Regularity results for the ordinary product stochastic pressure equation

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Abstract

We consider a stochastic pressure equation with lognormal coefficient with infinite dimensional noise. Using a White Noise framework, we study spatial and stochastic regularity of solutions of the stochastic pressure equation. We first establish that a particular class of weighted Chaos spaces can be characterized by Gaussian Sobolev type norms in the random argument under the Gaussian measure. Then, we use these results to prove that the solution of the stochastic pressure equation has the classical regularity in the spatial variable and a stochastic regularity on this class of weighted Chaos spaces.

Keywords: Stochastic pressure equation, White Noise Analysis, Stochastic Analysis, Weighted Chaos norms, Gaussian Sobolev norms. 60H15, 65N30,60H40,60H07, 60H30

1 Introduction

Uncertainty quantification techniques have gained the attention of researches in the last years. The theoretical and numerical treatments of stochastic partial differential equations are important for uncertainty quantification because the behavior of many interested random quantities is described by partial differential equations. In particular, we study elliptic partial differential equations which are important for the better understanding of many physical and engineering systems. In this paper we consider the equation

$$\left\{ \begin{array}{rcl} -\nabla_x \cdot (\kappa(x,\omega)\nabla_x u(x,\omega)) &=& f(x,\omega), \text{ for all } x \in D \\ u(x,\omega) &=& 0, \text{ for all } x \in \partial D, \end{array} \right. \quad (1)$$

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where $\log \kappa(x,\omega)$ is a Gaussian field and f is a (possible random) forcing term; see [26, 1, 3, 14, 12, 30, 2] among others. The presence of the log-normal coefficient $\kappa(x,\cdot)$ induces lack of uniform ellipticity and boundedness, and therefore, the analysis become very challenging. There are few works addressing this difficulty, we mention the work [13] where the White Noise framework analysis is carried out in the framework of Hilbert spaces, [8, 15] where Banach space frameworks are used for the analysis, [3, 28] where the Wick product formulation is considered, and [31, 23, 30] where weighted Wiener Chaos expansions and other modeling methodology are proposed.

Another level of difficulty is the infinite dimensional behavior of the Gaussian fields. We note that the assumption of finite dimensional noise, compared to the infinite dimensional noise case, in the coefficient κ and the forcing term f simplify the analysis a great deal, however, it has serious practical limitations. The dimension of the finite dimensional noise is often associated with truncated or finite dimensional approximations of Karhunen-Loève (KL) expansions or Chaos expansions. In some real-world applications the dimension of the noise may be very large, for instance, in applications related to flow in heterogeneous porous media. In this case, the coefficient κ represents the permeability of a porous medium that contain uncertainties at the fine resolution. Since permeability data are also collected at finest scales, such as core scales, detailed geological models are constructed to contain such scales. At these scales, we have to deal with large uncertainties associated with the fine grid information. Modeling this detailed geological system may require large dimensional set to parametrize the noise. Hence, robust error estimates and analysis for stochastic discretizations that take into account these fine-scale uncertainties are needed. In this case, it is more advantageous to work with infinite dimensional stochastic space due to a large dimension of the stochastic space. The White Noise analysis is a suitable framework to develop this infinite dimensional analysis. In this paper, the case of *infinite dimensional noise* is considered.

Regularity results for Wick product pressure equations with log-normal coefficient have been considered for several authors. We emphasize that in (1) we use the ordinary product $\kappa(x,\omega)\nabla_x u(x,\omega)$ rather then the Wick product, $\kappa(x,\omega)\diamond\nabla_x u(x,\omega)$. For regularity results of stochastic pressure equations of Wick type see [3] and references therein. We also mention [30] were new ways of introducing the Wick calculus in the pressure equation are explored. In [3], the authors find the Chaos expansion of the solution of the Wick product pressure equation, and calculate its stochastic regularity in the distributional sense using Kondratiev type norms, see [20, 18]. One of the main properties of the Wick product is that it simplifies the computation of Chaos expansion of the Wick product of two random functions when compared with the *ordinary product.*

In [13], it is considered the Problem (1) where $\kappa(x,\omega) := e^{W_\phi(x,\omega)} = e^{\langle\omega,\phi_x\rangle}$, $x\in D, \omega\in\mathcal{S}'$ and the exponent $W_\phi(x,\omega)$ is the smoothed White Noise process defined on the White Noise probability space $(\mathcal{S}',\mathcal{B}(\mathcal{S}'),\mu)$ and well-posedness results were established in tensor product of Hilbert spaces. Here $\mathcal{S}'$ is an appropriately chosen space of distributions; see Section 2 below. The tensor product space of the solution involves the usual $H_0^1(D)$ space and the Hilbert space $(L^2)_s$ (a weighted L^2 space in $\mathcal{S}'$ with a exponential type weight). The parameter s is related to the stochastic exponential decay (or growth) of the functions on the White Noise probability space. The $(L^2)_s$ space has two fundamental aspects, it circumvents the lack of uniform ellipticity and boundedness of the problem, and due to its Hilbert space structure, its norm can be computed easily using orthogonality relations. Furthermore, in the resulting tensor product space we can use orthogonal projections to analyze finite dimensional Galerkin type methods and to obtain a priori error estimates. For the a priori error estimates, as usual, some regularity of the solution is assumed. It is required that the solution of (1) to belong to a more regular tensor product space (in x and ω). The regular tensor space used in [13] involves the spaces $H^2(D)$, for the regularity in the x variable, and a *weighted Chaos space* (see [13, 22, 18, 16, 17, 25, 25, 20, 6, 9, 3] and references therein) for the regularity in the ω variable. The weighted Chaos spaces norms depend on the choice of a sequence of weights. The corresponding norm measures the decay of the coefficients in the Chaos expansion of a random function. We recall that the Chaos expansion of a random function is its expansion in terms of Fourier-Hermite orthogonal polynomials. The a priori error estimates in [13] are general and apply to any weighted Chaos space.

In this paper we study the joint spatial and stochastic regularity of solutions of (1) assuming similar regularity for the right-hand side $f(x,\omega)$ and the smoothed White Noise $W_\phi(x,\omega)$. A main issue is that the computation of the weighted Chaos norms turn out to be difficult when the Chaos expansion of the solution is not available. For solutions of (1), it is difficult to write a manageable expression for the Chaos expansion of the solution, either in terms of Fourier-Hermite polynomials or in terms of multiple Itô integrals, see [18, 16, 25]. On the other hand, *Gaussian Sobolev spaces* have been also used in the literature, [27, 10, 24]. The Gaussian Sobolev norms involve $(L^2)_s$ norms of derivative of random functions. In particular, Sobolev type norms turns out to be equivalent to some particular weighted Chaos norms. This equivalence is useful to obtain the regularity results required in the a priori error estimates provided in [13]. In particular, we prove Theorem 32 where we obtain that the solution of (1) has regularity H^2 in the spatial variable x, see Lemma 29, and stochastic regularity given by a particular weighted Chaos space, see Lemma 22. We first prove that the weighted Chaos space to be used in Theorem 32 can be characterized using Sobolev type norms in the ω variable for the Gaussian measure as in [10, 27, 24]; see Theorem 20. In particular, the weighted Chaos spaces used in Theorem 32 require norms of partial derivatives in the ω variable up to certain order to be bounded.

In Section 2 we introduce the White Noise framework to be used in the paper. In Section 3 we present some detail of the model problem formulation in the White Noise framework and summarize results from [13]. In Section

4 we define and characterize the tensor product spaces. Sections 5 and 6 are dedicated to the use of partial derivatives in the ω variable to compute weighted Chaos norms. In Section 7 we study the stochastic regularity of solutions and in Section 8 we obtain also the spatial regularity results. Final remarks are presented in Section 9.

2 Framework: White noise analysis

Let H be a real Hilbert space with inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$, and let A be an operator on H such that there exists an H-orthonormal basis $\{\eta_j\}_{j=1}^{\infty}$ satisfying (see Examples 2 and 3 below):

1) $A\eta_j = \lambda_j \eta_j$, $j = 1, 2, \ldots$.
2) $1 < \lambda_1 \le \lambda_2 \le \cdots$.
3) $\sum_{j=1}^{\infty} \lambda_j^{-2\theta} < \infty$ for some constant $\theta > 0$.

For $p > 0$ let $\mathcal{S}_p := \{\xi \in H;\ \ \|\xi\|_p < \infty\}$ where

$$\|\xi\|_p^2 := \|A^p \xi\|_H^2 = \sum_{j=0}^{\infty} \lambda_j^{2p} (\xi, \eta_j)_H^2,$$

and for $p < 0$ let $\mathcal{S}_p$ be defined as the dual space of $\mathcal{S}_{-p}$. It is easy to see that for $p < 0$ we also have $\|\cdot\|_p = \|A^p \cdot\|_H$ and the duality pairing between $\mathcal{S}_p$ and $\mathcal{S}_{-p}$ is an extension of the H inner product. We also define

$$\mathcal{S} = \cap_{p \ge 0} \mathcal{S}_p \text{ (with the projective limit topology)}$$

and let $\mathcal{S}'$ be defined as the dual space of $\mathcal{S}$, i.e., by considering the standard countably Hilbert space constructed from (H, A); see [20, 25].

Let $\mathcal{S}'$ be the probability space with the sigma-field $\mathcal{B}(\mathcal{S}')$ of Borel subsets of $\mathcal{S}'$. The probability measure μ is given by the Bochner-Minlos theorem and characterized by

$$E_\mu e^{i\langle \cdot, \xi \rangle} := \int_{\mathcal{S}'} e^{i\langle \omega, \xi \rangle} d\mu(\omega) = e^{-\frac{1}{2}\|\xi\|_H^2}, \text{ for all } \xi \in \mathcal{S}. \tag{2}$$

Here, the pairing $\langle \omega, \xi \rangle = \omega(\xi)$ is the action of $\omega \in \mathcal{S}'$ on $\xi \in \mathcal{S}$, and E_μ denotes the expectation with respect to the measure μ; see [25, 18, 16, 17, 20, 5] and references therein. The measure μ is often called the (normalized) *Gaussian measure* on $\mathcal{S}'$. We note that from (2) we have that for any function $\xi \in H$, the random variable $\omega \mapsto \langle \omega, \xi \rangle$ can be defined in the $L^2(\mu)$ sense and it is normally distributed with zero mean and variance $\|\xi\|_H^2$; see [18, 20, 25].

In what follows we use the notation (L^2) for the space $L^2(\mu)$. We always interpret properties in the "almost everywhere" or "almost surely" or "almost all" sense, therefore, we will sometimes omit this interpretation to make notation and formula less cumbersome.

Definition 1. *The 1-dimensional smoothed White Noise associated to H and A is the map $w : \mathcal{S} \times \mathcal{S}' \longrightarrow \mathbb{R}$ given by $w(\xi) = w(\xi, \omega) = \langle \omega, \xi \rangle$ for $\omega \in \mathcal{S}'$, $\xi \in \mathcal{S}$. Let $D \subset \mathbb{R}^d$. Using the 1-dimensional smoothed White Noise w we can construct a stochastic process, called the* smoothed White Noise process *$W_\phi(x, \omega)$, as follows:*

$$W_\phi(x, \omega) := w(\phi_x, \omega) = \langle \omega, \phi_x \rangle, \ x \in D, \ \omega \in \mathcal{S}',$$

where $\phi_x \in H$ for all $x \in D$. For each $x \in D$, $W_\phi(x, \cdot)$ is normally distributed with zero mean and for $x, \hat{x} \in D$ we can write

$$W_\phi(x, \omega) = \sum_{j=1}^{\infty} (\eta_j, \phi_x)_H \langle \omega, \eta_j \rangle$$

where the $\langle \omega, \eta_j \rangle$ are independent and identically standard normal distributions and it is easy to see that $E_\mu W_\phi(x, \cdot) W_\phi(\hat{x}, \cdot) = (\phi_x, \phi_{\hat{x}})_H$.

Example 2. *Let $D \subset \mathbb{R}^d$ and take $H = L^2(D)$ and $A = Q^{-1}$, where $Q : L^2(D) \to L^2(D)$ is the integral operator on $(D \times D)$ with kernel given by a covariance $C(x, \hat{x})$. In this case, for $x, \hat{x} \in D$ we define $\phi_x(\hat{x}) = \sum_{j=1}^{\infty} \lambda_j^{-1/2} \eta_j(x) \eta_j(\hat{x})$, where λ_j and η_j are the eigenvalues and eigenfunctions of A. It is easy to see that $E_\mu(W_\phi(x, \cdot) W_\phi(\hat{x}, \cdot)) = C(x, \hat{x})$.*

Example 3. *We can take $H = L^2(\mathbb{R}^d)$ and $A = A_1 \otimes \cdots \otimes A_d$ where $A_i = -\frac{d^2}{dx_i^2} + x_i^2 + 1$. The eigenfunctions of A_i are the ℓ-th Hermite function with associated eigenvalue 2ℓ, for all $\ell \in \mathbb{N}$. The η_j and λ_j are obtained by tensor product operations. Let $\phi_x(\hat{x}) = \phi(\hat{x} - x)$, $x \in D$ and $\hat{x} \in \mathbb{R}^d$, where the window ϕ can be chosen such that the diameter of the support of ϕ is the maximum distance which $W_\phi(x, \cdot)$ and $W_\phi(\hat{x}, \cdot)$ might be correlated; see [18].*

The following particular case of Fernique's Theorem will be used throughout this paper; see [27, 6, 19, 10, 11].

Lemma 4. *We have*

$$\int_{\mathcal{S}'} e^{s\|\omega\|_{-\theta}^2} d\mu(\omega) = \begin{cases} \prod_{j=1}^{\infty} \left(1 - \frac{2s}{\lambda_j^{2\theta}}\right)^{-\frac{1}{2}}, & s < \frac{\lambda_1^{2\theta}}{2} \\ +\infty, & s \geq \frac{\lambda_1^{2\theta}}{2}. \end{cases}$$

We note that Lemma 4 implies that $\int_{\mathcal{S}'} \|\omega\|_{-\theta}^2 d\mu < \infty$ which in turn implies that $\mu(\mathcal{S}_{-\theta}) = 1$. To see this, note that $\mathcal{S}' \setminus \mathcal{S}_{-\theta} = \{\omega : \|\omega\|_{-\theta}^2 = \infty\}$ and then $\mu(\mathcal{S}' \setminus \mathcal{S}_{-\theta}) > 0$ would imply that $\int_{\mathcal{S}'} \|\omega\|_{-\theta}^2 d\mu = \infty$ which gives a contradiction. Without further comments, we use that $\mu(\mathcal{S}_{-\theta}) = 1$ throughout this paper.

3 The problem and variational formulation

Given $\phi_x \in \mathcal{S}_\theta$ for all $x \in D$ we consider the following problem: For all $\omega \in \mathcal{S}'$, find $u(x, \omega; \phi)$ such that

$$\begin{cases} -\nabla_x \cdot (\kappa(x, \omega; \phi) \nabla_x u(x, \omega; \phi)) &= f(x, \omega), \text{ for all } x \in D \\ u(x, \cdot; \phi) &= 0, \text{ for all } x \in \partial D, \end{cases} \tag{3}$$

where

$$\kappa(x,\omega;\phi) := e^{W_\phi(x,\omega)} = e^{\langle \omega, \phi_x \rangle} \tag{4}$$

and the exponent $W_\phi(x,\omega)$ is the 1-dimensional smoothed White Noise process of Definition 1. Thus, κ is log-normal random process. Observe that for different maps $x \mapsto \phi_x \in \mathcal{S}_\theta$ there exists a different permeability function $\kappa(\cdot,\cdot,\phi)$ associated to it. We will omit, whenever there is no danger of confusion, the dependence of κ on the map $x \mapsto \phi_x$ just to make the notation less cumbersome.

Denote

$$C_\theta = C_\theta(\phi) := \sup_{x\in D} \|\phi_x\|_\theta. \tag{5}$$

Then we have for all $\epsilon > 0$ and almost sure all $\omega \in \mathcal{S}'$

$$\kappa_{\min}(\omega) := e^{-\frac{C_\theta^2}{2\epsilon}} e^{-\frac{\epsilon}{2}\|\omega\|^2_{-\theta}} \le \kappa(x,\omega) \le e^{\frac{C_\theta^2}{2\epsilon}} e^{\frac{\epsilon}{2}\|\omega\|^2_{-\theta}} =: \kappa_{\max}(\omega). \tag{6}$$

Define $\mathcal{U}_s^m$ as the space of functions $u : D \times \mathcal{S}' \to \mathbb{R}$ such that

$$\int_{\mathcal{S}'} \|u(\cdot,\omega)\|^2_{H^m(D)} e^{s\|\omega\|^2_{-\theta}} d\mu(\omega) < +\infty \tag{7}$$

with norm

$$\|u\|^2_{\mathcal{U}_s^m} := \int_{\mathcal{S}'} \|u(\cdot,\omega)\|^2_{H^m(D)} e^{s\|\omega\|^2_{-\theta}} d\mu(\omega)$$

and seminorm

$$|u|^2_{\mathcal{U}_s^m} := \int_{\mathcal{S}'} |u(\cdot,\omega)|^2_{H^m(D)} e^{s\|\omega\|^2_{-\theta}} d\mu(\omega).$$

Note that $\mathcal{U}_0^0 = L^2(D) \otimes (L^2)$ and in general $\mathcal{U}_s^m = H^m(D) \otimes (L^2)_s$ where

$$(L^2)_s := L^2(\mathcal{S}', e^{s\|\omega\|^2_{-\theta}} d\mu(\omega)) \tag{8}$$

with norm $\|v\|^2_{(L^2)_s} := \int_{\mathcal{S}'} |v(\omega)|^2 e^{s\|\omega\|^2_{-\theta}} d\mu$. We also define $\widehat{\mathcal{U}}_s^1 = H_0^1(D) \otimes (L^2)_s \subset \mathcal{U}_s^1$, i.e., the functions in $\mathcal{U}_s^1$ which vanish on ∂D almost sure in ω. By using a Poincaré inequality, the seminorm $|\cdot|_{\mathcal{U}_s^1}$ is a norm equivalent to $\|\cdot\|_{\mathcal{U}_s^1}$ in $\widehat{\mathcal{U}}_s^1$. Since the space $(L^2)_s$ is the dual of $(L^2)_{-s}$ and the $H^{-1}(D)$ is the dual of $H_0^1(D)$, we can identify the dual space of $\widehat{\mathcal{U}}_{-s}^1$ with $\mathcal{U}_s^{-1}$.

We note that $\kappa(x,\omega) > 0$ is neither bounded uniformly from above nor from away zero, hence, the bilinear

$$a(u,v) = \int_{D\times\mathcal{S}'} \kappa(x,\omega) \nabla u(x,\omega) \nabla v(x,\omega) dx d\mu \tag{9}$$

is neither continuous nor coercive on $\widehat{\mathcal{U}}_s^1 \times \widehat{\mathcal{U}}_{-s}^1$. According to [13], the coerciveness (the inf-sup condition) and boundedness of the bilinear form $a(\cdot,\cdot)$ can be circumvent by enlarging the space of test functions for v from $\widehat{\mathcal{U}}_{-s}^1$ to $\widehat{\mathcal{U}}_{-s-\epsilon}^1$ and by reducing the solution space for u from $\widehat{\mathcal{U}}_s^1$ to $\widehat{\mathcal{D}}_s^1$ where

$$\widehat{\mathcal{D}}_s^1 := \{u \in \widehat{\mathcal{U}}_s^1 : \sup_{v\in\widehat{\mathcal{U}}_{-s-\epsilon}^1\setminus\{0\}} \frac{a(u,v)}{|v|_{\mathcal{U}_{-s-\epsilon}^1}} < \infty\}.$$

The *weak formulation* of problem (3) is then introduced as follows:

$$\left\{\begin{array}{l}\text{Given } f \in \mathcal{U}_{s+\epsilon}, \text{ find } u \in \widehat{\mathcal{D}}^1_s \text{ such that}\\ a(u,v) = \langle f, v\rangle \text{ for all } v \in \widehat{\mathcal{U}}^1_{-s-\epsilon}\end{array}\right. \tag{10}$$

where the bilinear form a is defined in (9) and the duality pairing between $f \in \mathcal{U}^1_{s+\epsilon}$ and $v \in \widehat{\mathcal{U}}_{-s-\epsilon}$ is given by

$$\langle f, v\rangle = \int_{D\times\mathcal{S}'} f(x,\omega)v(x,\omega)dxd\mu.$$

Lemma 5 ([13] Existence and uniqueness of solutions)**.** *Let $\epsilon > 0$ and assume that $C_\theta = \sup_{x\in D} \|\phi_x\|_\theta < \infty$. Then for $f \in \mathcal{U}^{-1}_{s+\epsilon}$, there exists a unique solution $u \in \widehat{\mathcal{D}}^1_s \subset \widehat{\mathcal{U}}^1_s$ of Problem (10) and*

$$\|u\|_{\mathcal{U}^1_s} \leq Ce^{\frac{C_\theta^2}{2\epsilon}} \|f\|_{\mathcal{U}^{-1}_{s+\epsilon}}, \tag{11}$$

where $C = \sqrt{1 + C_{poin}}$ and C_{poin} is the Poincaré inequality constant which is independent of ϵ and θ.

Remark 6. *From Lemma 5, when $f \in \mathcal{U}^{-1}_0$ then for every $s < 0$ (take $\epsilon = -s$) the solution $u \in \widehat{\mathcal{U}}^1_s$. In order to have $u \in \widehat{\mathcal{U}}^1_0$ we need $f \in \mathcal{U}^{-1}_\epsilon$ for some $\epsilon > 0$. When the right-hand side f is deterministic or is given by a finite sum of Fourier-Hermite polynomials, we have the solution $u \in \widehat{\mathcal{U}}^1_s$ for every s satisfying $s < \frac{\lambda_1^{2\theta}}{2}$; see Definition 7 and Theorem 8.*

4 The Galerkin approximation

In the following we characterize the space $(L^2)_s$ defined in (8), and note that this is enough for characterizing the tensor product space $\mathcal{U}^m_s = H^m(D)\otimes(L^2)_s$.

We need to consider multi-index of arbitrary length. To simplify the notation, we regard multi-indices as elements of the space $(\mathbb{N}_0^{\mathbb{N}})_c$ of all sequences $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots)$ with elements $\alpha_j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and with compact support, i.e., with only finitely many $\alpha_j \neq 0$. We write $\boldsymbol{\mathcal{J}} = (\mathbb{N}_0^{\mathbb{N}})_c$. Given $\boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}$ define the order and length of $\boldsymbol{\alpha}$, denoted by $d(\boldsymbol{\alpha})$ and $|\boldsymbol{\alpha}|$ respectively, by

$$d(\boldsymbol{\alpha}) = \max\{j : \alpha_j \neq 0\} \quad \text{and} \quad |\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \ldots + \alpha_{d(\boldsymbol{\alpha})}.$$

We also introduce the σ-Hermite polynomials, $h_{\sigma^2,n}$, where $\sigma > 0$ and $n = 0, 1, 2, \ldots$. These polynomials can be defined by the generating function identity

$$e^{tx-\frac{1}{2}\sigma^2t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} h_{\sigma^2,n}(x). \tag{12}$$

When $\sigma^2 = 1$ we denote $h_{1,n}$ simply by h_n. Note that $h_{\sigma^2,n}(x) = \sigma^n h_n(x/\sigma)$ and $h'_{\sigma^2,n}(x) = n h_{\sigma^2,n-1}(x)$. The σ-Hermite polynomials are an orthogonal basis for $L^2(\mathbb{R}, e^{-\frac{1}{2\sigma^2}x^2}dx)$.

For $s < \frac{\lambda_1^{2\theta}}{2}$ define $\sigma_j = \sigma_j(s) = \left(1 - \frac{2s}{\lambda_j^{2\theta}}\right)^{-\frac{1}{2}}$, $j = 1, 2, \ldots$, and for $\boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}$ let

$$\boldsymbol{\sigma}^{\boldsymbol{\alpha}} = \boldsymbol{\sigma}^{\boldsymbol{\alpha}}(s) := \prod_{j=1}^{d(\boldsymbol{\alpha})} \sigma_j^{\alpha_j}(s)$$

and

$$\sigma_* = \sigma_*(s) := \int_{\mathcal{S}'} e^{s\|\omega\|_{-\theta}^2} d\mu(\omega).$$

From Lemma 4, $\sigma_* = \prod_{j=1}^{\infty} \sigma_j < \infty$ when $s < \frac{\lambda_1^{2\theta}}{2}$. Now we define the $\boldsymbol{\sigma(s)}$-Fourier-Hermite polynomials.

Definition 7. *Given $s < \frac{\lambda_1^{2\theta}}{2}$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots) \in \boldsymbol{\mathcal{J}}$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}(s) = (\sigma_1, \sigma_2, \ldots)$, define*

$$H_{\boldsymbol{\sigma}^2,\boldsymbol{\alpha}}(\omega) = \frac{1}{\sqrt{\sigma_*}} \prod_{j=1}^{d(\boldsymbol{\alpha})} h_{\sigma_j^2,\alpha_j}(\langle \omega, \eta_j \rangle); \quad \omega \in \mathcal{S}'.$$

We now state the Wiener-Chaos expansion theorem; see [10, 16, 18, 17, 25].

Theorem 8. *When $s < \frac{\lambda_1^{2\theta}}{2}$, the $\boldsymbol{\sigma}(s)$-Fourier-Hermite polynomials are orthogonal in $(L^2)_s$. Moreover,*

$$\|H_{\boldsymbol{\sigma}^2(s),\boldsymbol{\alpha}}\|_{(L^2)_s}^2 = \boldsymbol{\alpha}!\boldsymbol{\sigma}(s)^{2\boldsymbol{\alpha}}.$$

In addition, every polynomial in ω belongs to $(L)_s$ and every $u \in (L^2)_s$ can be represented as a Wiener-Chaos expansion

$$u = \sum_{\boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}} u_{\boldsymbol{\alpha},s} H_{\boldsymbol{\sigma}(s)^2,\boldsymbol{\alpha}} \quad \textit{with} \quad \|u\|_{(L^2)_s}^2 = \sum_{\boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}} \boldsymbol{\alpha}!\boldsymbol{\sigma}(s)^{2\boldsymbol{\alpha}} u_{\boldsymbol{\alpha},s}^2.$$

Remark 9. *The corresponding tensor product norm for $u \in \mathcal{U}_s^m$ with $s < \frac{\lambda_1^{2\theta}}{2}$ is given by*

$$\|u\|_{\mathcal{U}_s^m}^2 = \sum_{\boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}} \boldsymbol{\alpha}!\boldsymbol{\sigma}(s)^{2\boldsymbol{\alpha}} \|u_{\boldsymbol{\alpha},s}\|_{H^m(D)}^2,$$

where $u = \sum_{\boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}} u_{\boldsymbol{\alpha},s} H_{\boldsymbol{\sigma}(s)^2,\boldsymbol{\alpha}}$ with $u_{\boldsymbol{\alpha},s} \in H^m(D)$ for all $\boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}$.

Let $N, K \in \mathbb{N}_0$ and define

$$\boldsymbol{\mathcal{J}}^{N,K} = \{\boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}} : d(\boldsymbol{\alpha}) \leq K, \text{ and, } |\boldsymbol{\alpha}| \leq N\}$$

and

$$\mathcal{P}^{N,K} := \operatorname{span}\left\{H_{\boldsymbol{\sigma}(s)^2,\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}^{N,K}\right\} = \operatorname{span}\left\{\prod_{j=1}^{d(\boldsymbol{\alpha})} \langle \omega, \eta_j \rangle^{\alpha_j} : \boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}^{N,K}\right\},$$

i.e., $\mathcal{P}^{N,K}$ consists of polynomials in $\langle\omega,\eta_1\rangle,\ldots,\langle\omega,\eta_K\rangle$ of total degree at most N.

Note that when $s<\frac{\lambda_1^{2\theta}}{2}$, polynomials in ω belong to $(L^2)_s$. Let $X_0^h(D)\subset H_0^1(D)$ be the finite element space of piecewise linear and continuous functions with respect to a quasi-uniform triangulation of D with mesh size h.

For $N,K\in\mathbb{N}_0$ and $h>0$ define the following discrete spaces:

$$\mathcal{X}_s^{N,K,h}:=X_0^h(D)\otimes\mathcal{P}^{N,K}\subset\widehat{\mathcal{U}}_s^1\subset\mathcal{U}_s^1$$

and

$$\mathcal{Y}_s^{N,K,h}:=\left\{v:v(x,\omega)=\tilde{v}(x,\omega)e^{(s+\frac{\epsilon}{2})\|\Pi_K\omega\|^2_{-\theta}},\ \tilde{v}\in\mathcal{X}_s^{N,K,h}\right\}\subset\widehat{\mathcal{U}}^1_{-(s+\epsilon)},$$

where Π_K is the (H-orthogonal) projection on the span$\{\eta_1,\ldots,\eta_K\}$ is defined by $\Pi_K\omega:=\sum_{j=1}^K\langle\omega,\eta_j\rangle\eta_j$, for all $\omega\in\mathcal{S}'$. The discrete version of problem (10) is introduced as:

$$\begin{cases}\text{Find } u_s^{N,K,h}\in\mathcal{X}_s^{N,K,h} \text{ such that}\\ a(u_s^{N,K,h},v)=\langle f,v\rangle \text{ for all } v\in\mathcal{Y}_s^{N,K,h}.\end{cases}$$

The corresponding discrete inf-sup condition, resulting linear system and one spatial dimension numerical example are discussed in [13].

4.1 Weighted norms and a priori error estimates

In $(L^2)_s$ with $s<\frac{\lambda_1^{2\theta}}{2}$ we introduce the system of Hilbert norms

$$||u||^2_{p;\rho,s}:=\sum_{\boldsymbol{\alpha}\in\boldsymbol{\mathcal{J}}}\rho(\boldsymbol{\alpha},p)^2\boldsymbol{\alpha}!\boldsymbol{\sigma}(s)^{2\boldsymbol{\alpha}}u^2_{\boldsymbol{\alpha},s}, \tag{13}$$

where $u=\sum_{\boldsymbol{\alpha}\in\boldsymbol{\mathcal{J}}}u_{\boldsymbol{\alpha},s}H_{\sigma(s)^2,\boldsymbol{\alpha}}$. We assume that $\rho(\boldsymbol{\alpha},q)\geq\rho(\boldsymbol{\alpha},p)>0$ for all $q>p\geq0$ and that $\rho(\boldsymbol{\alpha},0)=1$ for all $\boldsymbol{\alpha}\in\boldsymbol{\mathcal{J}}$. Usually, the *weights* $\rho(\boldsymbol{\alpha},s)$ are the eigenvalues of some nonnegative operator in $(L^2)_s$ with the $\sigma(s)$-Fourier-Hermite polynomials as eigenfunctions; see [18, 17, 25, 20, 6, 9, 3].

For $p>0$ define the spaces $\mathcal{S}_{p,\rho,s}$ by

$$\mathcal{S}_{p;\rho,s}=\{v\in(L^2)_s:\|v\|_{p;\rho,s}<\infty\}. \tag{14}$$

For $p<0$ define $\mathcal{S}_{p,\rho,s}$ as the dual space of $\mathcal{S}_{-p;\rho,s}$. We have $\mathcal{S}_{0;\rho,s}=(L^2)_s$ and the inclusion $\mathcal{S}_{q;\rho,s}\subset\mathcal{S}_{p;\rho,s}$ holds for all $q>p$.

For examples of weights $\rho(\boldsymbol{\alpha},p)$ we refer to [10, 6, 16, 18, 20, 25, 24, 27]. We consider the following weight describe in the next example and to be used through the paper; see [6, 10, 17, 24, 27]

Example 10. *Given a multi-index* $\boldsymbol{\alpha}$ *we denote* $\langle \boldsymbol{\alpha}, \boldsymbol{\lambda} \rangle := \sum_{j=1}^{d(\boldsymbol{\alpha})} \alpha_j \lambda_j$. *Note that* $\langle \boldsymbol{\alpha}, \boldsymbol{\lambda} \rangle \geq 0$. *We introduce the weight* ρ *defined by,*

$$\rho(\boldsymbol{\alpha}, p)^2 = 1 + \langle \boldsymbol{\alpha}, \boldsymbol{\lambda} \rangle^{2p}, \quad p > 0, \text{ and } \rho(\boldsymbol{\alpha}, 0) = 1, \quad \boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}. \tag{15}$$

Norms $\| \cdot \|_{p;\rho,s}$ defined in (13) can also be extended to tensor products. The corresponding norms, using Example 10, for the tensor product spaces $\mathcal{U}^m_{p;\rho,s} := H^m(D) \otimes \mathcal{S}_{p,\rho,s}$ are defined by

$$||u||^2_{\mathcal{U}^m_{p;\rho,s}} = \sum_{\boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}} \left(1 + \langle \boldsymbol{\alpha}, \boldsymbol{\lambda} \rangle^{2p}\right) \boldsymbol{\alpha}! \boldsymbol{\sigma}(s)^{2\boldsymbol{\alpha}} \|u_{\boldsymbol{\alpha},s}\|^2_{H^m(D)}, \tag{16}$$

and we also introduce the seminorms,

$$|u|^2_{\mathcal{U}^m_{p;\rho,s}} := \sum_{\boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}} \left(1 + \langle \boldsymbol{\alpha}, \boldsymbol{\lambda} \rangle^{2p}\right) \boldsymbol{\alpha}! \boldsymbol{\sigma}(s)^{2\boldsymbol{\alpha}} |u_{\boldsymbol{\alpha},s}|^2_{H^m(D)}. \tag{17}$$

We have the following a priori error estimates. See [13].

Lemma 11. *Let* $s \in \mathbb{R}$ *and* $u \in \widehat{\mathcal{U}}^1_s$ *be the solution of (10) with* $\epsilon > 0$ *and* $f \in \mathcal{U}^{-1}_{s+\epsilon}$. *Assume that* $s + \tilde{\epsilon} + \epsilon < \frac{\lambda_1^{2\theta}}{2}$ *and* $-s - \epsilon < \frac{\lambda_{K+1}^{2\theta}}{2}$ *for some* $\tilde{\epsilon} > 0$. *Consider the weights* ρ *defined in (15). We have for all* $p > 0$ *and* $\ell \leq 2$ *that*

$$|u - u_s^{N,K,h}|_{\mathcal{U}^1_s} \leq C_* \Big\{ \max\left\{ \tfrac{1}{1+(N+1)\lambda_1}, \tfrac{1}{1+\lambda_{K+1}} \right\}^q |u|_{\mathcal{U}^1_{p;\rho,s+\tilde{\epsilon}+\epsilon}} + \hat{C} h^{\ell-1} \|u\|_{\mathcal{U}^\ell_{s+\tilde{\epsilon}+\epsilon}} \Big\},$$

where $C_* = C_*(s, \epsilon, \tilde{\epsilon}) = 1 + e^{\frac{C_\theta^2}{\epsilon}} e^{\frac{C_\theta^2}{\tilde{\epsilon}}} \prod_{j=K+1}^{\infty} \sigma_j(-s-\epsilon)$ *and* $\hat{C}$ *is is the Clement finite element interpolation constant on the space* $X_0^h(D)$.

5 Derivatives and Chaos weighted norms

The error estimates in Lemma 11 above is one the motivation for the regularity studies carried out in this paper. It turns out that the weighted norms in (13) are, in general, difficult to compute or estimate when the Chaos expansion is not explicitly available. In Section 5.1 we recall that the weighted norms can be written as a square integral, in the White Noise measure, using an operator acting on functions in $(L^2)_s$. In Section 5.2 we review the computation of $(L^2)_s$ norms of derivatives. Later on in Section 6 we will establish that some Chaos weighted norms can be computed using $(L^2)_s$ norms of partial derivatives.

5.1 Chaos weighted norms and the operator $\Gamma_\oplus(A)$

We consider the weighted norm (13) with the particular weight in (15). We can write

$$\begin{aligned} ||u||^2_{p;\rho,s} &= \sum_{\boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}} (1 + \langle \boldsymbol{\alpha}, \boldsymbol{\lambda} \rangle^{2p}) \boldsymbol{\alpha}! \boldsymbol{\sigma}(s)^{2\boldsymbol{\alpha}} u^2_{\boldsymbol{\alpha},s}, \\ &= ||u||^2_{(L^2)_s} + ||\Gamma_\oplus(A)^p u||^2_{(L^2)_s} \\ &= \int_{\mathcal{S}'} \left(|u(\omega)|^2 + |\Gamma_\oplus(A)^p u(\omega)|^2\right) e^{s\|\omega\|^2_{-\theta}} d\mu(\omega), \end{aligned}$$

where $\Gamma_{\oplus}(A)$ is the operator defined by

$$\Gamma_{\oplus}(A)H_{\boldsymbol{\sigma}^2,\boldsymbol{\alpha}} = \langle \boldsymbol{\alpha}, \boldsymbol{\lambda}\rangle H_{\boldsymbol{\sigma}^2,\boldsymbol{\alpha}}. \tag{18}$$

We point out that $\Gamma_{\oplus}(A^p) \neq \Gamma_{\oplus}(A)^p$ since $\Gamma_{\oplus}(A)^p H_{\boldsymbol{\sigma}^2,\boldsymbol{\alpha}} = \langle \boldsymbol{\alpha}, \boldsymbol{\lambda}\rangle^p H_{\boldsymbol{\sigma}^2,\boldsymbol{\alpha}}$ and $\Gamma_{\oplus}(A^p)H_{\boldsymbol{\sigma}^2,\boldsymbol{\alpha}} = \langle \boldsymbol{\alpha}, \boldsymbol{\lambda}^p\rangle H_{\boldsymbol{\sigma}^2,\boldsymbol{\alpha}}$. We observe that $||\Gamma_{\oplus}(A)^p \cdot ||^2_{(L^2)_s}$ is a norm in the space of function in $(L^2)_s$ with $u_{\mathbf{0}} = 0$ in its $\sigma(s)$-Fourier-Hermite expansion.

5.2 Derivatives and Gaussian Sobolev norms

Using partial derivative (as in the deterministic Sobolev spaces norms), we want to be able to compute a norm equivalent to the norm (13) with the weights ρ defined (15).

In this section we work with differential operators acting on $(L^2)_s$ and define Sobolev type norms for Gaussian measure; see [6, 10, 17, 20, 24, 27, 29, 32] and references therein.

Denote by $\partial_\ell u$ as the directional derivative of u in the direction of the $\ell-$th basis function $\eta_\ell \in \mathcal{S}$. Given $u \in (L^2)_s$ we have

$$\partial_\ell u(\omega) := \frac{d}{dt} u(\omega + t\eta_\ell)\Big|_{t=0}.$$

For any Fourier-Hermite polynomial $H_{\boldsymbol{\sigma}^2,\boldsymbol{\alpha}}$ with $\alpha_\ell > 0$ we have that

$$\partial_\ell H_{\boldsymbol{\sigma}^2,\boldsymbol{\alpha}}(\omega) = \partial_\ell \prod_{j=1}^{d(\boldsymbol{\alpha})} h_{\sigma_j^2,\alpha_j}(\langle \omega, \eta_j\rangle) = \alpha_\ell H_{\boldsymbol{\sigma}^2,\boldsymbol{\alpha}-\boldsymbol{\xi}_\ell}(\omega) \tag{19}$$

where $\boldsymbol{\xi}_\ell$ is the multi-index with one in the $\ell-$entry and zero in the other positions so that $\boldsymbol{\alpha} - \boldsymbol{\xi}_\ell = (\alpha_1, \ldots, \alpha_{\ell-1}, \alpha_\ell - 1, \alpha_{\ell+1}, \ldots)$. Here we have used that $h'_{\sigma^2,n} = n h_{\sigma^2,n-1}$, see (12). For $\alpha_\ell = 0$ define $\partial_\ell H_{\boldsymbol{\sigma}^2,\boldsymbol{\alpha}}(\omega) = 0$. Then for $u = \sum_{\boldsymbol{\alpha}\in\boldsymbol{\mathcal{J}}} u_{\boldsymbol{\alpha}} H_{\boldsymbol{\sigma}^2,\boldsymbol{\alpha}}$ such that $\partial_j u \in (L^2)_s$ we have

$$\partial_\ell u(\omega) = \sum_{\boldsymbol{\alpha}\in\boldsymbol{\mathcal{J}}} \alpha_\ell u_{\boldsymbol{\alpha}} H_{\boldsymbol{\sigma}^2,\boldsymbol{\alpha}-\boldsymbol{\xi}_\ell}(\omega) \tag{20}$$

and

$$||\partial_\ell u||^2_{(L^2)_s} = \sum_{\boldsymbol{\alpha}\in\boldsymbol{\mathcal{J}}} \alpha_\ell^2 u_{\boldsymbol{\alpha}}^2 \boldsymbol{\sigma}^{2\boldsymbol{\alpha}} (\boldsymbol{\alpha} - \boldsymbol{\xi}_\ell)! = \sum_{\boldsymbol{\alpha}\in\boldsymbol{\mathcal{J}}} \alpha_\ell u_{\boldsymbol{\alpha}}^2 \boldsymbol{\sigma}^{2\boldsymbol{\alpha}} \boldsymbol{\alpha}!, \tag{21}$$

where we have used that $\alpha_\ell(\boldsymbol{\alpha} - \boldsymbol{\xi}_\ell)! = \boldsymbol{\alpha}!$; see [10]. Analogously, for any Fourier-Hermite polynomial the $\boldsymbol{\gamma}$ partial derivative ∂^{γ} can be computed as

$$\begin{aligned} \partial^{\gamma} H_{\boldsymbol{\sigma}^2,\boldsymbol{\alpha}}(\omega) &= \prod_{\ell=1}^{d(\boldsymbol{\gamma})} \partial_\ell^{\gamma_\ell} \prod_{j=1}^{d(\boldsymbol{\alpha})} h_{\sigma_j^2,\alpha_j}(\langle \omega, \eta_j\rangle) & (22)\\ &= \prod_{j=1}^{d(\alpha)} \frac{\alpha_j!}{(\alpha_j - \gamma_j)!} h_{\sigma_j^2,\alpha_j-\gamma_j}(\langle \omega, \eta_j\rangle) = \frac{\boldsymbol{\alpha}!}{(\boldsymbol{\alpha} - \boldsymbol{\gamma})!} H_{\boldsymbol{\sigma}^2,\boldsymbol{\alpha}-\boldsymbol{\gamma}} & (23) \end{aligned}$$

for every multi-indexes $\boldsymbol{\gamma}$ and $\boldsymbol{\alpha}$ with $\boldsymbol{\gamma} \leq \boldsymbol{\alpha}$. Then for $u = \sum_{\boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}} c_{\boldsymbol{\alpha}} H_{\boldsymbol{\sigma}^2, \boldsymbol{\alpha}}$ we have

$$\partial^{\boldsymbol{\gamma}} u(\omega) = \sum_{\boldsymbol{\alpha} \geq \boldsymbol{\gamma}} \frac{\boldsymbol{\alpha}!}{(\boldsymbol{\alpha} - \boldsymbol{\gamma})!} u_{\boldsymbol{\alpha}} H_{\boldsymbol{\sigma}^2, \boldsymbol{\alpha} - \boldsymbol{\gamma}}(\omega). \tag{24}$$

This implies that the $(L^2)_s$ norm of $\partial^{\boldsymbol{\gamma}} u$ is given by

$$\begin{aligned} \|\partial^{\boldsymbol{\gamma}} u\|^2_{(L^2)_s} &= \sum_{\boldsymbol{\alpha} \geq \boldsymbol{\gamma}} \frac{\boldsymbol{\alpha}!^2}{(\boldsymbol{\alpha} - \boldsymbol{\gamma})!^2} u^2_{\boldsymbol{\alpha}} \boldsymbol{\sigma}^{2\boldsymbol{\alpha}} (\boldsymbol{\alpha} - \boldsymbol{\gamma})! \\ &= \sum_{\boldsymbol{\alpha} \geq \boldsymbol{\gamma}} \frac{\boldsymbol{\alpha}!}{(\boldsymbol{\alpha} - \boldsymbol{\gamma})!} u^2_{\boldsymbol{\alpha}} \boldsymbol{\sigma}^{2\boldsymbol{\alpha}} \boldsymbol{\alpha}!. \end{aligned} \tag{25}$$

Remark 12. *Recall that when $s = 0$ we have $\|\partial^{\boldsymbol{\gamma}} u\|^2_{(L^2)} = \int_{\mathcal{S}'} |\partial^{\boldsymbol{\gamma}} u(\omega)|^2 d\mu(\omega)$ and we refer to norms defined in terms of (L^2) norms of partial derivatives as Gaussian Sobolev norms. We will use the same terminology for the case $s \neq 0$.*

6 Equivalence of norms

This section is dedicated to prove that, using partial derivative, we can compute the norm $\|\cdot\|_{\frac{k}{2};\rho,s}$ defined in Example 10 when $k \in \mathbb{N}$. We will prove in Theorem 20 that for every $k \in \mathbb{N}$ we have

$$||u||^2_{\frac{k}{2};\rho,s} = ||u||^2_{(L^2)_s} + \sum_{i=1}^{k} \sum_{R \in P^{k,i}} \sum_{\ell_1, \ell_2, \ldots, \ell_i} \lambda_{\ell_1}^{2R_1} \ldots \lambda_{\ell_i}^{2R_i} \|\partial_{\ell_1} \ldots \partial_{\ell_i} u\|^2_{(L^2)_s} \tag{26}$$

where $P^{k,i}$ is a finite subset (of indexes) of $\mathbb{R}^i$ that will be described below. Here and below we will use the iterated summation notation

$$\sum_{\ell_1, \ell_2, \ldots, \ell_k} := \sum_{\ell_1 \in \mathbb{N}} \sum_{\ell_2 \in \mathbb{N}} \cdots \sum_{\ell_k \in \mathbb{N}}.$$

Similar result for the case $k = 1$ and $k = 2$ (with $s = 0$) can be found in [10] and the corresponding spaces are denoted by $W^{1,2}(H, \mu)$ and $W^{2,2}(H, \mu)$ respectively. Here, we generalize their results to any $k \in \mathbb{N}$ and $s < \lambda_1^{2\theta}/2$, see Theorem 20. Additionally, we introduce several intermediate results for general norms of derivatives which can be used for defining fractional derivatives. We note that Theorem 20 is a key tool for establishing the regularity theory in Section 7.

We define the $k-$th derivative as follows. See [10, 27, 24] and references therein.

Definition 13. *For $k \in \mathbb{N}$ and $p \in \mathbb{R}$ define*

$$D^k u(\omega) := \sum_{\ell_1, \ell_2, \ldots, \ell_k} \partial_{\ell_1} \ldots \partial_{\ell_k} u(\omega) \eta_{\ell_1} \otimes \ldots \otimes \eta_{\ell_k} \in (\mathcal{S}')^{\otimes k}$$

and with $\Gamma_{\oplus}(A)$ defined in (18) we set

$$\Gamma_{\oplus}(A)^{\frac{p}{2}} D^k u(\omega) := \sum_{\ell_1,\ell_2,\ldots,\ell_k} \Gamma_{\oplus}(A)^{\frac{p}{2}} \partial_{\ell_1} \ldots \partial_{\ell_k} u(\omega) \eta_{\ell_1} \otimes \ldots \otimes \eta_{\ell_k} \in (\mathcal{S}')^{\otimes k}.$$

We also use the convention $D^0 u = u$.

We will compute $(L_2)_s$-norms of derivatives according to the next definition.

Definition 14. *For $k \in \mathbb{N}$ and $\boldsymbol{q} = (q_1, \ldots, q_k) \in \mathbb{R}^k$ define*

$$\begin{aligned} \|D^k u\|^2_{\boldsymbol{q}} &= \|A^{q_1} \otimes \ldots \otimes A^{q_k} D^k u\|^2_{L^2(\mathcal{S}', (L^2)_s^{\otimes k})} \\ &= \sum_{\ell_1,\ell_2,\ldots,\ell_k} \lambda_{\ell_1}^{2q_1} \ldots \lambda_{\ell_k}^{2q_k} \|\partial_{\ell_1} \ldots \partial_{\ell_k} u\|^2_{(L^2)_s}. \end{aligned}$$

We also set $||D^0 u||^2 = ||u||^2_{(L^2)_s}$.

Now we prove some basic relations between derivatives in the ω variable and the operator $\Gamma_{\oplus}(A)$ defined in in (18). See [27] for related results.

Lemma 15. *For all $p, q \in \mathbb{R}$ we have the following relations*

$$(\Gamma_{\oplus}(A^q) + \lambda_\ell^q)^{\frac{p}{2}} \partial_\ell u = \partial_\ell \Gamma_{\oplus}(A^q)^{\frac{p}{2}} u, \tag{27}$$

$$\left(\Gamma_{\oplus}(A^q) + \lambda_{\ell_1}^q + \ldots + \lambda_{\ell_k}^q\right)^{\frac{p}{2}} \partial_{\ell_1} \partial_{\ell_2} \cdots \partial_{\ell_k} u = \partial_{\ell_1} \partial_{\ell_2} \cdots \partial_{\ell_k} \Gamma_{\oplus}(A^q)^{\frac{p}{2}} u, \tag{28}$$

and

$$(\Gamma_{\oplus}(A^q) + \langle \boldsymbol{\beta}, \boldsymbol{\lambda}^q \rangle)^{\frac{p}{2}} \partial^{\boldsymbol{\beta}} u = \partial^{\boldsymbol{\beta}} \Gamma_{\oplus}(A^q)^{\frac{p}{2}} u. \tag{29}$$

Proof. Since $\partial_\ell u = \sum_{\boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}} \alpha_\ell u_{\boldsymbol{\alpha}} H_{\boldsymbol{\sigma}(s)^2, \boldsymbol{\alpha} - \boldsymbol{\xi}_\ell}$, then

$$\begin{aligned} (\Gamma_{\oplus}(A^q) + \lambda_\ell^q)^{\frac{p}{2}} \partial_\ell u &= \sum_{\boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}} (\langle \boldsymbol{\alpha} - \boldsymbol{\xi}_\ell, \boldsymbol{\lambda}^q \rangle + \lambda_\ell^q)^{\frac{p}{2}} \alpha_\ell u_{\boldsymbol{\alpha}} H_{\boldsymbol{\sigma}^2, \boldsymbol{\alpha} - \boldsymbol{\xi}_\ell} \\ &= \sum_{\boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}} (\langle \boldsymbol{\alpha}, \boldsymbol{\lambda}^q \rangle - \lambda_\ell^q + \lambda_\ell^q)^{\frac{p}{2}} \alpha_\ell u_{\boldsymbol{\alpha}} H_{\boldsymbol{\sigma}^2, \boldsymbol{\alpha} - \boldsymbol{\xi}_\ell} \\ &= \sum_{\boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}}} \langle \boldsymbol{\alpha}, \boldsymbol{\lambda}^q \rangle^{\frac{p}{2}} \alpha_\ell u_{\boldsymbol{\alpha}} H_{\boldsymbol{\sigma}^2, \boldsymbol{\alpha} - \boldsymbol{\xi}_\ell} = \partial_\ell \Gamma_{\oplus}(A^q)^{\frac{1}{2}} u, \end{aligned}$$

which prove (27). Note that (28) follows easily from (27) and (29) is consequence of (28) and the notation $\langle \boldsymbol{\beta}, \boldsymbol{\lambda}^q \rangle = \sum_{j=1}^{d(\boldsymbol{\alpha})} \beta_j \lambda_j^q$. □

Lemma 16. *For $k \in \mathbb{N}$ and $\boldsymbol{q} \in \mathbb{R}^k$ we have*

$$\sum_{\ell_k} \lambda_{\ell_k}^{2q_k} \|D^{k-1} \partial_{\ell_k} u\|^2_{(q_1,\ldots,q_{k-1})} = \|D^k u\|^2_{(q_1,\ldots,q_k)}, \tag{30}$$

$$\|D\Gamma_{\oplus}(A^{2q_2})^{\frac{1}{2}} u\|^2_{q_1} = \|\Gamma_{\oplus}(A^{2q_2})^{\frac{1}{2}} Du\|^2_{q_1} + \|Du\|^2_{q_1+q_2} \tag{31}$$

and for $\boldsymbol{q} = (q_1, \ldots, q_k) \in \mathbb{R}^k$ *and* $t \in \mathbb{R}$ *we have*

$$\|D^k \Gamma_\oplus(A^{2t})^{\frac{1}{2}} u\|_{\boldsymbol{q}}^2 \quad = \quad \|\Gamma_\oplus(A^{2t})^{\frac{1}{2}} D^k u\|_{\boldsymbol{q}}^2 + \sum_{i=1}^{k} \|D^k u\|_{\boldsymbol{q}+t\boldsymbol{\xi}_i}^2 \tag{32}$$

where $\boldsymbol{q} + t\boldsymbol{\xi}_i = (q_1, \ldots, q_i + t, \ldots, q_k)$.

Proof. Equation (30) follows directly from Definition 14. We prove (31). Using Definitions 13 and 14 together with Equation (27),

$$\begin{aligned}
\|D\Gamma_\oplus(A^{2q_2})^{\frac{1}{2}} u\|_{q_1}^2 &= \sum_{\ell=1} \lambda_\ell^{2q_1} \|\partial_\ell \Gamma_\oplus(A^{2q_2})^{\frac{1}{2}} u\|_{(L^2)_s}^2 \\
&= \sum_{\ell=1} \lambda_\ell^{2q_1} \| \left(\Gamma_\oplus(A^{2q_2}) + \lambda_\ell^{2q_2}\right)^{\frac{1}{2}} \partial_\ell u\|_{(L^2)_s}^2 \\
&= \sum_{\ell=1} \lambda_\ell^{2q_1} \left(\|\Gamma_\oplus(A^{2q_2})^{\frac{1}{2}} \partial_\ell u\|_{(L^2)_s}^2 + \lambda_\ell^{2q_2} \|\partial_\ell u\|_{(L^2)_s}^2\right) \\
&= \|\Gamma_\oplus(A^{2q_2})^{\frac{1}{2}} Du\|_{q_1}^2 + \|Du\|_{q_2+q_1}^2.
\end{aligned}$$

To prove (32) observe that using (28) we get

$$\begin{aligned}
\|D^k \Gamma_\oplus(A^{2t})^{\frac{1}{2}} u\|_{\boldsymbol{q}}^2 &= \sum_{\ell_1,\ldots,\ell_k} \lambda_{\ell_1}^{2q_1} \cdots \lambda_{\ell_k}^{2q_k} \|\partial_{\ell_1} \cdots \partial_{\ell_k} \Gamma_\oplus(A^{2t})^{\frac{1}{2}} u\|_{(L^2)_s}^2 \\
&= \sum_{\ell_1,\ldots,\ell_k} \lambda_{\ell_1}^{2q_1} \cdots \lambda_{\ell_k}^{2q_k} \| \left(\Gamma_\oplus(A^{2t}) + \lambda_{\ell_1}^{2t} + \cdots + \lambda_{\ell_k}^{2t}\right)^{\frac{1}{2}} \partial_{\ell_1} \cdots \partial_{\ell_k} u\|_{(L^2)_s}^2 \\
&= \sum_{\ell_1,\ldots,\ell_k} \lambda_{\ell_1}^{2q_1} \cdots \lambda_{\ell_k}^{2q_k} \|\Gamma_\oplus(A^{2t})^{\frac{1}{2}} \partial_{\ell_1} \cdots \partial_{\ell_k} u\|_{(L^2)_s}^2 \\
&\quad + \sum_{\ell_1,\ldots,\ell_k} \lambda_{\ell_1}^{2q_1} \cdots \lambda_{\ell_k}^{2q_k} (\lambda_{\ell_1}^{2t} + \cdots + \lambda_{\ell_k}^{2t}) \|\partial_{\ell_1} \cdots \partial_{\ell_k} u\|_{(L^2)_s}^2 \\
&= \|\Gamma_\oplus(A^{2t})^{\frac{1}{2}} D^k u\|_{\boldsymbol{q}}^2 + \sum_{i=1}^{k} \|D^k u\|_{\boldsymbol{q}+t\boldsymbol{\xi}_i}^2.
\end{aligned}$$

□

The following result reveals the basic relation between norms of derivatives and the norm $||u||_{p;\rho,s}^2$ defined in (10) with weights in (15) for the values $p = 1/2$ and $p = 1$. This result will be used as the initial induction step in the proof of the equivalence of norms for any value of p half a positive integer; see Theorem 20.

Theorem 17. *For any* $k \in \mathbb{N}$ *and* $\boldsymbol{q} = (q_1, \ldots q_k) \in \mathbb{R}^k$ *we have*

$$\|\Gamma_\oplus(A^{2q_1})^{\frac{1}{2}} u\|_{(L^2)_s}^2 = \sum_{\ell=1}^{\infty} \lambda_\ell^{2q_1} ||\partial_\ell u||^2 = \|Du\|_{q_1}^2, \tag{33}$$

$$\|\Gamma_\oplus(A^{2q_k})^{\frac{1}{2}}D^{k-1}u\|^2_{(q_1,\ldots,q_{k-1})} = \|D^k u\|^2_{(q_1,q_2,\ldots,q_k)} \tag{34}$$

and we have the identities

$$\begin{aligned}\|\Gamma_\oplus(A^{2q_1})^{\frac{1}{2}}\Gamma_\oplus(A^{2q_2})^{\frac{1}{2}}u\|^2_{(L^2)_s} &= \|D^2u\|^2_{(q_1,q_2)} + \|\Gamma_\oplus(A^{2(q_1+q_2)})^{\frac{1}{2}}u\|^2_{(L^2)_s} \\ &= \|D^2u\|^2_{(q_1,q_2)} + \|Du\|^2_{q_1+q_2}. \end{aligned} \tag{35}$$

and

$$\begin{aligned}&\|\Gamma_\oplus(A^{2q_1})^{\frac{1}{2}}\Gamma_\oplus(A^{2q_2})^{\frac{1}{2}}\Gamma_\oplus(A^{2q_3})^{\frac{1}{2}}u\|^2_{(L^2)_s} = \|D^3u\|^2_{(q_1,q_2,q_3)} \\ &+\|D^2u\|^2_{(q_1+q_3,q_2)} + \|D^2u\|^2_{(q_1,q_2+q_3)} + \|D^2u\|^2_{(q_1+q_2,q_3)} \\ &+\|Du\|^2_{(q_1+q_2+q_3)} \end{aligned} \tag{36}$$

Proof. From Equations (20) and (21) we have that

$$\begin{aligned}\|Du\|^2_{q_1} &= \sum_{\ell=1}^{\infty}\lambda_\ell^{2q_1}\|\partial_\ell u\|^2_{(L^2)_s} \\ &= \sum_{\ell=1}^{\infty}\sum_{\alpha_\ell\geq 1}\alpha_\ell\lambda_\ell^{2q_1}u^2_{\boldsymbol{\alpha}}\boldsymbol{\sigma}^{2\boldsymbol{\alpha}}\boldsymbol{\alpha}! \\ &= \sum_{\boldsymbol{\alpha}\in\boldsymbol{\mathcal{J}}}\left(\sum_{\ell=1}^{d(\boldsymbol{\alpha})}\alpha_\ell\lambda_\ell^{2q_1}\right)u^2_{\boldsymbol{\alpha}}\boldsymbol{\sigma}^{2\boldsymbol{\alpha}}\boldsymbol{\alpha}! \\ &= \sum_{\boldsymbol{\alpha}\in\boldsymbol{\mathcal{J}}}\langle\boldsymbol{\alpha},\boldsymbol{\lambda}^{2q_1}\rangle u^2_{\boldsymbol{\alpha}}\boldsymbol{\sigma}^{2\boldsymbol{\alpha}}\boldsymbol{\alpha}! = ||\Gamma_\oplus(A^{2q_1})^{\frac{1}{2}}u||^2_{(L^2)_s},\end{aligned}$$

and hence (33) holds. To prove (34) observe that from (33) and (30) we get

$$\begin{aligned}&\|\Gamma_\oplus(A^{2q_k})^{\frac{1}{2}}D^{k-1}u\|^2_{(q_1,\ldots,q_{k-1})} \\ &\quad= \sum_{\ell_1,\ldots,\ell_{k-1}}\lambda_{\ell_1}^{2q_1}\ldots\lambda_{\ell_{k-1}}^{2q_{k-1}}\|\Gamma_\oplus(A^{2q_k})^{\frac{1}{2}}\partial_{\ell_1}\ldots\partial_{\ell_{k-1}}u\|^2_{(L^2)s} \\ &\quad= \sum_{\ell_1,\ldots,\ell_k}\lambda_{\ell_1}^{2q_1}\ldots\lambda_{\ell_k}^{2q_k}\|\partial_{\ell_1}\ldots\partial_{\ell_k}u\|^2_{(L^2)s} = \|D^ku\|^2_{(q_1,\ldots,q_k)}.\end{aligned}$$

To prove (35) observe that from (33), (31) and (34) we have

$$\begin{aligned}\|\Gamma_\oplus(A^{2q_1})^{\frac{1}{2}}\Gamma_\oplus(A^{2q_2})^{\frac{1}{2}}u\|^2_{(L^2)_s} &= \|D\Gamma_\oplus(A^{2q_2})^{\frac{1}{2}}u\|^2_{q_1} \\ &= \|\Gamma_\oplus(A^{2q_2})^{\frac{1}{2}}Du\|^2_{q_1} + \|Du\|^2_{q_1+q_2} \\ &= \|D^2u\|^2_{(q_1,q_2)} + \|\Gamma_\oplus(A^{2(q_1+q_2)})^{\frac{1}{2}}u\|^2_{(L^2)_s}.\end{aligned}$$

For the proof of (36), see Theorem 20 where we prove the general case. □

In order to write down the general version of formula (35) we shall introduce some notation. Consider the set of indexes $\{1, 2, \ldots, k\}$ and its set of partitions P^k; see Charalambides [7]. Recall that, given $i \in \mathbb{N}$, an $i-$partition of

$\{1, 2, \ldots, k\}$ is a decomposition of this set into i nonempty and disjoint subsets. We denote by $P^{k,i}$ the set of all $i-$partitions of $\{1, 2, \ldots, k\}$. It is well known that $\#(P^{k,i}) = S(k, i)$, the Stirling number of the second kind (which is also the number of distributions of k distinguishable balls into i indistinguishable urns). Let each i-partition $R = (R_1, \ldots, R_i) \in P^{k,i}$, be ordered in such a way that

$$\min R_1 < \min R_2 < \ldots < \min R_i.$$

To each i-partition and $\boldsymbol{q} = (q_1, \ldots, q_k) \in \mathbb{R}^k$ we associate a multi-index $R(\boldsymbol{q}) = (R_1(\boldsymbol{q}), \ldots, R_i(\boldsymbol{q})) \in \mathbb{R}^i$ defined by

$$R_{i'}(\boldsymbol{q}) = \sum_{i'' \in R_{i'}} q_{i''}, \qquad i' = 1, \ldots, i.$$

Example 18. *Let* $\boldsymbol{q} = (q_1, q_2, q_3)$ *and consider the* 2*-partition* $R = \{R_1 = \{1\}, R_2 = \{2, 3\}\}$. *Then* $R(\boldsymbol{q}) = (q_1, q_2 + q_3)$.

Example 19. *Let* $\boldsymbol{q} = (q, q, q, q)$ *and consider the* 3*-partition* $R = \{R_1 = \{1\}, R_2 = \{2, 3\}, R_3 = \{4\}\}$. *Then* $R(\boldsymbol{q}) = (q, 2q, q)$.

The following result gives a closed formula that allows us to compute the norm $||\cdot||^2_{p;\rho,s}$ using ω-partial derivatives . It shows the equivalence between the weighted Chaos norms, using the weight (15), and the Gaussian Sobolev norms, defined using $(L^2)_s$ norms of derivatives.

Theorem 20. *Let* $k \in \mathbb{N}$ *and* $\boldsymbol{q} = (q_1, q_2, \ldots, q_k) \in \mathbb{R}^k$. *We have*

$$\|\Gamma_{\oplus}(A^{2q_1})^{\frac{1}{2}} \ldots \Gamma_{\oplus}(A^{2q_k})^{\frac{1}{2}} u\|^2_{(L^2)_s} \quad = \quad \sum_{i=1}^{k} \sum_{R \in P^{k,i}} \|D^i u\|^2_{R(\boldsymbol{q})}. \tag{37}$$

In particular, if we take $\boldsymbol{q} = \frac{1}{2}\mathbf{1}_k$ *where* $\mathbf{1}_k := (1, \ldots, 1) \in \mathbb{N}^k$

$$\|\Gamma_{\oplus}(A)^p u\|^2_{(L^2)_s} \quad = \quad \sum_{i=1}^{k} \sum_{R \in P^{k,i}} \|D^i u\|^2_{R(\frac{1}{2}\mathbf{1}_k)}$$

and

$$||u||^2_{\frac{k}{2};\rho,s} = ||u||^2_{(L^2)_s} + ||\Gamma_{\oplus}(A)^p u||^2_{(L^2)_s} = ||u||^2_{(L^2)_s} + \sum_{i=1}^{k} \sum_{R \in P^{k,i}} \|D^i u\|^2_{R(\frac{1}{2}\mathbf{1}_k)}.$$

Proof. We proceed by induction on k. For $k = 1$ and $k = 2$ we already proved the result, see (33) and (35) of Theorem 17.

Assume that (37) is valid for the first $k \in \mathbb{N}$. Then we have

$$\begin{aligned} &\|\Gamma_{\oplus}(A^{2q_1})^{\frac{1}{2}} \ldots \Gamma_{\oplus}(A^{2q_{k+1}})^{\frac{1}{2}} u\|^2_{(L^2)_s} \\ &= \sum_{i=1}^{k} \sum_{R \in P^{(i)}} \|D^i \Gamma_{\oplus}(A^{2q_{k+1}})^{\frac{1}{2}} u||^2_{R(\boldsymbol{q})} \\ &= \sum_{i=1}^{k} \sum_{R \in P^{(i)}} \left(\|\Gamma_{\oplus}(A^{2q_{k+1}})^{\frac{1}{2}} D^i u\|^2_{R(q_1,\ldots,q_k)} + \sum_{i'=1}^{i} \|D^i u\|^2_{R(q_1,\ldots,q_k)+q_{k+1}\boldsymbol{\xi}_{i'}} \right) \end{aligned}$$

where we have used formula (32). Then, from (34) we get

$$
\begin{aligned}
&\|\Gamma_{\oplus}(A^{2q_1})^{\frac{1}{2}} \dots \Gamma_{\oplus}(A^{2q_{k+1}})^{\frac{1}{2}} u\|^2_{(L^2)_s} \\
&\quad = \sum_{i=1}^{k} \sum_{R \in P^{k,i}} \left(\|D^i u\|^2_{(R(q_1,\dots,q_k),q_{k+1})} + \sum_{i'=1}^{i} \|D^i u\|^2_{R(q_1,\dots,q_k)+q_{k+1}\boldsymbol{\xi}_{i'}} \right) \\
&\quad = \sum_{i=1}^{k+1} \sum_{R \in P^{k+1,i}} \|D^i u\|^2_{R(q_1,\dots,q_{k+1})}.
\end{aligned}
$$

To obtain the last equality we observe that the i-partitions $P^{k+1,i}$ of the set $\{1,\dots,k+1\}$ are of the form $\{R,\{k+1\}\}$ where $R \in P^{k,i-1}$ or $R = (R_1,\dots,R_{i'} \cup \{k+1\},\dots,R_i)$ for $1 \le i' \le i$ and $R \in P^{k,i}$. □

Remark 21. *Note that, given $r = (r_\ell)_{\ell=1}^i \in \mathbb{N}^i$, (see [7])*

$$
\#(\{R \in P^{k,i} : \frac{1}{2} r = R(\frac{1}{2}\mathbf{1}_k)\} = \prod_{j=1}^{i-1} \binom{\sum_{\ell=j}^{i} r_\ell - 1}{r_j - 1}.
$$

6.1 A remark on Kondratiev type norms

In this section we study another classical weighted norm. We select different weights in the general norm defined in (13). Given a multi-index $\boldsymbol{\alpha}$ we denote $\boldsymbol{\lambda}^{\boldsymbol{\alpha}} := \prod_{j=1}^{d(\boldsymbol{\alpha})} \lambda_j^{\alpha_j}$. Take $\nu \in [0,1)$ and

$$
\rho(\boldsymbol{\alpha}, p)^2 = (\boldsymbol{\alpha}!)^\nu \boldsymbol{\lambda}^{2p\boldsymbol{\alpha}}, \quad \boldsymbol{\alpha} \in \boldsymbol{\mathcal{J}} \tag{38}
$$

in (13). See [18, 20, 25]. Let us denote by $|||u|||^2_{p;\rho,s}$ the resulting weighted norm. Note that we can write

$$
|||u|||^2_{p;\rho,s} = ||\Gamma_{\otimes,\nu}(A)^p u||^2_{(L^2)_s} = \int_{\mathcal{S}'} |\Gamma_{\otimes,\nu}(A)^p u(\omega)|^2 e^{s\|\omega\|^2_{-\theta}} d\mu(\omega),
$$

where $\Gamma_{\otimes,\nu}(A)$ is the operator defined by $\Gamma_{\otimes,\nu}(A) H_{\boldsymbol{\sigma}^2,\boldsymbol{\alpha}} = (\alpha!)^\nu \boldsymbol{\lambda}^{\boldsymbol{\alpha}} H_{\boldsymbol{\sigma}^2,\boldsymbol{\alpha}}$. Note also that $\Gamma_{\otimes,0}(A^p) = \Gamma_{\otimes,0}(A)^p$. In the case of $\nu = 0$ and $s = 0$, $\Gamma_{\otimes,0}(A)$ is called the Second Quantization of A; see [17].

For a priori error estimates for Lemma 11 using $|||\cdot|||^2_{p;\rho,s}$, we refer to [13]. Now we show how to compute the norm $|||\cdot|||_{p;\rho,s}$ defined above for the case $\nu = 0$. We use the notation $(\boldsymbol{\lambda}^p - \mathbf{1})^{\boldsymbol{\gamma}} = \prod_{j=1}^{d(\boldsymbol{\gamma})} (\lambda_j^p - 1)^{\gamma_j}$. Recall that $1 < \lambda_1 \le \lambda_2 \le \dots$. We have

$$
\begin{aligned}
\boldsymbol{\lambda}^{2p\boldsymbol{\alpha}} &= \prod_{j=1}^{d(\boldsymbol{\alpha})} (\lambda_j^{2p} - 1 + 1)^{\alpha_j} = \prod_{j=1}^{d(\boldsymbol{\alpha})} \left(\sum_{\gamma_j \le \alpha_j} \binom{\alpha_j}{\gamma_j} (\lambda_j^{2p} - 1)^{\gamma_j} \right) \\
&= \sum_{\boldsymbol{\gamma} \le \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\gamma}} (\boldsymbol{\lambda}^{2p} - \mathbf{1})^{\boldsymbol{\gamma}}.
\end{aligned}
$$

Then

$$\begin{aligned}
\sum_{\gamma\in\mathcal{J}} \frac{(\boldsymbol{\lambda}^{2p}-\mathbf{1})^{\gamma}}{\gamma!}\|\partial^{\gamma}u\|^2_{(L^2)_s} &= \sum_{\gamma\in\mathcal{J}} \frac{(\boldsymbol{\lambda}^{2p}-\mathbf{1})^{\gamma}}{\gamma!} \sum_{\boldsymbol{\alpha}\geq\gamma} \frac{\boldsymbol{\alpha}!}{(\boldsymbol{\alpha}-\gamma)!} u^2_{\boldsymbol{\alpha}}\boldsymbol{\sigma}^{2\boldsymbol{\alpha}}\boldsymbol{\alpha}! \\
&= \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \left(\sum_{\gamma\leq\boldsymbol{\alpha}} \frac{\boldsymbol{\alpha}!}{\gamma!(\boldsymbol{\alpha}-\gamma)!}(\boldsymbol{\lambda}^{2p}-\mathbf{1})^{\gamma} \right) u^2_{\boldsymbol{\alpha}}\boldsymbol{\sigma}^{2\boldsymbol{\alpha}}\boldsymbol{\alpha}! \\
&= \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \boldsymbol{\lambda}^{2p\boldsymbol{\alpha}} u^2_{\boldsymbol{\alpha}}\boldsymbol{\sigma}^{2\boldsymbol{\alpha}}\boldsymbol{\alpha}! = |||u|||^2_{p;\rho,s}.
\end{aligned}$$

Summarizing we have $|||u|||^2_{p;\rho,s} = \sum_{\gamma\in\mathcal{J}} \frac{(\boldsymbol{\lambda}^{2p}-\mathbf{1})^{\gamma}}{\gamma!}\|\partial^{\gamma}u\|^2_{(L^2)_s}$. We conclude that the weighted norm with weight ρ defined in (38) requires all partial derivative of all orders being $(L^2)_s$ functions, while the weighted norm $||u||^2_{p;\rho,s}$ defined in (15) requires only a finite number of partial derivatives, see (26). Due to the technical difficulties in dealing with this infinity weighted sum, in this paper we consider only the norm $||u||^2_{p;\rho,s}$ to analyze and measure the stochastic regularity of the solution of the stochastic pressure equation.

7 Stochastic regularity

We recall the definition of the tensor product space,

$$\mathcal{U}^1_{p;\rho,s} = H^1(D) \otimes \mathcal{S}_{p;\rho,s}.$$

For $u(x,\omega) = \sum_{\boldsymbol{\alpha}\in\mathcal{J}} u_{\boldsymbol{\alpha},s}(x) H_{\boldsymbol{\sigma}(s)^2,\boldsymbol{\alpha}}(\omega)$, $(x,\omega)\in D\times\mathcal{S}'$ we denote

$$||u||^2_{\mathcal{U}^m_{p;\rho,s}} := \sum_{\boldsymbol{\alpha}\in\mathcal{J}} \boldsymbol{\alpha}!\rho(\boldsymbol{\alpha},p)^2\boldsymbol{\sigma}^{2\boldsymbol{\alpha}}\|u_{\boldsymbol{\alpha}}\|^2_{H^m(D)},$$

with ρ defined in (15). For $k\in\mathbb{N}$ and $\boldsymbol{q}\in\mathbb{R}^k$ we also introduce (see Definition 13)

$$\|D^k u\|^2_{m,\boldsymbol{q};s} = \sum_{\ell_1,\ell_2,\ldots,\ell_k} \lambda^{2q_1}_{\ell_1}\ldots\lambda^{2q_k}_{\ell_k}\|\partial_{\ell_1}\ldots\partial_{\ell_k}u\|^2_{H^m(D)\times(L^2)_s}.$$

From Theorem 20 and the definition of $\|\cdot\|_{\frac{k}{2};\rho,s}$ in (13) with ρ defined in (15) we have the equality

$$\|u\|^2_{\mathcal{U}^m_{\frac{k}{2};\rho,s}} = \|u\|^2_{\mathcal{U}^m_s} + \sum_{i=1}^{k}\sum_{R\in P^{k,i}} \|D^i u\|^2_{m,R(\frac{1}{2}\mathbf{1}_k);s}.$$

Now we study the behavior of the solution according to the regularity in the ω variable of the right-hand side data f. In the following result we control the norm of a ω-partial derivative of the solution in terms of the norm of the ω-partial derivatives of the forcing term. Before estating the result, we introduce needed notation. We defined the set $I(k,i)$ by

$$I^{k,i} = \left\{ \tau = (\tau_1, \dots, \tau_k); \quad \begin{array}{l} \text{such that } \cup_{i=1}^{k} \{\tau_i\} = \{1, \dots, k\} \text{ and for some} \\ i,\, 0 \le i \le k, \text{ we have } \tau_1 < \dots < \tau_i, \text{ and} \\ \tau_{i+1} < \dots < \tau_k. \end{array} \right\}. \tag{39}$$

Lemma 22. *Let $s \in \mathbb{R}$ and $\epsilon > 0$, and let $u \in \widehat{\mathcal{U}}_s^1$ be the solution of (10) with right-hand side $f \in \mathcal{U}_{s+2k\epsilon'+\epsilon}^{-1}$. Let us assume that for $k \in \mathbb{N}$ and $\boldsymbol{q} = (q_1, \dots q_k) \in \mathbb{R}^k$ and $\epsilon' > 0$*

$$\|D^i f\|^2_{-1,(q_{\tau_1},\dots,q_{\tau_i});s+2(k-i)\epsilon'+\epsilon} < \infty \quad \textit{for all} \quad 0 \le i \le k$$

and ϕ satisfies

$$\pi_{\boldsymbol{q}}(\phi) := \max_{1 \le i \le k} \max_{x \in D} \|\phi_x\|_{q_i} < \infty.$$

Then,

$$\|D^k \nabla u\|^2_{0,\boldsymbol{q};s} \le \tilde{C}(\epsilon, \epsilon', k, \phi) \left(\sum_{i=0}^{k} \sum_{\tau \in I^{k,i}} \|D^i f\|^2_{-1,(q_{\tau_1},\dots,q_{\tau_i});s+2(k-i)\epsilon'+\epsilon} \right) \tag{40}$$

where the constant $\tilde{C}(\epsilon, \epsilon', k, \phi)$ is given by

$$\tilde{C}(\epsilon, \epsilon', k, \phi) = 2^{k(k+1)} C^2 e^{\frac{C_\theta^2}{\epsilon}} \max\{1, \pi_{\boldsymbol{q}}(\phi) C e^{\frac{C_\theta^2}{\epsilon'}}\}^{2k}, \tag{41}$$

where $C = \sqrt{1 + C_{poin}}$ and C_{poin} is the Poincaré inequality constant which depends on D. The set $I^{k,i}$ is defined in (39).

Proof. We first show the theorem holds for $k = 1$, then we proceed by induction on the order of the derivatives k.

Assume that u is a solution of (10). For almost sure all ω we have for all $v \in H_0^1(D)$

$$\int_D e^{\langle \omega, \phi_x \rangle} \nabla u(x,\omega) \nabla v(x) dx = \int_D fv. \tag{42}$$

Note that $\partial_\ell e^{\langle \omega, \phi_x \rangle} = \langle \phi_x, \eta_\ell \rangle e^{\langle \omega, \phi_x \rangle}$. Taking partial derivative in (42) we get

$$\begin{aligned} \int_D e^{\langle \omega, \phi_x \rangle} \nabla \partial_\ell u(x,\omega) \nabla v(x) dx d\mu \quad &= \quad \int_D \partial_\ell f(x,\omega) v(x) dx d\mu \qquad (43) \\ &\quad - \int_D e^{\langle \omega, \phi_x \rangle} \langle \phi_x, \eta_\ell \rangle \nabla u(x,\omega) \nabla v(x) dx d\mu. \end{aligned}$$

Define $\Phi_\ell(x,\omega) = \langle \phi_x, \eta_\ell \rangle \nabla u(x,\omega)$ and by using similar arguments as in (47) below, we have $\Phi_\ell \in \mathcal{U}^0_{s+\epsilon'+\epsilon} = (L_2)_{s+\epsilon'+\epsilon}$. Integrating in $\mathcal{S}'$ we see that $\partial_\ell u$ is the solution of the weak problem

$$\left\{ \begin{array}{l} \text{Find } \partial_\ell u \in \widehat{\mathcal{U}}_s^1 \text{ such that} \\ a(\partial_\ell u, v) = G(v) \text{ for all } v \in \widehat{\mathcal{U}}^1_{-s-\epsilon} \end{array} \right. \tag{44}$$

where the right-hand side is defined by

$$G(v)=\int_{D\times\mathcal{S}'}\partial_\ell f(x,\omega)v(x,\omega)dxd\mu-\int_D e^{\langle\omega,\phi_x\rangle}\Phi_\ell(x,\omega)\nabla v(x,\omega)dxd\mu.$$

To bound $\|\partial_\ell u\|_{\mathcal{U}^1_s}$ we need first to estimate $\|G\|_{\mathcal{U}^{-1}_{s+\epsilon}}$. Note that from (6) and $2\langle\omega,\phi_x\rangle\le\epsilon'\|\omega\|^2_{-\theta}+C_\theta^2/\epsilon'$ we obtain

$$\begin{aligned}\int_D e^{\langle\omega,\phi_x\rangle}\Phi_\ell(x,\omega)\nabla v(x,\omega)dxd\mu\quad&\le\left(\int_D e^{2\langle\omega,\phi_x\rangle+(s+\epsilon)\|\omega\|^2_{-\theta}}\Phi_\ell(x,\omega)^2dxd\mu\right)^{\frac12}\\&\quad\left(\int_D e^{(-s-\epsilon)\|\omega\|^2_{-\theta}}\nabla v(x,\omega)^2dxd\mu\right)^{\frac12}\\&\le\quad e^{\frac{C_\theta^2}{2\epsilon'}}\|\Phi_\ell\|_{s+\epsilon'+\epsilon}\|v\|_{\mathcal{U}^1_{-s-\epsilon}}\end{aligned}$$

and then

$$\|G\|_{\mathcal{U}^{-1}_{s+\epsilon}}\quad\le\quad\|\partial_\ell f\|_{\mathcal{U}^{-1}_{s+\epsilon}}+e^{\frac{C_\theta^2}{2\epsilon'}}\|\Phi_\ell\|_{s+\epsilon'+\epsilon}.$$

Using this bound and Lemma 5 applied to the weak problem (44) we have that

$$\|\partial_\ell u\|_{\mathcal{U}^1_s}\le Ce^{\frac{C_\theta^2}{2\epsilon}}\left(\|\partial_\ell f\|_{\mathcal{U}^{-1}_{s+\epsilon}}+e^{\frac{C_\theta^2}{2\epsilon'}}\|\Phi_\ell\|_{s+\epsilon'+\epsilon}\right).\tag{45}$$

We can now estimate $||D^1u||^2_{1,q_1;s}$ in Definition (13) as follows:

$$\begin{aligned}||D^1u||^2_{1,q_1;s}\quad&=\quad\sum_{\ell=1}^\infty\lambda_\ell^{2q_1}||\partial_\ell\nabla u||^2_{(L^2)_s}\\&\le\quad 2C^2e^{\frac{C_\theta^2}{\epsilon}}\left(\sum_{\ell=1}^\infty\lambda_\ell^{2q_1}\|\partial_\ell f\|^2_{\mathcal{U}^{-1}_{s+\epsilon}}+e^{\frac{C_\theta^2}{\epsilon'}}\sum_{\ell=1}^\infty\lambda_\ell^{2q_1}||\Phi_\ell||^2_{s+\epsilon'+\epsilon}\right)\\&=\;2C^2e^{\frac{C_\theta^2}{\epsilon}}\Big(\|D^1f\|^2_{-1,q_1,s+\epsilon}+e^{\frac{C_\theta^2}{\epsilon'}}\sum_{\ell=1}^\infty\lambda_\ell^{2q_1}||\Phi_\ell||^2_{s+\epsilon'+\epsilon}\Big).\end{aligned}\tag{46}$$

To estimate the last term in (46) observe that

$$\begin{aligned}\sum_{\ell=1}^\infty\lambda_\ell^{2q_1}||\Phi_\ell||^2_{s+\epsilon'+\epsilon}\quad&=\quad\int_{\mathcal{S}'\times D}\sum_{\ell=1}^\infty\lambda_\ell^{2q_1}\langle\phi_x,\eta_\ell\rangle^2|\nabla u(x,\omega)|^2e^{(s+\epsilon'+\epsilon)\|\omega\|^2_{-\theta}}dxd\mu(\omega)\\&=\quad\int_{\mathcal{S}'\times D}\|\phi_x\|^2_{q_1}|\nabla u(x,\omega)|^2e^{(s+\epsilon'+\epsilon)|\omega|^2_{-\theta}}dxd\mu(\omega)\\&\le\quad\max_{x\in D}\|\phi_x\|^2_{q_1}\int_{\mathcal{S}'\times D}|\nabla u(x,\omega)|^2e^{(s+\epsilon'+\epsilon)|\omega|^2_{-\theta}}dxd\mu(\omega)\\&=\quad\pi^2_{q_1}\|u\|^2_{\mathcal{U}^1_{s+\epsilon'+\epsilon}}.\end{aligned}\tag{47}$$

Here and below, in order to simplify notation we have written $\pi_{q_1} = \pi_{q_1}(\phi)$. By inserting Equation (47) in (46) we obtain

$$||D^1 u||^2_{1,q_1;s} \leq 2C^2 e^{\frac{C_\theta^2}{\epsilon}} \left(\|D^1 f\|^2_{-1,q_1;s+\epsilon} + \pi_{q_1}^2 e^{\frac{C_\theta^2}{\epsilon'}} \|u\|^2_{\mathcal{U}^1_{s+\epsilon'+\epsilon}} \right).$$

Using the estimate $\|u\|^2_{\mathcal{U}^1_{s+\epsilon'+\epsilon}} \leq C^2 e^{C_\theta^2/\epsilon'} \|f\|^2_{\mathcal{U}^{-1}_{s+2\epsilon'+\epsilon}}$, see Lemma 5, we obtain

$$\begin{aligned} ||D^1 u||^2_{1,q_1;s} &\leq 2C^2 e^{\frac{C_\theta^2}{\epsilon}} \left(\|D^1 f\|^2_{-1,q_1;s+\epsilon} + \pi_{q_1}^2 C^2 e^{\frac{2C_\theta^2}{\epsilon'}} \|f\|^2_{\mathcal{U}^{-1}_{s+2\epsilon'+\epsilon}} \right) \\ &\leq 2C^2 e^{\frac{C_\theta^2}{\epsilon}} \max\{1, \pi_{q_1} C e^{\frac{C_\theta^2}{\epsilon'}}\}^2 \left(\|D^1 f\|^2_{-1,q_1;s+\epsilon} + \|f\|_{\mathcal{U}^{-1}_{s+2\epsilon'+\epsilon}} \right) \end{aligned}$$

which finish the proof for the case $k = 1$.

Now assume that the result holds valid for every $0 \leq i < k$. The main induction step argument is similar to the case $k = 1$. We will:

1. Deduce a weak problem whose solution is a partial derivative of order k of u; see (48).
2. Apply Lemma 5 to estimate the norm of each partial derivative of order k of u, and use Definition 13 to estimate the norm of $D^k u$ in term of lower order derivatives of u; see (50).
3. Use the induction argument; see (51).

Step 1. Using the Leibniz rule we have

$$\begin{aligned} &\partial_{\ell_1} \ldots \partial_{\ell_k} \left(e^{\langle \omega, \phi_x \rangle} \nabla u(x,\omega) \right) = \\ &e^{\langle \omega, \phi_x \rangle} \sum_{i=0}^{k} \sum_{\tau \in I^{k,i}} \left(\nabla \partial_{\ell_{\tau_1}} \cdots \partial_{\ell_{\tau_i}} u(x,\omega) \right) a_{\ell_{\tau_{i+1}}} \cdots a_{\ell_{\tau_k}} \\ &= e^{\langle \omega, \phi_x \rangle} \left(\partial_{\ell_1} \ldots \partial_{\ell_k} \nabla u(x,\omega) + \sum_{i=0}^{k-1} \sum_{\tau \in I^{k,i}} \Phi^{(i),\tau}_{\ell_{\tau_1} \cdots \ell_{\tau_k}}(x,\omega) \right) \end{aligned}$$

where $a_\ell(x) = \langle \phi_x, \eta_\ell \rangle$ and the set $I^{k,i}$ is defined in (39). We also have defined

$$\Phi^{(i),\tau}_{\ell_{\tau_1} \cdots \ell_{\tau_k}}(x,\omega) := \left(\nabla \partial_{\ell_{\tau_1}} \cdots \partial_{\ell_{\tau_i}} u(x,\omega) \right) a_{\ell_{\tau_{i+1}}} \cdots a_{\ell_{\tau_k}}.$$

From (42) we get

$$\begin{aligned} &\int_D e^{\langle \omega, \phi_x \rangle} \partial_{\ell_1} \ldots \partial_{\ell_k} \nabla u(x,\omega) \nabla v(x) dx d\mu = \int_D \partial_{\ell_1} \ldots \partial_{\ell_k} f(x,\omega) v(x) dx d\mu \\ &- \int_D e^{\langle \omega, \phi_x \rangle} \sum_{i=0}^{k-1} \sum_{\tau \in I^{k,i}} \Phi^{(i),\tau}_{\ell_{\tau_1} \ldots \ell_{\tau_k}}(x,\omega) \nabla v(x) dx d\mu \end{aligned}$$

As above we have that $\partial_{\ell_1} \dots \partial_{\ell_k} u$ is the solution of of the weak problem

$$\begin{cases} \text{Find } \partial_{\ell_1} \dots \partial_{\ell_k} u \in \widehat{\mathcal{U}}_s \text{ such that} \\ a(\partial_{\ell_1} \dots \partial_{\ell_k} u, v) = G(v) \text{ for all } v \in \widehat{\mathcal{U}}^1_{-s-\epsilon} \end{cases} \tag{48}$$

with a new right-hand side

$$\begin{aligned} G(v) &= \int_{\mathcal{S}' \times D} \partial_{\ell_1} \dots \partial_{\ell_k} f(x,\omega) \nabla v(x,\omega) dx d\mu - \\ & \sum_{i=0}^{k-1} \sum_{\tau \in I^{k,i}} \int_{\mathcal{S}' \times D} e^{\langle \omega, \phi_x \rangle} \Phi^{(i),\tau}_{\ell_{\tau_1} \dots \ell_{\tau_k}}(x,\omega) \nabla v(x,\omega) dx d\mu. \end{aligned} \tag{49}$$

Step 2. In order to estimate $||G||_{s+\epsilon}$ we estimate each term in (49) above. For each i and $\tau \in I^{k,i}$ we have

$$\int_{\mathcal{S}' \times D} e^{\langle \omega, \phi_x \rangle} \Phi^{(i),\tau}_{\ell_{\tau_1} \dots \ell_{\tau_k}}(x,\omega) \nabla v(x,\omega) dx d\mu \le e^{\frac{C_\theta^2}{2\epsilon'}} \|\Phi^{(i),\tau}_{\ell_{\tau_1} \dots \ell_{\tau_k}}\|_{s+\epsilon'+\epsilon} \|v\|_{\mathcal{U}^1_{-s-\epsilon}}.$$

Then from Lemma 5 applied to problem (48) with the right-hand side G in (49) we get

$$\|\partial_{\ell_1} \dots \partial_{\ell_k} u\|_{\mathcal{U}^1_s} \le C e^{\frac{C_\theta^2}{2\epsilon}} \left(\|\partial_{\ell_1} \dots \partial_{\ell_k} f\|_{\mathcal{U}^{-1}_{s+\epsilon}} + e^{\frac{C_\theta^2}{2\epsilon'}} \sum_{i=0}^{k-1} \sum_{\tau \in I^{k,i}} \|\Phi^{(i),\tau}_{\ell_{\tau_1} \dots \ell_{\tau_k}}\|_{s+\epsilon'+\epsilon} \right)$$

and using $\sum_{i=0}^{k-1} \sum_{\tau \in I^{k,i}} 1 = \sum_{i=0}^{k-1} \binom{k}{i} = 2^k - 1$, we obtain

$$\begin{aligned} ||D^k \nabla u||^2_{0,\boldsymbol{q};s} &= \sum_{\ell_1, \cdots, \ell_k} \lambda_{\ell_1}^{2q_1} \cdots \lambda_{\ell_k}^{2q_k} ||\partial_{\ell_1} \dots \partial_{\ell_k} u||^2_{\mathcal{U}^1_s} \\ &\le 2^k C^2 e^{\frac{C_\theta^2}{\epsilon}} \Big(\sum_{\ell_1, \cdots, \ell_k} \lambda_{\ell_1}^{2q_1} \cdots \lambda_{\ell_k}^{2q_k} \|\partial_{\ell_1} \dots \partial_{\ell_k} f\|^2_{\mathcal{U}^{-1}_{s+\epsilon}} + \\ & \quad e^{\frac{C_\theta^2}{\epsilon'}} \sum_{\ell_1, \cdots, \ell_k} \lambda_{\ell_1}^{2q_1} \cdots \lambda_{\ell_k}^{2q_k} \sum_{i=0}^{k-1} \sum_{\tau \in I^{k,i}} \|\Phi^{(i),\tau}_{\ell_{\tau_1} \dots \ell_{\tau_k}}\|^2_{s+\epsilon'+\epsilon} \Big) \\ &\le 2^k C^2 e^{\frac{C_\theta^2}{\epsilon}} \Big(\|D^k f\|^2_{-1,\boldsymbol{q};s+\epsilon} + \\ & \quad e^{\frac{C_\theta^2}{\epsilon'}} \sum_{i=0}^{k-1} \sum_{\tau \in I^{k,i}} \sum_{\ell_1, \cdots, \ell_k} \lambda_{\ell_1}^{2q_1} \cdots \lambda_{\ell_k}^{2q_k} \|\Phi^{(i),\tau}_{\ell_{\tau_1} \dots \ell_{\tau_k}}\|^2_{s+\epsilon'+\epsilon} \Big). \end{aligned}$$

Finally note that

$$\begin{aligned} & \lambda_{\ell_1}^{2q_1} \cdots \lambda_{\ell_k}^{2q_k} \|\Phi^{(i),\tau}_{\ell_{\tau_1} \dots \ell_{\tau_k}}\|^2_{(s+\epsilon'+\epsilon)} = \lambda_{\ell_{\tau_1}}^{2q_{\tau_1}} \cdots \lambda_{\ell_{\tau_k}}^{2q_{\tau_k}} \|\Phi^{(i),\tau}_{\ell_{\tau_1} \dots \ell_{\tau_k}}\|^2_{(s+\epsilon'+\epsilon)} \\ &= \int_{\mathcal{S}' \times D} \Big(\lambda_{\ell_{\tau_1}}^{2q_{\tau_1}} \cdots \lambda_{\ell_{\tau_i}}^{2q_{\tau_i}} |\partial_{\ell_{\tau_1}} \cdots \partial_{\ell_{\tau_i}} \nabla u(x,\omega)|^2 \Big) \cdot \Big(\lambda_{\ell_{\tau_{i+1}}}^{2q_{\tau_{i+1}}} \cdots \\ & \quad \lambda_{\ell_{\tau_{i+1}}}^{2q_{\tau_{i+1}}} a^2_{\ell_{\tau_{i+1}}}(x) \cdots a^2_{\ell_{\tau_k}}(x) \Big) e^{(s+\epsilon'+\epsilon)\|\omega\|^2_{-\theta}} d\mu dx \end{aligned}$$

implies

$$\sum_{\ell_1,\cdots,\ell_k} \lambda_{\ell_1}^{2q_1}\cdots\lambda_{\ell_k}^{2q_k}\|\Phi^{(i),\tau}_{\ell_{\tau_1}\cdots\ell_{\tau_k}}\|^2_{s+\epsilon'+\epsilon}$$
$$= \max_{x\in D}\|\phi_x\|^2_{q_{\tau_{i+1}}}\cdots\|\phi_x\|^2_{q_{\tau_k}}\|D^i\nabla u\|^2_{0,(q_{\tau_1},\ldots,q_{\tau_i}),s+\epsilon'+\epsilon}$$
$$\leq \pi_{\boldsymbol{q}}^{2(k-i)}\|D^i\nabla u\|^2_{0,(q_{\tau_1},\ldots,q_{\tau_i});s+\epsilon'+\epsilon}.$$

Here and below, in order to simplify the notation we have written $\pi_{\boldsymbol{q}} = \pi_{\boldsymbol{q}}(\phi)$. Summarizing

$$||D^k\nabla u||^2_{\boldsymbol{q};s} \leq 2^kC^2e^{\frac{C_\theta^2}{\epsilon}}\Big(\|D^kf\|^2_{-1,\boldsymbol{q};s+\epsilon} + e^{\frac{C_\theta^2}{\epsilon'}}\sum_{i=0}^{k-1}\pi_{\boldsymbol{q}}^{2(k-i)}\sum_{\tau\in I^{k,i}}\|D^i\nabla u\|^2_{0,(q_{\tau_1},\ldots,q_{\tau_i});s+\epsilon'+\epsilon}\Big). \tag{50}$$

Step 3. We have from the induction argument, i.e., (40) holds with k replaced i, s replaced by $s+\epsilon'+\epsilon$ and ϵ replaced by ϵ',

$$e^{\frac{C_\theta^2}{\epsilon'}}\sum_{i=0}^{k-1}\pi_{\boldsymbol{q}}^{2(k-i)}\sum_{\tau\in I^{k,i}}\|D^i\nabla u\|^2_{0,(q_{\tau_1},\ldots,q_{\tau_i});s+\epsilon'+\epsilon}$$
$$\leq e^{\frac{C_\theta^2}{\epsilon'}}\sum_{i=0}^{k-1}\pi_{\boldsymbol{q}}^{2(k-i)}2^{i(i+1)}C^2e^{\frac{C_\theta^2}{\epsilon'}}\max\{1,\pi_{\boldsymbol{q}}Ce^{\frac{C_\theta^2}{\epsilon'}}\}^{2i}\Big(\sum_{\tau\in I^{k,i}}\sum_{j=0}^{i}\sum_{\tau'\in I^{i,j}}\|D^jf\|^2_{-1,(q_{\tau_{\tau'_1}},\ldots,q_{\tau_{\tau'_j}});(s+\epsilon'+\epsilon)+(2(i-j)\epsilon'+\epsilon')}\Big) \tag{51}$$

Now we use the fact that the total number of terms in the sum $\sum_{i=0}^{k-1}\sum_{\tau\in I^{k,i}}$ is 2^k-1 to get

$$\sum_{i=0}^{k-1}\sum_{\tau\in I^{k,i}}\sum_{j=0}^{i}\sum_{\tau'\in I^{i,j}}\|D^jf\|^2_{-1,(q_{\tau_{\tau'_1}},\ldots,q_{\tau_{\tau'_j}}),s+2(i-j)\epsilon'+\epsilon}\leq$$
$$\sum_{i=0}^{k-1}\sum_{\tau\in I^{k,i}}\sum_{j=0}^{i}\sum_{\tau'\in I^{i,j}}\|D^jf\|^2_{-1,(q_{\tau_{\tau'_1}},\ldots,q_{\tau_{\tau'_j}}),s+2(k-j)\epsilon'+\epsilon}\leq$$
$$\sum_{i=0}^{k-1}(2^k-1)\sum_{\tau\in I^{k,j}}\|D^jf\|^2_{-1,(q_{\tau_1},\ldots,q_{\tau_j}),s+2(k-j)\epsilon'+\epsilon}. \tag{52}$$

Using that $Ce^{\frac{C_\theta^2}{\epsilon'}}>1$, hence $Ce^{\frac{C_\theta^2}{\epsilon'}}\leq(Ce^{\frac{C_\theta^2}{\epsilon'}})^{2(k-i)}$ for $0\leq i\leq k-1$, and together with (51) yields

$$e^{\frac{C_\theta^2}{\epsilon'}}\sum_{i=0}^{k-1}\pi_{\boldsymbol{q}}^{2(k-i)}\sum_{\tau\in I^{k,i}}\|D^i\nabla u\|^2_{0,(q_{\tau_1},\ldots,q_{\tau_i});s+\epsilon'+\epsilon}$$

$$
\begin{aligned}
&\leq (2^k-1)\max\{1,\pi_{\boldsymbol{q}}Ce^{\frac{C_\theta^2}{\epsilon'}}\}^{2k}\Big(\\
&\quad\sum_{i=0}^{k-1}2^{i(i+1)}\sum_{\tau\in I^{k,i}}\|D^if\|^2_{-1,(q_{\tau_1},\dots,q_{\tau_i}),s+2(k-i)\epsilon'+\epsilon}\Big)\\
&\leq 2^{(k-1)k}(2^k-1)\max\{1,\pi_{\boldsymbol{q}}Ce^{\frac{C_\theta^2}{\epsilon'}}\}^{2k}\Big(\\
&\quad\sum_{\tau\in I^{k,\ell}}\|D^if\|^2_{-1,(q_{\tau_1},\dots,q_{\tau_i}),s+2(k-i)\epsilon'+\epsilon}\Big). \qquad (53)
\end{aligned}
$$

Inserting (53) in (50) we get

$$
\begin{aligned}
||D^k\nabla u||^2_{\boldsymbol{q};s} &\leq 2^k(1+2^{(k-1)k}(2^k-1))C^2e^{\frac{C_\theta^2}{\epsilon}}\max\{1,\pi_{\boldsymbol{q}}Ce^{\frac{C_\theta^2}{\epsilon'}}\}^{2k}\Big(\\
&\sum_{i=0}^{k}\sum_{\tau\in I^{k,i}}\|D^if\|^2_{-1,(q_{\tau_1},\dots,q_{\tau_i});s+2(k-i)\epsilon'+\epsilon}\Big) \qquad (54)
\end{aligned}
$$

and (40) follows by using $2^k(1+2^{(k-1)k}(2^k-1))\leq 2^{k(k+1)}$. □

Corollary 23. *Let $s\in\mathbb{R}$ and $\epsilon>0$, and let $u\in\widehat{\mathcal{U}}^1_s$ be the solution of (10) with right-hand side $f\in\mathcal{U}^{-1}_{s+2k\epsilon'+\epsilon}$. Let us assume that for $k\in\mathbb{N}$ and $\boldsymbol{q}=(q_1,\dots q_k)\in\mathbb{R}^k$ we have $\|\phi_x\|_{q_i}<\infty$ for $1\leq i\leq k$ and $\|f\|_{\mathcal{U}^{-1}_{|\boldsymbol{q}|;\rho,t}}<\infty$ where $|\boldsymbol{q}|=\sum_{i=1}^k q_i$, ρ is defined by (15), $\epsilon'>0$ and $t=s+2k\epsilon'+\epsilon$. Then*

$$
\|D^k\nabla u\|^2_{0,\boldsymbol{q};s}<\tilde{C}(\epsilon,\epsilon',k,\phi)\|f\|^2_{\mathcal{U}^{-1}_{|\boldsymbol{q}|;\rho,t}} \qquad (55)
$$

where the constant $\tilde{C}$ is defined in (41).

Proof. From Lemma 22 and Lemma 5 we have

$$
\begin{aligned}
\|D^k\nabla u\|^2_{0,\boldsymbol{q};s} &\leq \tilde{C}(\epsilon,\epsilon',k,\phi)\left(\sum_{i=0}^{k}\sum_{\tau\in I^{k,\ell}}\|D^if\|^2_{-1,(q_{\tau_1},\dots,q_{\tau_i});t}\right)\\
&< \tilde{C}(\epsilon,\epsilon',k,\phi)\left(\sum_{i=0}^{k}\sum_{R\in P^{k,i}}\|D^if\|^2_{-1,R(\boldsymbol{q});t}\right)\\
&\leq \tilde{C}(\epsilon,\epsilon',k,\phi)\|f\|^2_{\mathcal{U}^{-1}_{|\boldsymbol{q}|;\rho,t}}
\end{aligned}
$$

□

The following result summarizes our stochastic regularity result.

Theorem 24. *Let $s\in\mathbb{R}$ and $\epsilon>0$, and let $u\in\widehat{\mathcal{U}}^1_s$ be the solution of (10) with right-hand side $f\in\mathcal{U}^{-1}_{s+\epsilon}$. Let us assume that for $k\in\mathbb{N}$ we have $\|\phi_x\|_{\frac{k}{2}}<\infty$ where ρ is defined by (15), $\epsilon'>0$ and $t=s+2k\epsilon'+\epsilon$. Then $u\in\widehat{\mathcal{U}}^1_{\frac{k}{2};\rho,s}$ and*

$$
|u|_{\mathcal{U}^1_{\frac{k}{2};\rho,s}}\leq\tilde{C}(\epsilon,\epsilon',k,\phi)B(k)\|f\|^2_{\mathcal{U}^{-1}_{\frac{k}{2};\rho,t}}
$$

where (the Bell number) $B(k)$ is the total number of partitions of the set $\{1,2,\dots,k\}$.

Proof. According to Corollary 23 we see that

$$\begin{aligned}\|\nabla u\|^2_{\mathcal{U}^0_{\frac{k}{2};\rho,s}} &= \sum_{i=1}^{k}\sum_{R\in P^{k,i}} \|D^i\nabla u\|^2_{0,R(\frac{1}{2}\mathbf{1}_k);\rho,s}\\ &< \sum_{i=1}^{p}\sum_{R\in P^{k,i}} \tilde{C}(\epsilon',i,\phi,D)\|f\|^2_{\mathcal{U}^{-1}_{|R(\frac{1}{2}\mathbf{1}_k)|;\rho,t}}\\ &\leq \tilde{C}(\epsilon',k,\phi,D)B(k)\|f\|^2_{\mathcal{U}^{-1}_{\frac{k}{2};\rho,t}},\end{aligned}$$

where we have used that $|R(\frac{1}{2}\mathbf{1}_k)| = \frac{k}{2}$ for all R partition of the set $\{1,2,\dots,k\}$. □

Remark 25. *Bounds for the Bell numbers $B(k)$ are know. It is know that $B(k) = \frac{1}{e}\sum_{i=1}^{\infty}\frac{i^k}{i!} < (\frac{0.792k}{\ln(k+1)})^k$. See [4] and references therein.*

Remark 26. *In the special case of f being a polynomial in ω, i.e., a finite sum of Fourier-Hermite polynomials with coefficients in $H^{-1}(D)$ we can easily verify that $f\in\mathcal{U}^{-1}_{p;\rho,s}$ for all p and all $s < \frac{\lambda_1^{2\theta}}{2}$.*

Next we present a result that can be directly applied to bound the first term in the a priori error estimate in Lemma 11.

Corollary 27. *Let the conditions of Theorem 24 hold with $k = 2p$ and $\epsilon' = \frac{\tilde{\epsilon}}{2p}$. Then*

$$|u|_{\mathcal{U}^1_{p;\rho,s+\tilde{\epsilon}+\epsilon}} \leq \tilde{C}(\epsilon,\frac{\tilde{\epsilon}}{2p},2p,\phi)B(2p)\|f\|^2_{\mathcal{U}^{-1}_{p;\rho,s+\tilde{\epsilon}+\epsilon}},$$

where the constant $\tilde{C}$ is defined in (41).

8 Spatial regularity

In this section we will study the spatial regularity of the solution of (10).

Fix ω and take partial derivatives with respect to spatial coordinates x_i. In particular $\frac{\partial\langle\omega,\phi_x\rangle}{\partial x_i} = \langle\omega,(\frac{\partial\phi_x}{\partial x_i})\rangle$, hence,

$$|\frac{\partial\langle\omega,\phi_x\rangle}{\partial x_i}| \leq \max_{x\in D}\|\frac{\partial\phi_x}{\partial x_i}\|_\theta\ \|\omega\|_{-\theta} = \tilde{\tilde{C}}_\theta(\phi)\ \|\omega\|_{-\theta}$$

where we have denoted

$$\tilde{\tilde{C}}_\theta(\phi) = \max_{1\leq i\leq d}\max_{x\in D}\|\frac{\partial\phi_x}{\partial x_i}\|_\theta. \tag{56}$$

Since $\frac{\partial\kappa(x,\omega)}{\partial x_i} = \langle\omega,(\frac{\partial\phi_x}{\partial x_i})\rangle e^{\langle\omega,\phi_x\rangle}$, we obtain

$$|\frac{\partial\kappa}{\partial x_i}(x,\omega)| \leq \tilde{\tilde{C}}_\theta(\phi)\|\omega\|_{-\theta}\ e^{\frac{C_\theta^2}{2\epsilon}} e^{\frac{\epsilon}{2}\|\omega\|^2_{-\theta}}. \tag{57}$$

Remark 28. *In general we have*

$$\|\frac{\partial \phi_x}{\partial x_i}\|_\theta^2 = \sum_{j=1}^{\infty} \lambda_j^{2\theta} (\frac{\partial \phi_x}{\partial x_i}, \eta_j)_H^2.$$

For the particular case of Example 3 we have $\phi_x(\cdot) = \phi(\cdot - x)$, hence, if we consider for simplicity the uni-dimensional case $H = L^2(\mathbb{R})$, we have

$$(\frac{\partial \phi_x}{\partial x}, \eta_j)_H = \int_{\mathbb{R}} \frac{\partial \phi}{\partial x}(\hat{x} - x)\eta_j(\hat{x})d\hat{x} = -\int_{\mathbb{R}} \frac{\partial \phi}{\partial \hat{x}}(\hat{x} - x)\eta_j(\hat{x})d\hat{x} =$$
$$\int_{\mathbb{R}} \phi(\hat{x} - x)\frac{\partial}{\partial \hat{x}}\eta_j(\hat{x})d\hat{x} = \sqrt{\frac{j}{2}}(\frac{\partial \phi_x}{\partial x}, \eta_{j-1})_H - \sqrt{\frac{j+1}{2}}(\frac{\partial \phi_x}{\partial x}, \eta_{j+1})_H,$$

where we have used a recursive relation of derivative of Hermite functions. Using that $\lambda_j = 2j$, we obtain $\tilde{\check{C}}_\theta(\phi) \leq \check{C} C_{\theta+\frac{1}{2}}$. For the case of Example 2, we have neither the recursive relation nor η_j vanishes on ∂D; see [12] for issues on the regularity of the η_j and the decaying of the $1/\lambda_j$.

The following result is a particular case of Theorem 9.1, page 184 of Ladyzhenskaya and Ural′tseva [21].

Lemma 29. *Consider the following elliptic problem*

$$\begin{cases} -\nabla \cdot (\mu(x)\nabla u(x)) &= f(x), \text{ for } x \in D \\ u(x) &= 0, \text{ on } \partial D. \end{cases} \tag{58}$$

Suppose that:

1. *There is constants such that*

$$0 < \mu_{\min} \leq \mu(x) \leq \mu_{\max} \quad \text{for all } x \in D \text{ and that } \partial D$$

   and

$$\|\frac{\partial \mu}{\partial x_j}\|_{L_q(D)} \leq \mu_{\max} \quad \text{with } q > d$$

2. *∂D is piecewise smooth with curvature bounded below by a number K (See [21] page 174 and 175).*
3. *The domain D is of class W_q^2 or that D can be topologically mapped into a parallelepiped by a function in $W_q^2(\mathbb{R}^d)$ with nonzero Jacobian.*

Then the problem has unique solution in $H^2(D) \cap H_0^1(D)$ if $f \in L^2(D)$.

Corollary 30. *Under the assumption of Lemma 29 for the Domain D we have that for almost all $\omega \in \Omega$ the weak solution $u(\cdot, \omega)$ of Problem (3) is an element of $H^2(D)$ if $f(\cdot, \omega) \in L^2(D)$*

Now we only need to bound the H^2 norm of $u(\cdot,\omega)$ in terms of ω. We next apply the second fundamental inequality, Lemma 8.1 in page 175 of [21] applied to the class of coefficients we consider in this paper. In order to simplify the presentation we assume that D is a nondegenerate $d-1$ dimension polyhedron.

Lemma 31. *Assume that D is a nondegenerate $d-1$ dimensional polyhedron. For every function $v \in H^2 \cap H_0^1$ we have that for every $\hat{\epsilon} > 0$ and almost all $\omega \in \mathcal{S}'$*

$$|v|^2_{H^2(D)} \leq 2e^{\frac{C_\theta^2}{\hat{\epsilon}}} e^{\hat{\epsilon}\|\omega\|^2_{-\theta}} \int_D (\nabla \cdot \kappa \nabla v)^2 + \frac{8d}{\hat{\epsilon}} \tilde{\tilde{C}}_\theta(\phi)\, e^{\frac{4C_\theta^2}{\hat{\epsilon}}} e^{5\hat{\epsilon}\|\omega\|^2_{-\theta}} \int_D |\nabla v|^2 \quad (59)$$

where C_θ is defined in (5) and $\tilde{\tilde{C}}_\theta$ in (56).

Proof. Note that it is enough prove the result for smooth functions v. Assume $v \in C^2(\overline{D})$ and $v = 0$ on ∂D. We have

$$(\nabla \cdot \kappa \nabla v)^2 = (\nabla \kappa \cdot \nabla v)^2 + 2\kappa(\nabla \kappa \cdot \nabla v)(\Delta v) + \kappa^2 (\Delta v)^2. \quad (60)$$

Using two integration by parts and $v = 0$ on ∂D we have

$$\int_D (\Delta v)^2 = |v|^2_{H^2(D)} + \int_{\partial D} \Delta v \nabla v \cdot \boldsymbol{\eta} - \sum_{i=1}^d \partial_i v \nabla(\partial_i v) \cdot \boldsymbol{\eta} = |v|^2_{H^2(D)}. \quad (61)$$

To see the boundary integral vanish it is enough to compute this integral in each face of D. Let F be a face of D. We can assume $F \subset R^{d-1} \times \{0\}$. Then $\boldsymbol{\eta} = (0, \ldots, 1) \in \mathbb{R}^d$. Then

$$\int_F \Delta v \nabla v \cdot \boldsymbol{\eta} - \sum_{i=1}^d \partial_i v \nabla(\partial_i v) \cdot \boldsymbol{\eta} = \int_F \Delta v \partial_d v - \sum_{i=1}^d \partial_i v \partial^2_{di} v \quad (62)$$

Since $v = 0$ on F we have that $\partial_i v = 0$ and $\partial_{ij} v = 0$, on F, $i = 1, \ldots, d-1$. Then

$$\int_F \Delta v \nabla v \cdot \boldsymbol{\eta} - \sum_{i=1}^d \partial_i v \nabla(\partial_i v) \cdot \boldsymbol{\eta} = \int_F \partial_{dd} v \partial_d v - \partial_d v \partial^2_{dd} v = 0. \quad (63)$$

Now, observe that

$$\int_D (\nabla \kappa \cdot \nabla v)^2 \leq d \max_{1 \leq i \leq d} \|\partial_i \kappa\|^2_\infty \int_D |\nabla v|^2 \quad (64)$$

and with $\delta = \frac{\kappa^2_{\min}}{2\kappa_{\max}} > 0$ we have

$$\begin{aligned} 2\int_D \kappa(\nabla \kappa \cdot \nabla v)(\Delta v) &\leq \kappa_{\max} \left(\delta \int_D (\Delta v)^2 + \frac{1}{\delta} \int_D (\nabla \kappa \cdot \nabla v)^2 \right) \\ &\leq \delta \kappa_{\max} |v|^2_{H^2(D)} + d \max_{1 \leq i \leq d} \|\partial_i \kappa\|^2_\infty \frac{\kappa_{\max}}{\delta} \int_D |\nabla v|^2 \\ &\leq \frac{\kappa^2_{\min}}{2} |v|^2_{H^2(D)} + 2d \max_{1 \leq i \leq d} \|\partial_i \kappa\|^2_\infty \frac{\kappa^2_{\max}}{\kappa^2_{\min}} \int_D |\nabla v|^2 \quad (65) \end{aligned}$$

By combining (61), (64) and (65) we get

$$\begin{aligned}|v|^2_{H^2(D)} &\le \frac{1}{\kappa^2_{\min}}\int_D (\nabla\cdot\kappa\nabla v)^2 + \\ &\quad +\frac{1}{2}|v|^2_{H^2(D)} + 2d\max_{1\le i\le d}\|\partial_i\kappa\|^2_\infty\left(\frac{1}{\kappa^2_{\min}}+\frac{\kappa^2_{\max}}{\kappa^4_{\min}}\right)\int_D|\nabla v|^2\end{aligned}$$

and then

$$|v|^2_{H^2(D)} \le \frac{2}{\kappa^2_{\min}}\int_D (\nabla\cdot\kappa\nabla v)^2 + 4d\max_{1\le i\le d}\|\partial_i\kappa\|^2_\infty\left(\frac{1}{\kappa_{\min}}+\frac{\kappa^2_{\max}}{\kappa^2_{\min}}\right)\int_D|\nabla v|^2.$$

Finally, using (6), we see that

$$\begin{aligned}|v|^2_{H^2(D)} &\le 2e^{\frac{C_\theta^2}{\hat\epsilon}}e^{\hat\epsilon\|\omega\|^2_{-\theta}}\int_D (\nabla\cdot\kappa\nabla v)^2 + \\ &\quad 4d\max_{1\le i\le d}\|\partial_i\kappa\|^2_\infty\left(1+e^{\frac{3C_\theta^2}{\hat\epsilon}}e^{3\hat\epsilon\|\omega\|^2_{-\theta}}\right)\int_D|\nabla v|^2\end{aligned}$$

and using (57) we get

$$\begin{aligned}|v|^2_{H^2(D)} &\le 2e^{\frac{C_\theta^2}{\hat\epsilon}}e^{\hat\epsilon\|\omega\|^2_{-\theta}}\int_D (\nabla\cdot\kappa\nabla v)^2 + \\ &\quad 4d\tilde{\tilde{C}}_\theta(\phi)\|\omega\|^2_{-\theta}e^{\frac{C_\theta^2}{\hat\epsilon}}e^{\hat\epsilon\|\omega\|^2_{-\theta}}\left(1+e^{\frac{3C_\theta^2}{\hat\epsilon}}e^{3\hat\epsilon\|\omega\|^2_{-\theta}}\right)\int_D|\nabla v|^2\end{aligned}$$

and using that $\hat\epsilon\|\omega\|^2_{-\theta} < e^{\hat\epsilon|\omega\|^2_{-\theta}}$, (59) follows. □

We establish a bound for the second term in the a priori error estimate 11.

Theorem 32. *Let $s \in \mathbb{R}$ and $\epsilon > 0$, and let $u \in \widehat{\mathcal{U}}^1_s$ be the solution of (10) with right-hand side $f \in \mathcal{U}^{-1}_{s+\epsilon}$. Assume that D is a nondegenerate $d-1$ dimensional polyhedron, $f \in \mathcal{U}^0_{s+\tilde\epsilon+\epsilon+\hat\epsilon}$ and $f \in \mathcal{U}^{-1}_{s+\tilde\epsilon+2\epsilon+5\hat\epsilon}$ for $\tilde\epsilon, \epsilon, \hat\epsilon$ positive. Then, $u \in \mathcal{U}^2_{s+\epsilon\tilde+\hat\epsilon}$ and*

$$\begin{aligned}|u|^2_{\mathcal{U}^2_{s+\tilde\epsilon+\epsilon+\hat\epsilon}} &\le 2e^{\frac{C_\theta^2}{\hat\epsilon}}e^{\hat\epsilon\|\omega\|^2_{-\theta}}||f||^2_{\mathcal{U}^0_{s+\tilde\epsilon+\epsilon+\hat\epsilon}} + \\ &\quad \frac{8d}{\hat\epsilon}C^2\tilde{\tilde{C}}_\theta(\phi)\,e^{C_\theta^2(\frac{4}{\hat\epsilon}+\frac{1}{\epsilon})}e^{(4\hat\epsilon+\epsilon)\|\omega\|^2_{-\theta}}\|f\|^2_{\mathcal{U}^{-1}_{s+\tilde\epsilon+2\epsilon+5\hat\epsilon}}.\end{aligned}$$

where $C = \sqrt{1+C_{poin}}$ and C_{poin} is the Poincaré inequality constant which depends on D, C_θ is defined in (5) and $\tilde{\tilde{C}}_\theta$ in (56).

Proof. Corollary 30 and $f \in \mathcal{U}^0_{s+\tilde\epsilon+\epsilon+\hat\epsilon}$ imply that for almost all $\omega \in \mathcal{S}'$, $u(\cdot,\omega) \in H^2(D)$. The bound (66) follows by first replacing v by u in (59), then multiply (59) by $e^{(s+\tilde\epsilon+\epsilon)\|\omega\|^2_{-\theta}}$ and integrate in $\mathcal{S}'$, then use $\nabla\cdot\kappa\nabla u = f$ to obtain the first term of the right-hand side of (66), and use Lemma 5 to obtain the second term. □

9 Final remarks

We presented regularity results for stochastic elliptic equations with lognormal coefficient κ. We obtained joint spatial and stochastic regularity of solutions of the ordinary product pressure equation assuming similar regularity for the right-hand side $f(x, \omega)$ and stochastic process $\log(\kappa(x, \omega))$. Standard assumptions on the spatial domain D are also used. The main results in Theorem 24 and Theorem 32 which state that the solution of the pressure equation with regular data has classical H^{1+r} regularity in the spatial variable x and stochastic regularity given by a particular weighted Chaos space. To compute regular norms of function in the stochastic variable we use the White Noise framework and directional derivatives. This resulting norm require norms of partial derivatives in the ω variable up to certain order to be bounded. The fact this norm is equivalent to a weighted Chaos space norm is proved in Theorem 20.

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