The Onset of Dominance in Balls-in-Bins Processes with Feedback

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Abstract

Consider a balls-in-bins process in which each new ball goes into a given bin with probability proportional to $f(n)$, where n is the number of balls currently in the bin and f is a fixed positive function. It is known that these so-called *balls-in-bins processes with feedback* have a monopolistic regime: if $f(x) = x^p$ for $p > 1$, then there is a finite time after which one of the bins will receive all incoming balls.

Our goal in this paper is to quantify the onset of monopoly. We show that the initial number of balls is large and bin 1 starts with a fraction $\alpha > 1/2$ of the balls, then with very high probability its share of the total number of balls never decreases significantly below α. Thus a bin that obtains more than half of the balls at a "large time" will most likely preserve its position of leadership. However, the probability that the winning bin has a non-negligible advantage after n balls are in the system is $\sim$ const. $\times\, n^{1-p}$, and the number of balls in the losing bin has a power-law tail. Similar results also hold for more general functions f.

1 Introduction

Consider a discrete-time Markov process with B bins, each one of which containing $I_i(m) > 0$ balls at time m for each $m \in \{0, 1, 2, \ldots\}$ and $i \in \{1, \ldots, B\}$. Its evolution is as follows: at each time $m > 0$, a ball is added to a bin i_m, so that $I_{i_m}(m) = I_{i_m}(m-1) + 1$ and $I_j(m) = I_j(m-1)$ for all $i \in \{1, \ldots, B\}\backslash\{i_m\}$, and the random choice of bin i_m has distribution

$$\mathbf{Pr}\left(i_m = i \mid \{I_j(m-1) \,:\, 1 \leq j \leq B\}\right) = \frac{f(I_i(m-1))}{\sum_{j=1}^{B} f(I_j(m-1))} \quad (1 \leq i \leq B), \tag{1}$$

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for a fixed positive function $f : \mathbf{N} \to (0, +\infty)$. This recipe specifies what we call a *balls-in-bins process with feedback function f and B bins*, and one should notice that when f is an increasing function — the case that has been mostly considered in the literature — there is a tendency that *the rich get richer*: the more balls a bin has, the more likely it is to receive the next ball.

This class of processes[1] was proposed by Drinea, Frieze and Mitzenmacher [4] as a model for competing products in an Economy, as well as a simpler variant of so-called preferential-attachment models for large networks (see [1] for a survey). That paper concentrates on the special case where $f(x) = x^p$ for some parameter $p > 0$. The authors prove that when $p > 1$, there almost surely exists one bin that gets all but a negligible fraction of the balls in the large-time limit; whereas for $p < 1$, the asymptotic fractions of balls in each bin are all the same. The $p = 1$ case is the classic Pólya Urn model, for which it has been long known that the number of balls in each bin converges almost surely to a non-degenerate random variable, and thus the process has different *regimes* depending on whether $p < 1$, $p = 1$ or $p > 1$.

However, stronger results on the $p > 1$ case are available. A paper by Khanin and Khanin [5] introduced what amounts to the same process as a model for neuron growth, and proved the following stronger result: if $p > 1$, there almost surely is some bin that gets *all but finitely many balls*. That is to say, consider the following event, in which bin i is the only one to receive balls after some finite time M (we call this *monopoly by bin i*).

$$\begin{aligned} \mathsf{Mon}_i &\equiv \{\exists M \in \mathbf{N}\, \forall m \geq M \;\; \forall j \in [B] \;\; j \neq i \Rightarrow I_m(j) = I_M(j)\} && (2) \\ &= \{\exists M \in \mathbf{N}\, \forall m \geq M \;\; i_m = i\}. && (3) \end{aligned}$$

The result of [5] – or rather, a straightforward extension of it proven in [10, 7] – says that

Theorem 1 (From [5, 10, 7]) *If $\{I_m\}_{m=0}^{+\infty}$ is a balls-in-bins process with B bins and feedback function $f = f(x) > 0$ satisfying*

$$\sum_{j=1}^{+\infty} \frac{1}{f(j)} < +\infty. \tag{4}$$

satisfies

$$\mathbf{Pr}\left(\exists i \in [B] \,:\, \mathsf{Mon}_i\right) = 1.$$

In particular, this holds for $f(x) = x^p$, $p > 1$.

This is a much stronger statement than the one that Drinea et al. proved in [4], and we take it as our starting point in the present paper. We sketch a proof of Theorem 1 in Section 3.2 below, since it helps to build some intuition for our own results.

[1] A longer background discussion is available from the first author's PhD thesis [7].

Our main goal is to quantify the monopoly phenomenon in the case where there are $B = 2$ bins. As powerful as Theorem 1 is, it tells us nothing about *how fast* the system approaches this asymptotic regime. Informally, we will be interested in questions like: suppose we throw in a million balls into $B = 2$ bins. Should we expect to find a bin with 10% more balls than the other bin? And in case that does happen, is the leading bin likely to lose its lead as we add more and more balls into the system? This paper treats rigorous forms of those problems, for a broad (but not entirely general) class of feedback functions. A summary of our results is given below.

1. *Imbalanced start.* Start the process with a total of $t \gg 1$ balls in the two bins and at least $\alpha = 52\%$ of the balls in bin 2. We show in Section 5 that with very high probability, there is no future time at which bin 2 will have $\beta = 51\%$ of the total number of balls in this balls-in-bins process. A similar result holds for any other $\alpha > \beta > 1/2$.

2. *Balls in the losing bin.* Let L be the number of balls that go into the *losing* bin, i.e. the bin that does not achieve monopoly, when the initial number of balls is fixed. We prove in Section 6 that the distribution of L has a *heavy tail.* More concretely, if $f(x) = x^p$ for some $p > 1$, $\mathbf{Pr}(L > n) \sim c_p \times n^{1-p}$ for large n.

3. *The time until imbalance.* We show in Section 7 that for fixed initial conditions, the probability that the losing bin has at least αn balls at time n (for $\alpha < 1/2$ fixed) also decays slowly in n. In particular, in the case $f(x) = x^p$, this probability is $\sim c'_p \times n^{1-p}$ for n large; that is, we have a power law with the same exponent as in 2.. An extension of this result is presented in Section 8.

The picture that emerges from these results is that *it takes a long time before a clear leader emerges, but once it does, it is likely to stick.* One indication of the heavy-tail part of our results was in the numerical simulations of [4], which indicated that the a clear leader of the process took a long time to emerge. Reference [4] also showed that once the leader emerges, it achieves dominance very quickly, but our results regrading this (cf. 1.) are stronger.

Our main technical tool has been employed in [5, 10] and other works, and seems to have originated in Davis' work on reinforced random walks [3]. We shall *embed* the discrete-time process we are interested in into a continuous-time process built from exponentially distributed random variables. The most salient feature of this so-called *exponential embedding* is that arrival times at different bins are independent and have an explicit distribution. This greatly simplifies calculations and permits the use of Chernoff-like bounds that we develop below.

The remainder of the paper is organized as follows. We discuss preliminary material in Section 2. Section 3 rigorously introduces the exponential embedding process and discusses its key properties (including Theorem 1). In Section 4 we detail the assumptions we make on our feedback functions f, while also deriving some consequences of those assumptions. The next three sections correspond to items 1.—3. . Section 8 discusses possible extensions to our results and some related work.

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2 Preliminaries

Set notation. Throughout the paper, $\mathbf{N} = \{1, 2, 3, \ldots\}$ is the set of non-negative integers, $\mathbf{R}^+ = [0, +\infty)$ is the set of non-negative reals, and for any $k \in \mathbf{N}\backslash\{0\}$ $[k] = \{1, \ldots, k\}$.

Balls-in-bins. Formally, a balls-in-bins process with feedback function $f : \mathbf{N} \to (0 + \infty)$ and $B \in \mathbf{N}$ bins is a discrete-time Markov chain $\{(I_1(m), \ldots, I_B(m))\}_{m=0}^{+\infty}$ with state space $\mathbf{N}^B$ and transition probabilities given in the Introduction (see (1)). We will usually refer to the index $i_m \in [B]$ as *the bin that receives a ball at time* m.

If $B = 2$, $t \in \mathbf{N}$ and $0 \leq \alpha \leq 1$, we sometimes denote by $[t, \alpha]$ the state $(\lceil \alpha t \rceil, t - \lceil \alpha t \rceil) \in \mathbf{N}^2$, i.e. there are t bins and an α-fraction of them is in bin 1 (with rounding). This alternative notation will be used whenever convenient.

For any B, if E is an event of the process and $u \in \mathbf{N}^B$, $\mathbf{Pr}_u(E)$ is the probability of E when the initial conditions are set to u. The same notation will be used for the exponential embedding defined in Section 3.

Exponential random variables. $X =^d \exp(\lambda)$ means that X is a random variable with exponential distribution with rate $\lambda > 0$, meaning that $X \geq 0$ and

$$\mathbf{Pr}(X > t) = e^{-\lambda t} \quad (t \geq 0).$$

The shorthand $\exp(\lambda)$ will also denote a generic random variable with that distribution. Some elementary but extremely useful properties of those random variables include

1. *Lack of memory.* Let $X =^d \exp(\lambda)$ and $Z \geq 0$ be independent from X. The distribution of $X - Z$ conditioned on $X > Z$ is still equal to $\exp(\lambda)$.

2. *Minimum property.* Let $\{X_i =^d \exp(\lambda_i)\}_{i=1}^m$ be independent. Then

$$X_{\min} \equiv \min_{1 \leq i \leq m} X_i =^d \exp(\lambda_1 + \lambda_2 + \ldots \lambda_m)$$

and for all $1 \le i \le m$

$$\mathbf{Pr}\left(X_i = X_{\min}\right) = \frac{\lambda_i}{\lambda_1 + \lambda_2 + \ldots \lambda_m} \tag{5}$$

3. *Multiplication property.* If $X =^d \exp(\lambda)$ and $\eta > 0$ is a fixed number, $\eta X =^d \exp(\lambda/\eta)$.

4. *Moments and transforms.* If $X =^d \exp(\lambda)$, $r \in \mathbf{N}$ and $t \in \mathbf{R}$,

$$\mathbf{Ex}\left[X^r\right] = \frac{r!}{\lambda^r}, \tag{6}$$

$$\mathbf{Ex}\left[e^{\sqrt{-1}tX}\right] = \frac{1}{1 - \frac{\sqrt{-1}t}{\lambda}} \tag{7}$$

$$\mathbf{Ex}\left[e^{tX}\right] = \begin{cases} \frac{1}{1-\frac{t}{\lambda}} & (t < \lambda) \\ +\infty & (t \ge \lambda) \end{cases} \tag{8}$$

3 The exponential embedding

3.1 Definition and key properties

Let $f : \mathbf{N} \to (0, +\infty)$ be a function, $B \in \mathbf{N}$ and $(a_1, \ldots, a_B) \in \mathbf{N}^B$. We define below a continuous-time process with state space $(\mathbf{N} \cup \{+\infty\})^B$ and initial state $(a_1, \ldots, a_B)$ as follows. Consider a set $\{X(i,j) : i \in [B], j \in \mathbf{N}\}$ of independent random variables, with $X(i,j) =^d \exp(f(j))$ for all $(i,j) \in [B] \times \mathbf{N}$, and define

$$N_i(t) \equiv \sup\left\{ n \in \mathbf{N} : \sum_{j=a_i}^{n-1} X(i,j) \le t \right\} \quad (i \in [B], t \in \mathbf{R}^+ = [0, +\infty)), \tag{9}$$

where by definition $\sum_{j=i}^{k}(\ldots) = 0$ if $i > k$. Thus $N_i(0) = a_i$ for each $i \in [B]$, and one could well have $N_i(T) = +\infty$ for some finite time T (indeed, that *will* happen for our cases of interest); but in any case, the above defines a continuous-time stochastic process, and in fact the $\{N_i(\cdot)\}_{i=1}^{B}$ processes are independent. Each one of this processes is said to correspond to *bin* i, and each one of the times

$$X(i, a_i), X(i, a_i) + X(i, a_i + 1), X(i, a_i) + X(i, a_i + 1) + X(i, a_i + 2), \ldots$$

is said to be an *arrival time at bin* i. As in the balls-in-bins process, we imagine that each arrival correspond to a ball being placed in bin i.

In fact, we *claim* that this process is related as follows to the balls-in-bins process with feedback function f, B bins and initial conditions $(a_1, \ldots, a_B)$.

Theorem 2 (Proven in [3, 5, 10, 7, 8]) *Let the $\{N_i(\cdot)\}_{i\in[B]}$ process be defined as above. One can order the arrival times of the B bins in increasing order (up to their first accumulation point, if they do accumulate) so that $T_1 < T_2 < \dots$ is the resulting sequence. The distribution of*

$$\{I_m = (N_1(T_m), N_2(T_m), \dots, N_B(T_m))\}_{m\in\mathbf{N}}$$

is the same as that of a balls-in-bins process with feedback function f and initial conditions $(a_1, a_2, \dots, a_B)$.

One can prove this result[2] as follows. First, notice that the *first arrival time* T_1 is the minimum of $X(j, a_j)$, $(1 \le j \le B)$. By the minimum property presented above, the probability that bin i is the one at which the arrival happens is like the first arrival probability in the corresponding balls-in-bins process with feedback:

$$\mathbf{Pr}\left(X(i, a_i) = \min_{1\le j\le B} X(j, a_j)\right) = \frac{f(a_i)}{\sum_{j=1}^{B} f(a_j)}. \tag{10}$$

More generally, let $t \in \mathbf{R}^+$ and condition on $(N_i(t))_{i=1}^B = (b_i)_{i=1}^B \in \mathbf{N}^B$, with $b_i \ge a_i$ for each i (in which case the process has not blown up). This amounts to conditioning on

$$\forall i \in [B] \ \sum_{j=a_i}^{b_i-1} X(i, b_i) \le t < \sum_{j=a_i}^{b_i} X(i, b_i).$$

From the lack of memory property of exponentials, one can deduce that the first arrival after time t at a given bin i will happen at a $\exp(f(b_j))$-distributed time, independently for different bins. This almost takes us back to the situation of (10), with b_i replacing a_i, and we can similarly deduce that bin i gets the next ball with the desired probability,

$$\frac{f(b_i)}{\sum_{j=1}^{B} f(b_j)}.$$

3.2 The occurrence of monopoly

The exponential embedding yields a "Book proof" of Theorem 1. Without loss of generality, we can assume that $B = 2$ (the general case follows from comparing pairs of bins). The notation we use comes from the previous section, and we also employ the version of the balls-in-bins process given by $I_m = (N_i(T_m))_{i=1,2}$ (cf. Theorem 2).

Under the condition that

$$\sum_{j=1}^{+\infty} \frac{1}{f(j)} < +\infty, \tag{11}$$

[2] The exact attribution of this result is somewhat confusing. Ref. [5] cites the work of Davis [3] on reinforced random walks, where it is in turn attributed to Rubin.

one has

$$F_i \equiv \sum_{j=a_i}^{+\infty} X(i,j) < +\infty \text{ almost surely } (i = 1,2). \tag{12}$$

Indeed, the terms in F_i are all positive and since $X(i,j) =^d \exp(f(j))$ for all i, j,

$$\mathbf{Ex}[F_i] = \sum_{j=a_i}^{+\infty} \frac{1}{f(j)} < +\infty \text{ by (11)}.$$

It is also easy to see that the F_i's are independent and have no point masses in their distributions. Thus with probability 1, either $F_1 < F_2$ or $F_2 < F_1$.

Suppose that the first alternative holds. Since $\sum_{j=a_i}^{N-1} X(i,j) \nearrow F_i$ as $N \to +\infty$, we can deduce that

$$\exists n_2 \in \mathbf{N} \quad \forall n \geq a_1 \quad A_{1,n} \equiv \sum_{j=a_1}^{n-1} X(1,j) < F_1 < A_{2,n_2} \equiv \sum_{\ell=a_2}^{n_2-1} X(2,j). \tag{13}$$

The sequence $\{A_{1,n}\}_{n\in\mathbf{N}}$ is composed of arrival times; that is to say, it is a subsequence of $\{T_m\}_{m\in\mathbf{N}}$. Moreover, that sequence converges to F_1. It follows that the first accumulation point of the sequence $\{T_m\}_{m\in\mathbf{N}}$ is at most F_1, and that $T_m \leq F_1$ for all m. But since $A_{2,n_2} > F_1$, this implies that for all times m of the discrete-time process,

$$I_m(2) = N_2(T_m) < n_2.$$

Thus if $F_1 < F_2$, there exists a finite $n_2 \in \mathbf{N}$ such that $I_m(2) < n_2$ for all m, i.e. bin 2 never has more than n_2 balls. This implies that bin 1 *must* achieve monopoly whenever $F_1 < F_2$.

If $F_1 > F_2$, the same reasoning shows that bin 2 achieves monopoly. As pointed out above, with probability 1 either $F_1 < F_2$ or $F_2 > F_1$; thus the proof is finished.

Remark 1 *It is not hard to show that $F_i = +\infty$ almost surely if $\sum_j f(j)^{-1} = +\infty$, and in that case monopoly has probability* 0*. The interested reader can see [5, 7, 8] for details.*

Remark 2 *Assume that bin* 1 *achieves monopoly, as in the proof above. In this case, all arrivals of the* continuous-time *process at bin* 2 *after time F_1 do not actually happen in the embedded* discrete-time *process $\{I_m = (I_m(1), I_m(2))\}$. We call these "ghost events" a* fictitious continuation *of our process. This very useful device is akin to the continuation of a Galton-Watson process beyond its extinction time (see e.g. [2]) and is equally useful in calculations and proofs.*

4 Assumptions on feedback functions and a large-deviations bound

The purpose of this rather technical section is two-fold. First, we spell out the technical assumptions on the feedback function f that we need in our proofs. Nothing seems to actually *require* these assumptions, but they facilitate certain estimates that we employ in the proofs.

The second purpose is to present a large-deviations bound on random sums such as $\sum_{j=m}^{+\infty} X(i,j)$. One can check that the *variance* of such a sum decreases as $m \to +\infty$; we show below that these sums are in fact close to their means with all-but-exponential probability.

Some readers might wish to skip the proofs in this section on a first reading.

4.1 Valid feedback functions

The feedback functions we allow in our results satisfy the following definition.

Definition 1 *An increasing function $f : \mathbf{N} \to (0,+\infty)$ with $f(1) = 1$[3] is said to be a* valid feedback function *if it can be extended to a piecewise C^1 function $g : \mathbf{R}^+ \cup \{0\} \to (0,+\infty)$ with the following property: if $(\ln g(\cdot))'$ is the right-derivative of $\ln g$, and $h(x) \equiv x(\ln g(x))'$ (for $x \in \mathbf{R}^+ \cup \{0\}$),*

1. $\liminf_{x\to+\infty} h(x) > 1$*;*
2. $\lim_{x\to+\infty} x^{-1/4} h(x) = 0$*;*
3. *there exist $C > 0$ and $x_0 \in \mathbf{R}^+$ such that for all $\epsilon \in (0,1)$ and all $x \geq x_0$*

$$\sup_{x \leq t \leq x^{1+\epsilon}} \left| \frac{h(t)}{h(x)} - 1 \right| \leq C\,\epsilon. \tag{14}$$

With slight abuse of notation, we will always assume that f is defined over $\mathbf{R}^+ \cup \{0\}$ and is piecewise C^1. We will also call h the characteristic exponent *of f.*

Functions with exponential growth (such as $f(x) = 2^x$) or with oscillations fail to satisfy Definition 1. On the other hand, requiring that f be increasing seems natural, and the smoothness assumption still leaves us with plenty of interesting examples of feedback functions. For instance, any function in one of the following categories

$$\begin{aligned} f(x) &= x^p \ (x \geq 1, p > 1), & (15)\\ f(x) &= x^{p \ln^\alpha x} \ (x \geq 1, p > 1, \alpha > 0), & (16)\\ f(x) &= x^p \ln(x+3) \ (x \geq 1, p > 1). & (17) \end{aligned}$$

[3]The requirement that $f(1) = 1$ is just a normalization condition, as it does not change the process.

are all smooth. The "canonical case" where $f(x) = x^p$ $(x \geq 1)$ explains the terminology for the characteristic exponent: in that case, $h(x) \equiv p$ for all $x > 1$.

4.2 Consequences of the definition

Let us now define the quantity

$$S_r(n,m) \equiv \sum_{j=n}^{m-1} \frac{1}{f(n)^r} \quad (r \in \mathbf{N}\backslash\{0\};\, n \in \mathbf{N}, m \in \mathbf{N} \cup \{+\infty\}) \tag{18}$$

for some $f : \mathbf{N} \to (0, +\infty)$, and also let $S_r(n) \equiv S_r(n, +\infty)$. If $f(x) = x^p$, then for $n \gg 1$ a simple calculation shows that for $m > n \gg 1$

$$S_r(n,m) \sim \int_n^m \frac{dx}{f(x)^r} = \frac{n^{1-rp}}{(rp-1)}.$$

The main content of the following lemma is that a similar result holds for any valid f, if p is replaced by the characteristic exponent h. In particular, any valid f satisfies the monopoly condition in Theorem 1.

Lemma 1 *Assume that f is a valid feedback function with characteristic exponent h. Define*

$$M_r(n) = \int_n^{+\infty} \frac{dx}{f(x)^r} \qquad (r \in \mathbf{N}\backslash\{0\}, n \in \mathbf{N}).$$

Then, as $n \to +\infty$

$$S_r(n) \sim M_r(n) \sim \frac{n}{(rh(n)-1)f(n)^r}.$$

and for all fixed $\rho > 1$ there exists $c = c_\rho < 1$ that is a decreasing function of ρ such that

$$S_r(\lceil \rho n \rceil) \leq c_\rho \frac{n}{(rh(n)-1)f(n)} \sim c_\rho S_r(n) \text{ for all large } n.$$

Thus for any $\rho > 1$

$$S_r(n) - S_r(\lceil \rho n \rceil) = \Omega\left(\frac{n}{(rh(n)-1)f(n)}\right).$$

Before we present the proof of Lemma 1, we state two other lemmas. They follow directly from the fact that $h(x) \ll x^{1/4}$ by assumption, and we omit their proofs.

Lemma 2 *For any fixed $r \in \mathbf{N}\backslash\{0\}$, $S_1(n) \gg e^{-n^{1/4}}$ as $n \to +\infty$.*

Lemma 3 *For any bounded function $w : \mathbf{N} \to \mathbf{R}$, $f(n + w(n)) \sim f(n)$ and $h(n + w(n)) \sim h(n)$ as $n \to +\infty$.*

Proof: [of Lemma 1] Essentially the same proof appears in the last result in [6], but we reproduce the argument here for convenience. Under our assumptions, $F(\cdot) \equiv f(\cdot)^r$ is a valid feedback function with characteristic exponent $rh(\cdot)$. Clearly, the lemma holds for f iff it holds for F with $r = 1$. It follows that it suffices to prove the lemma in the case $r = 1$, which is what we do below.

We can assume that n is large enough, so that $\inf_{x\geq n} h(x) = c > 1$, and moreover

$$\forall x \geq n, \forall \epsilon > 0 \quad \sup_{x\leq t\leq x^{1+\epsilon}} \left|\frac{h(t)}{h(x)} - 1\right| \leq C\epsilon \tag{19}$$

which follows from the assumptions on h in Definition 1. We start by noting that for all $y > x \geq n$

$$\frac{f(y)}{f(x)} \geq \left(\frac{y}{x}\right)^c \tag{20}$$

To see this, it suffices to notice that

$$\ln\frac{f(y)}{f(x)} = \int_x^y (\ln f(u))'\,du = \int_x^y \frac{h(u)}{u}\,du \geq (\inf_{x\leq u\leq y} h(u))\,\ln\frac{y}{x} \geq c\ln\frac{y}{x}.$$

Inequality (20) implies that $f(x) \gg x$ as $x \to +\infty$. This justifies the following integration by parts procedure.

$$\begin{aligned} M_1(n) &= \int_n^{+\infty} \frac{dx}{\exp(\ln f(x))} && (21)\\ &= \left.\frac{x}{f(x)}\right|_{x=n}^{x\to+\infty} + \int_n^{+\infty} \frac{x(\ln f(x))'\,dx}{\exp(\ln f(x))} && (22)\\ &= -\frac{n}{f(n)} + \int_n^{+\infty}\frac{h(x)\,dx}{f(x)} && (23)\end{aligned}$$

where in the last line we plugged in the definition of h. We now *claim* that

$$\int_n^{+\infty}\frac{h(x)\,dx}{f(x)} \sim h(n)\int_n^{+\infty}\frac{dx}{f(x)} = h(n)\,M_1(n) \text{ as } n \to +\infty. \tag{24}$$

We will prove (24) below, but first we show how it implies the lemma. Employing (24) with equations (21) to (23) shows that

$$M_1(n) \sim \frac{n}{(h(n)-1)f(n)} \text{ as } n \to +\infty, \tag{25}$$

since $h(n) \geq c > 1$ for n large (as discussed above). By the smoothness assumption, $h(n)/n \to 0$ as $n \to +\infty$, and therefore

$$\frac{n}{(h(n)-1)f(n)} \gg \frac{1}{f(n)} \text{ as } n \to +\infty. \tag{26}$$

Now notice that, since f is increasing,

$$-\frac{1}{f(n)} \leq M_1(n) - S_1(n) \leq 0,$$

hence $|M_1(n) - S_1(n)| \leq f(n)^{-1}$ and by equations (25) and (26),

$$S_1(n) \sim M_1(n) \sim \frac{n}{(h(n)-1)f(n)} \text{ as } n \to +\infty,$$

as desired. Moreover, if $\rho > 1$ is fixed, equations (19) and (20) imply that, for large n,

$$h(\lceil \rho n \rceil) \sim h(n) \text{ and } f(\lceil \rho n \rceil) \geq \rho^c f(n) \text{ with } c \text{ as above,}$$

whence

$$\frac{\rho n}{(h(\lceil \rho n \rceil)-1)f(\lceil \rho n \rceil)} \leq (1+o(1))\rho^{1-c}\frac{n}{(h(n)-1)f(n)},$$

from which the desired statement about $S_r(\lceil \rho n \rceil)$ follows.

We now prove (24). Choosing an arbitrary (but fixed) $\epsilon > 0$, we first show that

$$\text{as } n \to +\infty, \int_{n^{1+\epsilon}}^{+\infty} \frac{h(x)\,dx}{f(x)} \ll \int_n^{+\infty} \frac{h(x)\,dx}{f(x)}. \tag{27}$$

Indeed,

$$\begin{aligned}
\int_{n^{1+\epsilon}}^{+\infty} \frac{h(x)\,dx}{f(x)} &= (1+\epsilon)\int_n^{+\infty} \frac{h(u^{1+\epsilon})\,u^\epsilon\,du}{f(u^{1+\epsilon})} &(28)\\
&\leq (1+\epsilon)(1+C\epsilon)\int_n^{+\infty} \frac{h(u)\,u^\epsilon\,du}{f(u^{1+\epsilon})} &(29)\\
&\leq (1+\epsilon)(1+C\epsilon)\int_n^{+\infty} \frac{h(u)\,u^\epsilon\,du}{f(u)u^{c\epsilon}} &(30)\\
&\leq (1+\epsilon)(1+C\epsilon)n^{(1-c)\epsilon}\int_n^{+\infty} \frac{h(u)\,du}{f(u)} &(31)\\
&\ll \int_n^{+\infty} \frac{h(u)\,du}{f(u)}, &(32)
\end{aligned}$$

where the first line is a change of variables, the second line employs (19), the third line uses (20) applied to $x = u$ and $y = u^{1+\epsilon}$, and the remaining lines follow from $c > 1$. The sequence of equations proves (27), which implies in particular that, for n large,

$$\int_n^{+\infty} \frac{h(x)\,dx}{f(x)} \sim \int_n^{n^{1+\epsilon}} \frac{h(x)\,dx}{f(x)}$$

But another use of (19) implies that

$$(1-C\epsilon)h(n) \leq \frac{\int_n^{n^{1+\epsilon}} \frac{h(x)\,dx}{f(x)}}{\int_n^{n^{1+\epsilon}} \frac{dx}{f(x)}} \leq (1+C\epsilon)h(n), \tag{33}$$

and, similarly to (27), one can show that

$$\text{as } n \to +\infty, \int_{n^{1+\epsilon}}^{+\infty} \frac{dx}{f(x)} \ll \int_n^{+\infty} \frac{dx}{f(x)}. \tag{34}$$

Thus we conclude that, as $n \to +\infty$,

$$1 - C\epsilon - o(1) \leq \frac{\int_n^{+\infty} \frac{h(x)\,dx}{f(x)}}{h(n) \int_n^{+\infty} \frac{dx}{f(x)}} \leq 1 + C\epsilon + o(1). \tag{35}$$

Since $\epsilon > 0$ is arbitrary, (24) follows, and the proof is finished. □

4.3 A large-deviations estimate

If bin 1 starts with a_1 balls, the time until it has $b_1 > a_1$ balls in the continuous-time process is

$$\sum_{j=a_1}^{b_1-1} X(1,j) \tag{36}$$

and the time until bin 1 acquires infinitely many balls is

$$\sum_{j=a_1}^{+\infty} X(1,j). \tag{37}$$

The latter is a sum of independent $\exp(f(j))$ random variables, and it converges whenever $\sum_j f(j)^{-1} < +\infty$ (cf. Section 3.2). We *claim* that the sum in fact concentrate very strongly around its mean.

Lemma 4 *Let f be a valid feedback function. Assume that $\{V_j =^d \exp(f(j))\}_{j\in\mathbf{N}}$ be a sequence of independent random variables, and define (for $n < m \leq +\infty$)*

$$A_{n,m} \equiv \sum_{j=n}^{m-1} \left(V_j - \frac{1}{f(j)}\right)$$

Then there exists a constant $C = C_f > 0$ such that for all large enough $n, m \in \mathbf{N}$ and all $t \in \mathbf{R}^+$

$$\mathbf{Pr}\left(A_{n,m} \; > \; t\sqrt{S_2(n,m)}\right) \;\leq\; Ce^{-t} \tag{38}$$

$$\mathbf{Pr}\left(A_{n,m} < -t\sqrt{S_2(n,m)}\right) \;\leq\; Ce^{-t} \tag{39}$$

What Lemma 4 means to us is that $\sum_{j=a_1}^{m} X(1,j)$ can be though of as "almost constant" in many calculations. This will be put to use in all of our main proofs. In fact, the following corollary will suffice.

Corollary 1 *Let f and $\{V_j\}_{j\in\mathbf{N}}$ be as above, and define*

$$B_n \equiv \sum_{j=n}^{+\infty} V_j.$$

Then $\mathbf{Ex}[B_n] = S_1(n)$ and there exists a $C' = C'_f > 0$ a such that

$$\forall n \in \mathbf{N}, \quad \mathbf{Pr}\left(\left|\frac{B_n}{S_1(n,m)} - 1\right| > \frac{C'}{n^{\frac{1}{4}}}\right) \leq C'e^{-n^{\frac{1}{4}}}$$

Proof: [of Lemma 4] We will only prove the first inequality in the case $m = +\infty$, since all other proofs are similar. The technique we employ is fairly standard and is commonly used in other proofs of Chernoff-type large deviation inequalities [2].

Fix any $0 < s \leq f(n)/2 = \min_{j\geq n} f(j)/2$ and notice that, by the standard Bernstein's trick, the formulae in Section 2, the inequality "$1 + x \leq e^x$", and some simple calculations

$$\begin{aligned}
\mathbf{Pr}\left(A_n \; > \; t\sqrt{S_2(n)}\right) &= \mathbf{Pr}\left(e^{s\,A_n} > e^{s\,t\sqrt{S_2(n)}}\right) && (40)\\
&\leq e^{-s\,t\sqrt{S_2(n)}}\mathbf{Ex}\left[e^{\sum_{j\geq n} s\left(V_j - \frac{1}{f(j)}\right)}\right] && (41)\\
&= e^{-s\,t\sqrt{S_2(n)}}\prod_{j\geq n}\mathbf{Ex}\left[e^{s\left(V_j - \frac{1}{f(j)}\right)}\right] && (42)\\
&= e^{-s\,t\sqrt{S_2(n)}}\prod_{j\geq n}\frac{e^{-\frac{s}{f(j)}}}{1 - \frac{s}{f(j)}} && (43)\\
&= e^{-s\,t\sqrt{S_2(n)}} \times && (44)\\
&\quad \prod_{j\geq n} e^{-\frac{s}{f(j)}}\left(1 + \frac{s}{f(j)} + \frac{s^2}{f(j)^2}\frac{1}{1 - \frac{s}{f(j)}}\right) && (45)
\end{aligned}$$

$$\leq \quad e^{-s\,t\sqrt{S_2(n)}} \prod_{j \geq n} \exp(2\frac{s^2}{f(j)^2}) \tag{46}$$

$$= \quad \exp(2s^2\, S_2(n) - s\,t\sqrt{S_2(n)}) \tag{47}$$

We now set

$$s \equiv \frac{1}{\sqrt{S_2(n)}}.$$

If we show that this choice is permissible (i.e. that $1/\sqrt{S_2(n)} \leq f(n)/2$ for n large), we will have finished the proof. But notice that

$$\frac{1}{S_2(n)} \sim \frac{(2h(n)-1)f(n)^2}{n} \ll f(n)^2,$$

since $h(n) \ll n^{1/4}$ by assumption. We deduce that there is some $n_0 = n_0(f)$ such that for all $n \geq n_0$ $1/S_2(n) \leq f(n)^2/4$, from which the lemma follows. □

Proof: [of Corollary 1] Recall that in this case $0 < B_n < +\infty$, since B_n is positive and has infinite expectation. Hence, if A_n is as in Lemma 4, $B_n = A_n + S_1(n)$. The Corollary follows from Lemma 4 by the choice of $t \equiv n^{1/4}$ and recalling Lemma 1, which implies that $S_2(n)^{1/2} \leq C' S_1(n)/\sqrt{n}$. □

5 Imbalanced start

We now start the discussion of the first of our main results. Recall the notation $[t, \alpha]$ for the state of a two-bin balls-in-bins process with feedback (cf. Section 2). Our interest in this section will be in processes with two bins, started from $[t, \alpha]$ with t large, $\alpha \in [0, 1/2)$ fixed, and f smooth. To state this theorem, we need a definition.

Definition 2 *Let $\beta \in (0, 1/2)$ and $T \in \mathbf{N}$ be given, and consider a balls-in-bins process with two bins.* $\mathsf{HasMoreThan}(\beta, T)$ *is the event that there are more than βT balls in bin 1, at the moment when there is a total of T balls in both bins.*

Theorem 3 *Suppose that f is a valid feedback function. Then for all $0 < \alpha < \beta < 1/2$, there exists a constant $c > 0$ depending only on α, β and f such that for all large enough $t \in \mathbf{N}$,*

$$\mathbf{Pr}_{[t,\alpha]}\left(\exists T \geq t,\, \mathsf{HasMoreThan}(\beta, T)\right) \leq e^{-ct^c}.$$

In particular, if the initial conditions are $[t, \alpha]$ as above, bin 2 achieves monopoly with all-but-exponentially-small probability.

Recalling that the family $f(x) = x^p$, $p > 1$ is valid, we deduce the following corollary.

Corollary 2 *Suppose that one has a balls-in-bins process with feedback function $f(x) = x^p$, $p > 1$. Then for all $0 < \alpha < \beta < 1/2$ there exists a constant $c > 0$ depending on α, β and p for which*

$$\mathbf{Pr}_{[t,\alpha]}\left(\exists T \geq t,\, \mathsf{HasMoreThan}(\beta, T)\right) \leq e^{-ct^c} \textit{ for all large enough } t.$$

In particular, for large t a process started from state $[t, \alpha]$ will result in monopoly by bin 2 with high probability.

Proof: [of Theorem 3] Recall the definition of the exponential embedding: for $i = 1, 2$, $X(i, 0)$ parameterizes the time until the first arrival at bin i, while $X(i, j)$ (for $j > 0$) parameterizes the time between the $(j-1)$th and jth arrivals at bin i. We begin by showing the following fact, that will also be useful in later chapters of the dissertation.

Claim 1 *Consider a balls-in-bins process with feedback function f and initial conditions (x, y) with $x + y = t$. Let $\beta \in (0, 1)$ and $T \geq t$ be given. Then*

$$\mathsf{HasMoreThan}(\beta, T) = \left\{ \sum_{j=x}^{\lceil \beta T \rceil - 1} X(1, j) < \sum_{\ell=y}^{T - \lceil \beta T \rceil - 1} X(2, \ell) \right\}. \tag{48}$$

Proof: [of Claim 1]Let A be the event in the RHS of (48). In the first part of our proof, we assume that $\mathsf{HasMoreThan}(\beta, T)$ occurs, and show that this implies the occurrence of A. Consider the time τ_T when the total number of balls in the continuous time process reaches T. At that time, the number of balls in bin 1 (respectively 2) is larger than $\lceil \beta T \rceil$ (resp. smaller than $T - \lceil \beta T \rceil$), by assumption, so

$$\sum_{j=x}^{\lceil \beta T \rceil - 1} X(1, j) \leq \tau_T < \sum_{\ell=y}^{T - \lceil \beta T \rceil - 1} X(2, \ell),$$

which implies the occurrence of A. Conversely, assume that A occurs, and let τ_T be as above. Then, because

$$\sum_{j=x}^{\lceil \beta T \rceil - 1} X(1, j) < \sum_{\ell=y}^{T - \lceil \beta T \rceil - 1} X(2, \ell),$$

the number of balls in bin 2 at time $\sum_{j=x}^{\lceil \beta T \rceil - 1} X(1, j)$ is smaller than $T - \lceil \beta T \rceil$, so that

$$\sum_{j=x}^{\lceil \beta T \rceil - 1} X(1, j) < \tau_T.$$

This implies that at time τ_T, the number of balls at bin 1 is at least $\lceil \beta T \rceil$, which implies the occurrence of $\mathsf{HasMoreThan}(\beta, T)$. □

We now continue the proof of Theorem 3, using the result in the claim to write

$$\mathbf{Pr}_{[t,\alpha_t]}\left(\exists T \geq t,\, \mathsf{HasMoreThan}(\beta, T)\right) \tag{49}$$

$$\leq \sum_{T=t}^{+\infty} \mathbf{Pr}\left(\sum_{j=\lceil \alpha t\rceil}^{\lceil \beta T\rceil -1} X(1,j) < \sum_{\ell=t-\lceil \alpha t\rceil}^{T-\lceil \beta T\rceil -1} X(2,\ell)\right). \tag{50}$$

We can omit from the sum the terms for which $T - \lceil \beta T\rceil \leq t - \lceil \alpha t\rceil$, for the corresponding events are empty. Our goal now is to estimate the probability of each of the remaining events in the RHS of (49). Fix a number T for which

$$T - \lceil \beta T\rceil > t - \lceil \alpha t\rceil, \tag{51}$$

and consider the corresponding event.

$$A_{t,T} \equiv \left\{\sum_{j=\lceil \alpha t\rceil}^{\lceil \beta T\rceil -1} X(1,j) < \sum_{t-\lceil \alpha t\rceil}^{T-\lceil \beta T\rceil -1} X(2,\ell)\right\} \tag{52}$$

By Corollary 1, if t is large enough, there is an event $E_{t,T}$ with probability at least $\geq 1 - 8e^{2-\lceil \alpha t\rceil^{1/4}}$ inside of which

$$\left|\frac{\sum_{j=\lceil \alpha t\rceil}^{+\infty} X(1,j)}{S_1(\lceil \alpha t\rceil)} - 1\right| \leq D\,t^{-\frac{1}{4}}, \tag{53}$$

$$\left|\frac{\sum_{j=\lceil \beta T\rceil}^{+\infty} X(1,j)}{S_1(\lceil \beta T\rceil)} - 1\right| \leq D\,t^{-\frac{1}{4}}, \tag{54}$$

$$\left|\frac{\sum_{\ell=t-\lceil \alpha t\rceil}^{+\infty} X(2,\ell)}{S_1(t-\lceil \alpha t\rceil)} - 1\right| \leq D\,t^{-\frac{1}{4}}, \tag{55}$$

$$\left|\frac{\sum_{\ell=T-\lceil \beta T\rceil}^{+\infty} X(2,\ell)}{S_1(T-\lceil \beta T\rceil)} - 1\right| \leq D\,t^{-\frac{1}{4}}, \tag{56}$$

for a constant D that does not depend on t. We now *claim* that if t is large enough, for all T satisfying (51),

$$A_{t,T} \cap E_{t,T} = \emptyset. \tag{57}$$

Notice that this implies that

$$\mathbf{Pr}_{[t,\alpha_t]}\left(\exists T \geq t,\, \mathsf{HasMoreThan}(\beta, T)\right) \leq \sum_{T\geq t} \mathbf{Pr}\left(A_{t,T}\right) \tag{58}$$

$$\leq \sum_{T\geq t} \mathbf{Pr}\left(E_{t,T}^c\right) \leq \sum_{T\geq t} 8e^{2-\lceil \alpha t\rceil^{1/4}}, \tag{59}$$

from which the Theorem follows directly.

To prove the claim, we recall the asymptotic formulae for $S_1(n)$ in Lemma 1. According to this formula, there exists a function $r_0 = r_0(n)$ that decreases to 0 as $n \to +\infty$ such that

$$1 - r_0(n) \leq \frac{1}{S_1(n)} \left(\frac{n}{(h(n)-1)f(n)} \right) \leq 1 + r_0(n). \tag{60}$$

Putting this together with equations (53) – (56) and employing Lemma 3[4], we deduce that inside the event $E_{t,T}$,

$$\begin{aligned}
\left| \frac{\sum_{j=\lceil \alpha t \rceil}^{+\infty} X(1,j)}{\alpha t/(h(\alpha t)-1)f(\alpha t)} - 1 \right| &\leq r_1(t), & (61)\\
\left| \frac{\sum_{j=\lceil \beta T \rceil}^{+\infty} X(1,j)}{\beta T/(h(\beta T)-1)f(\beta T)} - 1 \right| &\leq r_1(t), & (62)\\
\left| \frac{\sum_{\ell=t-\lceil \alpha t \rceil}^{+\infty} X(2,\ell)}{(1-\alpha)t/(h((1-\alpha)t)-1)f((1-\alpha)t)} - 1 \right| &\leq r_1(t), & (63)\\
\left| \frac{\sum_{\ell=T-\lceil \beta T \rceil}^{+\infty} X(2,\ell)}{(1-\beta)T/(h((1-\beta)T)-1)f((1-\beta)T)} - 1 \right| &\leq r_1(t), & (64)
\end{aligned}$$

where $r_1(t)$ also decreases to 0 as $t \to +\infty$.

Plugging those bounds into the definition of $A_{t,T}$, we deduce that in the event $A_{t,T} \cap E_{t,T}$

$$\begin{aligned}
&\frac{(1-r_1(t))\alpha t}{(h(\alpha t)-1)f(\alpha t)} - \frac{(1+r_1(t))\beta T}{(h(\beta T)-1)f(\beta T)} & (65)\\
&\leq \sum_{j=\lceil \alpha t \rceil}^{\lceil \beta T \rceil - 1} X(1,j) \leq \sum_{t-\lceil \alpha t \rceil}^{T-\lceil \beta T \rceil - 1} X(2,\ell) & (66)\\
&\leq \frac{(1+r_1(t))(1-\alpha)t}{(h((1-\alpha)t)-1)f((1-\alpha)t)} - \frac{(1-r_1(t))(1-\beta)T}{(h((1-\beta)T)-1)f((1-\beta)T)}. & (67)
\end{aligned}$$

One can check, using the last assertion of Lemma 1, that there is a constant $u < 1$ for which

$$\frac{\alpha t}{(h(\alpha t)-1)f(\alpha t)} > c \frac{(1-\alpha)t}{(h((1-\alpha)t)-1)f((1-\alpha)t)} \quad \text{and} \tag{68}$$

[4]This lemma guarantees that $f(n+w(n)) \sim f(n)$ and $h(n+w(n)) \sim h(n)$ for n large, as long as $|w(n)| \ll n$. It is needed here in conjunction with Lemma 1 to deduce that for $\eta \in (0,1)$ fixed and n large,

$$S_1(\lceil \eta n \rceil) \sim \frac{\lceil \eta n \rceil}{(h(\lceil \eta n \rceil)-1)f(\lceil \eta n \rceil)} \sim \frac{\eta n}{(h(\eta n)-1)f(\eta n)}.$$

$$\frac{\beta T}{(h(\beta T)-1)f(\beta T)} \quad > \quad c\,\frac{(1-\beta)T}{(h((1-\beta)T)-1)f((1-\beta)T)} \text{ for large } t, \tag{69}$$

and it follows from (65) that

$$A_{t,T}\cap E_{t,T} \Rightarrow \left[\frac{\alpha t}{(h(\alpha t)-1)f(\alpha t)} - \frac{(1-\alpha)t}{(h((1-\alpha)t)-1)f((1-\alpha)t)}\right] \tag{70}$$

$$\leq (1+o(1))\left[\frac{\beta T}{(h(\beta T)-1)f(\beta T)} - \frac{(1-\beta)T}{(h((1-\beta)T)-1)f((1-\beta)T)}\right], \tag{71}$$

with a $o(1)$ term that only depends on t (and not on T). One can easily deduce from Lemma 1 and its proof that $A_{t,T}\cap E_{t,T} \Rightarrow$

$$\frac{\alpha t}{(h(\alpha t)-1)f(\alpha t)} - \frac{(1-\alpha)t}{(h((1-\alpha)t)-1)f((1-\alpha)t)} \quad \sim \quad \int_{\alpha t}^{(1-\alpha)t}\frac{dx}{f(x)}, \tag{72}$$

$$\frac{\beta T}{(h(\beta T)-1)f(\beta T)} - \frac{(1-\beta)T}{(h((1-\beta)T)-1)f((1-\beta)T)} \quad \sim \quad \int_{\beta T}^{(1-\beta)T}\frac{dx}{f(x)}.$$

Therefore, we conclude

$$A_{t,T}\cap E_{t,T} \Rightarrow \int_{\alpha t}^{(1-\alpha)t}\frac{dx}{f(x)} \leq (1+o(1))\int_{\beta T}^{(1-\beta)T}\frac{dx}{f(x)}. \tag{73}$$

We now prove that $A_{t,T}\cap E_{t,T}=\emptyset$ by showing that the RHS of the above cannot hold for any large t and all corresponding T. To this end, we employ estimate (20) from the proof of Lemma 1, which states that

$$\forall y\geq x \;\; \frac{f(y)}{f(x)} \geq \left(\frac{y}{x}\right)^c, \text{ where } c\equiv \inf_{x'\geq x} h(x') \tag{74}$$

we will only employ this inequality for large $x<y$, which means we can assume $c>1$, because of the validity of f. Then, for any T satisfying (51),

$$\int_{\beta T}^{(1-\beta)T}\frac{dx}{f(x)} \quad \leq \quad \left(\frac{t}{T}\right)^c \int_{\beta T}^{(1-\beta)T}\frac{dx}{f\left(\frac{t}{T}x\right)} \tag{75}$$

$$= \quad \left(\frac{t}{T}\right)^{c-1} \int_{\beta t}^{(1-\beta)t}\frac{dy}{f(y)} \tag{76}$$

$$= \quad \left((1+o(1))\frac{1-\beta}{1-\alpha}\right)^{c-1} \int_{\beta t}^{(1-\beta)t}\frac{dy}{f(y)}. \tag{77}$$

Here, we employed (74) for the second line, the substitution $x\mapsto Ty/t$ in the third line, and (51) on the third, with (once again) a $o(1)$ term that is uniform over t. Since $c>1$

and $\beta > \alpha$ (hence $1-\beta < 1-\alpha$), we conclude that there exists a constant $0 < d < 1$ such that for all large enough t and all T satisfying (51),

$$\int_{\beta T}^{(1-\beta)T} \frac{dx}{f(x)} \leq d \int_{\beta t}^{(1-\beta)t} \frac{dy}{f(y)}. \tag{78}$$

To conclude, note that $[\alpha, 1-\alpha] \supset [\beta, 1-\beta]$, hence the above implies

$$\int_{\beta T}^{(1-\beta)T} \frac{dx}{f(x)} \leq d \int_{\alpha t}^{(1-\alpha)t} \frac{dx}{f(x)}. \tag{79}$$

This shows that the RHS of (73) cannot occur for large enough t and any corresponding T, which proves the claim in (57) that $A_{t,T} \cap E_{t,T} = \emptyset$ and finishes the proof. $\square$

6 The number of balls in the losing bin

We now prove the first of our two heavy-tail results. First, we recall the definition of L.

Definition 3 *Let f be a feedback function satisfying the monopoly condition $\sum_j f(j)^{-1} < +\infty$, so that in the corresponding balls-in-bins process there almost surely is one bin that receives all but finitely many balls. For a two-bin process with feedback function f, the* losing number *L is the (almost surely finite) number of balls that go into the remaining bin.*

Our heavy tails result for L is stated and proved below.

Theorem 4 *The following holds for all valid feedback functions f. For any fixed initial conditions (x,y), there exists a sequence $\{c_n\}_{n\in\mathbf{N}}$ converging to a limit $c > 0$ (depending only on x, y and f) such that, as $n \to +\infty$,*

$$|\mathbf{Pr}_{(x,y)}(L > n) - c_n S_1(n)| = O\left(e^{2-n^{1/4}} + n^{-1/4} S_1(n) + S_1(n)^3\right).$$

In particular,

$$\mathbf{Pr}_{(x,y)}(L > n) \sim c\, S_1(n) \sim c\, \frac{n}{(h(n)-1)f(n)},$$

where $h(n)$ is the characteristic exponent of f (cf. Definition 1 and Lemma 1).

In the case $f(n) = n^p$, $p > 1$, $S_1(n) \sim n^{1-p}/(p-1)$ for n large, and as a consequence we have the following corollary.

Corollary 3 *For $f(n) \sim n^p$ with $p > 1$ and $(x,y) \in \mathbf{N}^2$ fixed, as $n \to +\infty$*

$$\mathbf{Pr}_{(x,y)}(L > n) \sim \frac{c}{(p-1)\, n^{p-1}},$$

with c as above.

Proof: [of Theorem 4]We will assume without loss of generality that $x \geq y$. First, notice that for any $n > x$

$$\{L > n\} = \left\{ \sum_{\ell=y}^{n} X(2,\ell) < \sum_{j=x}^{\infty} X(1,j) < \sum_{\ell=y}^{\infty} X(2,\ell) \right\} \tag{80}$$

$$\bigcup \left\{ \sum_{j=x}^{n} X(1,j) < \sum_{\ell=y}^{\infty} X(2,\ell) < \sum_{j=x}^{\infty} X(1,j) \right\}. \tag{81}$$

Indeed, $L > n$ if and only if bin 1 explodes first, but bin 2 has at least n balls when that happens, or vice-versa. Define

$$\Delta_n \equiv \sum_{j=x}^{n-1} X(1,j) - \sum_{\ell=y}^{n-1} X(2,\ell) \tag{82}$$

$$A_n^{(1)} \equiv \sum_{j=n}^{\infty} X(1,j) \tag{83}$$

$$A_n^{(2)} \equiv \sum_{\ell=n}^{\infty} X(2,\ell) \tag{84}$$

These random variables are independent, and one can rewrite

$$\{L > n\} = \{0 < \Delta_n < A_n^{(2)}\} \cup \{0 > \Delta_n > -A_n^{(1)}\} \tag{85}$$

$A_n^{(1)}$ and $A_n^{(2)}$ have the same distribution and are independent of Δ_n, and the distribution of Δ_n has no point masses. It follows that

$$\mathbf{Pr}_{(x,y)}(L > n) = \mathbf{Pr}\left(0 < \Delta_n < A_n^{(2)}\right) + \mathbf{Pr}_{(x,x)}\left(0 > \Delta_n > -A_n^{(1)}\right) \tag{86}$$

$$= \mathbf{Pr}(0 < |\Delta_n| < A_n) \tag{87}$$

$$= \mathbf{Pr}(|\Delta_n| \leq A_n) \tag{88}$$

where we let $A_n \equiv A_n^{(1)}$ for simplicity. We now employ one of our concentration results, Corollary 1. According to this result, as applied to $A_n \equiv \sum_{j=n}^{+\infty} X(1,j)$, there exists a constant C' depending only on f such that for all large enough n

$$\mathbf{Pr}\left(\left| \frac{A_n}{S_1(n)} - 1 \right| \geq \frac{C'}{n^{1/4}} \right) \leq 2e^{2-n^{1/4}}. \tag{89}$$

Plugging this into the previous equation yields

$$\mathbf{Pr}\left(|\Delta_n| \leq (1 - C' n^{-1/4}) S_1(n)\right) - 2e^{2-n^{1/4}} \tag{90}$$

$$\leq \mathbf{Pr}_{(x,y)}(L > n) \leq \mathbf{Pr}\left(|\Delta_n| \leq (1 + C' n^{-1/4}) S_1(n)\right) + 2e^{2-n^{1/4}}. \tag{91}$$

We now show that there exist a sequence $\{c_n\}_{n\in\mathbf{N}}$ and a constant $C > 0$, both of which depend only on f, x and y, such that

$$|\mathbf{Pr}\,(|\Delta_n| \leq \epsilon) - c_n\epsilon| \leq C\epsilon^3, \tag{92}$$

and $c_n \to c$ for some real-valued c. This inequality implies that, as $n \to +\infty$,

$$|\mathbf{Pr}\left(|\Delta_n| \leq (1 \pm C'n^{-1/4})S_1(n)\right) - c_n\, S_1(n)| \leq C'n^{-1/4}S_1(n) + C[(1 \pm C'n^{-1/4})S_1(n)]^3.$$

Notice that this *almost* finishes the proof, since we still need to show that the limit c is positive. To prove (92), let $\psi_n(\cdot)$ be the characteristic function of Δ_n.

$$\begin{aligned}
\psi_n(t) &= \mathbf{Ex}\left[\exp(\sqrt{-1}t\Delta_n)\right] &(93)\\
&= \mathbf{Ex}\left[\exp\left[\sum_{j=x}^{n-1} X(1,j) - \sum_{\ell=y}^{n-1} X(2,\ell)\right]\right] &(94)\\
&= \left(\prod_{j=x}^{n-1} \frac{1}{1 - \frac{\sqrt{-1}t}{f(j)}}\right) \times \left(\prod_{j=y}^{n-1} \frac{1}{1 + \frac{\sqrt{-1}t}{f(\ell)}}\right) &(95)\\
&= \left(\prod_{\ell=y}^{x-1} \frac{1}{1 + \frac{\sqrt{-1}t}{f(j)}}\right) \times \left(\prod_{j=x}^{n-1} \frac{1}{1 + \frac{t^2}{f(j)^2}}\right) &(96)
\end{aligned}$$

Clearly, for all $n \in \mathbf{N}$ Δ_n has a smooth distribution, hence the inversion formula for characteristic functions (a.k.a. Fourier inversion formula) [11] implies

$$\begin{aligned}
&\forall -\infty < a \leq b < +\infty \;\; \mathbf{Pr}\,(a \leq \Delta_n \leq b) &(97)\\
&= \frac{1}{2\pi} \lim_{T\nearrow+\infty} \int_{-T}^{T} \psi_n(t) \left(\frac{e^{-\sqrt{-1}ta} - e^{-\sqrt{-1}tb}}{\sqrt{-1}t}\right) dt. &(98)
\end{aligned}$$

In the present setting, we use this formula with $b = -a = \epsilon$, to prove (92), noting that, since ψ_n is integrable, we can dispense with the limit in T.

$$\begin{aligned}
\epsilon^{-1}\mathbf{Pr}\,(|\Delta_n| \leq \epsilon) &= \frac{1}{\pi} \lim_{T\nearrow+\infty} \int_{-T}^{T} \psi_n(t) \left(\frac{e^{+\sqrt{-1}\epsilon t} - e^{-\sqrt{-1}\epsilon t}}{2\sqrt{-1}\epsilon t}\right) dt &(99)\\
&= \frac{1}{\pi} \int_{-\infty}^{+\infty} \psi_n(t) \frac{\sin(\epsilon t)}{\epsilon t}\, dt &(100)
\end{aligned}$$

Now notice that, quite crudely,

$$\forall s \in \mathbf{R},\, |\frac{\sin(s)}{s} - 1| \leq Cs^2$$

for some constant $C > 0$. Applying this inequality with $s = \epsilon t$, one obtains

$$\left|\pi\epsilon^{-1}\mathbf{Pr}\left(|\Delta_n| \le \epsilon\right) - \int_{-\infty}^{+\infty} \psi_n(t)\,dt\right| \le \int_{-\infty}^{+\infty} |\psi_n(t)| \left|\frac{\sin(\epsilon t)}{\epsilon t} - 1\right| dt \tag{101}$$

$$\le C\epsilon^2 \int_{-\infty}^{+\infty} |t^2\,\psi_n(t)|\,dt. \tag{102}$$

For $n = x+3$, $|t^2\psi_n(t)|$ is of order $1/t^2$, so the above integral converges; for $n > x+3$, the integrand is even smaller. Hence, we can guarantee that, for a possibly larger $C > 0$,

$$\left|\mathbf{Pr}\left(|\Delta_n| \le \epsilon\right) - \left(\frac{1}{\pi}\int_{-\infty}^{+\infty} \psi_n(t)\,dt\right)\epsilon\right| \le C\epsilon^3 \text{ uniformly over } n. \tag{103}$$

Moreover, since $|\psi_{n+1}(t)| \le |\psi_n(t)|$ for all t, the Dominated Convergence Theorem implies that

$$c_n \equiv \frac{1}{\pi}\int_{-\infty}^{+\infty} \psi_n(t)\,dt \to c \equiv \frac{1}{\pi}\int_{-\infty}^{+\infty} \left(\prod_{\ell=y}^{x-1} \frac{1}{1+\frac{\sqrt{-1}t}{f(j)}}\right) \times \left(\prod_{j=x}^{n-1} \frac{1}{1+\frac{t^2}{f(j)^2}}\right) dt. \tag{104}$$

Together with the previous equation, this implies (92).

Our last step is to prove that $c > 0$. To see this, consider first the case $x = y$. In this case, the formula for c in (104) becomes

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} \left(\prod_{j=x}^{+\infty} \frac{1}{1+\frac{t^2}{f(j)^2}}\right) dt.$$

The product in the integrand converges to a positive limit for all $t \in \mathbf{R}$, since $\sum f(j)^{-1} < +\infty$; hence, the value of the integral is positive, and we are done in this case.

We now consider the case $y < x$. Clearly, $c = \lim_{n\to+\infty} S_1(n)^{-1}\mathbf{Pr}\left(L > n\right)$ is a real number. Moreover, notice that there is a positive probability α that in the process started from (x,y), bin 2 receives the first $x-y$ balls, thus evolving to state (x,x). Conditioned on that happening, the probability of $L > n$ is $\mathbf{Pr}_{(x,x)}\left(L > n\right) \sim c' S_1(n)$ for some $c' > 0$ (as shown above). But then

$$c = \lim_{n\to+\infty} \frac{\mathbf{Pr}_{(x,y)}\left(L > n\right)}{S_1(n)} \tag{105}$$

$$\ge \mathbf{Pr}_{(x,y)}\left(\text{bin 2 gets first } x-y \text{ balls}\right) \tag{106}$$

$$\times \lim_{n\to+\infty} \frac{\mathbf{Pr}_{(x,y)}\left(L > n \mid \text{bin 2 gets first } x-y \text{ balls}\right)}{S_1(n)}$$

$$= \alpha \lim_{n\to+\infty} \frac{\mathbf{Pr}_{(x,x)}\left(L > n\right)}{S_1(n)} = \alpha c' > 0 \tag{107}$$

which proves that c is positive even when x and y differ, thus finishing the proof. □

7 The time until imbalance

The strategy used to prove Theorem 4 was very simple. After the event $\{L > n\}$ was written down in terms of the exponential embedding random variables, A_n was approximated by its expectation, and the distribution of Δ_n near the origin via Fourier transform techniques. As we shall see below, our last theorem in this section has a similar proof. First, we need a definition.

Definition 4 *Given a time $T \in \mathbf{N}$, a number $0 < \beta < 1/2$ and a balls-in-bins process with two bins, the event* $\mathsf{LoserHasMoreThan}(q, T)$ *holds if at the time the number of balls in both systems reaches T, the number of balls in the losing bin is at least βT.*

Our second heavy-tails result can now be properly stated.

Theorem 5 *For all valid functions f it holds that for all fixed initial conditions $(x, y) \in \mathbf{N}^2$ and $0 < \alpha < 1/2$ there is a constant $c > 0$ depending only on f, x and y such that*

$$\mathbf{Pr}_{(x,y)}\left(\mathsf{LoserHasMoreThan}(\alpha, n)\right) \sim c[S_1(\lfloor \alpha n \rfloor) - S_1(n - \lfloor \alpha n \rfloor)] \text{ as } n \to +\infty.$$

Moreover, we can take c to be the same constant (depending on f, x and y) that appears in the proof of Theorem 4.

In the case $f(x) = x^p$ for $p > 1$, the estimate of $S_1(n)$ by an integral implies the following corollary.

Corollary 4 *For $f(n) = n^p$ with $p > 1$ and $(x, y) \in \mathbf{N}^2$, $0 < \alpha < 1/2$ fixed, as $n \to +\infty$*

$$\mathbf{Pr}_{(x,y)}\left(\mathsf{LoserHasMoreThan}(\alpha, N)\right) \sim \frac{c\,(\alpha^{1-p} - (1-\alpha)^{1-p})}{(p-1)\,n^{p-1}},$$

with c as above.

Proof: [of Theorem 5] This proof is only slightly more complicated than that of Theorem 4, and follows the same outline. We will again assume that $x \geq y$, and write the event under consideration in terms of the random variables in the definition of the exponential embedding. To do that, first notice that $\mathsf{LoserHasMoreThan}(\alpha, T)$ occurs if and only if both bins have more than αT balls at the time when the total number of balls in the system is T. Indeed, if one of the bins has less than $\alpha T < T/2$ balls, then this must necessarily be the losing bin, and $\mathsf{LoserHasMoreThan}(\alpha, T)$ cannot occur in this case. Conversely, if both bins have at least αT balls, then in particular the losing bin has $\geq \alpha T$ balls, and $\mathsf{LoserHasMoreThan}(\alpha, T)$ occurs.

The event that bin 1 has at least αT balls when T balls are present in the system is precisely the event $\mathsf{HasMoreThan}(\alpha, T)$ defined in Definition 2, which was shown in Claim 1 to be equal to

$$\mathsf{HasMoreThan}(\alpha, n) = \left\{ \sum_{j=x}^{\lfloor \alpha n \rfloor - 1} X(1,j) < \sum_{\ell=y}^{n-\lfloor \alpha n \rfloor - 1} X(2,\ell) \right\}. \tag{108}$$

Similarly, the event than bin 2 has at least αT balls when there are T balls in the system is precisely $\mathsf{HasMoreThan}(\alpha, T)$ with the roles of the two bins reversed, and can thus be written down as

$$\left\{ \sum_{\ell=y}^{\lfloor \alpha n \rfloor - 1} X(2,\ell) < \sum_{j=x}^{n-\lfloor \alpha n \rfloor - 1} X(1,j) \right\}. \tag{109}$$

We conclude that

$$\begin{aligned}
& \mathsf{LoserHasMoreThan}(\alpha, n) & (110)\\
= & \left\{ \sum_{j=x}^{\lfloor \alpha n \rfloor - 1} X(1,j) < \sum_{\ell=y}^{n-\lfloor \alpha n \rfloor - 1} X(2,\ell) \right\} & (111)\\
& \bigcap \left\{ \sum_{\ell=y}^{\lfloor \alpha n \rfloor - 1} X(2,\ell) < \sum_{j=x}^{n-\lfloor \alpha n \rfloor - 1} X(1,j) \right\} & (112)\\
= & \left\{ \sum_{j=x}^{\lfloor \alpha n \rfloor - 1} X(1,j) - \sum_{j=y}^{\lfloor \alpha n \rfloor - 1} X(2,\ell) < \sum_{\ell=\lfloor \alpha n \rfloor}^{n-\lfloor \alpha n \rfloor - 1} X(2,\ell) \right\} & (113)\\
& \bigcap \left\{ \sum_{\ell=y}^{\lfloor \alpha n \rfloor - 1} X(2,\ell) - \sum_{j=x}^{\lfloor \alpha n \rfloor - 1} X(1,j) < \sum_{j=\lfloor \alpha n \rfloor}^{n-\lfloor \alpha n \rfloor - 1} X(1,j) \right\}, & (114)
\end{aligned}$$

where for the last equality we used the fact that $\alpha < 1/2$, which guarantees $\lceil \alpha n \rceil < n/2$ for all large n, to ensure that the sums from $\lceil \alpha n \rceil$ to $n - \lceil \alpha n \rceil$ are non-empty. Defining

$$\begin{aligned}
\Sigma_n &= \sum_{j=x}^{\lfloor \alpha n \rfloor - 1} X(1,j) - \sum_{j=y}^{\lfloor \alpha n \rfloor - 1} X(2,\ell), & (115)\\
B_n^{(1)} &= \sum_{j=\lfloor \alpha n \rfloor}^{n-\lfloor \alpha n \rfloor - 1} X(1,j), & (116)\\
B_n^{(2)} &= \sum_{\ell=\lfloor \alpha n \rfloor}^{n-\lfloor \alpha n \rfloor - 1} X(2,\ell), & (117)
\end{aligned}$$

we can rewrite $\mathsf{LoserHasMoreThan}(\alpha, n)$ as

$$\mathsf{LoserHasMoreThan}(\alpha, n) = \{-B_n^{(1)} < \Sigma_n < B_n^{(2)}\}. \tag{118}$$

The random variable Σ_n equals $\Delta_{\lceil \alpha n \rceil}$, as defined in the proof of Theorem 4, and the $B_n^{(i)}$'s are akin to the $A_n^{(i)}$'s in that proof. Similarly to that proof, we note that $B_n^{(1)}$ and $B_n^{(2)}$ are independent, identically distributed and independent from Σ_n. Since Σ_n has no point-masses in its distribution,

$$\begin{aligned} & \mathbf{Pr}_{(x,y)} \left(\mathsf{LoserHasMoreThan}(\alpha, n)\right) && (119)\\ = \; & 1 - \mathbf{Pr}\left(-B_n^{(1)} > \Sigma_n\right) - \mathbf{Pr}\left(\Sigma_n > B_n^{(2)}\right) && (120)\\ = \; & 1 - \mathbf{Pr}\left(-B_n > \Sigma_n\right) - \mathbf{Pr}\left(\Sigma_n > B_n\right) && (121)\\ = \; & \mathbf{Pr}\left(|\Sigma_n| < B_n\right), && (122) \end{aligned}$$

where $B_n = B_n^{(1)}$. Now notice that

1. B_n concentrates around its mean. Indeed,

   $$B_n = \left[\sum_{j=\lceil \alpha n \rceil}^{+\infty} X(1,j)\right] - \left[\sum_{j=n-\lceil \alpha n \rceil}^{+\infty} X(1,j)\right].$$

   Applying Corollary 1 to each bracketed term and noticing that

   $$S_1(n - \lceil \alpha n) \rceil \leq S_1(\lceil \alpha n \rceil),$$

   we conclude that

   $$\mathbf{Pr}\left(|B_n - (S_1(\lceil \alpha n \rceil) - S_1(n - \lceil \alpha n \rceil))| \leq \frac{2C'}{n^{1/4}} S_1(\lceil \alpha n \rceil)\right) \leq 4e^{2-n^{1/4}} \tag{123}$$

   for some constant C' depending only on f and all large enough n.

2. The estimates on Δ_n in (92) imply that there exists a constant depending only on f such that for all $\epsilon > 0$ and all n large enough

   $$\left|\mathbf{Pr}\left(|\Sigma_n| < \epsilon\right) - c_{\lceil \alpha n \rceil}\epsilon\right| \leq C\epsilon^3 \tag{124}$$

   for the same sequence $\{c_m\}_{m \in \mathbf{N}}$ converging to $c > 0$ appearing in the proof of Theorem 4.

Putting those results together we conclude that

$$\left|\mathbf{Pr}\left(|\Sigma_n| \leq B_n\right) - c_{\lceil \alpha n\rceil}[S_1(\lceil \alpha n\rceil) - S_1(n - \lceil \alpha n\rceil)]\right| \quad (125)$$

$$\leq O\left(\frac{1}{n^{1/4}}[S_1(\lceil \alpha n\rceil) + S_1(n - \lceil \alpha n\rceil)] + [S_1(\lceil \alpha n\rceil) - S_1(n - \lceil \alpha n\rceil)]^3 + e^{-n^{1/4}}\right) \quad (126)$$

$$= o\left([S_1(\lceil \alpha n\rceil) - S_1(n - \lceil \alpha n\rceil)]\right) \quad (127)$$

which implies that

$$\mathbf{Pr}_{(x,y)}\left(\mathsf{LoserHasMoreThan}(\alpha, n)\right) = \mathbf{Pr}\left(|\Delta_n| \leq B_n\right) \sim c\,[S_1(\lceil \alpha n\rceil) - S_1(n - \lceil \alpha n\rceil)]. \quad (128)$$

This is precisely the desired result. □

8 Extensions, related results and open problems

- The proof of Theorem 4 generalizes directly to the following statement.

  Theorem 6 *Let $q : \mathbf{N} \to \mathbf{N}$ be a function and f be a valid feedback function such that*

  $$S_1^2\left(\left\lceil \frac{n - q(n)}{2}\right\rceil, \left\lfloor \frac{n + q(n)}{2}\right\rfloor\right) \gg n^{\gamma} S_2\left(\left\lceil \frac{n - q(n)}{2}\right\rceil, \left\lfloor \frac{n + q(n)}{2}\right\rfloor\right) \quad (129)$$

  as $n \to +\infty$, for some constant $\gamma > 0$ depending only on f. Then for any fixed $(x, y) \in \mathbf{N}^2$ there exists a constant $c > 0$ depending only on f, x and y such that for $n \gg 1$

  $$\mathbf{Pr}_{(x,y)}\left(|I_1(n - (x + y)) - I_2(n - (x + y))| \leq q(n)\right) \sim c\, S_1\left(\left\lceil \frac{n - q(n)}{2}\right\rceil, \left\lfloor \frac{n + q(n)}{2}\right\rfloor\right).$$

  Theorem 4 is a special case of this result when $q(n) = \alpha n$, and the rôle of (129) is to show that $\sum_{j=(n-q(n))/2}^{(n+q(n))/2} X(i, 1)$ concentrates around its mean (cf. Lemma 4). We omit the proof, but note the following corollary (take $q(n) = \lambda\sqrt{n}$.)

  Corollary 5 *Assume $f(n) = n^p$ for $p > 1$. Then the probability that the losing bin has at least $(n - \lambda\sqrt{n})/2$ balls at the time when the total number of balls is n is asymptotic to*

  $$(const.) \times n^{1/2-p} \quad (n \gg 1),$$

  with a constant that depends on the initial conditions, p and λ.

We mention this corollary because it relates to a result of [6], where it is shown that for a large initial number of balls n, the probability of monopoly by bin 1 has non-trivial behavior when $|I_1(0) - I_2(0)| = \Theta(\sqrt{n})$.

- We do not know any result more precise that Theorem 3 about the behavior of $\mathbf{Pr}_{[t,\alpha]}(\mathsf{HasMoreThan}(\beta, T))$, but a related question has been addressed. Let $f(x) = x^p$ and recall that Mon_1 is the event that bin 1 achieves monopoly. Assume we start with bin 1 losing, from state $[t, \alpha]$, $0 < \alpha < 1/2$. We prove in [9] that

$$\mathbf{Pr}_{[t,\alpha]}(\mathsf{Mon}_1) = \exp(c_p(\alpha)t + o(t))$$

for some negative, smooth function $c_p(\cdot) < 0$. Moreover, we show that conditioned on Mon_1, the fraction of balls in the first bin approximately follows the solution to a deterministic ODE.

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