* (4.5) *for* $c = c_o$. *Then,* M, $\mathbf{u}^-$ *and* $\mathbf{u}^+$ *satisfy* (4.5) *for all* $c \in [c_o, +\infty[$ *and also* (5.1). *Moreover, for all* $\mathbf{u}$ *such that* $\lim_{x\to-\infty} \mathbf{u}(x) = \mathbf{u}^-$, $\lim_{x\to+\infty} \mathbf{u}(x) = \mathbf{u}^+$ *and* $\mathrm{TV}(\mathbf{u}) \le \kappa_\gamma M$,

$$\left\| S_t^c \mathbf{u} - S_t u \right\|_{\mathbf{L}^1} \le C \cdot \frac{1}{c^2} \cdot t\,,$$

where the constant C *depends only on* M, $\mathbf{u}^\pm$ *and* $\mathrm{TV}(\mathbf{u})$.

Proof. To prove the first statement, simply observe that for all $c > c_o$

$$\int_0^\rho \frac{\sqrt{p'(r)}}{r + \frac{p(r)}{c^2}}\, dr > \int_0^\rho \frac{\sqrt{p'(r)}}{r + \frac{p(r)}{c_o{}^2}}\, dr \quad \text{and} \quad \int_0^\rho \frac{\sqrt{p'(r)}}{r}\, dr > \int_0^\rho \frac{\sqrt{p'(r)}}{r + \frac{p(r)}{c_o{}^2}}\, dr\,.$$

Thanks to the constructions of the SRS provided by Theorem 4.1, the latter statement follows from [3, Corollary 2.5]. □

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Dipartimento di Matematica, Università degli Studi di Torino, Via Carlo Alberto 10, 10123 Torino, Italy
E-mail address: danela00@hotmail.com

Dipartimento di Matematica, Università degli Studi di Brescia, Via Branze 38, 25123 Brescia, Italy
E-mail address: rinaldo@ing.unibs.it

Instituto de Matemtica Pura e Aplicada-IMPA, Estrada Dona Castorina 110, 22460-320, Rio de Janeiro, Brazil
E-mail address: hermano@impa.br