# NEWHOUSE PHENOMENA AND HOMOCLINIC CLASS

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Abstract. We show that there exists a generic subset $R$ among the $C^1$ diffeomorphisms set which are $C^1$ far away from tangency, such that for $f \in R$ and any non-trivial chain recurrent class $C$ of $f$, if $C \bigcap P_0^* \neq \phi$ then $C$ is a homoclinic class contains index 1 periodic point and there are a family of sources converge to $C$ in Hausdorff topology.

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## 1. Introduction

In the middle of last century, with many remarkable work, hyperbolic diffeomorphisms have been understood very well, but soon people discovered that the set of hyperbolic diffeomorphisms are not dense among differential dynamics, two kinds of counter examples were described, one associated with heterdimension cycle was given by R.Abraham and Smale [3] and then given by Shub [40] and Mañé [28], another counter example associated with homoclinic tangency was given by Newhouse [31] [32]. In fact, Newhouse got an open set $\mathcal{U} \subset C^2(M)$ where $dim(M) = 2$ such that there exists a $C^2$ generic subset $R \subset \mathcal{U}$ and for any $f \in R$ , $f$ has infinite sinks or sources. Such complicated phenomena (there exist an open set $\mathcal{U}$ in $C^r(M)$ and a generic subset $R \subset \mathcal{U}$, such that any $f \in R$ has infinite sinks or sources) is called $C^r$ Newhouse phenomena today, and we say $C^r$ Newhouse phenomena happens at $\mathcal{U}$.

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In last 90's, some new examples of Newhouse phenomena were found, [33] generalized Newhouse phenomena to high dimensional manifold ($dimM > 2$) but with the same topology $C^r(r > 1)$. [7] used a new tool 'Blender' to show the existence of $C^1$ Newhouse phenomena on manifold with $dim(M) > 2$. Until now, all the construction of $C^r$ Newhouse phenomena relate closely with homoclinic tangency, more precisely, all the open set $\mathcal{U}$ given by the construction above which happens Newhouse phenomena there will have $\mathcal{U} \subset \overline{HT}$. We hope that it's a necessary condition for $C^r$ Newhouse phenomena happens at $\mathcal{U}$. Pujals states it as a conjecture.

**Conjecture** (Pujals): *If $C^r$ Newhouse phenomena happens at $\mathcal{U}$, then $\mathcal{U}$ is contained in $\overline{HT^r}$.*

When $r = 1$ and $M$ is a compact surface, with Mañé's work [29], Pujals' conjecture is equivalent with the famous $C^1$ Palis strong conjecture.

$C^1$ **Palis strong conjecture** : *Diffeomorphisms of $M$ exhibiting either a homoclinic tangency or heterodimensional cycle are $C^1$ dense in the complement of the $C^1$ closure of hyperbolic systems.*

In the remarkable paper [36] they proved $C^1$ Palis strong conjecture on $C^1(M)$ when $M$ is a boundless compact surface, so in such case Pujals' conjecture is right. In [37] they gave many relations between $C^2$ Newhouse phenomena and $\overline{HT^1}$. In this paper we just consider $C^1$ Newhouse phenomena, and we show that if $C^1$ Newhouse phenomena happens in an open set $\mathcal{U} \subset C^1(M)\backslash\overline{HT^1}$, it should have some special properties with [7]'s example, in fact, in [7] they found an open set $\mathcal{U} \subset \overline{(HT^1)}$ and there exists a generic subset $R \subset \mathcal{U}$ such that any $f \in R$ has infinite sinks or sources stay near a chain recurrent class, and such class does not contain any periodic points, such kind of chain recurrent class is called aperiodic class now. Here we proved that in $\overline{HT}^c$, if there exists Newhouse phenomena, the sinks or sources will just stay near a special kind of homoclinic class.

**Theorem 1** *There exists a generic subset $R \subset C^1(M)\backslash\overline{HT^1}$, such that for $f \in R$ and $C$ is any non-trivial chain recurrent class of $f$, if $C \bigcap P_0^* \neq \phi$, $C$ should be a homoclinic class containing index 1 periodic points and $C$ is an index 0 fundamental limit.*

Theorem 1 means that if we want to disprove the existence of Newhouse phenomena in $C^1(M)\overline{HT}$, we just need study the homoclinic class containing index 1 periodic point.

In §3 we'll state some generic properties. In §4 we'll introduce a special minimal non-hyperbolic set and theorem 1 will be proved in §5.

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## 2. Notations and definitions

Let $M$ be a compact boundless Riemannian manifold, since when $M$ is a surface [36] has proved that hyperbolic diffeomorphisms are open and dense in $C^1(M) \setminus \overline{HT}$, we suppose $dim(M) = d > 2$ in this paper. Let $Per(f)$ denote the set of periodic points of $f$ and $\Omega(f)$ the non-wondering set of $f$, for $p \in Per(f)$, $\pi(p)$ means the period of $p$. If $p$ is a hyperbolic periodic point, the index of $p$ is the dimension of the stable bundle. We denote $Per_i(f)$ the set of the index $i$ periodic periodic points of $f$, and we call a point $x$ is an index $i$ preperiodic point of $f$ if there exists a family of diffeomorphisms $g_n \xrightarrow{C^1} f$, where $g_n$ has an index $i$ periodic point $p_n$ and $p_n \longrightarrow x$. $P_i^*(f)$ is the set of index $i$ preperiodic point of $f$, it's easy to know $\overline{P_i(f)} \subset P_i^*(f)$.

Let $\Lambda$ be an invariant compact set of $f$, we say $\Lambda$ is an index $i$ fundamental limit if there exists a family of diffeomorphisms $g_n$ $C^1$ converging to $f$, $p_n$ is an index $i$ periodic point of $g_n$ and $Orb(p_n)$ converge to $\Lambda$ in *Hausdorff* topology. So if $\Lambda(f)$ is an index $i$ fundamental limit, we have $\Lambda(f) \subset P_i^*(f)$.

For two points $x, y \in M$ and some $\delta > 0$, we say there exists a $\delta$-pseudo orbit connects $x$ and $y$ means that there exist points $x = x_0, x_1, \cdots, x_n = y$ such that $d(f(x_i), x_{i+1}) < \delta$ for $i = 0, 1, \cdots, n-1$, we denote it $x \underset{\delta}{\dashv} y$. We say $x \dashv y$ if for any $\delta > 0$ we have $x \underset{\delta}{\dashv} y$ and denote $x \vdash\dashv y$ if $x \dashv y$ and $y \dashv x$. A point $x$ is called a chain recurrent point if $x \vdash\dashv x$. $CR(f)$ denotes the set of chain recurrent points of $f$, it's easy to know that $\vdash\dashv$ is an closed equivalent relation on $CR(f)$, and every equivalent class of such relation should be compact and is called chain recurrent class. Let $K$ be a compact invariant set of $f$, if $x, y$ are two points in $K$, we'll denote $x \underset{K}{\dashv} y$ if for any $\delta > 0$, we have a $\delta$ -pseudo orbit in $K$ connects $x$ and $y$. If for any two points $x, y \in K$ we have $x \underset{K}{\dashv} y$, we call $K$ a chain recurrent set. Let $C$ be a chain recurrent class of $f$, we call $C$ is an aperiodic class if $C$ does not contain periodic point.

Let $\Lambda$ be an invariant compact set of $f$, for $0 < \lambda < 1$ and $1 \le i < d$, we say $\Lambda$ has an index $i - (l, \lambda)$ dominated splitting if we have a continuous invariant splitting $T_\Lambda M = E \oplus F$ where $dim(E_x) = i$ for any $x \in \Lambda$ and $\parallel Df^l|_E(x) \parallel \cdot \parallel Df^{-l}|_F(f^l x) \parallel < \lambda$ for all $x \in \Lambda$. For simplicity, sometimes we just call $\Lambda(f)$ has an index $i$ dominated splitting. A compact invariant set can have many dominated splittings, but for fixed $i$, the index $i$ dominated splitting is unique.

We say a diffeomorphism $f$ has $C^r$ tangency if $f \in C^r(M)$, $f$ has hyperbolic periodic point $p$ and there exists a non-transverse intersection between $W^s(p)$ and $W^u(p)$. $HT^r$ is the set of the diffeomorphisms which have $C^r$ tangency, usually we just use $HT$ denote $HT^1$. We call a diffeomorphism $f$ is far away from tangency if $f \in C^1(M) \setminus \overline{HT}$. The following proposition shows the relation between dominated splitting and far away from tangency.

**Proposition 2.1.** *([42]) $f$ is $C^1$ far away from tangency if and only if there exists $(l, \lambda)$ such that $P_i^*(f)$ has index $i - (l, \lambda)$ dominated splitting for $0 < i < d$.*

Usually dominated splitting is not a hyperbolic splitting, Mañé showed that in some special case, one bundle of the dominated splitting is hyperbolic.

**Proposition 2.2.** *([29]) Suppose $\Lambda(f)$ has an index $i$ dominated splitting $E \oplus F$ $(i \neq 0)$, if $\Lambda(f) \bigcap P_j^*(f) = \phi$ for $0 \le j < i$, then $E$ is a contracting bundle.*

## 3. Generic properties

For a topology space $X$, we call a set $R \subset X$ is a generic subset of $X$ if $R$ is countable intersection of open and dense subsets of $X$, and we call a property is a generic property of $X$ if there exists some generic subset $R$ of $X$ holds such property. Especially, when $X = C^1(M)$ and $R$ is a generic subset of $C^1(M)$, we just call $R$ is $C^1$ generic, and we call any generic property of $C^1(M)$ 'a $C^1$ generic property' or 'the property is $C^1$ generic'.

Here we'll state some well known $C^1$ generic properties.

**Proposition 3.1.** *There is a $C^1$ generic subset $R_0$ such that for any $f \in R_0$, one has*

1) *$f$ is Kupka-Smale (every periodic point $p$ in $Per(f)$ is hyperbolic and the invariant manifolds of periodic points are everywhere transverse).*
2) *$CR(f) = \Omega = \overline{Per(f)}$.*
3) *$P_i^*(f) = \overline{P_i(f)}$*
4) *any chain recurrent set is the Hausdorff limit of periodic orbits.*
5) *any index $i$ fundamental limit is the Hausdorff limit of index $i$ periodic orbits of $f$.*
6) *any chain recurrent class containing a periodic point $p$ is the homoclinic class $H(p, f)$.*
7) *Suppose $C$ is a homoclinic class of $f$, and $j_0 = min\{j : C \bigcap Per_j(f) \neq \phi\}$, $j_1 = max\{j : C \bigcap Per_j(f) \neq \phi\}$, then for any $j_0 \leq j \leq j_1$, we have $C \bigcap Per_j(f) \neq \phi$.*

By proposition 3.1, for any $f$ in $R_0$, every chain recurrent class $C$ of $f$ is either an aperiodic class or a homoclinic class. If $\#C = \infty$, we call $C$ is non-trivial.

Let $R = R_0 \setminus \overline{HT}$, we'll show that the generic subset $R$ of $\overline{HT}^c$ will satisfy theorem 1.

## 4. A special minimal set

Let $f \in R$, $C$ is a non-trivial chain recurrent class of $f$, and $j_0 = \min\{j : C \bigcap P_j^* \neq \phi\}$.

**Definition 4.1.** *: An invariant compact subset $\Lambda$ of $f$ is called minimal if all the invariant compact subset of $\Lambda$ are just $\Lambda$ and $\phi$. An invariant compact subset $\Lambda$ of $f$ is called minimal index $j$ fundamental limit if $\Lambda$ is an index $j$ fundamental limit and any invariant compact subset $\Lambda_0 \subsetneq \Lambda$ is not an index $j$ fundamental limit.*

**Lemma 4.2.** *If $C \bigcap P_j^* \neq \phi$, there always exists a minimal index $j$ fundamental limit in $C$.*

**Proof** Let $H = \{\tilde{\Lambda} : \tilde{\Lambda} \subset C$ is an index $j$ fundamental limit$\}$ and we order $H$ by inclusion. Suppose $x \in C \bigcap P_j^*$, then there exist $g_n \xrightarrow{C^1} f$, $p_n$ is index $j$ periodic point of $g_n$ and $p_n \longrightarrow x$. Denote $\Lambda_x = \lim Orb(P_n)$, then $\Lambda_x$ is an index $j$ fundamental limit. It's easy to know $\Lambda_x$ is a chain recurrent set and $\Lambda_x \subset C$, so $\Lambda_x \in H$. It means $H \neq \phi$.

Let $H_\Gamma = \{\Lambda_\lambda : \lambda \in \Gamma\}$ be a totally ordered chain of $H$. Then $\Lambda_\infty = \bigcap_{\lambda \in \Gamma} \Lambda_\lambda$ is a compact invariant set, in fact, there exists $\{\lambda_i\}_{i=1}^\infty$ such that $\Lambda_{\lambda_i} \supset \Lambda_{\lambda_{i+1}}$ and $\Lambda_\infty = \bigcap_{i=1}^\infty \Lambda_{\lambda_i}$.

We claim that $\Lambda_\infty$ is an index $j$ fundamental limit also.

**Proof of the claim** From generic property 5) of proposition 3.1 and $f \in R$, for any $\varepsilon > 0$, there exists periodic point $p_i$ such that $p_i \in Per_j(f)$ and $d_H(Orb(p_i), \Lambda_{\lambda_i}) < \frac{\varepsilon}{2}$. When $i$ is big enough, we'll have $d_H(\Lambda_{\lambda_i}, \Lambda_\infty) < \frac{\varepsilon}{2}$, so for any $\varepsilon > 0$, there exists $p_i \in Per_j(f)$such that $d_H(Orb(p_i), \Lambda_\infty) < \varepsilon$. □

Now by Zorn's lemma, there exists a minimal index $j$ fundamental limit in $C$. □

Suppose $\Lambda$ is a minimal index $j_0$ fundamental limit of $C$, the main aim of this section is the following lemma.

**Lemma 4.3.** *Suppose $f \in R$, $C$ is a non-trivial chain recurrent class of $f$, $j_0 = \min\{j : C \bigcap P_j^* \neq \phi\}$. Let $\Lambda$ be any minimal index $j_0$ fundamental limit in $C$, then*

a) *either $\Lambda$ is a non-trivial minimal set with partial hyperbolic splitting $T|_\Lambda M = E^s_{j_0} \oplus E^c_1 \oplus E^u_{j_0+2}$,*
b) *or $C$ contains a periodic point with index $j_0$ or $j_0 + 1$ and $C$ is an index $j_0$ fundamental limit.*

We postpone the proof of lemma 4.3 to §4.4, before that, I'll give or introduce some results at first. In §4.1 I'll give a proof of Shaobo Gan's lemma, in §4.2 I'll introduce Liao's selecting lemma and prove a weakly selecting lemma, in §4.3 I'll introduce a powerful tool 'transition' given by [BDP].

4.1. **Shaobo Gan's lemma.** Let $GL(d)$ be the group of linear isomorphisms of $R^d$, we call $\xi$ a periodic sequence of linear map if $\xi : Z \longrightarrow Gl(d)$ is a sequence of isomorphisms of $R^d$ and there exists $n_0 \geq 1$ such that $\xi_{j+n_0} = \xi_j$ for all $j$. We denote $\pi(\xi) = min\{n : \xi_{j+n} = \xi_j$ for all $j\}$ the period of $\xi$, and we call $\xi$ has index $i$ if the map $\prod_{j=0}^{\pi(\xi)-1} \xi_j$ is hyperbolic and has index $i$, we say $\xi$ is contracting if $\xi$ has index $d$. We denote $E^{s(u)}$ the stable (unstable) bundle of $\xi$.

Suppose $\eta$ is a periodic sequence of linear maps also, we call $\eta$ is an $\varepsilon$-perturbation of $\xi$ if $\pi(\eta) = \pi(\xi)$ and $\| \eta_j - \xi_j \| \leq \varepsilon$ for any $j$.

Let $\{\xi^\alpha\}_{\alpha \in \mathcal{A}}$ be a family of periodic sequence of linear maps with index $i$, we call it is bounded if there exists $K > 0$ such that for any $\alpha \in \mathcal{A}$ and any $j \in \mathbb{Z}$, we have $\| \xi_j^{(\alpha)} \| < K$. For a family of bounded periodic sequences of linear maps $\{\xi^\alpha\}_{\alpha \in \mathcal{A}}$, we say it's index stable if $\xi^{(\alpha)}$ has index $i$ for all $\alpha \in \mathcal{A}$, and there exists $\varepsilon_0 > 0$ such that $\#\{\alpha |$ there exists $\eta^{(\alpha)}$ is $\varepsilon_0$-perturbation of $\xi^{(\alpha)}$ and $\eta^{(\alpha)}$ has index different with $i\} < \infty$. Especially, if it's index 0 stable, we call $\xi^{(\alpha)}|_{\alpha \in \mathcal{A}}$ is uniformly contracting.

Suppose $f \in C^1(M)$ and $\{p_n(f)\}$ is a family of hyperbolic periodic points of $f$ with index $i$, we say $p_n(f)$ is index $i$ stable if $\{Df|_{Orb(p_n)}\}_{n=1}^\infty$ is index $i$ stable and $\lim_{n\to\infty} \pi(p_n) = \infty$.

**Remark 4.4.** *Pliss has proved that if $\{p_n(f)\}$ is index $i$ stable, then $i \neq 0, d$.*

The following lemma was given by Shaobo Gan, and the proof comes from him also.

**Lemma 4.5.** *([15]) $f \in C^1(M)$, suppose $\{p_n(f)\}$ is index $i$ stable, then there exists a subsequence $\{p_{n_j}\}_{j=1}^\infty$ such that $p_{n_j}$ and $p_{n_{j+1}}$ are homoclinic related.*

Here we just prove the following weaker statement of Gan's lemma.

**Lemma 4.6.** *( Weaker statement of Gan's lemma) Suppose $f \in R$, $\Lambda$ is a non-trivial chain recurrent set of $f$, $\{p_n(f)\}$ is index $i$ stable and $\lim\limits_{n\to\infty} Orb(p_n) = \Lambda$, then there exists a subsequence $\{p_{n_j}(f)\}_{j=1}^{\infty}$ such that $p_{n_j}(f)$ and $p_{n_{j+1}}(f)$ are homoclinic related.*

Before we prove lemma 4.6, we'll give a few lemmas which will be used in the proof.

**Lemma 4.7.** *Suppose $A=\begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$ is a hyperbolic linear map with index $i$ $(i \neq 0, d)$, where $B \in GL(R^i)$ is a contracting map and $D \in GL(R^{d-i})$ is a expanding map. If there exists $B' \in GL(R^i)$ an $\varepsilon$-perturbation of $B$ and $B'$ has index different with $i$, then $A'=\begin{pmatrix} B' & C \\ 0 & D \end{pmatrix}$ is an $\varepsilon$-perturbation of $A$ with index different with $i$. In fact, we'll have $ind(A') = ind(B')$.*

With lemma 4.7, the following lemma is obviously.

**Lemma 4.8.** *Suppose $\{\xi^{(n)}\}_{n=1}^{\infty}$ is index $i$ stable, then $\{\xi^{(n)}|_{E^s(\xi^{(n)})}\}_{n=1}^{\infty}$ is stable contracting, and at the same time, $\{\xi^{(n)}|_{E^u(\xi^{(n)})}\}_{n=1}^{\infty}$ is stable expanding.*

In [29] Mañé has given a necessary condition for bounded stable contracting sequence.

**Lemma 4.9.** *(Mañé) If $\{\xi^{(n)}\}_{n=1}^{\infty}$ is stable contracting and bounded, then there exist $N_0, l_0, 0 < \lambda_0 < 1$ such that if $\pi(\xi^{(n)}) > N_0$ we'll have*

$$\prod_{j=0}^{[\frac{\pi(\xi_n)}{l_0}]-1} \| \prod_{t=0}^{l_0-1} \xi^{(n)}_{(jl_0+t)+s} \| \le \lambda_0^{[\frac{\pi(\xi^{(n)})}{l_0}]}$$

*for any $0 \le s < \pi(\xi^{(n)})$.*

**Proof of lemma 4.6**: Since $\Lambda \subset P_i^*$ and $f$ is far away from tangency, by proposition 2.1, $\Lambda$ has an index $i-(l,\lambda)$ dominated splitting $T|_\Lambda M = E \oplus F$. In order to make the proof more simiplier, here we just suppose $l = 1$. Choose a small open neighborhood $U$ of $\Lambda$, when $U$ is small enough, $\widetilde{\Lambda} = \bigcap\limits_{j\in\mathbb{Z}} f^j(\overline{U})$ has an index $i-(1,\widetilde{\lambda})$ dominated splitting $T_{\widetilde{\Lambda}}M = \widetilde{E} \oplus \widetilde{F}$ where $\lambda < \widetilde{\lambda} < 1$ and $\widetilde{E}|_\Lambda = E$, $\widetilde{F}|_\Lambda = F$.

Since $\lim\limits_{n\to\infty} Orb(P_n) = \Lambda$, we can always suppose $Orb(p_n) \subset \overline{U}$, so $Orb(P_n) \subset \widetilde{\Lambda}$ and $E^s|_{Orb(p_n)} = \widetilde{E}|_{Orb(p_n)}$, $F^u|_{Orb(p_n)} = \widetilde{F}|_{Orb(p_n)}$.

By lemma 4.8, we know that $\{Df|_{E^s(Orb(p_n))}\}_{n=1}^{\infty}$ is stable contracting and $\{Df|_{E^u(Orb(p_n))}\}_{n=1}^{\infty}$ is stable expanding. By lemma 4.9, there exist$N_0, l_0, 0 < \lambda_0 < 1$ such that if $\pi(p_n(f)) > N_0$, we have

$$\prod_{j=0}^{[\frac{\pi(p_n)}{l_0}]-1} \|Df^{l_0}|_{E^s(f^{jl_0}p_n)}\| \le \lambda_0^{[\frac{\pi(p_n)}{l_0}]} \tag{4.1}$$

$$\prod_{j=0}^{[\frac{\pi(p_n)}{l_0}]-1} \|Df^{-l_0}|_{F^u(f^{-jl_0}p_n)}\| \le \lambda_0^{[\frac{\pi(p_n)}{l_0}]} \tag{4.2}$$

Since $\lim\limits_{n\to\infty} Orb(p_n) = \Lambda$ and $\Lambda$ is not trivial, we have $\lim\limits_{n\to\infty} \pi(p_n) \longrightarrow \infty$, then we can always suppose all the $p_n$ satisfy (4.1) and (4.2). For simplicity, we suppose $l_0 = 1$ here.

Choose some $\varepsilon > 0$ and $\lambda_1 < 1$ such that $\max\{\widetilde{\lambda},\ \lambda_0\} + \varepsilon < \lambda_1^2 < \lambda_1 < 1$. Now we'll state Pliss lemma in a special context.

**Lemma 4.10.** *(Pliss[34]) Given $0 < \lambda_0 + \varepsilon < \lambda_1 < 1$ and $Orb(P_n) \subset \widetilde{\Lambda}$ such that for some $m \in \mathbb{N}$, we have $\prod_{j=0}^{t-1} \|Df|_{E^s(f^j p_n))}\| \le (\lambda_0 + \varepsilon)^t$ for all $s \ge m$, there exists a sequence $0 \le n_1 < n_2 < \cdots$ such that $\prod_{j=n_r}^{t-1} \|Df|_{E^s(f^j p_n))}\| \le \lambda_1^{t-n_r}$ for all $t \ge n_r$, $r = 1, 2, \cdots$.*

**Remark 4.11.** *The sequence $\{n_j\}_{j=1}^{\infty}$ we get above is called the $\lambda_1$-hyperbolic time for bundle $E^s|_{Orb(p_n)}$. By* (4.1),(4.2)*, when $n$ is big enough, $P_n$ will satisfy the assumption of Pliss lemma, so by lemma 4.10, there exists $q_n^+ \in Orb(p_n)$ such that $\prod_{j=0}^{t-1} \|Df|_{E^s(f^j q_n^+)}\| \le \lambda_1^t$ and $q_n^- \in Orb(p_n)$ such that $\prod_{j=0}^{t-1} \|Df^{-1}|_{F^u(f^{-j} q_n^-)}\| \le \lambda_1^t$ for all $t > 0$.*

Let's denote

$$S_{n,+} = \{m \in \mathbb{Z} : \prod_{j=0}^{s-1} \|Df|_{E^s(f^{m+j} p_n)}\| \le \lambda_1^s \text{ for all } s > 0\},$$

$$S_{n,-} = \{m \in \mathbb{Z} : \prod_{j=0}^{s-1} \|Df^{-1}|_{F^u(f^{m-j} p_n)}\| \le \lambda_1^s \text{ for all } s > 0\}.$$

Then $S_{n,+}$ is the set of $\lambda_1$ hyperbolic time for bundle $E^s|_{Orb(p_n)}$ and $S_{n,-}$ is the set of hyperbolic time for bundle $F^u|_{Orb(p_n)}$. From remark 4.11, the set $S_{n,+}$ and $S_{n,-}$ are not empty. We denote $S_n = S_{n,+} \bigcap S_{n,-}$.

**Lemma 4.12.** *$S_n \neq \phi$.*

**Proof**: Here for $a, b \in \mathbb{Z}$ and $a < b$, we denote $(a, b)_{\mathbb{Z}} = \{c |\ c \in \mathbb{Z} \text{ and } a < c < b\}$.

Now suppose the lemma is false, we can choose $\{b_{n,s},\ b_{n,s+1}\} \subset S_{n,-}$ such that we have $b_{n,s+1} > b_{n,s}$, $(b_{n,s}, b_{n,s+1})_{\mathbb{Z}} \bigcap S_{n,-} \neq \phi$ and $a_{n,t} \in (b_{n,s}, b_{n,s+1})_{\mathbb{Z}} \bigcap S_{n,+}$, then $b_{n,s}, b_{n,s+1} \notin S_{n,+}$.

We claim that for $0 < k \le b_{n,s+1} - b_{n,s} - 1$, we have $\prod_{j=0}^{k-1} \|Df^{-1}|_{F^u(f^{b_{n,s}+j+1} p_n)}\| \ge \lambda_1^k$.

**Proof of the claim**: We'll use induction to give a proof.

When $k = 1$, since $b_{n,s} + 1 \notin S_{n,-}$, we have $\|Df^{-1}|_{F^u(f^{b_{n,s}+1} p_n)}\| > \lambda_1$.

Now suppose the claim is true for all $1 \le k \le k_0 - 1$ where $1 < k_0 \le b_{n,s+1} - b_{n,s} - 1$, and we suppose the claim is false for $k_0$, it means that

$$\prod_{j=0}^{k_0-1} \|Df^{-1}|_{F^u(f^{b_{n,s}+j+1} p_n)}\| \le \lambda_1^{k_0}. \tag{4.3}$$

Then by the assumption above that the claim is true for $1 \le k \le k_0 - 1$, we have

$$\prod_{j=0}^{k-1} \|Df^{-1}|_{F^u(f^{b_{n,s}+j+1} p_n)}\| \ge \lambda_1^k \tag{4.4}$$

By (4.3) and (4.4), we get that $\prod_{j=k}^{k_0-1}\|Df^{-1}|_{F^u(f^{b_{n,s}+j+1}p_n)}\|<\lambda_1^{k_0-k}$ for $1\le k\le k_0-1$. It's equivalent to say that

$$\prod_{j=0}^{k-1}\|Df^{-1}|_{F^u(f^{b_{n,s}+k_0-j}p_n)}\|<\lambda_1^k \quad \text{for } 1\le k\le k_0-1 \tag{4.5}$$

By (4.3) and (4.5), we get that

$$\prod_{j=0}^{k-1}\|Df^{-1}|_{F^u(f^{b_{n,s}+k_0-j}p_n)}\|\le\lambda_1^k \quad \text{for } 1\le k\le k_0 \tag{4.6}$$

When $k>k_0$,by (4.6) and the fact $b_{n,s}\in S_{n,-}$, we have

$$\prod_{j=0}^{k-1}\|Df^{-1}|_{F^u(f^{b_{n,s}+k_0-j})}\|=\prod_{j=0}^{k_0-1}\|Df^{-1}|_{F^u(f^{b_{n,s}+k_0-j})}\|\cdot\prod_{j=0}^{k-k_0-1}\|Df^{-1}|_{F^u(f^{b_{n,s}-j})}\|<\lambda_1^{k_0}\cdot\lambda_1^{k-k_0}=\lambda_1^k,$$

it means $b_{n,s}+k_0\in S_{n,-}$, it's a contradiction since $b_{n,s}+k_0\in(b_{n,s},b_{n,s+1})_{\mathbb{Z}}$, so we finished the induction. □

By the claim above, for $0<k\le b_{n,s+1}-b_{n,s}-1$, we have

$$\prod_{j=0}^{k-1}\|Df^{-1}|_{F^u(f^{b_{n,s}+j+1}p_n)}\|>\lambda_1^k. \tag{4.7}$$

Since on $\widetilde{\Lambda}$, $\widetilde{E}\oplus\widetilde{F}$ is an index $i-(1,\widetilde{\lambda})$ dominated splitting, we have

$$\prod_{j=0}^{k-1}(\|Df|_{\widetilde{E}(f^{b_{n,s}+j}p_n)}\|\cdot\|Df^{-1}|_{\widetilde{F}(f^{b_{n,s}+j+1}p_n)}\|)<\widetilde{\lambda}^k.$$

By (4.7) and $\widetilde{E}|_{Orb(p_n)}=E^s|_{Orb(p_n)}$,$\widetilde{F}|_{Orb(p_n)}=F^u|_{Orb(p_n)}$, we'll get

$$\prod_{j=0}^{k-1}\|Df|_{E^s(f^{b_{n,s}+j}p_n)}\|<\frac{\widetilde{\lambda}^k}{\lambda_1^k}\underset{(\widetilde{\lambda}<\lambda_1^2<1)}{<}\lambda_1^k \quad \text{for } 1<k\le b_{n,s+1}-b_{n,s}-1. \tag{4.8}$$

When $k>b_{n,s+1}-b_{n,s}-1$, let $k=(a_{n,t}-b_{n,s})+(k-a_{n,t})$, by (4.8) and $a_{n,t}\in S_{n,+}$,

$$\begin{aligned}\prod_{j=0}^{k-1}\|Df|_{E^s(f^{b_{n,s}+j}p_n)}\| &= \prod_{j=0}^{a_{n,t}-b_{n,s}-1}\|Df|_{E^s(f^{b_{n,s}+j}p_n)}\|\cdot\prod_{j=0}^{k-a_{n,t}-1}\|Df|_{E^s(f^{a_{n,t}+j}p_n)}\|\\ &< \lambda_1^{a_{n,t}-b_{n,s}}\cdot\lambda_1^{k-a_{n,t}}=\lambda_1^{k-b_{n,s}}\end{aligned} \tag{4.9}$$

By (4.8) and (4.9), we get $b_{n,s}\in S_{n,+}$, so $S_{n,+}\bigcap S_{n,-}\ne\phi$, it's a contradiction with our assumption, so we finished the proof of lemma 4.12. □

Now let's continue the proof of lemma 4.6, we need the following two lemmas to show that for $a_n\in S_n$, the point $f^{a_n}(p_n)$ will have uniform size of stable manifold and unstable manifold.

Let $I_1=(-1,1)^i$ and $I_\varepsilon=(-\varepsilon,\ \varepsilon)^i$, denote by $Emb^1(I,M)$ the set of $C^1$-embedding of $I_1$ on $M$, recall by [21] that $\widetilde{\Lambda}$ has a dominated splitting $\widetilde{E}\oplus\widetilde{F}$ implies the following.

**Lemma 4.13.** *There exist two continuous function $\Phi^{cs}:\ \widetilde{\Lambda}\longrightarrow Emb^1(I,M)$ and $\Phi^{cu}:\ \widetilde{\Lambda}\longrightarrow Emb^1(I,M)$ such that, with $W^{cs}_\varepsilon(x)=\Phi^{cs}(x)I_\varepsilon$ and $W^{cu}_\varepsilon(x)=\Phi^{cu}(x)I_\varepsilon$, the following properties hold:*

a) $T_xW^{cs}_\varepsilon=\widetilde{E}(x)$ *and* $T_xW^{cu}_\varepsilon=\widetilde{F}(x)$,

b) *For all* $0<\varepsilon_1<1$, *there exists* $\varepsilon_2$ *such that* $f(W^{cs}_{\varepsilon_2}(x))\subset W^{cs}_{\varepsilon_1}(f(x))$ *and* $f^{-1}(W^{cu}_{\varepsilon_2}(x))\subset W^{cu}_{\varepsilon_1}(f^{-1}(x))$.

c) *For all* $0<\varepsilon<1$, *there exists* $\delta>0$ *such that if* $y_1,\ y_2\in\widetilde{\Lambda}$ *and* $d(y_1,\ y_2)<\delta$, *then* $W^{cs}_\varepsilon(y_1)\pitchfork W^{cu}_\varepsilon(y_2)\neq\phi$.

**Corollary 4.14.** *([36]) For any* $0<\lambda<1$, *there exists* $\varepsilon>0$ *such that for* $x\in\widetilde{\Lambda}$ *which satisfies* $\prod_{j=0}^{n-1}\|Df|_{\widetilde{E}(f^jx)}\|\le\lambda^n$ *for all* $n>0$, *then* $diam(f^n(W^{cs}_\varepsilon))\longrightarrow 0$, *i.e. the central stable manifold of* $x$ *with size* $\varepsilon$ *is in fact a stable manifold.*

Now for $\lambda_1$, using corollary 4.14, we can get an $\varepsilon>0$. It means that for any $a_n\in S_n$, denote $q_n=f^{a_n}(p_n)$, then $W^{cs}_\varepsilon(q_n)$ is a stable manifold and $W^{cu}_\varepsilon(q_n)$ is an unstable manifold. For this $\varepsilon>0$, use c) of lemma 4.13, we can fix a $\delta$. Choose a subsequence $\{n_i\}$ such that $d(q_{n_i},q_{n_{i+1}})\le\delta$, then by c) of lemma 4.13, we know $W^{cu}_\varepsilon(q_{n_i})\pitchfork W^{cs}_\varepsilon(q_{n_{i+1}})\neq\phi$ and $W^{cu}_\varepsilon(q_{n_{i+1}})\pitchfork W^{cs}_\varepsilon(q_{n_i})\neq\phi$. Since the local central stable manifold and local central unstable manifold of $q_{n_i}$ have dynamical meaning, we know that $Orb(q_{n_i})$ and $Orb(q_{n_{i+1}})$ are homoclinic related. □

**Remark 4.15.** *In the proof of lemma 4.6 we suppose the set* $\Lambda$ *has 1-step dominated splitting, that means* $l=1$, *and we suppose* $l_0=1$ *there also, they are just in order to make the proof more simplier. In the rest part of the paper, usually we don't use such assumption any more, if we use it we'll point out.*

Now let's consider a sequence of periodic points which are not index stable.

**Lemma 4.16.** *Suppose* $f\in R$, $\lim_{n\to\infty}g_n=f$, $\{p_n(g_n)\}_{n=1}^\infty$ *is a family of index* $i$ *periodic points* ($i\neq\ 0,\ d$) *and* $\lim_{n\to\infty}\pi(p_n)\longrightarrow\infty$. *If there exist* $\lambda_n\longrightarrow 1^-$ *and* $\lim_{n\to\infty}l_n\longrightarrow\infty$ *such that* $\lim_{n\to\infty}\frac{\pi(p_n)}{l_n}\longrightarrow\infty$ *and* $\prod_{j=0}^{[\frac{\pi(p_n)}{l_n}]-1}\|Dg_n^{l_n}|_{E^s(g_n^{jl_n}(p_n))}\|\ge\lambda_n^{[\frac{\pi(p_n)}{l_n}]}$, *then for any* $\varepsilon>0$ *and* $N>0$, *there exists an* $n_0>N$ *and* $g'_{n_0}$ *is an* $\varepsilon$*-perturbation of* $g_{n_0}$ *such that* $p_{n_0}(g_{n_0})$ *is an index* $i-1$ *periodic point of* $g'_n$.

**Proof**: Fix $N$, consider the periodic sequence of linear maps $\{\xi^n:\ \xi^n=Dg_n|_{E^s(Orb(p_n))}\}_{n\ge N}$, they are all contracting maps. We claim that $\{\xi^n\}$ are not stable contracting.

**Proof of the claim**: If $\{\xi^n\}$ is stable contracting, by lemma 4.9, there exist $N_0,\ l_0,\ 0<\lambda_0<1$ such that if $\pi(\xi^n)>N_0$, we have

$$\prod_{j=0}^{[\frac{\pi(p_n)}{l_0}]-1}\|Dg_n^{l_0}|_{E^s(g_n^{jl_0}p_n)}\|\le\lambda_0^{[\frac{\pi(p_n)}{l_0}]}\tag{4.10}$$

Choose some $N_1$ big enough such that for $n\ge N_1$, we have $\lambda_n\ge\ \lambda^*>\ \lambda_0$ for some $\lambda^*\in\ (\lambda_0,1)$, then by $\lim_{n\to\infty}\frac{\pi(p_n)}{l_n}\longrightarrow\infty$ and $\lim_{n\to\infty}l_n\longrightarrow\infty$, when $n$ is big enough, we have $\pi(p_n)\gg\ l_n\gg\ \max\{l_0,N_0\}$ and from $\prod_{j=0}^{[\frac{\pi(p_n)}{l_0}]-1}\|Dg_n^{l_n}|_{E^s(g_n^{jl_n}p_n)}\|\ge\lambda_n^{[\frac{\pi(p_n)}{l_n}]}>(\lambda^*)^{[\frac{\pi(p_n)}{l_n}]}$, we'll get $\prod_{j=0}^{[\frac{\pi(p_n)}{l_0}]-1}\|Dg_n^{l_0}|_{E^s(g_n^{jl_0}p_n)}\|\ge\lambda_0^{[\frac{\pi(p_n)}{l_n}]}>\lambda_0^{[\frac{\pi(p_n)}{l_0}]}$, It's a contradiction with (4.10). □

Since $\{\xi^n\}_{n\geq N}$ isn't stable contracting, for $\varepsilon>0$, there exists a sequence $\{n_i\}$ and $\{\eta^{n_i}\}$ such that $\eta^{n_i}$ is an $\varepsilon$-perturbation of $\xi^{n_i}$ and $\eta^{n_i}$ has index smaller than $i$. Since $\{\xi^{n_i}\}$ is bounded and $\lim_{n\to\infty}\pi(p_n)\longrightarrow\infty$, by [10]'s work, for $n_i$ big enough, we can in fact get $\eta^{n_i}$ with index $i-1$. By lemma 4.7, there exists $\{A|_{Orb(p_n)}\}_{n\geq 0}$ an $\varepsilon$-perturbation of $\{Dg_n|_{Orb(p_n)}\}$ such that $\{A|_{Orb(p_n)}\}$ has index $i-1$. Now we need the following version of Franks lemma.

**Lemma 4.17.** *(Franks lemma) $p_n$ is a periodic point of $g_n$, $A|_{Orb(p_n)}$ is an $\varepsilon$-perturbation of $\{Dg_n|_{Orb(p_n)}\}$, then for any neighborhood $U$ of $Orb(p_n)$, there exists $g_n'$ such that $g_n'\equiv g_n$ on $(M\setminus U)\bigcup Orb(p_n)$, $d_{C^1}(g_n,g_n')<\varepsilon$ and $\{Dg_n'|_{orb(p_n)}\}=\{A|_{Orb(p_n)}\}$.*

From Franks lemma, we can change the derivative map along $T_{Orb(p_{n_i})}M$ to be $\{A|_{Orb(p_n)}\}$ and get a new map $g_{n_i}'$ such that $p_{n_i}(g_{n_i})$ is index $i-1$ periodic point of $g_{n_i}'$. □

4.2. **Weakly selecting lemma.** Liao's selecting lemma is a powerful shadowing lemma for non-uniformly hyperbolic system, with it, we can not only get a lot of periodic points like what the standard shadowing lemma can do, we can even let the periodic points have hyperbolic property as weak as we like. Liao at first used this lemma to study minimal non-hyperbolic set and proved the $\Omega$-stable conjecture for diffeomorphisms in dimension 2 and for flow without singularity in dimension 3. [16] [17] [19] [41] use the same idea proved structure($\Omega$) stability conjecture for flows without singularity in any dimension. Until now, the most important papers about selecting lemma are [18],[44], [45] and there contain more details about selecting lemma.

In this subsection and the next, we'll show what will happen if all the conditions in weakly selecting lemma are satisfied. The main result in this subsection is lemma 4.21 (The weakly selecting lemma). Now let's state the selecting lemma at first.

**Proposition 4.18.** *(Liao) Let $\Lambda$ be a compact invariant set of $f$ with index $i-(l,\lambda)$ dominated splitting $E^{cs}\oplus F^{cu}$. Assume that*

- a) *there is a point $b\in\Lambda$ satisfying $\prod_{j=0}^{n-1}\|Df^l|_{E^{cs}(f^{jl}b)}\|\geq 1$ for all $n\geq 1$.*
- b) *(The tilda condition) there are $\lambda_1$ and $\lambda_2$ with $\lambda<\lambda_1<\lambda_2<1$ such that for any $x\in\Lambda$ satisfying $\prod_{j=0}^{n-1}\|Df^l|_{E^{cs}(f^{jl}x)}\|\geq{\lambda_2}^n$ for all $n\geq 1$, $\omega(x)$ contains a point $c\in\Lambda$ satisfying $\prod_{j=0}^{n-1}\|Df^l|_{E^{cs}(f^{jl}c)}\|\leq\lambda_1^n$ for all $n\geq 1$.*

*Then for any $\lambda_3$ and $\lambda_4$ with $\lambda_2<\lambda_3<\lambda_4<1$ and any neighborhood $U$ of $\Lambda$, there exists a hyperbolic periodic orbit $Orb(q)$ of $f$ of index $i$ contained entirely in $U$ with a point $q\in Orb(q)$ such that*

$$\prod_{j=0}^{m-1}\|Df^l|_{E^{cs}(f^{jl}q)}\|\leq\lambda_4^m,\quad for\ m=1,\cdots,\ \pi_l(q) \tag{4.11}$$

$$\prod_{j=\pi_l(q)-m}^{\pi_l(q)-1}\|Df^l|_{E^{cs}(f^{jl}q)}\|\geq\lambda_3^{\pi_l(q)-m+1}\quad for\ m=1,\cdots,\ \pi_l(q) \tag{4.12}$$

*where $\pi_l(q)$ is the period of $q$ for the map $f^l$. The similar assertion for $F^{cu}$ holds respecting $f^{-1}$.*

**Remark 4.19.** *It's easy to know* $\pi(q) \geq \pi_l(q)$. *Since* $f^{l\cdot\pi_l(q)}(q) = q$, *it's obviously that* (4.11) *and* (4.12) *are true for all* $m \in \mathbb{N}$. *In the selecting lemma, when* $\lambda_3$ *and* $\lambda_4$ *are fixed, we can indeed find a sequence of periodic points* $\{q_n\}$ *satisfying* (4.11) *and* (4.12) *and* $\overline{\lim_{n\to\infty}} Orb(q_n) \subset \Lambda$. *If* $f$ *is a Kupuka-Smale diffeomorphism, especially when* $f \in R$, *we can let* $\lim_{n\to\infty} \pi_l(q_n) \longrightarrow \infty$, *then we'll have* $\lim_{n\to\infty} \pi(q_n) \longrightarrow \infty$ *at the same time.*

**Corollary 4.20.** $f \in R$, *let* $\Lambda$ *be a compact chain recurrent set of* $f$ *with index* $i - (l_0, \lambda)$ *dominated splitting* $E^{cs} \oplus F^{cu}$ $(1 \leq\ i \leq\ d-1)$. *Assume that the splitting satisfies all the condition of selecting lemma for all* $l_n = nl_0$ $(n \in \mathbb{N})$ *but with the same parameters* $\lambda\ < \lambda_1\ < \lambda_2\ < 1$, *then for any sequence* $\{(\lambda_{n,3},\ \lambda_{n,4})\}_{n=1}^{\infty}$ *satisfying* $\lambda_2 < \lambda_{1,3} < \lambda_{1,4} < \lambda_{2,3} < \lambda_{2,3} < \cdots$ *where* $\lambda_{n,3} \longrightarrow 1^-$, *there exists a family of periodic points* $\{q_n(f)\}$ *with index* $i$ *such that*

a) $\lim_{n\to\infty} \pi_{l_n}(q_n(f)) \longrightarrow \infty$.

b)

$$\prod_{j=0}^{m-1} \|Df^{l_n}|_{E^s(f^{jl_n}q_n)}\| \leq \lambda_{n,4}^m \tag{4.13a}$$

$$\prod_{j=\pi_{l_n}(q_n)-m}^{\pi_{l_n}(q_n)-1} \|Df^{l_n}|_{E^s(f^{jl_n}q_n)}\| \geq \lambda_{n,3}^m \quad \textit{for } m \in \mathbb{N} \tag{4.13b}$$

c) $\overline{\lim_{n\to\infty}} Orb(q_n) \subset \Lambda$.

d) $\Lambda \subset H(q_n(f))$ *for all* $n$.

**Proof** : At first, let's fix $\lambda_2 < \lambda_{1,3} < \lambda_{1,4} < 1$ and a small neighborhood $U$ of $\Lambda$ small enough such that the maximal invariant set $\widetilde{\Lambda}$ of $\overline{U}$ has index $i - (l_0, \widetilde{\lambda})$ dominated splitting with $\widetilde{\lambda} < \lambda_2$, we denote the dominated splitting still by $E_i^{cs} \oplus F_{i+1}^{cu}$. (If $q$ is an index $i$ periodic point in $\widetilde{\Lambda}$, then we denote $E_i^{cs} \oplus F_{i+1}^{cu}|_{Orb(q)} = E^s \oplus F^u|_{Orb(q)}$). Now using selecting lemma, with remark 4.19, we can find a family of periodic points $\{q_{1,m}(f)\}_{m=1}^{\infty}$ with index $i$ satisfying b), $\overline{\lim_{n\to\infty}}(q_{1,m}) \subset \Lambda$, $\lim_{m\to\infty} \pi_{l_1}(q_{1,n}) \longrightarrow \infty$ and $Orb(q_{1,m}(f)) \subset \widetilde{\Lambda}$.

Since $\widetilde{\Lambda}$ has an index $i - (l_1, \widetilde{\lambda})$ dominated splitting $E_{\widetilde{\Lambda}}^{cs} \oplus F_{\widetilde{\Lambda}}^{cu}$, from (4.13b) we can know

$$\prod_{j=\pi_{l_1}(q_{1,m})-t+1}^{\pi_{l_1}(q_{1,m})} \|Df^{-l_1}|_{F^{cu}(f^{jl_1}q_{1,m})}\| \leq \widetilde{\lambda}^t / \prod_{j=\pi_{l_1}(q_{1,m})-t}^{\pi_{l_1}(q_{1,m})-1} \|Df^{l_1}|_{E^{cs}(f^{jl_1}q_{1,m})}\| \leq (\frac{\widetilde{\lambda}}{\lambda_{1,3}})^t \text{ for } {\scriptstyle (t\in\mathbb{N})},$$

it equivalent with

$$\prod_{j=0}^{m-1} \|Df^{-l_1}|_{F^{cu}(f^{-jl_1}q_{1,m})}\| \leq (\frac{\widetilde{\lambda}}{\lambda_{1,3}})^m \quad \text{for } m \in \mathbb{N}. \tag{4.14}$$

From (4.13a), (4.13b), by lemma 4.13, Corollary 4.14 and $\frac{\widetilde{\lambda}}{\lambda_{1,3}} < 1$, we can know that for some $\varepsilon_1$, $q_{1,n}$ will have uniformly size of stable manifold $W_{\varepsilon_1}^s(q_{1,n})$ and uniform size of unstable manifold $W_{\varepsilon_1}^u(q_{1,n})$ and there exists a subsequence $\{q_{1,n_j}\}_{j=1}^{\infty}$ such that they are homoclinic related with each other, so $H(q_{1,n_1}) = H(q_{1,n_2}) = \cdots$, with $\overline{\lim_{j\to\infty}} Orb(q_{1,n_j}) \subset \Lambda$, we know $\Lambda \bigcap H(q_{1,n_j}) \neq \phi$. Since $f \in R$, $H(q_{1,n_j})$ should be a chain recurrent class. Because $\Lambda$ is a chain recurrent set, we have $\Lambda \subset H(q_{1,n_j})$, let $q_1 = q_{1,n_j}$ for some $j$ big enough, then $q_1$ satisfies $a)$, $d)$.

Now consider $0<\lambda_2<\lambda_{2,3}<\lambda_{2,4}<1$, $E^{cs}_\Lambda\oplus F^{cu}_\Lambda$ is obviously an index $i-(l_2,\lambda)$ dominated splitting of $\Lambda$ and by the assumption, the splitting satisfy the conditions of selecting lemma for $l_2,\ \lambda<\lambda_1<\lambda_2<1$, so repeat the above argument, we can get a family of periodic points $\{q_{2,n}(f)\}_{n=1}^{\infty}$ satisfying $b)$, $d)$, $\overline{\lim_{n\to\infty}} Orb(q_{2,n})\subset\Lambda$, $\Lambda\subset H(q_{2,1},f)=\cdots=H(q_{2,n},f)=\cdots$ and $\lim_{n\to\infty}\pi_{l_2}(q_{2,n}(f))\longrightarrow\infty$. When $n_0$ is big enough, we'll have $\pi_{l_2}(q_{2,n_0})>\pi_{l_1}(q_1)$ and $Orb(q_{2,n_0})$ is near $\Lambda$ more than $Orb(q_1)$. Let $q_2=q_{2,n_0}$, continue the above argument for $l_n$ and $\lambda_2<\lambda_{n,3}<\lambda_{n,4}<1$, we can get $\{q_n\}_{n=1}^{\infty}$ which we need. $\square$

The following weakly selecting lemma shows when the conditions of the above corollary will be satisfied.

**Lemma 4.21.** *(Weakly selecting lemma) Let $f\in R$, $\Lambda$ be a compact invariant set of $f$ with index $i-(l_0,\ \lambda)$ dominated splitting $E^{cs}\oplus F^{cu}$ $(1\le i\le d-1)$. Assume that*

a) *(Non-hyperbolic condition) the bundle $E^{cs}$ is not contracting,*
b) *(Strong tilda condition) there are $\lambda_2<1$ and $l_0'>1$ such that for any $x\in\Lambda$, $\omega(x)$ contains a point $c\in\Lambda$ satisfying $\prod_{j=0}^{n-1}\|Df^{l_0'}|_{E^{cs}(f^{jl_0'}c)}\|\le\lambda_2^n$ for all $n\ge1$.*

*Then for any $l_n=n\cdot(l_0\cdot l_0')$ and any sequence $\{(\lambda_{n,3},\lambda_{n,4})\}_{n=1}^{\infty}$ satisfying $max\{\lambda^{l_0'},\lambda_2\}<\lambda_{1,3}<\lambda_{1,4}<\cdots<\lambda_{n,3}<\lambda_{n,4}<\cdots$ where $\lambda_{n,3}\longrightarrow1^-$, there exists a family of periodic points $\{q_n(f)\}$ with index $i$ such that*

- $\lim_{n\to\infty}\pi_{l_n}(q_n(f))\longrightarrow\infty$
- $\prod_{j=0}^{m-1}\|Df^{l_n}|_{E^s(f^{jl_n}q_n)}\|\le\lambda_{n,4}^m$ *and* $\prod_{j=\pi_{l_n}(q_n)-m}^{\pi_{l_n}(q_n)-1}\|Df^{l_n}|_{E^s(f^{jl_n}q_n)}\|\ge\lambda_{n,3}^m$ *for* $m\ge1$
- $\overline{\lim_{n\to\infty}} Orb(q_n)\subset\Lambda$
- $\Lambda\subset H(q_n(f))$ *for* $n\ge1$.

**Proof** Since $E^{cs}_\Lambda\oplus F^{cu}_\Lambda$ is a $(l_0,\lambda)$ dominated splitting and $l_1=l_0\cdot l_0'$, it should be a $(l_1,\lambda^{l_0'})$ dominated splitting also. Choose $\lambda_2',\ \lambda_1$ such that $\max\{\lambda^{l_0'},\ \lambda_2\}<\lambda_1<\lambda_2'<\lambda_{1,3}$, we'll show that the splitting $E^{cs}_\Lambda\oplus F^{cu}_\Lambda$ and the $l_1,\ \lambda^{l_0'}<\lambda_1<\lambda_2'<1$ will satisfy all conditions of corollary 4.20, equivalent, we'll show the splitting $E^{cs}_\Lambda\oplus F^{cu}_\Lambda$, $l_n$ and $\lambda^{l_0'}<\lambda_1<\lambda_2'<1$ will satisfy the condition of selecting lemma for all $n\ge1$.

0) Since $E^{cs}_\Lambda\oplus F^{cu}_\Lambda$ is a $(l_1,\ \lambda^{l_0'})$ dominated splitting and $l_n=n\cdot l_1$, $E^{cs}_\Lambda\oplus F^{cu}_\Lambda$ is a $(l_n,\ \lambda^{l_0'})$ dominated splitting also.
1) Here we need the following lemma:

   **Lemma 4.22.** *Let $\Lambda$ be a compact invariant set of $f$, $E^{cs}_\Lambda$ is an continuous invariant bundle on $\Lambda$, and $dim(E^{cs}(x))=i$ for any $x\in\Lambda$ where $i\ne0$, suppose $l\in\mathbb{N}$, if for any $x\in\Lambda$, there exists an $n$ such that $\prod_{j=0}^{n-1}\|Df^l|_{E^{cs}(f^{jl}x)}\|<1$, then $E^{cs}_\Lambda$ is a contracting bundle.*

   Since we know $E^{cs}_\Lambda$ is continuous but not contracting, so for any $l_n$, there exists $b_n$, such that $\prod_{j=0}^{n-1}\|Df^l_n|_{E^{cs}(f^{jl_n}b_n)}\|\ge1$ for all $m\ge1$.

2) For any $x \in \Lambda$, $\omega(x)$ contains a point $c_n \in \Lambda$ such that $\prod_{j=0}^{nl_0m-1} \|Df^{l_0'}|_{E^{cs}(f^{jl_0'}c_n)}\| \leq \lambda_2^{nl_0m}$ for all $m \geq 1$, since

$$\prod_{j=0}^{nl_0m-1} \|Df^{l_0'}|_{E^{cs}(f^{jl_0'}c_n)}\| \geq \prod_{j=0}^{m-1} \|Df^{nl_0l_0'}|_{E^{cs}(f^{jnl_0l_0'}c_n)}\| = \prod_{j=0}^{m-1} \|Df^{l_n}|_{E^{cs}(f^{jl_n}c_n)}\|,$$

we have that $\prod_{j=0}^{m-1} \|Df^{l_n}|_{E^{cs}(f^{jl_n}c_n)}\| \leq \lambda_2^{mnl_0} \leq \lambda_2^m$ for all $m \geq 1$.

**Remark 4.23.** *In b) of weakly selecting lemma, we don't give any restriction on $x$, so b) is in fact more stronger than the tilda condition, that's why we call the condition b) in weakly selecting lemma the strong tilda condition.*

By 0), 1), 2) above and corollary 4.20, we proved the lemma. □

4.3. **Transition.** Transition was introduced in [6] at first, there they consider a special linear system with a special property called transition and use it to study homoclinic class. Here I prefer to use a little different way to state it, the notation and definition are basically copy from [6]. The main result in this subsection is corollary 4.26. We begin by giving some definitions.

Given a set $\mathcal{A}$, a word with letters in $\mathcal{A}$ is a finite sequence of $\mathcal{A}$, its length is the number of letters composing it. The set of words admits a natural semi-group structure: the product of the word $[a] = (a_1, \cdots, a_n)$ by $[b] = (b_1, \cdots, b_l)$ is $[a] \cdot [b] = (a_1, \cdots, a_n, b_1, \cdots, b_l)$. We say that a word $[a]$ is not a power if $[a] \neq [b]^k$ for every word $[b]$ and $k > 1$.

Here we'll use some special words. Let's fix $f \in C^1(M)$, for any $x \in Per(f)$, we write $[x] = (f^{\pi(x)-1}(x)), \cdots, x)$ and $\{x\} = (Df(f^{\pi(x)-1}(x)), \cdots, Df(x))$. We call a word $[a] = (a_k, \cdots, a_1)$ with letters in $M$ is a finite $\varepsilon$-pseudo orbit if $d(f(a_i), a_{i+1}) \leq \varepsilon$ for $1 \leq i \leq k-1$, if $\varepsilon = 0$, that means $f(a_i) = a_{i+1}$ for $1 \leq i \leq k-1$, then we call $[a]$ is a finite segment of orbit. We always denote $\{a\} = (Df(a_k), \cdots, Df(a_1))$.

Suppose we have a finite orbit $[a] = (a_n, \cdots, a_1)$ and an $\varepsilon$-pseudo orbit $[b] = (b_l, \cdots, b_1)$, we say $[b]$ is $\delta$-shadowed by $[a]$ if $n = l$ and $d(a_i, b_i) \leq \varepsilon$ for $1 \leq i \leq n$. We say $\{a\}$ is $\delta$-close to $\{b\}$ if $n = l$ and $\|Df(a_i) - Df(b_i)\| \leq \delta$ for $1 \leq i \leq n$.

Suppose $H(p, f)$ is a non-trivial homoclinic class, we say $H(p, f)$ has $\varepsilon$-transition property if : for any finite hyperbolic periodic points $p_1, \cdots, p_n$ in $H(p, f)$ which are homoclinic related with each other, there exist finite orbits $[t^{i,j}] = (t^{i,j}_{k(i,j)}, \cdots, t^{i,j}_1)$ for any $(i, j) \in \{1, \cdots, n\}^2$ where $k(i, j)$ is the length of $[t^{i,j}]$, such that, for every $m \in \mathbb{N}$, $l = (i_1, \cdots, i_m) \in \{1, \cdots, n\}^m$ and $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{N}^m$ where the word $((i_1, \alpha_1), \cdots, (i_m, \alpha_m))$ with letters in $\mathbb{N} \times \mathbb{N}$ is not a power, the pseudo orbit $[w(l, \alpha)] = [t^{i_m,i_1}] \cdot [p_{i_m}]^{\alpha_m} \cdot [t^{i_{m-1},i_m}] \cdot [p_{i_{m-1}}]^{\alpha_{m-1}} \cdot \cdots \cdot [t^{i_1,i_2}] \cdot [p_{i_1}]^{\alpha_1}$ is an $\varepsilon$-pseudo orbit and there is a periodic orbit $Orb(q(l, \alpha)) \subset H(p, f)$ such that:

a) the length of $[w(l, \alpha)]$ is $\pi(q(l, \alpha))$ and $[q(l, \alpha)]$ $\varepsilon$-shadow the pseudo orbit $[w(l, \alpha)]$.
b) the word $\{q(l, \alpha)\}$ is $\varepsilon$-close to $\{w(l, \alpha)\}$.
c) there exists a word $\{\widetilde{t}^{t_j,t_{i+1}}\} = (T^{i_j,i_{j+1}}_{k(i_j,i_{j+1})}, \cdots, T^{i_j,i_{j+1}}_1)$ with letters in $GL(R^d)$ $\varepsilon$ close to $\{t^{i_j,t_{j+1}}\}$, let $T^{i_j,i_{j+1}} = T^{i_j,i_j+1}_{k(i_j,i_j+1)} \cdot \cdots \cdot T^{i_j,i_j+1}_1$, then

$$T^{i_j,i_{j+1}}(E^s(q_{i_j})) = E^s(q_{i_{j+1}}), \quad , T^{i_j,i_{j+1}}(E^u(q_{i_j})) = E^u(q_{i_{j+1}}).$$

We say $H(p,f)$ has transition property if $H(p,f)$ has $\varepsilon$-transition property for any $\varepsilon > 0$.

**Lemma 4.24.** *([6]) $f \in C^1(M)$, suppose $p$ is an index $i$ $(i \neq 0, d)$ hyperbolic periodic point of $f$, then $H(p,f)$ has transition property.*

**Lemma 4.25.** *$f \in R$, suppose $p$ is an index $i$ $(i \neq 0, d)$ hyperbolic periodic point of $f$ and $H(p,f)$ is not trivial. Suppose there exists a family of periodic point $\{p_n\}_{n=1}^{\infty}$ with index $i$ in $H(p,f)$ homoclinic related with $p$ and $l_n \longrightarrow \infty$, $\lambda_n \longrightarrow 1^-$ such that $\pi_{l_n}(p_n) \longrightarrow \infty$ and $\prod_{j=0}^{\pi_{l_n}(p_n)-1} \|Df^{l_n}|_{E^s(f^{jl_n}(p_n))}\| \geq \lambda_n^{\pi_{l_n}(p_n)}$, then $H(p,f)$ is an index $i-1$ fundamental limit.*

**Proof** : We claim that we can find $q_n(g_n)$ is periodic point of $g_n$ with index $i$ such that:

1) $\lim_{n\to\infty} g_n = f$.
2) $Orb_{g_n}(q_n)$ is periodic orbit of $f$ also $(f|_{Orb_{g_n}(q_n)} = g_n|_{Orb_{g_n}(q_n)})$, so we just denote it $Orb(q_n)$, then we have $Orb(q_n) \subset H(p,f)$ and $\lim_{n\to\infty} Orb(q_n) = H(p,f)$.
3) $\lim_{n\to\infty} \frac{\pi(q_n)}{l_n} \longrightarrow \infty$
4) $\prod_{j=0}^{[\frac{\pi(q_n)}{l_n}]-1} \|Dg_n^{l_n}|_{E^s_{g_n}(g_n^{jl_n}(q_n))}\| \geq \lambda_n^{[\frac{\pi(q_n)}{l_n}]}$

**Proof of the claim**: Choose $\varepsilon_n \longrightarrow 0^+$, let's fix $n_0$ at first and choose an $\varepsilon > 0$ such that $\lambda_{n_0} + 2\varepsilon < 1$. There exists $N_0 \gg n_0$ such that for any $n \geq N_0$, we'll have $l_n \gg l_{n_0}$ and $\lambda_n > \lambda_{n_0} + 2\varepsilon$, then by $\prod_{j=0}^{\pi_{l_n}(p_n)-1} \|Df^{l_n}|_{E^s(f^{jl_n}p_n)}\| \geq \lambda_n^{\pi_{l_n}(p_n)}$, we have $\prod_{j=0}^{ml_{n_0}\pi_{l_n}(p_n)-1} \|Df^{l_n}|_{E^s(f^{jl_n}p_n)}\| \geq \lambda_n^{ml_{n_0}\pi_{l_n}(p_n)}$ for $m \geq 1$, then we get

$$\prod_{j=0}^{ml_n\pi_{l_n}(p_n)-1} \|Df^{l_{n_0}}|_{E^s(f^{jl_{n_0}}p_n)}\| \geq (\lambda_{n_0} + 2\varepsilon)^{ml_{n_0}\pi_{l_n}(p_n)} \text{ for } m \geq 1. \tag{4.15}$$

Since $f \in R$, there exists a family of periodic points $\{q_i'\}_{i=1}^{N}$ with index $i$, which are $\varepsilon_{n_0}$-dense in $H(p,f)$ and they are homoclinic related with $p$ and $\{p_n\}_{n=1}^{\infty}$. Now use $\varepsilon_{n_0}$-transition property for $\{q_0'(= p_{N_0}), q_1', \cdots, q_N'\}$, then for $\{i,j\} \in \{0,1,\cdots,N\}^2$, there exists finite orbit $[t^{i,j}] = (t^{i,j}_{k(i,j)}, \cdots, t^{i,j}_1)$ such that for $l = (0,1,\cdots,N)$ and $\alpha_m = (m \cdot l_{n_0}, 1, \cdots, 1)$, the pseudo orbit $[w(l,\alpha_m)] = [t^{N,0}] \cdot [q_N'] \cdot \cdots \cdot [t^{0,1}] \cdot [q_0']^{m \cdot l_{N_0} \frac{l_{N_0} \cdot \pi_{l_{N_0}}(p_{N_0})}{\pi(p_{N_0})}}$ is an $\varepsilon_{n_0}$-pseudo orbit and is $\varepsilon_{n_0}$-shadowed by periodic orbit $[q(l,\alpha_m)]$ whose index is $i$, where $Orb(q(l,\alpha_m))) \subset H(p,f)$ and $\{q(l,\alpha_m)\}$ is $\varepsilon_{n_0}$-near $\{w(l,\alpha_m)\}$.

Consider the word $\{\widetilde{w}(l,\alpha_m)\} = \{\widetilde{t}^{N,0}\} \cdot \{q_N'\} \cdot \cdots \cdot \{\widetilde{t}^{N,0}\} \cdot \{q_0'\}^{ml_0}$, it's $\varepsilon_{n_0}$ near $\{w(l,\alpha_m)\}$, so $\{\widetilde{w}(l,\alpha_m)\}$ is $2\varepsilon_{n_0}$ near with $\{q(l,\alpha_m)\}$. Now use lemma 4.17 (Franks lemma), we can get a $C^1$ diffeomorphism $g_{(l,\alpha_m)}$ such that $d(g_{(l,\alpha_m)}, f) < 2\varepsilon_{n_0}$, $Orb_f(q(l,\alpha_m))$ is also orbit of $g_{(l,\alpha_m)}$, and $\{Dg_{(l,\alpha_m)}|_{Orb(q(l,\alpha_m))}\} = \{\widetilde{w}(l,\alpha_m)\}$. By c) of transition property, $E_f^{s(u)}(q_0')$ is invariant bundle of $\{\widetilde{w}(l,\alpha_m)\}$, so they are invariant bundle of $g_{l,\alpha_m}$, that means $Dg_{(l,\alpha_m)}^{\pi(q(l,\alpha_m))}(E_f^s(q_0')) = E_f^s(q_0')$ and $Dg_{(l,\alpha_m)}^{\pi(q(l,\alpha_m))}(E_f^u(q_0')) = E_f^u(q_0')$. It's easy to know when $m$ is big enough, $E_f^{s(u)}(q_0')$ is stable(unstable) bundle for $g_{(l,\alpha_m)}$, so when $m$ is big enough, $q_{(l,\alpha_m)}$ would be an index $i$ hyperbolic periodic point of $g_{(l,\alpha_m)}$.

Now choose $m$ big enough and let $q_{n_0} = q(l, \alpha_m)$, $g_{n_0} = g_{(l,\alpha_m)}$, it's easy to know $q_{n_0}$, $g_{n_0}$ satisfy 1), 2). About 3), let's notice that $\pi(q_n) \geq ml_{n_0}$ and $m$ can be chosen arbitrary big. 4) comes from (4.15) and $m$ is big enough. □

Now let's continue the proof of lemma 4.25, by the above claim and lemma 4.16, for any $\varepsilon > 0$ and $N > 0$, there exist an $n_0 > N$ and $g'_{n_0}$ is $\varepsilon$-perturbation of $g_{n_0}$ such that $Orb(q_{n_0})$ is index $i-1$ periodic orbit of $g'_{n_0}$ and $Orb(q_{n_0})$ is $\varepsilon_{n_0}$-dense in $H(p, f)$. Since $\varepsilon$ and $\varepsilon_{n_0}$ can be arbitrarily small, we get that $\lim_{n\to\infty} g'_{n_j} = f$, $Orb(q_{n_j})$ is index $i-1$ periodic orbit of $g'_{n_j}$ and $\lim_{j\to\infty} Orb(q_{n_j}) = H(p, f)$, so $H(p, f)$ is an index $i-1$ fundamental limit. □

Then main result of this subsection is the following corollary.

**Corollary 4.26.** *$f \in R$, $C$ is a chain recurrent class of $f$, $\Lambda$ is compact invariant set of $f$ with index $i-(l,\lambda)$ dominated splitting $E^{cs} \oplus F^{cu}$ ($1 \leq i \leq d$) and assume they satisfy all the assumption of weakly selecting lemma, then $C$ contains index $i$ periodic point and $C$ is an index $i-1$ funadamental limit.*

**Proof** : It's just a corollary from Lemma 4.21 (weakly selecting lemma) and lemma 4.25. □

4.4. **Proof of lemma 4.3. Proof** : When $\Lambda$ is trivial ($\#(\Lambda) < \infty$), $\Lambda$ is a periodic orbit, since $\Lambda$ is an index $j_0$-fundamental limit, it should be an index $j_0$ hyperbolic periodic orbit, so $C$ contains an index $j_0$ periodic point and it's an index $j_0$ fundamental limit.

Now we suppose $\Lambda$ is not trivial, by generic property 5 of proposition 3.1, there exists a family of index $j_0$ periodic points $\{p_n(f)\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} Orb(p_n(f)) = \Lambda$. Since $\Lambda$ is not trivial, we have $\pi(p_n(f)) \longrightarrow \infty$.

If $\Lambda$ isn't an index $j_0+1$ fundamental limit, we know that $\{p_n(f)\}$ is index $j_0$ stable, then by lemma 4.6 (Gan's lemma), there exits a subsequence $\{p_{n_i}(f)\}_{i=1}^{\infty}$ such that $p_{n_i}(f)$ and $p_{n_j}(f)$ are homoclinic related, so $H(p_{n_1}, f) = H(p_{n_2}, f) = \cdots$, especially, by $\lim_{n\to\infty} Orb(p_n(f)) = \Lambda$, we know that $\Lambda \subset H(p_{n_1}, f)$, by generic property 6) of proposition 3.1, $C = H(p_{n_1}, f)$, so $C$ contains index $j_0$ periodic point and it's an index $j_0$ fundamental limit.

So from now, we suppose $\Lambda$ is an index $j_0+1$ fundamental limit also, then $\Lambda \subset P^*_{j_0} \bigcap P^*_{j_0+1}$, since $f$ is far away from tangency, by proposition 2.1, $\Lambda$ has an index $j_0$ dominated splitting $E^{cs}_{j_0}(\Lambda) \oplus E^{cu}_{j_0+1}(\Lambda)$ and an index $j_0+1$ dominated splitting $E^{cs}_{j_0+1}(\Lambda) \oplus E^{cu}_{j_0+2}(\Lambda)$. Let $E^c_1(\Lambda) = E^{cu}_{j_0+1}(\Lambda) \bigcap E^{cs}_{j_0+1}(\Lambda)$, then on $\Lambda$ we have the following dominated splitting: $T|_\Lambda M = E^{cs}_{j_0}(\Lambda) \oplus E^c_1(\Lambda) \oplus E^{cu}_{j_0+2}(\Lambda)$. Since $C \bigcap P^*_j = \phi$ for $j < j_0$, by proposition 2.2, $E^{cs}_{j_0}$ is in fact contracting, so we prefer denoting it $E^s_{j_0}$. Now on $\Lambda$ we have the dominated splitting $T|_\Lambda M = E^s_{j_0}(\Lambda) \oplus E^c_1(\Lambda) \oplus E^{cu}_{j_0+2}(\Lambda)$.

**Remark 4.27.** *Since $\Lambda$ is index $j_0$ fundamental limit, $E^c_1(\Lambda)$ is not contracting, that means that the bundle $(E^s_{j_0} \oplus E^c_1)|_\Lambda$ is not contracting also.*

When $j_0 + 1 = d$, especially, the dominated splitting on $\Lambda$ should be $T|_\Lambda M = E^s_{j_0}(\Lambda) \oplus E^c_1(\Lambda)$. In this case, if $\Lambda$ is not minimal, there exists an $x_0 \in \Lambda$ such that $\omega(x_0) \subsetneq \Lambda$. By the definition of $\Lambda$ and $j_0 = d-1$, $\omega(x_0)$ is an index $d$ fundamental limit but not index $j$ fundamental limit for $j < d$. With the generic property (5) of proposition 3.1, $\omega(x_0)$ can be converged by a family of sinks $\{p_n(f)\}$, by remark 4.4, $\pi(p_n(f))$ should be bounded ( If it's not bounded, there exist $p_{n_0}(f)$ and $g_{n_0} \overset{C^1}{\sim} f$ such that $g_{n_0}|_{Orb_f(p_{n_0}(f))} = f|_{Orb_f(p_{n_0}(f))}$ and $Orb(p_{n_0}(f))$ is a periodic orbit of $g$ with index smaller than $d$, that means $\omega(x_0)$ is an fundamental limit with index smaller than $d$, it's a contradiction). That means $\omega(x_0)$ is trivial, so it's a periodic orbit. Since $f$ is a Kupuka-Smale diffeomorphism and $\omega(x_0)$ is an index $d$ fundamental limit, we can know that $\omega(x_0)$ is an index $d$ hyperbolic periodic orbit, then $C$ contains a sink, it means $C$ itself is just the orbit of sink and $C = \omega(x_0)$, that's a contradiction with $C$ is not trivial, so we proved $\Lambda$ is minimal when $j_0 + 1 = d$.

Now we just consider $j_0 + 1 < d$, we claim that with all the assumptions above on $\Lambda$, then either $\Lambda$ is minimal, or $C$ contains periodic points with index $j_0 + 1$ and $C$ is an index $j_0$ fundamental limit.

**Proof of claim**: Suppose $\Lambda$ is not minimal, it means that there exists $x_0 \in \Lambda$ such that $\omega(x_0) \neq \Lambda$. Consider the set of compact chain recurrent subset of $\Lambda$: $\{\Lambda_\alpha : \ \Lambda_\alpha \subsetneq \Lambda\}_{\alpha\in\mathcal{A}}$, since $\omega(x_0) \in \{\Lambda_\alpha\}_{\alpha\in\mathcal{A}}$, $\mathcal{A} \neq \phi$, by generic property (4) of proposition 3.1, $\Lambda_\alpha$ is a fundamental limit. By the definition of $j_0$ and $\Lambda$, $\Lambda_\alpha$ is an index $j_\alpha$ fundamental limit with $j_\alpha \geq j_0 + 1$. Denote $\mathcal{B}$=$\{\beta \in \mathcal{A}$, $\Lambda_\beta$ is not an index $j$ fundamental limit for $j > j_0 + 1\}$.

**Remark 4.28.** : *For any $\beta \in \mathcal{B}$, $\Lambda_\beta$ is an index $j_0+1$ fundamental limit, on $\Lambda_\beta$ we have an index $j_0+1$ dominated splitting $E^{cs}_{j_0+1}(\Lambda_\beta) \oplus E^{cu}_{j_0+2}(\Lambda_\beta)$. Since we have $\Lambda_\beta \bigcap P^*_j \neq \phi$ for all $j \neq j_0+1$, by proposition 2.2, the index $j_0 + 1$ dominated splitting is in fact a hyperbolic splitting, that means $\Lambda_\beta$ is a hyperbolic set.*

Now we divide the proof of the claim to three subcases: $\#(B) = 0$, $\#(B) = N_1 < \infty$ and $\#(B) = \infty$.

Case A: $\#(B) = 0$.

That means for all $\alpha \in \mathcal{A}$, $\Lambda_\alpha$ is an index $j_\alpha$ fundamental limit for some $j_\alpha > j_0 + 1$.

Now we need the following two results.

**Lemma 4.29.** *([45]) Assume $f \in R$, let $\Lambda$ be an index $i$ fundamental limit of $f$ ($1 \leq \ i \leq \ d-1$), $E^{cs}_i(\Lambda) \oplus E^{cu}_{i+1}(\Lambda)$ is an index $i - (l, \lambda)$ dominated splitting on $\Lambda$ given by proposition 2.1, then*

1) *either for any $\mu \in (\lambda, 1)$, there exists $c \in \Lambda$ such that $\prod_{j=0}^{n-1} \|Df^l|_{E^{cs}_i(f^{jl}c)}\| \leq \mu^n$ for $n \geq 1$,*
2) *or $E^{cs}_i$ splits into a dominated splitting $V^{cs}_{i-1} \oplus V^c_1$ with $dim(V^c_1) = 1$ such that for any $\mu \in (\lambda, 1)$, there is $c' \in \Lambda$ such that $\prod_{j=0}^{n-1} \|Df^l|_{V^{cs}_{i-1}(f^{jl}c')}\| \leq \mu^n$ for all $n \geq 1$.*

**Lemma 4.30.** *Let $\Lambda$ be an invariant compact set of $f$, with two dominated splitting $E^{cs} \oplus F^{cu}$ and $\widetilde{E}^{cs} \oplus \widetilde{F}^{cu}$, if $dim(E^{cs}) \leq dim(\widetilde{E}^{cs})$, then $E^{cs} \subset \widetilde{E}^{cs}$.*

Choose $\mu_0 \in (\lambda, 1)$, since $\Lambda_\alpha$ is an index $j_\alpha$ fundamental limit, proposition 2.1 gives an index $j_\alpha - (l, \lambda)$ dominated splitting $E^{cs}_{j_\alpha} \oplus F^{cu}_{j_\alpha+1}$ on $\Lambda_\alpha$.

If 1) of lemma 4.29 is true for $\Lambda_\alpha$, then there exists $c \in \Lambda_\alpha$ such that $\prod_{j=0}^{n-1} \|Df^l|_{E^{cs}_{j_\alpha}(f^{jl}c)}\| \leq \mu_0^n$ for $n \geq 1$. On $\Lambda_\alpha$ we have another dominated splitting $(E^s_{j_0} \oplus E^c_1) \oplus E^{j_0+2}_{cu}$ induced from $\Lambda$. Since $dim(E^s_{j_0} \oplus E^c_1) = j_0 + 1 < j_\alpha = dim(E^{cs}_{j_\alpha})$, be lemma 4.30, $E^s_{j_0} \oplus E^c_1 \subset E^{cs}_{j_\alpha}$, so we have $\prod_{j=0}^{n-1} \|Df^l|_{E^s_{j_0} \oplus E^c_1(f^{jl}c)}\| \leq \mu_0^n$ for $n \geq 1$.

If 2) of lemma 4.29 is true for $\Lambda_\alpha$, then there exists $c'$ such that $\prod_{j=0}^{n-1} \|Df^l|_{V^{cs}_{j_\alpha-1}(f^{jl}c')}\| \leq \mu_0^n$ for $n \geq 1$, recall that $dim(E^s_{j_0} \oplus E^c_1) = j_0 + 1 \leq j_\alpha - 1 = dim(V^{cs}_{j_\alpha - 1})$, by lemma 4.30, $E^s_{j_0} \oplus E^c_1 \subset V^{cs}_{j_\alpha}(\Lambda_\alpha)$, so we have $\prod_{j=0}^{n-1} \|Df^l|_{E^s_{j_0} \oplus E^c_1(f^{jl}c')}\| \leq \mu_0^n$ for $n \geq 1$.

**Remark 4.31.** *: By the above arguments, we know that for any $\alpha \in \mathcal{A} \setminus B$, and for any $\mu_0 \in (\lambda, 1)$, there exists $c \in \Lambda_\alpha$ such that*

$$\prod_{j=0}^{n-1} \|Df^l|_{E^s_{j_0} \oplus E^c_1(f^{jl}c)}\| \leq \mu_0^n \quad \textit{for } n \geq 1. \tag{4.16}$$

By remark 4.27and remark 4.31, the index $j_0 + 1 - (l, \lambda)$ dominated splitting $(E^s_{j_0} \oplus E^c_1) \oplus E^{cu}_{j_0+2}$ on $\Lambda$ satisfies all the conditions of weakly selecting lemma, by corollary 4.26, $C$ contains index $j_0 + 1$ periodic point and $C$ is an index $j_0$ fundamental limit.

Case B: $\#(B) = N_1 < \infty$

Let $\mathcal{B}$=$\{\beta_1, \cdots, \beta_{N_1}\}$, fix $\lambda < \mu_0 < 1$, then by the argument in case A, for any $\beta \in \mathcal{A} \setminus B$, there exists $c \in \Lambda$ satisfies (4.16).

For $\beta_i \in \mathcal{B}$, $\Lambda_{\beta_i}$ should be a hyperbolic set where the bundle $E^s_{j_1} \oplus E^c_1|_{\Lambda_{\beta_i}}$ is a contracting bundle, so there exists $l'$ such that for any $x \in \Lambda_{\beta_i}$, $\|Df^{l'}|_{(E^s_{i_0} \oplus E^c_1)(x)}\| < 1/2$.

Let $l_0 = l \cdot l'$ and $1 > \mu_1 > \max\{\mu_0, \frac{1}{2}\}$, then for any $\Lambda_\alpha$ $(\alpha \in \mathcal{A})$, there exists a point $c \in \Lambda_\alpha$ such that $\prod_{j=0}^{n-1} \|Df^{l_0}|_{E^s_{j_0} \oplus E^c_1(f^{jl_0}c)}\| \leq \mu_1^n$. With remark 4.27, the index $j_0 + 1 - (l, \lambda)$ dominated splitting $(E^s_{j_0} \oplus E^c_1) \oplus E^{cu}_{j_0+2}$ on $\Lambda$ satisfies all the conditions of weakly selecting lemma, by corollary 4.26, $C$ contains index $j_0 + 1$ periodic point and $C$ is an index $j_0$ fundamental limit.

Case C: $\#(B) = \infty$

In remark 4.28, we have shown that for any $\beta \in \mathcal{B}$, $\Lambda_\beta$ is a hyperbolic chain recurrent set with index $j_0 + 1$. Then there exists a family of periodic points $\{p_{\beta,n}\}_{n=1}^\infty$ in $C$ with index $j_0 + 1$ and $\lim_{n\to\infty} Orb(p_{\beta,n}) = \Lambda_\beta$ (by shadowing lemma). If $\Lambda_\beta$ is trivial, that means it's an index $j_0 + 1$ periodic orbit, we can let $Orb(p_{\beta,n}) = \Lambda_\beta$ for $n \geq 1$; if $\Lambda_\beta$ is not trivial, we can let $\pi(p_{\beta,n}) \longrightarrow \infty$.

We have the following two subcases.

- Subcase $C.1$: There exists $\delta > 0$ such that for any $\Lambda_\beta,\ \beta \in \mathcal{B}$, there exists a family of periodic points $\{p_{\beta,n}\}_{n=1}^\infty$ such that $\lim_{n\to\infty} Orb(p_{\beta,n}) = \Lambda_\beta$ and $\mid Df^{\pi(p_{\beta,n})}|_{E^c_1(p_{\beta,n})} \mid < e^{-\delta\pi(p_{\beta,n})}$.
- Subcase $C.2$: For any $\frac{1}{m} > 0$, there exist $\beta_m \in \mathcal{B}$ and a family of periodic points $\{p_{\beta_m,n}\}_{n=1}^\infty$ satisfying $\lim_{n\to\infty} Orb(p_{\beta_m,n}) = \Lambda_\beta$ and $\mid Df^{\pi(p_{\beta_m,n})}|_{E^c_1(p_{\beta_m,n})} \mid > e^{-\frac{1}{m}\pi(p_{\beta_m,n})}$.

In the subcase $C.1$, let's fix $1 > \mu_1 > \mu_0 > e^{-\delta}$. For $\beta \in \mathcal{B}$, recall that $dim(E_1^c(\Lambda)) = 1$ and $\mid Df^{\pi(p_{\beta,n})}|_{E_1^c(p_{\beta,n})} \mid < e^{-\delta\pi(p_{\beta,n})}$, we'll get $\prod_{i=0}^{\pi(p_{\beta,n})-1} \mid Df|_{E_1^c(p_{\beta,n})} \mid < e^{-\delta\pi(p_{\beta,n})}$, that means for any $s \geq 1$, we have $\prod_{i=0}^{s\pi(p_{\beta,n})-1} \mid Df|_{E_1^c(p_{\beta,n})} \mid < e^{-s\delta\pi(p_{\beta,n})}$ for $s \geq 1$. By lemma 4.10 (Pliss lemma) there exists $x_{\beta,n} \in Orb(p_{\beta,n})$ such that $\mid Df^s|_{E_1^c(x_{\beta,n})} \mid = \prod_{i=0}^{s-1} \mid Df|_{E_1^c(f^i(x_{\beta,n}))} \mid < \mu_0^s$ for $s \geq 1$. Suppose $\lim_{n\to\infty} x_{\beta,n} \longrightarrow c_\beta$ where $c_\beta \in \Lambda_\beta$, then $\prod_{i=0}^{s-1} \mid Df|_{E_1^c(f^i(c_\beta))} \mid < \mu_0^s$ for $s \geq 1$. Notice that $E_{j_0}^s|_\Lambda$ is dominated by $E_1^c|_\Lambda$ and $\mu_1 > \mu_0$, there exists $l' \gg 1$ doesn't depend on $\beta$ such that $\prod_{i=0}^{t-1} \|Df^{l'}|_{E_1^c \oplus E_{j_0}^s(f^{il'}(c_\beta))}\| < \mu_1^t$ for $t \geq 1$.

For $\alpha \in \mathcal{A} \setminus B$, by the argument in case A, there exists $c_\alpha \in \mathcal{A}_\alpha$ such that $\prod_{i=0}^{t-1} \|Df^{l_0}|_{E_1^c \oplus E_{j_0}^s(f^{il_0}(c_\alpha))}\| < \mu_1^t$ for $t \geq 1$.

Let $l_1 = l' \cdot l_0$, then for any $\alpha \in \mathcal{A}$, there exists $c_\alpha \in \mathcal{A}$ such that $\prod_{i=0}^{t-1} \|Df^{l_1}|_{E_1^c \oplus E_{j_0}^s(f^{il_1}(c_\alpha))}\| < \mu_1^t$ for $t \geq 1$. With remark 4.27, the index $j_0 + 1 - (l, \lambda)$ dominated splitting $(E_{j_0}^s \oplus E_1^c) \oplus E_{j_0+2}^u$ on $\Lambda$ satisfies all the conditions of weakly selecting lemma. By Corollary 4.26, $C$ contains index $j_0 + 1$ periodic point and $C$ is an index $j_0$ fundamental limit.

In the subcase $C.2$, since $\Lambda_{\beta_m}$ is a hyperbolic set, we can always suppose $\{p_{\beta_m,n}\}_{n=1}^\infty$ is homoclinic related with each other and $p_{\beta_m,n} \in C$, so $C$ contains index $j_0 + 1$ periodic points. Now we'll show $C$ is an index $j_0$ fundamental limit also.

We claim that there exists a subsequence $\{\beta_{m_t}\}_{t=1}^\infty \subset \{\beta_m\}$ and for every $\beta_{m_t}$ there exists $p_{\beta_{m_t},n_t} \in \{p_{\beta_{m_t},n}\}_{n=1}^\infty$ such that $\lim_{t\to\infty} \pi(p_{\beta_{m_t},n_t}) \longrightarrow \infty$.

**Proof of the claim**: Let $\mathcal{B}_0$=$\{\beta_m :\ \Lambda_{\beta_m}$ is given in subcase $C.2$ and $\Lambda_{\beta_m}$ is not trivial. $\}$

If $\#(\mathcal{B}_0) = \infty$, then for any $\beta_{m_t} \in \mathcal{B}_0$, by $\Lambda_{\beta_{m_t}}$ is not trivial, we'll have $\lim_{n\to\infty} \pi(p_{\beta_{m_t},n}) \longrightarrow \infty$, so when n is big enough, we can let $\pi(p_{\beta_{m_t},n})$ arbitrarily big.

If $\#(\mathcal{B}_0) < \infty$, then for $\beta_m \notin \mathcal{B}_0$, $\Lambda_{\beta_m}$ is an index $j_0 + 1$ periodic orbit and $Orb(p_{\beta_m,n}) \equiv \Lambda_{\beta_m}$ for $n \geq 1$. Since $f$ is a Kupka-Smale diffeomorphism, the number of periodic points with fixed boundary of period should be finite, by the fact $\#(\mathcal{B} \setminus \mathcal{B}_0) = \infty$, there are infinite of $m$ such that $\Lambda_m$ is index $j_0 + 1$ periodic orbits, then we can choose $\Lambda_{\beta_m}$ is an index $j_0 + 1$ periodic orbit with arbitrarily big period. $\square$

Now for simiplicity, we denote $p_{\beta_{m_t},n_t}$ by $p_{\beta_m,n_m}$.

For $\{p_{\beta_m,n_m}\}_{m=1}^\infty$, we have $\lim_{m\to\infty} \pi(p_{\beta_m . n_m}) \longrightarrow \infty$ and

$$|Df^{\pi(p_{\beta_m,n_m})}|_{E_1^c(p_{\beta_m,n_m})}| > e^{-\frac{1}{m}\pi(p_{\beta_m,n_m})}. \tag{4.17}$$

Choose $\{l_m\}_{m=1}^\infty$ carefully, we'll have $\lim_{m\to\infty} l_m \longrightarrow \infty$, $\lim_{m\to\infty} \frac{\pi(p_{\beta_m,n_m})}{l_m} \longrightarrow \infty$ and $\frac{l_m}{m} \longrightarrow 0^+$ (after replacing $\{p_{\beta_m,n_m}\}_{m=1}^\infty$ by a subsequence, we can always do this). Since $\pi_{l_m}(p_{\beta_m,n_m}) \geq \frac{\pi(p_{\beta_m,n_m})}{l_m}$, we have

$$\lim_{m\to\infty} \pi_{l_m}(p_{\beta_m,n_m}) \longrightarrow \infty. \tag{4.18}$$

By (4.17) and the fact $l \cdot \pi_l(p)$ is always a multiple of $\pi(p)$ for any period point $p$ and $l \geq 1$, we have

$$|Df^{l_m \cdot \pi_{l_m}(p_{\beta_m, n_m})}|_{E_1^c(p_{\beta_m,n_m})}| > e^{-\frac{1}{m} l_m \cdot \pi_{l_m}(p_{\beta_m,n_m})},$$

it's equivalent with

$$\prod_{i=0}^{\pi_{l_m}(p_{\beta_m,n_m})-1} \|Df^{l_m}|_{E_1^c(f^{il_m}(p_{\beta_m,n_m}))}\| \geq e^{-\frac{l_m}{m} \cdot \pi_{l_m}(p_{\beta_m,n_m})},$$

then we get

$$\prod_{i=0}^{\pi_{l_m}(p_{\beta_m,n_m})-1} \|Df^{l_m}|_{(E_1^c \oplus E_{j_0}^s)(f^{il_m}(p_{\beta_m,n_m}))}\| \geq e^{-\frac{l_m}{m} \cdot \pi_{l_m}(p_{\beta_m,n_m})},$$

since $\lim\limits_{m\to\infty} \frac{l_m}{m} \longrightarrow 0^+$ and by (4.18), lemma 4.25 tells us $C$ is an index $j_0$ fundamental limit, this finishes the proof of the claim. □

Now let's continue the proof of lemma 4.3, by the above argument, we can suppose $\Lambda$ is minimal, not trivial, it's an index $j_0$ and $j_0 + 1$ fundamental limit with dominated splitting $E_{j_0}^s \oplus E_1^c \oplus E_{j_0+2}^{cu}|_\Lambda$ where $E_{j_0+2}^{cu}(\Lambda) \neq \phi$.

If $E_{j_0+2}^{cu}(\Lambda)$ is not expanding, by lemma 4.22, we can know that there exists a point $b \in \Lambda$ such that $\prod\limits_{i=0}^{n-1} \|Df^{-l}|_{E_{j_0+2}^{cu}(f^{(i+1)l}b)}\| \geq 1$, since $(E_{j_0}^s \oplus E_1^c) \oplus E_{j_0+2}^{cu}|_\Lambda$ is an index $j_0 + 1 - (l, \lambda)$ fundamental limit, it means that

$$\prod_{i=0}^{n-1} \|Df^l|_{E_{j_0}^s \oplus E_1^c(f^{il}(b))}\| / \prod_{i=0}^{n-1} \|Df^{-l}|_{E_{j_0+2}^{cu}(f^{(i+1)l}(b))}\| \leq \lambda^n, \quad \text{for } n \geq 1,$$

so $\prod\limits_{i=0}^{n-1} \|Df^l|_{E_{j_0}^s \oplus E_1^c(f^{il}(b))}\| \leq \lambda^n$ for all $n \geq 1$. Since $\Lambda$ is minimal, the index $j_0 + 1$ dominated splitting on $\Lambda$ satisfies strong tilda condition, by remark 4.27, it also satisfies the non-hyperbolic condition, so it satisfies all the conditions of weakly selecting lemma, then by corollary 4.26, $C$ contains index $j_0 + 1$ periodic point and it's an index $j_0$ fundamental limit. □

## 5. Proof of theorem 1

In order to prove theorem 1, we need the following lemma whose proof has been postponed to the end of this section.

**Lemma 5.1.** *Let $f \in R$, $C$ is any non-trivial chain recurrent class of $f$, suppose $\Lambda \subset C$ is a non-trivial minimal set with a codimension-1 partial hyperbolic splitting $T_\Lambda M = E_1^c \oplus E_2^u$ where $dim(E_1^c|_\Lambda) = 1$ and is not contracting, then $C$ is a homoclinic class containing index 1 periodic point and $C$ is an index 0 fundamental limit.*

**Remark 5.2.** *in* [9]*, they show that for $f \in R$, if $C$ is a chain recurrent class of $f$ with a codimension-1 dominated splitting $T_C M = E_1^c \oplus E_2^u$ where $dim(E_1^c|_C) = 1$ and $E_1^c|_C$ is not hyperbolic, then $C$ should be a homoclinic class. We generalize this result to minimal set with Crovisier's work on central curves.*

**Proof of theorem 1**: Suppose $C \bigcap P_0^* \neq \phi$, let $\Lambda$ be an minimal index 0 fundamental limit, then $\Lambda$ is not trivial ( if $\Lambda$ is trivial, $\Lambda$ should be an orbit of source, then $C$ itself is source also, that contradicts with $C$ is not trivial)). By lemma 4.3, either $C$ is a homoclinic class containing index 1 periodic point and

$C$ is an index 0 fundamental limit or $\Lambda$ is a non-trivial minimal set with codimension-1 partial hyperbolic splitting $T_\Lambda M = E_1^c \oplus E_2^u$ where $E_1^c|_\Lambda$ is not trivial. In the first case we've proved theorem 1, in the second case, by lemma 5.1, we also proved theorem 1. □

In §5.1, we'll introduce some properties for codimension-1 partial hyperbolic splitting set, in §5.2 we'll introduce Crovisier's central model for the invariant compact set with partial hyperbolic splitting whose central bundle is 1-dimension and non-hyperbolic. In §5.3 I'll give the proof of lemma 5.1.

5.1. **Some properties for codimension-1 partial hyperbolic splitting.** Let $f \in R$, $\Lambda$ is a given non-trivial minimal set of $f$ with a codimension-1 partial hyperbolic splitting $T_\Lambda M = E^u \oplus E_1^c$, where $dim(E_1^c(\Lambda)) = 1$ and the bundle $E_1^c|_\Lambda$ is not hyperbolic. In this section we always suppose the dominated splitting is 1-step and the bundle $E^u$ is 1-step expanding, it means that there exists $0 < \lambda < 1$ such that for any $v^u \in E^u(x)$, $v^c \in E_1^c(x)$ where $|v^u| = |v^c| = 1$, $x \in \Lambda$, we have $\frac{|Df(v^c)|}{|Df(v^u)|} < \lambda$, $|Df(v^u)| > \lambda^{-1}$. Fix a small neighborhood $U_0$ of $\Lambda$, then the maximal invariant set $\Lambda_0 = \bigcap_{j=-\infty}^{\infty} f^j(\overline{U_0})$ has also a codimension-1 partial hyperbolic splitting $\widetilde{E^u} \oplus \widetilde{E_1^c}$, the dominated splitting is 1-step and the bundle $\widetilde{E^u}|_{\Lambda_0}$ is also 1-step expanding. We say $E_1^c(\Lambda)$ has an $f$-orientation if $E_1^c|_\Lambda$ is orientable and $Df$ preserves the orientation. If $E_1^c|_\Lambda$ has an $f$-orientation, we choose $U_0$ small enough such that $\widetilde{E_1^c}(\Lambda)$ has an $f$-orientation also.

Here we should notice the reader that in this section, all the argument will take place just in $U_0$, and we can suppose $U_0$ is small enough such that it satisfies all the properties which we need.

When $U_0$ is small enough, we can extend the bundle $\widetilde{E^u}|_{\Lambda_0}$ and $\widetilde{E_1^c}|_{\Lambda_0}$ to $\overline{U_0}$ such that for any $x \in \overline{U_0}$, $T_xM = \widetilde{E^u}(x) \oplus \widetilde{E_1^c}(x)$, and if $E_1^c|_\Lambda$ is orientable, $\widetilde{E_1^c}|_{\overline{U_0}}$ is orientable also. In fact, no matter $\widetilde{E_1^c}|_{\overline{U_0}}$ is orientable or not, we can always locally define an orientation of $\widetilde{E_1^c}|_{\overline{U_0}}$, it means that there exists $\delta_0 > 0$ such that for any $x \in \overline{U_0}$, we can give an orientation for the bundle $\widetilde{E_1^c}|_{B_{\delta_0}(x)} \bigcap \overline{U}_0$.

For every point $x \in \overline{U_0}$, we define two kinds of cones on its tangent space $C_a^i(x) = \{v | v \in T_xM$, there exists $v' \in \widetilde{E^i}(x)$ such that $d(\frac{v}{|v|}, \frac{v'}{|v'|}) < a\}_{i=c,u}$. When $a$ small enough, $C_a^c \bigcap C_a^u = \phi$, $Df(C_a^u(x)) \subset C_a^u(f(x))$ and $Df^{-1}(C_a^c(x)) \subset C_a^c(f^{-1}(x))$ for $x \in \Lambda_0$.

We say a submanifold $D^i$ $(i = c, u)$ tangents with cone $C_a^i$ if $dim(D^i) = d - 1$ when $i = u$ and $dim(D^i) = 1$ when $i = c$ and for $x \in D^i$, $T_xD^i \subset C_a^i(x)$. For simplicity, sometimes we call it $i$-disk, especially when $i = c$, we just call $D^c$ a central curve. We say an $i$-disk $D^i$ has centrer $x$ with size $\delta$ if $x \in D^i$, and respecting the Riemannian metric restricting on $D^i$, the ball centered on $x$ with radius $\delta$ is in $D^i$. We say an $i$-disk $D^i$ has center $x$ with radius $\delta$ if $x \in D^i$, and respecting the Riemannian metric restricting on $D^i$, the distance between any point $y \in D^i$ and $x$ is smaller than $\delta$.

The following lemma shows some well-known results, it depends on a simple fact: locally the splitting $\widetilde{E_1^c} \oplus \widetilde{E^u}|_{\overline{U}_0}$ looks like linear. [9] 's subsection 4.1 gives many details about such view, from lemma 4.8 in [9], it would be very easy to get the following properties, so here we 'll not give a proof.

**Lemma 5.3.** *: Let $f \in R$, $\Lambda$ is a non-trivial minimal set of $f$ with a codimension-1 partial hyperbolic splitting $T_\Lambda M = E_1^c \oplus E^u$ where the bundle $E_1^c|_\Lambda$ is not hyperbolic. $U_0, \delta_0, C_a^u, C_a^c$ are defined by the above argument. Let $U$ be any small neighborhood of $\Lambda$ satisfying $\overline{U} \subset U_0$, there exist two neighborhoods $U_2, U_1$ of $\Lambda$ such that $\Lambda \subset U_2 \subset \overline{U_2} \subset U_1 \subset \overline{U_1} \subset U \subset U_0$ and there exist $a_0$ small enough and $0 < \delta_{1,3} < \delta_{1,2} < \delta_{1,1} < \delta_0/2$ such that they satisfy the following properties:*

P1 *For any* $x \in \overline{U_2}$*, we have* $B_{2\delta_{1,1}}(x) \subset U_1$*, and for any* $x \in \overline{U_1}$*, we have* $B_{2\delta_{1,1}}(x) \subset U$*, then any* $i$*-disk* $D^i$ $(i = c, u)$ *with center* $x \in \overline{U_1}$ *and radius* $2\delta_{1,1}$ *will have* $D^i \subset U$*.*

*For any* $x \in \overline{U_1}$*,* $\widetilde{E_1^c}|_{B_{2\delta_{1,1}}(x)}$ *is orientable, we can choose an orientation and call the direction right, then the orientation of* $\widetilde{E_1^c}|_{B_{2\delta_{1,1}}(x)}$ *will give an orientation for central curves in* $B_{2\delta_{1,1}}(x)$*. We suppose* $\delta_{1,1}$ *is small enough such that any central curve in* $B_{2\delta_{1,1}}(x)$ *will not intersect with itself.*

*For two points* $y_1, y_2 \in B_{2\delta_{1,1}}(x)$*, we say* $y_1$ *is on the x-right of* $y_2$ *if there exists a central curve* $l \subset B_{2\delta_{1,1}}(x)$ *connects* $y_1$ *and* $y_2$ *and in* $l$*,* $y_1$ *is on the right of* $y_2$*. Then since any central curve in* $B_{2\delta_{1,1}}(x)$ *is not self-intersection,* $y_2$ *is not on x-right of* $y_1$ *anymore. Usually, we just simply call* $y_1$ *is on the right of* $y_2$*.*

P2) *Let* $\Lambda_1 = \bigcap_{i=-\infty}^{\infty} f^i(\overline{U_1})$*, apply lemma 4.13 on* $\Lambda_1$*, we can get the following two kinds of submanifolds families: the local unstable manifolds* $W^{uu}_{loc}(x)_{\ x\in\Lambda_1}$ *and the local central curves* $W^c_{loc}(x)_{\ x\in\Lambda_1}$*.*

*Choose* $\delta_{1,1}$ *properly ( small enough) we can suppose* $W^i_{loc}(x)_{\ (i=uu,c)}$ *has size* $\delta_{1,1}$*, let* $W^i_{\delta_{1,1}}(x)$ *be the ball in* $W^i_{loc}(x)$ *with central* $x$ *and radius* $\delta_{1,1}$*, then we have* $W^i_{\delta_{1,1}}(x)_{\ (x\in\Lambda_1, i=c,uu)}$ *always tangents with cone* $C^i_{a_0}$*.*

*In fact, for* $\Lambda_1^+ = \bigcap_{i=0}^{\infty} f^i(\overline{U_1})$*, any* $x \in \Lambda_1^+$ *will have uniform size of unstable manifold* $W^{uu}_{\delta_{1,1}}(x)$ *which tangents with cone* $C^{uu}_{a_0}$*.*

P3) *By the property of strong unstable manifolds, for* $y_1, y_2 \in \Lambda_1^+$*, if we have* $W^{uu}_{\delta_{1,1}/2}(y_1) \bigcap W^{uu}_{\delta_{1,1}/2}(y_2) \neq \phi$*, then* $y_1 \in W^{uu}_{\delta_{1,1}}(y_2)$ *and* $y_2 \in W^{uu}_{\delta_{1,1}}(y_1)$*. There exists* $0 < \lambda < 1$ *such that for any smooth curve* $l \subset W^{uu}_{\delta_{1,1}}(x)$ *where* $x \in \Lambda_1^+$*, we'll have* $length(f^{-1}(l)) < \lambda \cdot length(l)$*.*

P4) *For any central curve* $D^c$ *and u-disk* $D^u$ *in* $U$ *with centers in* $\Lambda_1$ *and radius smaller than* $2\delta_{1,1}$*, we have* $\#\{z|\ z \in D^c \bigcap D^u\} \leq 1$*. If* $D^c \bigcap D^u \neq \phi$*, then they are transverse intersect with each other.*

P5) *For any* $x \in \overline{U_1}$*,* $y \in B_{\delta_{1,3}}(x) \bigcap \Lambda_1$*,* $D^i_{\delta_{1,2}}$ *is an i-disk with center* $y$ *and radius* $\delta_{1,2}$*, then* $D^i_{\delta_{1,2}} \subset B_{\delta_{1,1}}(x)$*.*

*For* $z \in B_{\delta_{1,3}}(x)$ *and* $l^{c+}_{\delta_{1,2}}(z)$ *is a central curve at the right of* $z$ *with length* $\delta_{1,2}$ *and* $z$ *is one of its extreme points, suppose* $l^{c-}_{\delta_{1,2}}(z)$ *is a central curve at the left of* $z$ *with length* $\delta_{1,2}$ *and* $z$ *is one of its extreme points, let* $l^c_{\delta_{1,2}}(z) = l^{c+}_{\delta_{1,2}}(z) \bigcup l^{c-}_{\delta_{1,2}}(z)$*, then* $\#\{l^c_{\delta_{1,2}}(z) \bigcap W^{uu}_{\delta_{1,2}}(y)\} = 1$ *and they are transverse intersect. Suppose* $z \notin W^{uu}_{\delta_{1,2}}(y)$*, then if* $l^{c+}_{\delta_{1,2}} \bigcap W^{uu}_{\delta_{1,2}}(y) \neq \phi$*, we say* $z$ *is at x-left of* $y$*; if* $l^{c-}_{\delta_{1,2}} \bigcap W^{uu}_{\delta_{1,2}}(y) \neq \phi$*, we say* $z$ *is at x-right of* $y$*. It's easy to show when* $z$ *is at x-right of* $y$*, it's not at x-right of* $y$ *anymore.*

*For simplicity, we just call* $z$ *at the left of* $W^{uu}_{loc}(y)$ *or the right of* $W^{uu}_{loc}(y)$*.*

P6) *For any* $x \in \overline{U_1}$*, any* $\delta < \delta_{1,2}$*, there exists* $\delta^* \ll \delta$ *such that for* $y \in B_{\delta^*}(x) \bigcap \Lambda_1$*, if we have* $z \in B_{\delta^*}(x) \bigcap \Lambda_1$ *also, then* $\#\{l^c_\delta(z) \bigcap W^{uu}_{\delta_{1,2}}(y)\} = 1$ *and they are transverse intersect (*$l^c_\delta(z)$ *is defined in P5).*

P7) *For any* $0 < \delta^* < 2\delta_{1,1}$*, there exists a* $\delta^{**}$ *such that if* $\Gamma$ *is a central curve in* $\overline{U_1}$ *with* $length(\Gamma) < 2\delta_{1,1}$*, for* $x, y \in \Gamma$ *and suppose the segment in* $\Gamma$ *connecting* $x$ *and* $y$ *has length bigger than* $\delta^*$*, then* $d(x, y) > \delta^{**}$*.*

P8) *For any $x \in \overline{U_1}$, any central curve $l$ in $B_{\delta_{1,2}}(x)$ will have length smaller than $\delta_{1,1}$.*

*For $y \in B_{\delta_{1,2}}(x) \bigcap \Lambda_1^+$, we can let $W^{uu}_{\delta_{1,1}}(y) \bigcap B_{\delta_{1,2}}(x)$ always just have one connected components, and $W^{uu}_{\delta_{1,1}/2}(y)$ divides $B_{\delta_{1,2}}(x)$ into two connected components: the left part and the right part.*

*If $z_1, z_2 \in B_{\delta_{1,2}}(x)$ are on the different side of $B_{\delta_{1,2}}(x) \bigcap W^{uu}_{\delta_{1,1}/2}(y)$ and there is a central curve $l \subset B_{\delta_{1,2}}(x)$ connecting them, then $\#\{l \bigcap W^{uu}_{\delta_{1,1}/2}(y)\} = 1$.*

P9) *Let $x \in \overline{U_1}$, suppose $y_1, y_2 \in B_{\delta_{1,2}}(x) \bigcap \Lambda_1^+$ and there exists a central curve $l$ in $B_{\delta_{1,2}}(x)$ connects them, so by P8) length$(l) < \delta_{1,1}$, now we know $W^{uu}_{\delta_{1,1}/2}(y_1) \bigcap W^{uu}_{\delta_{1,1}/2}(y_2) = \phi$ (otherwise $y_1 \in W^{uu}_{\delta_{1,1}}(y_2)$, then $\#\{l \bigcap W^{uu}_{\delta_{1,1}}(y_1)\} \geq 2$, it contradicts with P4), it means $W^{uu}_{\delta_{1,1}/2}(y_1)$ and $W^{uu}_{\delta_{1,1}/2}(y_2)$ divide $B_{\delta_{1,2}}(x)$ into three connected components. Suppose $y_1$ is at $x$-left of $y_2$, then for any point $z \in \Lambda_1^+$ which are on the left of $W^{uu}_{\delta_{1,1}/2}(y_2) \bigcap B_{\delta_{1,2}}(x)$ and on the right of $W^{uu}_{\delta_{1,1}/2}(y_1) \bigcap B_{\delta_{1,2}}(x)$, we have $W^{uu}_{\delta_{1,1}/2}(z) \bigcap W^{uu}_{\delta_{1,1}/2}(y_i) = \phi$ $_{(i=1,2)}$ and $W^{uu}_{\delta_{11}/2}(z) \bigcap l \neq \phi$.*

P10) *A $C^1$ curve $\Gamma$ in $\overline{U_1}$ is called a central segment if $f^i(\Gamma) \subset \overline{U_1}$ for all $i \in \mathbb{Z}$ and it always tangents with $C^c_{a_0}$. Then $\Gamma \subset \Lambda_1$ and it's easy to know that for any $x \in \Gamma$, we have $T_x\Gamma = \widetilde{E^c_1}(x)$. On $\Gamma$ we have normally hyperbolic splitting $\widetilde{E^c_1} \oplus \widetilde{E^u}|_\Gamma$ since $T_x\Gamma = \widetilde{E^c_1}(x)$, by the property of normally hyperbolic manifold, $\bigcup_{x\in\Gamma} W^{uu}_{\delta_{1,1}/2}(x)$ is a submanifold (dim $= d$) with boundary, we denote it $W^u_{\delta_{1,1}/2}(\Gamma)$.*

P11) *For any $\varepsilon > 0$, if we have a family of central segment $\{\Gamma_n\}_{n=1}^\infty$ with length$(\Gamma_n) > \varepsilon$, there exists $\delta > 0$ such that vol$(W^u_{\delta_{1,1}/2}(\Gamma_n)) > \delta$, so we can find $n_i \neq n_j$ such that $W^u_{\delta_{1,1}/2}(\Gamma_{n_i}) \bigcap W^u_{\delta_{1,1}/2}(\Gamma_{n_j}) \neq \phi$.*

5.2. **Crovisier's central model.** In this subsection, let's fix $U, U_1, U_2, \Lambda_1, \delta_0/2 > \delta_{1,1} > \delta_{1,2} > \delta_{1,3} > 0$, and $a_0$ given by lemma 5.3, we'll introduce Crovisier's central model. By his work, we can get some dynamical property for the central curve $W^c_{\delta_{1,1}}(x)$ where $x \in \Lambda_1$. The main result in this subsection is lemma 5.11.

**Definition 5.4.** *A central model is a pair $(\widetilde{K}, \widetilde{f})$ where*

a) *$\widetilde{K}$ is a compact metric space called the base of the central model.*
b) *$\widetilde{f}$ is a continuous map from $\widetilde{K} \times [0,1]$ into $\widetilde{K} \times [0,\infty)$*
c) *$\widetilde{f}(\widetilde{K} \times \{0\}) = \widetilde{K} \times \{0\}$*
d) *$\widetilde{f}$ is a local homeomorphism in a neighborhood of $\widetilde{K} \times \{0\}$ : there exists a continuous map $g :\ \widetilde{K} \times [0,1] \longrightarrow \widetilde{K} \times [0,\infty)$ such that $\widetilde{f} \circ \widetilde{g}$ and $\widetilde{g} \circ \widetilde{f}$ are identity maps on $\widetilde{g}^{-1}(\widetilde{K} \times [0,1])$ and $\widetilde{f}^{-1}(\widetilde{K} \times [0,1])$ respectively.*
e) *$\widetilde{f}$ is a skew product: there exits two map $\widetilde{f}_1 :\ \widetilde{K} \longrightarrow \widetilde{K}$ and $\widetilde{f}_2 :\ \widetilde{K} \times [0,1] \longrightarrow [0,\infty)$ respectively such that for any $(x,t) \in \widetilde{K} \times [0,1]$, one has $\widetilde{f}(x,t) = (\widetilde{f}_1(x), \widetilde{f}_2(x,t))$.*

*$f$ general doesn't preserve $\widetilde{K} \times [0,1]$, so the dynamics outside $\widetilde{K} \times \{0\}$ is only partially defined.*

The central model $(\widetilde{K}, \widetilde{f})$ has a chain recurrent central segment if it contains a segment $I = \{x\} \times [0,a]$ contained in a chain recurrent class of $f|_{\widetilde{K}\times[0,1]}$.

A subset $S \subset \widetilde{K} \times [0,1]$ of a product $\widetilde{K} \times [0,\infty)$ is a strip if for any $x \in \widetilde{K}$, the intersection $S \bigcap \{x\} \times [0,\infty)$ is a non-trivial interval.

In his remarkable paper [13], Crovisier got the following important result.

**Lemma 5.5.** *([13] Proposition 2.5) Let $(\widetilde{K}, \widetilde{f})$ be a central model with a chain transitive base, then the two following properties are equivalent:*

a) *There is no chain recurrent central segment.*
b) *There exists some strip $S$ in $\widetilde{K} \times [0,1]$ that is arbitrarily small neighborhood of $\widetilde{K} \times \{0\}$ and it's a trapping region for $\widetilde{f}$ or $\widetilde{f}^{-1}$ : either $\widetilde{f}(Cl(S)) \subset Int(S)$ or $\widetilde{f}^{-1}(Cl(S)) \subset Int(S)$.*

**Remark 5.6.** *If the central model $(\widetilde{K}, \widetilde{f})$ has a chain recurrent central segment and $\widetilde{K} \times \{0\}$ is transitive, from Crovisier's proof, we can know for any small neighborhood $V$ of $\widetilde{K} \times \{0\}$, there exists a segment $x \times [0,a]_{a \neq 0}$ contained in the same chain recurrent class of $\widetilde{f}|_V$ with $\widetilde{K} \times \{0\}$.*

An open strip $S \subset \widetilde{f} \times [0,1]$ satisfying $\widetilde{f}(Cl(S)) \subset Int(S)$ or $\widetilde{f}^{-1}(Cl(S)) \subset Int(S)$ will be called a trapping strip.

**Definition 5.7.** *Let $f$ be a diffeomorphism of a manifold $M$, $\Lambda, \Lambda_1, U, U_0, U_1, U_2, a_0, \delta_0/2 > \delta_{1,1} > \delta_{1,2} > \delta_{1,3} > 0$ are given in §5.1, where $\Lambda_1$ is a partial hyperbolic invariant compact set of $f$ having a 1-dimensional central bundle. A central model $(\widetilde{\Lambda_1}, \widetilde{f})$ is a central model for $(\Lambda_1, f)$ if there exists a continuous map $\pi : \ \widetilde{\Lambda_1} \times [0, \infty) \longrightarrow M$ such that:*

a) *$\pi$ semi-conjugate $\widetilde{f}$ and $f$ : $f \circ \pi = \pi \circ \widetilde{f}$ on $\widetilde{\Lambda}_1 \times [0,1]$*
b) *$\pi(\widetilde{\Lambda}_1 \times \{0\}) = \Lambda_1$*
c) *The collection of map $t \longrightarrow \pi(\widetilde{x}, t)$ is a continuous family of $C^1$ embedding of $[0, \infty)$ into $M$, parameterized by $\widetilde{x} \in \widetilde{\Lambda_1}$.*
d) *For any $\widetilde{x} \in \widetilde{\Lambda_1}$, the curve $\pi(\widetilde{x}, [0, \infty)) \subset U$ has length bigger than $\delta_{1,2}$ but smaller than $2\delta_{1,1}$, it's tangent at the point $x = \pi(\widetilde{x}, 0) \in \Lambda_1$ to the central bundle and it's a central curve ( that means the curve $\pi(\widetilde{x}, [0, \infty))$ tangents with the central cone $C^c_{a_0}$).*

**Remark 5.8.** *From now, if $(\widetilde{\Lambda}_1, \widetilde{f})$ is a central model for $(\Lambda_1, f)$ and $\pi$ is the projection map, we'll denote the central model as $(\widetilde{\Lambda}_1, \widetilde{f}, \pi)$. Here I should notice the reader that $\pi$ in this paper has two different meanings, one denote the period of periodic point and another denote the projection map of central model. If there is any confusion, I'll point out.*

The following lemma shows the relation between central model and a set with codimension-1 partial hyperbolic splitting.

**Lemma 5.9.** *([Cr2]) $\Lambda, \Lambda_1, U, U_1$ are given in §5.1, then there exists a central model $(\widetilde{\Lambda}_1, \widetilde{f}, \pi)$ for $(\Lambda_1, f)$. Let's denote $\widetilde{\Lambda} \subset \widetilde{\Lambda}_1$ satisfies $\pi^{-1}(\Lambda) \bigcap (\widetilde{\Lambda}_1 \times \{0\}) = \widetilde{\Lambda} \times \{0\}$, then $(\widetilde{\Lambda}, \widetilde{f}, \pi)$ is a central model for $(\Lambda, f)$, and $\widetilde{\Lambda} \times \{0\}$ is minimal.*

**Remark 5.10.** 1) *When the cental bundle $\widetilde{E^c_1}(\Lambda_1)$ has an $f$-orientation ( it means that $\widetilde{E^c_1}(\Lambda_1)$ is orientable and $Df$ preserves such orientation), we call the orientation 'right', then we can get two central models $(\widetilde{\Lambda^+_1}, \widetilde{f}^+, \pi^+)$ and $(\widetilde{\Lambda^-_1}, \widetilde{f}^-, \pi^-)$ for $(\Lambda_1, f)$, we call them the right model and the left model, where $\pi^i$ $_{(i=+,-)}$ is a bijection between $\widetilde{\Lambda}^i_1 \times \{0\}$ and $\Lambda_1$, and for $\widetilde{x}^i \in \widetilde{\Lambda}^i_1$, $\pi(\widetilde{x}^i \times [0, \infty))$ is a half of central curve at the right ($i = +$) or left ($i = -$) of $x = \pi(\widetilde{x}^i \times \{0\})$.*

2) *If $f$ doesn't preserve any orientation of $\widetilde{E_1^c}(\Lambda_1)$, then $\pi: \widetilde{\Lambda}_1 \longrightarrow \Lambda_1$ is two-one: any point $x \in \Lambda_1$ has two preimages $\widetilde{x}^-$ and $\widetilde{x}^+$ in $\widetilde{\Lambda}_1$, the homeomorphism $\sigma$ of $\widetilde{\Lambda}_1$ which exchanges the preimages $\widetilde{x}^+$ and $\widetilde{x}^-$ of any point $x \in \Lambda_1$ commutes with $\widetilde{f}$.*

   *In § 5.1, we know any point $x \in \Lambda_1$ has a local orientation, then $\pi(\widetilde{x}^+ \times [0,\infty))$ is a central curve on the right of $x$, $\pi(\widetilde{x}^- \times [0,\infty))$ is on the left of $x$, the union of them is a central curve with central at $x$ and radius $\delta_{1,1}$.*

The following lemma is the main result in this subsection, it's similar with [Cr]'s proposition 3.6, but a little stronger.

**Lemma 5.11.** *$f \in R$, $\Lambda$ is a non-trivial minimal set with a codimension-1 partial hyperbolic splitting $E_1^c \oplus E^u$ where $dim(E_1^c(\Lambda)) = 1$ and $E_1^c(\Lambda)$ is not hyperbolic. Let $U, U_1, \Lambda_1$ be given in §5.1, by lemma 5.9, $(\Lambda_1, f)$ has a central model $(\widetilde{\Lambda}_1, \widetilde{f}, \pi)$, then we can choose $U_1$ properly such that*

a) *either $(\widetilde{\Lambda}_1, \widetilde{f}, \pi)$ has a trapping region,*
b) *or $\Lambda$ is contained in a homoclinic class $C$, $C$ contains periodic points with index 1 and it's an index 0 fundamental limit.*

**Proof** : Let $\widetilde{\Lambda} \subset \widetilde{\Lambda}_1$ satisfy $\widetilde{\Lambda} \times \{0\} = \pi^{-1}(\Lambda) \bigcap \widetilde{\Lambda}_1 \times \{0\}$, then $(\widetilde{\Lambda}, \widetilde{f}, \pi)$ is a central model for $(\Lambda, f)$. Since now, we just denote $\widetilde{\Lambda} \times \{0\}$ by $\widetilde{\Lambda}$.

At first, let's suppose $(\widetilde{\Lambda}, \widetilde{f}, \pi)$ has no trapping region, then by remark 5.6, for any small neighborhood $V$ of $\widetilde{\Lambda}$ in $\widetilde{\Lambda} \times [0,1]$, there exists a chain recurrent central segment $x \times I$ in $V$ respecting the map $\widetilde{f}$. By Crovisier's result ([Cr], proposition 3.6), there exits a family of periodic points $\{p_n\}$ such that they all belong to the same chain recurrent class with $\Lambda$ and $\lim_{n\to\infty} Orb(p_n) = \Lambda$, so $\Lambda \subset H(p_n, f)_{n\geq 1}$. When $n$ is big enough, $Orb(p_n) \subset \Lambda_1$, so $Orb(p_n)$ has a codimension-1 partial hyperbolic splitting $\widetilde{E}_1^c \oplus \widetilde{E}^u|_{Orb(p_n)}$, that means $p_n$ is an index 1 or 0 periodic point.

Now we claim that $H(p_n, f)$ is an index 0 fundamental limit.

**Proof of the claim**: The argument is exactly the same with the case $C$ in the proof of lemma 4.3, so here we just give a sketch of the proof, we divide the proof to two cases.

A) : there exists $\delta > 0$ such that for any $p_n$, we have $|Df^{\pi(p_n)}|_{\widetilde{E_1^c}(p_n)}| < e^{-\delta\pi(p_n)}$.
B) : for any $\frac{1}{m} > 0$, there exists $p_{n_m}$ such that $|Df^{\pi(p_{n_m})}|_{\widetilde{E_1^c}(p_{n_m})}| > e^{-\frac{1}{m}\pi(p_{n_m})}$.

In the first case, we use weakly selecting lemma, in case B, we use lemma 4.25. □

Now we suppose $(\widetilde{\Lambda}, \widetilde{f}, \pi)$ has a trapping region $S$, we can suppose $\widetilde{f}(Cl(s)) \subset Int(S)$ always. Choose $\widetilde{\Lambda}_2$ an open neighborhood of $\widetilde{\Lambda}$ in $\widetilde{\Lambda}_1$ small enough, we can get an open strip $S_2$ for $\widetilde{\Lambda}_2$ (here open respect $\widetilde{\Lambda}_2 \times [0,1]$) such that:

a) for any $\widetilde{x} \in \widetilde{\Lambda}$, $\widetilde{x} \times [0,1] \bigcap S = \widetilde{x} \times [0,1] \bigcap S_2$,
b) for any $\widetilde{x} \in \widetilde{\Lambda}_2$ and $\widetilde{f}(\widetilde{x}) \in \widetilde{\Lambda}_2$, we have $\widetilde{f}(Cl((\widetilde{x} \times [0,1]) \bigcap S_2)) \subset (\widetilde{f}(\widetilde{x}) \times [0,1]) \bigcap S_2$.

Choose $U^*$ neighborhood of $\Lambda$ small enough, let $\Lambda^* = \bigcap_{-\infty}^{\infty} f^i(\overline{U}^*)$, then $\Lambda^* \subset \Lambda_1$ and let $\widetilde{\Lambda}^* \subset \widetilde{\Lambda}_1$ satisfies $\widetilde{\Lambda}^* = \pi^{-1}(\Lambda^*) \bigcap \widetilde{\Lambda}_1$, we'll have $\widetilde{\Lambda}^* \subset \widetilde{\Lambda}_2$. Then consider the central model $(\widetilde{\Lambda}^*, \widetilde{f}, \pi)$ for $(\Lambda^*, f)$, $S_2 \bigcap (\widetilde{\Lambda}^* \times [0,1])$ is a trapping region for $(\widetilde{\Lambda}^*, \widetilde{f}, \pi)$.

Now replace $U_1$ by $U^*$ and $\Lambda_1$ by $\Lambda^*$, we get a trapping region for $(\widetilde{\Lambda}_1, \widetilde{f}, \pi)$. □

5.3. **Proof of lemma 5.1.** Now we suppose $\Lambda$ is a non-trivial minimal set with a codimension-1 partial hyperbolic splitting $E_1^c \oplus E^u$ where $dim(E_1^c) = 1$ and $E_1^c(\Lambda)$ is not hyperbolic. We divide the proof of lemma 5.1 into two cases: $E_1^c(\Lambda)$ has an $f$-orientation or not.

**Proof of lemma 5.1** ( $E_1^c(\Lambda)$ has an $f$-orientation)

Let $U_0$ be the small neighborhood of $\Lambda$ given in §5.1 such that we can extend the splitting $E_1^c \oplus E^u|_\Lambda$ to $\overline{U}_0$, we denote the splitting $T_xM = \widetilde{E}_1^c \oplus \widetilde{E}^u$ $(x \in \widetilde{U}_0)$. Suppose $U$ is any small neighborhood of $\Lambda$ such that $\overline{U} \subset U_0$, then from lemma 5.3, we can get open sets $U_2, U_1$ and $\Lambda_1 = \bigcap_{i=-\infty}^{\infty} f^i(\overline{U}_1)$, $a_0 > 0, 0 < \delta_{1,3} < \delta_{1,2} < \delta_{1,1} < \delta_0/2$ such that they satisfy properties P1-P11 of lemma 5.3 there.

Since $E_1^c(\Lambda)$ has an $f$-orientation, $\widetilde{E}_1^c(\Lambda_1)$ has an $f$-orientation also, by remark 5.10 we get two central models: the right central model $(\widetilde{\Lambda}_1^+, \widetilde{f}^+, \pi^+)$ and the left central model $(\widetilde{\Lambda}_1^-, \widetilde{f}^-, \pi^-)$, where for any $\widetilde{x}^+ \in \widetilde{\Lambda}_1^+$, $\pi^+(\widetilde{x}^+ \times [0,\infty))$ is a central curve at the right of $x = \pi^+(\widetilde{x}^+ \times \{0\})$ and $\delta_{1,2} < length(\pi^+(\widetilde{x}^+ \times [0,\infty))) < 2\delta_{1,1}$, so $\pi^+(\widetilde{x}^+ \times [0,\infty)) \subset B_{2\delta_{1,1}}(x) \subset U$. For any $\widetilde{x}^- \in \widetilde{\Lambda}^-$, we have the similar property.

At first, we consider the right central model $(\widetilde{\Lambda}_1^+, \widetilde{f}^+, \pi^+)$, if the right central model doesn't have trapping region, by lemma 5.11, $\Lambda$ is contained in a homoclinic class $H(p,f)$ which contains an index 1 periodic point and the homoclinic class is an index 0 fundamental limit, then we've proved lemma 5.1, so now we suppose that there exists a trapping region $S^+$ for the right central model. By the similar argument for the left central model, we can suppose it has a trapping region $S^-$ also.

**Claim**: $\Lambda$ is an index 0 fundamental limit.

**Proof of the claim:** If $\Lambda$ is not an index 0 fundamental limit, since $\Lambda$ has a codimension-1 dominated splitting, $\Lambda$ should be an index 1 fundamental limit. By generic property 5 of proposition 3.1, there exists a family of index 1 periodic points $\{p_n\}$ such that $\lim_{n\to\infty} Orb(p_n) = \Lambda$ and they are index stable, then by Gan's lemma, there exists a subsequence of periodic points $\{p_{n_m}\}_{m=1}^{\infty}$ in $C$. Now with the same argument of the case $C$ in the proof of lemma 4.3, we can show $\Lambda$ satisfies weakly selecting lemma, by weakly selecting lemma 4.21, $\Lambda$ is an index 0 fundamental limit, that's a contradiction. $\square$

Since $\Lambda$ is an index 0 fundamental limit, by generic property P5 of proposition 3.1, there exists a family of sources $\{p_n\}_{n=1}^{\infty}$ of $f$ satisfying $\lim_{n\to\infty} Orb(p_n) = \Lambda$. We can suppose $Orb(p_n) \subset U_2$ always and let $\widetilde{p}_n^i \in \widetilde{\Lambda}_1^i$ $_{(i=+,-)}$ such that $\pi^{(i)}(\widetilde{p}_n^i \times \{0\}) = p_n$, then $(\widetilde{f}^i)^{\pi(p_n)}(\widetilde{p}_n^i) = \widetilde{p}_n^i$. Denote $\widetilde{p}_n^{+(-)} \times I_n^{+(-)} = (\widetilde{p}_n^{+(-)} \times [0,\infty)) \bigcap S^{+(-)}$ and $\gamma_n^{+(-)} = \pi^{+(-)}(\widetilde{p}^{+(-)} \times I_n^{+(-)})$, let $\gamma_= \gamma_n^+ \bigcup \gamma_n^-$, then $\gamma_n$ is a central curve with center at $p_n$. Since $length(\gamma_n^{+(-)}) < 2\delta_{1,1}$, we have $\gamma_n \subset B_{2\delta_{1,1}}(p_n) \subset U_1$.

We've suppose $S^\pm$ is a trapping, then $\widetilde{f}^{+(-)}(\overline{S^{+(-)}}) \subset Int(S^{+(-)})$ or $(\widetilde{f}^{+(-)})^{-1}(\overline{S^{+(-)}}) \subset Int(S^{+(-)})$. In the first case, we say the trapping region is 1-step contracting, in the second case we say it's 1-step expanding. When $S^i$ is 1-step contracting case, we have $(\widetilde{f}^i)^{\pi(p_n)}(\widetilde{p}_n^i \times \overline{I}_n^i) \subset \widetilde{p}_n^i \times I_n^i$, so $f^{\pi(p_n)}(\overline{\gamma_n^i}) \subset \gamma_n^i$ for $i = +,-$ and there exists $\delta > 0$ doesn't depend on $n$ such that $length(\gamma_n^i \setminus f^{\pi(p_n)}(\overline{\gamma_n^i})) > \delta$ for all $n \geq 1$. If $S^i$ is 1-step expanding, we'll still have $length(\gamma_n^i \setminus f^{-\pi(p_n)}(\overline{\gamma_n^i})) > \delta$ for all $n \geq 1$.

Since $\gamma_n^i$ is either expanding or contracting for $f^{\pi(p_n)}$, let $\Gamma_n^i = \bigcap_{j=-\infty}^{\infty} f^{j\pi(p_n)}(\gamma_n^i)$ $_{(i=+,-)}$, we'll have $f^{\pi(p_n)}(\Gamma_n^i) = \Gamma_n^i$ $_{(i=+,-)}$ where $\Gamma_n^i$'s extreme points are periodic points. When $\Gamma_n^i$ is not trivial, we denote $q_n^i$ $_{(i=+,-)}$ the extreme periodic point different with $p_n$, if $\Gamma_n^i$ is trivial, we just let $q_n^i = p_n$. We let $\Gamma_n = \Gamma_n^+ \bigcup \Gamma_n^-$ and $h_n^i = \gamma_n^i \setminus \Gamma_i^n$ $_{(i=+,-)}$, then $\Gamma_n \subset \Lambda_1$, $h_n^i \subset U_1$. It's easy to know that $h_n^i$ is in the stable(unstable) manifold of $q_n^i$ if $S^i$ is 1-step contracting(expanding). And since $f$ is Kupka-Smale diffwomorphism, $f^{\pi(p_n)}|_{\Gamma_n}$ is also a Kupka-Smale diffeomorphisms and just has finite sinks and sources (respect $f^{\pi(p_n)}|_{\Gamma_n}$).

**Lemma 5.12.** *If $\Gamma_n \bigcap \Gamma_m \neq \phi$, then $\Gamma_n \bigcap \Gamma_m$ is a connected central curve, and $\Gamma_n \bigcup \Gamma_m$ is a central segment.*

**Proof** : We need prove some lemmas at first.

**Lemma 5.13.** *let $x \in \Gamma_n \bigcap \Gamma_m$ and $x$ is not a periodic point, $x_1 \in \Gamma_n$ is the nearest periodic point at the left of $x$ and $x_2 \in \Gamma_n$ is the nearest periodic point at the right of $x$. Denote $I_n \subset \Gamma_n$ the segment connecting $x_1$ and $x_2$, then $I_n \subset \Gamma_m$.*

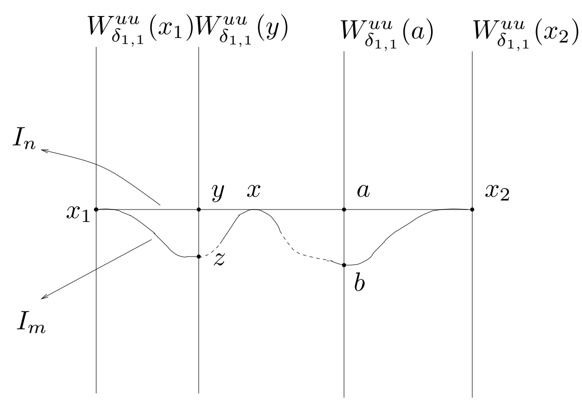
**Proof** : By the assumption, $f^{\pi(p_n)}$ has no any other fixed point in $I_n$, so for $x_1$ and $x_2$, one of them is sink for $f^{\pi(p_n)}|_{\Gamma_n}$ and another is source for $f^{\pi(p_n)}|_{\Gamma_n}$. We suppose $x_1$ is the source, then $\lim_{i\to\infty} f^{i\pi(p_n)}(x) \longrightarrow x_2$ and $\lim_{i\to\infty} f^{-i\pi(p_n)}(x) \longrightarrow x_1$. Since $\Gamma_m$ is a periodic central segment with period $\pi(p_m)$ and $x \in \Gamma_m$, we have $f^{i\pi(p_n)\pi(p_m)}(x) \in \Gamma_m$ for all $i \in \mathbb{Z}$, so $x_2 = \lim_{i\to\infty} f^{i\pi(p_n)\pi(p_m)}(x) \in \Gamma_m$ and $x_1 = \lim_{i\to\infty} f^{-i\pi(p_n)\pi(p_m)}(x) \in \Gamma_m$.

Now denote $I_m$ the central segment in $\Gamma_m$ connecting $x_1$ and $x_2$.

We claim that $I_n = I_m$.

**Proof of the claim**: If it's not true, there exists $y \in Int(I_n)$, $z \in W^{uu}_{\delta_{1,1}}(y) \bigcap I_m$ and $z \neq y$.

For any $\varepsilon > 0$, consider $a = f^{i\pi(p_n)\pi(p_m)}(y)$ where $i$ is very big, then $a \in I_n$ and it's near $x_2$ very much. Let $b \in W^{uu}_{\delta_{1,1}}(a) \bigcap I_m$, recall that $I_n$ and $I_m$ are tangent at $\widetilde{E^c_1}(x_2)$, when $i$ is big enough, there exists a curve $l$ in $W^{uu}_{\delta_{1,1}}(a)$ connecting $a$ and $b$ with $length(l) < \varepsilon$.




Now it's easy to know $f^{-i\pi(p_n)\pi(p_m)}(b) \in W^{uu}_{\delta_{1,1}}(y) \bigcap \Gamma_m$. By P4 of lemma 5.3, $\#\{W^{uu}_{\delta_{1,1}}(y) \bigcap \Gamma_m\} = 1$, so $f^{-i\pi(p_n)\pi(p_m)}(b) = z$, then $f^{-i\pi(p_n)\pi(p_m)}(l)$ is a curve connecting $y$ and $z$, by P3 of lemma 5.3, we'll have $length(f^{-i\pi(p_n)\pi(p_m)}(l)) < \varepsilon \cdot \lambda^{i\pi(p_n)\pi(p_m)}$.

Since $\varepsilon$ can be chosen arbitrarily small, we get $y = z$, that's a contradiction. $\square$

By the claim, we finish the proof of lemma 5.13. □

We still need the following result.

**Lemma 5.14.** *Let $x \in \Gamma_n \bigcap \Gamma_m$ and $x$ be a fixed point of $f^{\pi(p_n)}|_{\Gamma_n}$ and $f^{\pi(p_m)}|_{\Gamma_m}$, suppose $\Gamma_n$ and $\Gamma_m$ both have points on the right of $x$. Let $x_n \in \Gamma_n$ be the nearest fixed point of $f^{\pi(p_n)}|_{\Gamma_n}$ on the right of $x$ and $x_m \in \Gamma_m$ be the nearest fixed point of $f^{\pi(p_m)}|_{\Gamma_m}$ on the right of $x$. Denote $I_n \subset \Gamma_n$ the central segment in $\Gamma_n$ connecting $x$ and $x_n$, $I_m \subset \Gamma_m$ the central segment in $\Gamma_m$ connecting $x$ and $x_m$, then $I_n = I_m$.*

**Proof** : At first, we claim that either $W^{uu}_{\delta_{1,1}}(x_n) \bigcap I_m \neq \phi$ or $W^{uu}_{\delta_{1,1}}(x_m) \bigcap I_n \neq \phi$.
**Proof of the claim**: Suppose $W^{uu}_{\delta_{1,1}}(x_n) \bigcap I_m \neq \phi$, we know that $x_m$ is on the left of $W^{uu}_{\delta_{1,1}}(x_n)$, recall that $x_m$ is on the right of $x$, so by P9 of lemma 5.3, $W^{uu}_{\delta_{1,1}}(x_m) \bigcap I_n \neq \phi$. □

Now we suppose $W^{uu}_{\delta_{1,1}}(x_n) \bigcap I_m = y \neq \phi$, then $y \in I_m \setminus \{x\}$, it's easy to know $f^{-i\pi(p_n)\pi(p_m)}(y) \in W^{uu}_{\delta_{1,1}}(x_n) \bigcap I_m$ for $i \geq 1$, so $f^{-i\pi(p_n)\pi(p_m)}(y) = y$. But $\lim_{i\to\infty} f^{-i\pi(p_n)\pi(p_m)}(y) \longrightarrow x_n$, so $x_n = y$. It means that $x_n \in I_m \setminus \{x\}$, so $x_n = x_m$. By the same argument in lemma 5.13, we can prove $I_n = I_m$. □

Now let's continue the proof of lemma 5.12.

Let $\Gamma = \Gamma_n \bigcap \Gamma_m$, $x \in \Gamma$ be the left extreme point of $\Gamma$, then by lemma 5.13, $x$ should be a periodic point and on the left of $x$, there doesn't contain points of at least one of the segment $\Gamma_n$ or $\Gamma_m$. Let $y \in \Gamma$ be the right extreme point of $\Gamma$, then on the right of $y$, there doesn't contain points of at least one of the segments $\Gamma_n$ or $\Gamma_m$.

When $x = y$, $\Gamma_n$ and $\Gamma_m$ are on different side of $x$, $\Gamma_n \bigcup \Gamma_m$ is obviously a central segment.

When $x \neq y$, let $I$ be the maximal central curve in $\Gamma$ containing $x$, let $z$ be the right extreme point in $I$, by lemma 5.13, $z$ should be a periodic point. If $z \neq y$, $y$ is on the right of $z$ and $y \in \Gamma_n \bigcap \Gamma_m$, so by lemma 5.14, $I$ will contain a central segment on the right of $z$, that's a contradiction with the maximalicity of $I$, so $z = y$. It means that $I = \Gamma_n \bigcap \Gamma_m$ is an interval, and $x, y$ are its extreme points on the left and right, and $\Gamma_n$ and $\Gamma_m$ can not both have points on the left of $x$, they can not both have points on the right of $y$ also, it's easy to see now that $\Gamma_n \bigcup \Gamma_m$ is a central curve. □

Now we divide the proof of lemma 5.1 to three cases depending on the contracting or expanding properties of the two central models.

**Case A**: *Two central models have 1-step expanding properties.*

In this case, for any $\gamma_n$, we have $f^{-i}(\gamma_n) \in U_1$ for $i \geq 1$, it means $\gamma \subset \Lambda_1^+$, and any $x \in \gamma_n$ will have uniform size of unstable manifold $W^{uu}_{\delta_{1,1}}(x)$. Let $W^u_{\delta_{1,1}/2}(\gamma_n) = \bigcup_{x\in\gamma_n} W^{uu}_{\delta_{1,1}/2}(x)$, by the property of normally hyperbolic submanifold, $W^u_{\delta_{1,1}/2}(\gamma_n)$ is a submanifold ($dim = d$) with boundary, it's easy to know that $W^u_{\delta_{1,1}/2}(\gamma_n)$ has uniform size, that means there exists an $\varepsilon > 0$ such that $B_\varepsilon(p_n) \subset W^u_{\delta_{1,1}/2}(\gamma_n)$ for all $n \geq 1$. Suppose $\lim_{n\to\infty} p_n = p \in \Lambda$, then when $n$ is big enough, $p \in B_\varepsilon(p_n) \subset W^u_{\delta_{1,1}/2}(\gamma_n)$, then $\lim_{i\to\infty} f^{-i\pi(p_n)}(p_n) \longrightarrow$ some periodic point $z \in \Gamma_n$, so $z \in \Lambda$. Bust $\Lambda$ is a non-trivial minimal set of $f$, that's a contradiction. □

**Case B**: *Left central model is 1-step contracting and the right central model is 1-step expanding.*

Let's consider $\gamma_n^+$, with the same argument in case A, it has uniform size of unstable manifold $W^u_{\delta_{1,1}/2}(\gamma_n^+) = \bigcup_{x\in\gamma_n^+} W^{uu}_{\delta_{1,1}/2}(x)$ ( it's because $length(\gamma_n^+) > length(h_n^+) > \delta$), so there exists an $\varepsilon > 0$ such that $Vol(W^u_{\delta_{1,1}/2}(\gamma_n^+)) > \varepsilon$.

Now we claim that for any sequence $\{n_i\}_{i=1}^\infty$, there exists $i_0$ and a sequence $i_0 < i_1 < i_2 < \cdots$ such that for any $j > 0$, $W^u_{\delta_{1,1}/2}(\gamma^+_{n_{i_j}}) \bigcap W^u_{\delta_{1,1}/2}(\gamma^+_{n_{i_0}}) \neq \phi$.

**Proof of the claim**: Suppose that the claim is not true, then we can find a subsequence $\{n_{i_j}\}_{j=1}^\infty$ such that $W^u_{\delta_{1,1}/2}(\gamma^+_{n_{i_{j_0}}}) \bigcap W^u_{\delta_{1,1}/2}(\gamma^+_{n_{i_j}}) = \phi$ for $j_0 \in \mathbb{N}$ and $j > j_0$, it's a contradiction with $Vol(W^u_{\delta_{1,1}/2}(\gamma^+_{n_i})) > \varepsilon$, since we'll have $Vol(M) > \sum_j Vol(W^u_{\delta_{1,1}/2}(\gamma^+_{n_{i_j}})) = \infty$. $\square$

By the above claim, we can find a subsequence $\{n_i\}_{i=1}^\infty$ such that for any $i_0 \in \mathbb{N}^+$, we can get $W^u_{\delta_{1,1}/2}(\gamma^+_{n_i}) \bigcap W^u_{\delta_{1,1}/2}(\gamma^+_{n_{i_0}}) \neq \phi$ for $i \geq i_0$. Since $f$ is a Kupka-Smale diffeomorphism, on $\Gamma_{n_i}$ it just has finite periodic points. So when we fix $i_0$, we can let $i$ big enough such that $p_{n_i} \notin \gamma_{n_{i_0}}$. It means that we can choose a subsequence $\{(\Gamma_{n_i}, \Gamma_{m_i})\}_{i=0}^\infty$ such that $p_{m_i} \notin \Gamma_{n_i}$, $W^u_{\delta_{1,1}/2}(\gamma^+_{n_i}) \bigcap W^u_{\delta_{1,1}/2}(\gamma^+_{m_i}) \neq \phi$ and $\lim_{i\to\infty}(p_{n_i}) = \lim_{i\to\infty}(p_{m_i}) = x_0$ for some $x_0 \in \Lambda$.

Since $W^u_{\delta_{1,1}/2}(\gamma^+_{n_i}) \bigcap W^u_{\delta_{1,1}/2}(\gamma^+_{m_i}) \neq \phi$, suppose $y_i \in W^u_{\delta_{1,1}/2}(\gamma^+_{n_i}) \bigcap W^u_{\delta_{1,1}/2}(\gamma^+_{m_i})$, then

$$\lim_{j\to\infty} f^{-j\pi(p_{n_i})\pi(p_{m_i})}(y_i) \longrightarrow \Gamma^+_{n_i} \text{ and } \lim_{j\to\infty} f^{-j\pi(p_{n_i})\pi(p_{m_i})}(y_i) \longrightarrow \Gamma^+_{m_i},$$

so $\Gamma^+_{n_i} \bigcap \Gamma^+_{m_i} \neq \phi$, by lemma 5.12, $\Gamma_{n_i} \bigcup \Gamma_{m_i}$ is a central segment.

For simplicity, we suppose $p_{m_i}$ is on the right of $p_{n_i}$ for all $i \in \mathbb{N}$, the proof of the other case is similar. Since $p_{m_i} \notin \Gamma_{n_i}$ and $\Gamma_i = \Gamma_{n_i} \bigcup \Gamma_{m_i}$ is a central curve. $p_{m_i}$ is on the right of $q^+_{n_i}$ also. Recall that $q^+_{n_i}$ is a source for $f^{\pi(p_{n_i})}|_{\Gamma_{n_i}}$, and $h^+_{n_i}$ belongs to its basin, so $h^+_{n_i} \bigcap W^{uu}_{\delta_{1,1}/2}(p_{m_i}) = \phi$.

**Remark 5.15.** *: We don't know $h^+_{n_i} \subset \Gamma_{m_i}$ here.*

We know that $h^+_{n_i}$ is a central curve on the right of $q^+_{n_i}$ with length bigger than $\delta$, by property P6 of lemma 5.3, there exists a $\delta^*$ such that $d(q^+_{n_i}, p_{m_i}) > \delta^*$.( Since if $d(q^+_{n_i}, p_{m_i}) < \delta^*$, we have $l^+_\delta(q^+_{n_i}) \bigcap W^{uu}_{\delta_{1,1}/2} \neq \phi$ where $l^+_\delta(q^+_{n_i})$ is any central curve at the right of $q^+_{n_i}$ with length $\delta$ and $q^+_{n_i}$ is the left extreme point of it, with the fact that $p_{m_i}$ is on the right of $q^+_{n_i}$, we'll have $h^+_n \bigcap W^{uu}_{\delta_{1,1}/2}(p_{m_i}) \neq \phi$, that's a contradiction because $h^+_{n_i} \subset W^u(q^+_{n_i})$). So especially, in the central segment $\Gamma_i$, the distance between $p_{n_i}$ and $p_{m_i}$ is bigger than $\delta^*$. By property P7 of lemma 5.3, there exists $\delta^{**} > 0$ such that $d(p_{n_i}, p_{m_i}) > \delta^{**}$, it's a contradiction with $\lim_{i\to\infty}(p_{n_i}) = \lim_{i\to\infty}(p_{m_i}) = x_0 \in \Lambda$. $\square$

**Case C**: *The two central models have 1-step contracting properties.*

In this case, replace by a subsequence, we can suppose for $\{\Gamma_n\}_{n=1}^\infty$, we have $p_n \notin \bigcup_{i<n} \Gamma_i$.

**Lemma 5.16.** *There exists $n_0$ big enough such that for any $n_1, n_2 > n_0$, $n_1 \neq n_2$, we always have $W^u_{\delta_{1,1}/2}(\Gamma_{n_1}) \bigcap W^u_{\delta_{1,1}/2}(\Gamma_{n_2}) = \phi$.*

**Proof** Suppose the lemma is not true, then we can choose $n_1$ and $n_2$ arbitrarily big and satisfying $W^u_{\delta_{1,1}/2}(\Gamma_{n_1})\bigcap W^u_{\delta_{1,1}/2}(\Gamma_{n_2})\neq\phi$, then it's easy to know $\Gamma_{n_1}\bigcap\Gamma_{n_2}\neq\phi$ and $\Gamma_{n_1}\bigcup\Gamma_{n_2}$ is a central curve. We can suppose $n_2>n_1$, then by the assumption of $\{\Gamma_n\}_{n=1}^\infty$, we have $p_{n_2}\notin\Gamma_{n_1}$.

We just suppose $p_{n_2}$ is on the right of $p_{n_1}$, since $\Gamma=\Gamma_{n_1}\bigcup\Gamma_{n_2}$ is a central curve and $p_{n_2}\notin\Gamma_{n_1}$, we can know $p_{n_2}$ is on the right of $q^+_{n_1}$ also, and $q^+_{n_1}\in\Gamma_{n_2}$.

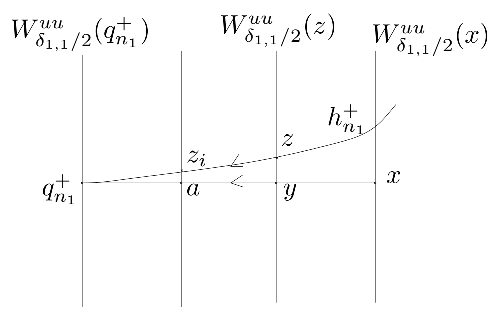
We know that there exists a $\delta>0$ such that $length(h_n^{+(-)})>\delta$ for all $n\geq 1$. And for such $\delta$, by proposition P6 of lemma 5.3, there exists $0<\delta^*\ll\delta$ such that for any $x,y\in\Lambda_1$, if $d(x,y)<\delta^*$, we have $\#\{W^{uu}_{\delta_{1,1}/2}(x)\bigcap l^c_\delta(y)\}=1$ where $l^c_\delta(y)$ is a central curve with center $y$ and on the two sides of $y$ both have length $\delta$.

Suppose $x\in\Gamma_m$ is the nearest periodic point on the right side of $q^+_{n_1}$, and let $I\subset\Gamma_m$ the central segment in $\Gamma_m$ connecting $q^+_{n_1}$ and $x$.

Now we claim that $length(I)>\delta^*$.

**Proof of the claim**: If $length(I)\leq\delta^*$, then $d(q^+_{n_1},x)\leq\delta^*$ also. By the facts that $x$ is on the right of $q^+_{n_i}$ and $h^+_{n_1}$ is a central curve with length bigger than $\delta$, we have $h^+_{n_1}\bigcap W^{uu}_{\delta_{1,1}/2}(x)\neq\phi$. Then for any $y\in Int(I)$, $W^{uu}_{\delta_{1,1}/2}(y)\bigcap h^+_{n_1}\neq\phi$.

It's easy to know $I\nsubseteq h^+_{n_1}$ since $h^+_{n_1}$ contains no periodic point, so there exists $z\in h^+_{n_1}$ such that $W^{uu}_{\delta_{1,1}/2}(z)\bigcap Int(I)=y\neq z$.




Because the two central models are 1-step contracting, $q^+_{n_1}$ is a sink for $f^{\pi(p_{n_1})}|_{\Gamma_{n_1}}$, then it's also a sink for $f^{\pi(p_{n_1})\pi(p_{n_2})}|_\Gamma$ where $\Gamma=\Gamma_{n_1}\bigcup\Gamma_{n_2}$. We can choose $i$ big enough, such that $z_i=f^{i\pi(p_{n_1})\pi(p_{n_2})}(z)$ near $q^+_{n_1}$ very much, let $a_i=W^{uu}_{\delta_{1,1}/2}(z_i)\bigcap I$. Since $h^+_{n_1}$ and $I$ are tangent at $q^+_{n_1}$ on $\widetilde{E^c_1}(q^+_{n_1})$, for any $\varepsilon>0$, when $i$ big enough, there exists a curve $l\subset W^{uu}_{\delta_{1,1}/2}(z_i)$ connecting $a_i$ and $z_i$ and $length(l)<\varepsilon$. Since $f^{-i\pi(p_{n_1})\pi(p_{n_2})}(a_i)\in W^{uu}_{\delta_{1,1}/2}(z)\bigcap I$, that means $f^{-i\pi(p_{n_1})\pi(p_{n_2})}(a_i)=y$ and $f^{-i\pi(p_{n_1})\pi(p_{n_2})}(l)$ is a curve connecting $z$ and $y$. By property P3 of lemma 5.3, $length(f^{-i\pi(p_{n_1})\pi(p_{n_2})}(l))<\varepsilon\lambda^i$. Since $i$ can be chosen arbitrarily big, we can get $y=z$, that's a contradiction. □

Since $length(I)>\delta^*$, the segment in $\Gamma$ connecting $p_{n_1}$ and $p_{n_2}$ will have length bigger than $\delta^*$ also, by property P7 of lemma 5.3, there exists $\delta^{**}>0$ such that $d(p_{n_1},p_{n_2})>\delta^{**}$. But recall that $\lim\limits_{n\to\infty}p_n\longrightarrow x_0\in\Lambda$ and $n_1,n_2$ can be chosen arbitrarily big, we can get $d(p_{n_1},p_{n_2})<\delta^{**}$, that's a contradiction. □

With lemma 5.16, we can chosen $\{\Gamma_n\}_{n=1}^\infty$ such that if $n\neq m$, $W^u_{\delta_{1,1}/2}(\Gamma_n)\bigcap W^u_{\delta_{1,1}/2}(\Gamma_n)=\phi$. Then by property P11 of lemma 5.3, $\lim\limits_{n\to\infty}length(\Gamma_n)=0$.

Choose $n_0$ big enough such that for $m \geq n_0$, $d(p_m, p_{n_0}) < \delta^*/4$ and $length(\Gamma_m) < \delta^*/4$, we can suppose $p_m$ is on the right of $p_{n_0}$, then by $W^u_{\delta_{1,1}/2}(\Gamma_n) \bigcap W^u_{\delta_{1,1}/2}(\Gamma_n) = \phi$, we know that $p_m$ is on the right of $q^+_{n_0}$ and $q^-_m$ is on the right of $q^+_{n_0}$ also.

Since $d(q^+_{n_0}, q^-_m) \leq d(q^+_{n_0}, p_{n_0}) + d(q^-_m, p_m) + d(p_{n_0}, p_m) < length(\Gamma_{n_0}) + \delta^*/4 + length(\Gamma_m) < \delta^*$, by Property P6 of lemma 5.3 and $length(h^+_{n_0}) > \delta$, $length(h^-_m) > \delta$, we can get $h^+_{n_0} \pitchfork W^{uu}_{\delta_{1,1}/2}(q^-_m) \neq \phi$ and $h^+_m \pitchfork W^{uu}_{\delta_{1,1}/2}(q^+_{n_0}) \neq \phi$. Recall that $h^+_{n_0} \subset W^s(q^+_{n_0})$ and $h^-_m \subset W^s(q^-_m)$, we can know $q^+_{n_0}$ and $q^-_m$ are in the same homoclinic class.

When $m \longrightarrow \infty$, by $length(\Gamma_m) \longrightarrow 0$ and $\lim_{m\to\infty} p_m \longrightarrow x_0 \in \Lambda$, we have $q^-_m \longrightarrow x_0$ also, so $x \in H(q^+_{n_0}, f)$ and then $\Lambda \subset H(q^+_{n_0}, f)$.

Now we'll prove $H(q^+_{n_0}, f)$ is an index 0 fundamental limit.

Recall that $Orb(q^+_{n_0}) \subset U$ and $U$ can be chosen arbitrarily small, so in fact we've proved that there exists a family of periodic points $q_n$ with index 1 such that $\lim_{n\to\infty} Orb(q_n) = \Lambda$ and $\Lambda \subset H(q_1, f) = H(q_2, f) = \cdots$.

By the same argument with case C in the proof of lemma 4.3, we can prove $H(q_1, f)$ is an index 0 fundamental limit. □

Now let's keep on proving the other case of lemma 5.1.

**Proof of lemma 5.1**($E^c_1(\Lambda)$ has no any $f$-orientation):

In this case, we just have one central model, but locally we still have orientation for $\widetilde{E^c_1}(\Lambda_1)$, and the two sides have the same dynamical property: they are both 1-step expanding or they are both 1-step contracting. All the other argument is the same with the case where $E^c_1(\Lambda)$ has an $f$-orientation. □

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