HOMOGENIZATION OF DEGENERATE POROUS MEDIUM TYPE EQUATIONS IN ERGODIC ALGEBRAS

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Abstract. We consider the homogenization problem for general porous medium type equations of the form $u_t = \Delta f(x, \frac{x}{\varepsilon}, u)$. The flux function $f(x, y, \cdot)$ may be of two different types. In the type 1 case, $f(x, y, \cdot)$ is strictly increasing; this is a mildly degenerate case. In the type 2 case, $f(x, y, \cdot)$ has the form $h(x, y)F(u) + S(x, y)$, where $F(u)$ is just a nondecreasing function; this is a strongly degenerate case. We address both, the Cauchy problem and the initial-boundary value problem, with null boundary condition. The homogenization is carried out in the general context of ergodic algebras. As far as the authors know, homegenization of such degenerate quasilinear parabolic equations is addressed here for the first time. We also review the existence and stability theory for such equations and establish new results needed for the homogenization analysis. Further, we include some new results on algebras with mean value, specially a new criterion establishing the null measure of level sets of elements of the algebra, which is useful in connection with the homogenization of porous medium type equations in the type 2 case.

1. Introduction

In this paper we consider the homogenization of a porous medium type equation of the general form

$$u_t = \Delta f(x, \frac{x}{\varepsilon}, u),$$

where f is a continuous function of (x, y, u) and $f(x, y, \cdot)$ is locally Lipschitz continuous, uniformly in (x, y), and may be of two different types. In the type 1 case, $f(x, y, \cdot)$ is strictly increasing; this is a mildly degenerate case. In the type 2 case, $f(x, y, u)$ has the form $h(x, y)F(u) + S(x, y)$, where $F(u)$ is just a nondecreasing function, which is *not* strictly increasing; this is a strongly degenerate case. We consider both, the Cauchy problem and the initial-boundary value problem with null boundary condition. Concerning the Cauchy problem, the discussion here largely extends the corresponding one in [3] concerning the homogenization of the particular non-degenerate type of such equations where $f(x, y, u) = f(u) + V(y)$, with f smooth and $f'(u) > 0$. Here, for the Cauchy problem, we assume that $f = f(y, u)$ does not depend on the "slow" variable x, so that $f(y, u) = h(y)F(u) + S(y)$ in the type 2 case, and for each u, $f(\cdot, u)$ belongs to a given general ergodic algebra $\mathcal{A}(\mathbb{R}^n)$; in particular, $h, S \in \mathcal{A}(\mathbb{R}^n)$, with $h > \delta_0 > 0$, if f is of type 2. Also, we restrict the initial data to "well-prepared" ones, that is, functions of the form $g(\frac{x}{\varepsilon}, \varphi_0(x))$, where $\varphi_0 \in L^\infty(\mathbb{R}^n)$, and, for each $y \in \mathbb{R}^n$, $g(y, \cdot)$ is the inverse of $f(y, \cdot)$, in the type 1 case, or $g(y, v) = G((v - S(y))/h(y))$, in the type 2 case, where G is the left-continuous function such that $F(G(v)) = v$. Both restrictions, namely, the fact that the flux function does not depend on the "slow" variable x, and the fact that the initial data is "well-prepared", have to do with limitations of the technique used in the homogenization process, which goes back to the work of E and Serre [14], on the periodic homogenization of the one-dimensional conservation law $u_t + (f(u) - V(\frac{x}{\varepsilon}))_x = 0$ (see [2], for a multidimensional extension to almost periodic homogenization), which in turn is inspired in the work of DiPerna [12] on the uniqueness of measure-valued solutions of scalar conservation laws. The technique is otherwise very powerful in that it applies to any ergodic algebra, and it implies the existence of strong correctors, that is, the existence of an oscillatory profile $U(x, \frac{x}{\varepsilon}, t)$, through

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which the weak convergence of the homogenizing sequence of solutions $u^\varepsilon(x,t)$ is converted into a strong convergence to 0 in $L^1_{\rm loc}(\mathbb{R}^{n+1}_+)$ of the difference $u^\varepsilon(x,t)-U(x,\frac{x}{\varepsilon},t)$. It strongly relies on the concept of two-scale Young measures first introduced by W. E in [13] as a nonlinear extension of the concept of two-scale convergence introduced by Nguetseng in [25] and further developed by Allaire in [1]. The extensions of two-scale Young measures to almost periodic functions and to general algebras with mean value (algebras w.m.v., for short) were established in [2] and [3], respectively.

Concerning the initial-boundary value problem with null boundary condition on a bounded domain Ω, the discussion in this paper largely extends the corresponding one in [17], where we consider the special non-degenerate case where $f(x,y,u)=f(u)+V(x,y)$, with f smooth and $f'(u)>0$. Here, we consider general flux functions $f(x,y,u)$ of type 1, where $f(x,y,\cdot)$ is strictly increasing, or type 2, where $f(x,y,u)=h(x,y)F(u)+S(x,y)$, and F is as above. As opposite to the case of the Cauchy problem, for this problem, $f(x,y,u)$ must depend on the "slow" variable x since we need to impose that $f(x,y,0)=0$ for all $(x,y)\in\partial\Omega\times\mathbb{R}^n\times\mathbb{R}$, because of the boundary condition. The method applied in this case is completely different from the one for the Cauchy problem and it is based on the conversion of the homogenization of the nonlinear parabolic equation into the homogenization of a particular type of fully nonlinear parabolic equation and we use ideas that go back to Evans [15] and Ishii [18], among others. Here, the initial data do not need to be "well-prepared" and, as already mentioned, the flux function depends also on x. On the other hand, we have to restrict the homogenization analysis to regular algebras with mean value. The latter is a concept introduced here whose largest representative so far known is the Fourier-Stieltjes algebra introduced in [17]. As pointed out in [17], the Fourier-Stieltjes algebra *strictly* contains the algebra of perturbed almost periodic functions, whose elements can be written as the sum of an almost periodic function and a continuous function vanishing at infinity. Here, we prove that a regular algebra w.m.v. is ergodic. We recall that the theory of algebras w.m.v. and ergodic algebras was first developed by Zhikov and Krivenko in [30] (see also [19]).

The results in this paper indicate, in general, that the homogenization of the porous medium type equation above leads to the homogenized equation

$$u_t=\Delta\bar f(x,u),$$

where $\bar f(x,u)$ is defined by the formula

$$u=\fint_{\mathbb{R}^n} g(x,y,\bar f(x,u))\,dy,$$

where g is such that, for each (x,y), $g(x,y,\cdot)$ is the inverse of $f(x,y,\cdot)$, in the type 1 case, or $g(x,y,v)=G((v-S(x,y))/h(x,y))$ is the type 2 case, where G is as above. For the definition of $\bar f$ by the last formula to make sense, in the type 2 case, it is necessary to have, for all α in the set of discontinuities of G,

$$\mathfrak{m}\left(\{z\in\mathcal{K}\,:\,\alpha h(z)+S(z)=q\}\right)=0,$$

for all $q\in\mathbb{R}$, where $\mathcal{K}$ is the compact space associated with the algebra w.m.v. $\mathcal{A}(\mathbb{R}^n)$ and $\mathfrak{m}$ is the corresponding probability measure in $\mathcal{K}$ (see, Theorem 2.1 below, established in [3]). This leads us to the question about necessary conditions for the vanishing of the measure of level sets in $\mathcal{K}$ of an element of $\mathcal{A}(\mathbb{R}^n)$, which is the subject of a general result on algebras w.m.v. proved herein (see Lemma 2.2 below). To illustrate this problem, we briefly exhibit here a very simple example in the periodic context. So, let us consider the homogenization of the strongly degenerate equation

$$u_t=\Delta(F(u)+\psi_0(\frac{x}{\varepsilon})),$$

where

$$F(u)=\begin{cases}u+\frac12, & u<-\frac12\\ 0, & -\frac12\le u\le\frac12,\\ u-\frac12, & u>\frac12,\end{cases}$$

and $\psi_0 : \mathbb{R} \to \mathbb{R}$ is the periodic function of period 4 defined for $x \in [-2, 2]$ by

$$\psi_0(x) = \begin{cases} -x-2, & -2 \le x \le -1, \\ x, & -1 \le x \le 1, \\ -x+2, & 1 \le x \le 2. \end{cases}$$

Such nonlinear flux function is a prototype for models of the so called Stefan problem (see, e.g., [11]). Our homogenization analysis of such strongly degenerate equations implies in this simple case that the homogenized equation is

$$u_t = \Delta \bar{f}(u),$$

where

$$\bar{f}(u) = \begin{cases} u + \frac{1}{2}, & u < -\frac{3}{2}, \\ \frac{2}{3}u, & -\frac{3}{2} \le u \le \frac{3}{2}, \\ u - \frac{1}{2}, & u > \frac{3}{2}. \end{cases}$$

So, although the equations to be homogenized are strongly degenerate, the homogenized equation is nondegenerate. The reason for this is basically the fact that the level sets of the periodic function ψ_0 defined above have Lebesgue measure zero. As remarked after the proof of Lemma 2.2 below, if $\mathcal{A}(\mathbb{R}^n)$ is an algebra w.m.v. containing the periodic function ψ_0, then small perturbations of the form $\psi = \psi_0 + \delta\psi_1$, with $\psi_1 \in \mathcal{A}(\mathbb{R}^n)$, will satisfy the zero measure condition on the level sets in $\mathcal{K}$, which yields a similar nice behavior of the homogenized equation.

Before concluding this introduction, we would like to mention that in this paper we also make a detailed review and provide some new results on the existence and stability theory for degenerate parabolic equations of the type considered here. We do that because we need some specific results that are not proved elsewhere, also just to introduce some notations used later on, as well as in order to have our work the most self-contained possible.

This paper is organized as follows. In Section 2 we recall the concepts of algebra w.m.v., generalized Besicovitch space and ergodic algebra. We also recall a general result established in [3] which relates such algebras and the translation operators acting on them with the continuous functions defined on certain compact spaces and certain groups of homeomorphisms of these compact spaces. In Section 3, we introduce the concept of regular algebra w.m.v., prove that these are ergodic algebras, and that this concept includes the Fourier-Stieltjes spaces $\mathrm{FS}(\mathbb{R}^n)$. In Section 4, we briefly recall the general result of [3] on the existence of two-scale Young measures associated with a given algebra w.m.v. In Section 5, we provided a self-contained discussion about the well-posedness of the Cauchy problem and the initial-boundary value problem with null boundary condition for degenerate porous medium type equations. In Section 6, we consider the homogenization problem for porous medium type equations defined in all $\mathbb{R}^n$ and we analize the case of the Cauchy problem. Finally, in Section 7, we consider the homogenization problem for porous medium type equations defined in a bounded domain and we analize the case of the initial-boundary value problem with null boundary condition.

2. Ergodic Algebras

In this section we recall some basic facts about algebras with mean values and ergodic algebras that will be needed for the purposes of this paper. To begin with, we recall the notion of mean value for functions defined in $\mathbb{R}^n$.

Definition 2.1. Let $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. A number $M(g)$ is called the *mean value of g* if

$$\lim_{\varepsilon \to 0} \int_A g(\varepsilon^{-1}x)\, dx = |A| M(g) \tag{2.1}$$

for any Lebesgue measurable bounded set $A \subseteq \mathbb{R}^n$, where $|A|$ stands for the Lebesgue measure of A. This is the same as saying that $g(\varepsilon^{-1}x)$ converges, in the duality with L^∞ and compactly supported functions, to the constant $M(g)$. Also, if $A_t := \{x \in \mathbb{R}^n : t^{-1}x \in A\}$ for $t > 0$ and $|A| \neq 0$, (2.1) may be written as

$$\lim_{t\to\infty} \frac{1}{t^n|A|} \int_{A_t} g(x)\, dx = M(g). \tag{2.2}$$

We will also use the notation $⨍_{\mathbb{R}^n} g\, dx$ for $M(g)$.

As usual, we denote by $\mathrm{BUC}(\mathbb{R}^n)$ the space of the bounded uniformly continuous real-valued functions in $\mathbb{R}^n$.

We recall now the definition of algebra with mean value introduced in [30].

Definition 2.2. Let $\mathcal{A}$ be a linear subspace of $\mathrm{BUC}(\mathbb{R}^n)$. We say that $\mathcal{A}$ is an *algebra with mean value* (or *algebra w.m.v.*, in short), if the following conditions are satisfied:

(A) If f and g belong to $\mathcal{A}$, then the product fg belongs to $\mathcal{A}$.
(B) $\mathcal{A}$ is invariant under the translations $\tau_y : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto x + y$, $y \in \mathbb{R}^n$, that is, if $f \in \mathcal{A}$, then $\tau_y f \in \mathcal{A}$, for all $y \in \mathbb{R}^n$, where $\tau_y f := f \circ \tau_y$, $f \in \mathcal{A}$.
(C) Any $f \in \mathcal{A}$ possesses a mean value.
(D) $\mathcal{A}$ is closed in $\mathrm{BUC}(\mathbb{R}^n)$ and contains the unity, i.e., the function $e(x) := 1$ for $x \in \mathbb{R}^n$.

Remark 2.1. Condition (D) was not originally in [30] but its inclusion does not change matters since any algebra satisfying (A), (B) and (C) can be extended to an algebra satisfying (A)–(D) in an unique way modulo isomorphisms.

For the development of the homogenization theory in algebras with mean value, as it is done in [30, 19] (see also [7, 3]), in similarity with the case of almost periodic functions, one introduces, for $1 \le p < \infty$, the space $\mathcal{B}^p$ as the abstract completion of the algebra $\mathcal{A}$ with respect to the Besicovitch seminorm

$$|f|_p := \left(⨍_{\mathbb{R}^n} |f|^p\, dx \right)^{1/p}$$

Both the action of translations and the mean value extend by continuity to $\mathcal{B}^p$, and we will keep using the notation $\tau_y f$ and $M(f)$ even when $f \in \mathcal{B}^p$. Furthermore, for $p > 1$ the product in the algebra extends to a bilinear operator from $\mathcal{B}^p \times \mathcal{B}^q$ into $\mathcal{B}^1$, with q equal to the dual exponent of p, satisfying

$$|fg|_1 \le |f|_p |g|_q.$$

In particular, the operator $M(fg)$ provides a nonnegative definite bilinear form on $\mathcal{B}^2$.

Since there is an obvious inclusion between elements of this family of spaces, we may define the space $\mathcal{B}^\infty$ as follows:

$$\mathcal{B}^\infty = \{f \in \bigcap_{1\le p<\infty} \mathcal{B}^p \,:\, \sup_{1\le p<\infty} |f|_p < \infty\},$$

We endow $\mathcal{B}^\infty$ with the (semi)norm

$$|f|_\infty := \sup_{1\le p<\infty} |f|_p.$$

Obviously the corresponding quotient spaces for all these spaces (with respect to the null space of the seminorms) are Banach spaces, and in the case $p = 2$ we obtain a Hilbert space. We denote by $\stackrel{\mathcal{B}^p}{=}$, the equivalence relation given by the equality in the sense of the $\mathcal{B}^p$ semi-norm. We will keep the notation $\mathcal{B}^p$ also for the corresponding quotient spaces.

Remark 2.2. A classical argument going back to Besicovitch [4] (see also [19], p.239) shows that the elements of $\mathcal{B}^p$ can be represented by functions in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$, $1 \le p < \infty$.

We next recall a result established in [3] which provides a connection between algebras with mean value and the algebra $C(\mathcal{K})$ of continuous functions on a certain compact (Hausdorff) topological space. We state here only the parts of the corresponding result in [3] that will be used in this paper.

Theorem 2.1 (cf. [3]). *For an algebra $\mathcal{A}$, we have:*

(i) *There exist a compact space $\mathcal{K}$ and an isometric isomorphism i identifying $\mathcal{A}$ with the algebra $C(\mathcal{K})$ of continuous functions on $\mathcal{K}$.*

(ii) *The translations $\tau_y : \mathbb{R}^n \to \mathbb{R}^n$, $\tau_y x = x + y$, induce a family of homeomorphisms $T(y) : \mathcal{K} \to \mathcal{K}$, $y \in \mathbb{R}^n$, satisfying the group properties $T(0) = I$, $T(x + y) = T(x) \circ T(y)$, such that the mapping $T : \mathbb{R}^n \times \mathcal{K} \to \mathcal{K}$, $T(y, z) := T(y)z$, is continuous.*

(iii) *The mean value on $\mathcal{A}$ extends to a Radon probability measure $\mathfrak{m}$ on $\mathcal{K}$ defined by*

$$\int_{\mathcal{K}} i(f)\, d\mathfrak{m} := \fint_{\mathbb{R}^n} f\, dx, \qquad f \in \mathcal{A},$$

which is invariant by the group of homeomorphisms $T(y) : \mathcal{K} \to \mathcal{K}$, $y \in \mathbb{R}^n$, that is, $\mathfrak{m}(T(y)E) = \mathfrak{m}(E)$ for all Borel sets $E \subseteq \mathcal{K}$.

(iv) *For $1 \le p \le \infty$, the Besicovitch space $\mathcal{B}^p / \overset{\mathcal{B}^p}{=}$ is isometrically isomorphic to $L^p(\mathcal{K}, \mathfrak{m})$.*

A function $f \in \mathcal{B}^2$ is said to be *invariant* if $\tau_y f \overset{\mathcal{B}^2}{=} f$, for all $y \in \mathbb{R}^n$. More clearly, $f \in \mathcal{B}^2$ is invariant if

$$M\big(|\tau_y f - f|^2\big) = 0, \qquad \text{for all } y \in \mathbb{R}^n. \tag{2.3}$$

The concept of ergodic algebra is then introduced as follows.

Definition 2.3. An algebra w.m.v. $\mathcal{A}$ is called an *ergodic algebra* if any invariant function f belonging to the corresponding space $\mathcal{B}^2$ is equivalent (in $\mathcal{B}^2$) to a constant.

A very useful alternative definition of ergodic algebra is also given in [19], p. 247, and shown therein to be equivalent to Definition 2.3. We state that as the following lemma.

Lemma 2.1 (cf. [19]). *Let $\mathcal{A} \subseteq \mathrm{BUC}(\mathbb{R}^n)$ be an algebra w.m.v.. Then $\mathcal{A}$ is ergodic if and only if*

$$\lim_{t\to\infty} M_y \left(\Big| \frac{1}{|B(0;t)|} \int_{B(0;t)} f(x + y)\, dx - M(f) \Big|^2 \right) = 0 \qquad \forall f \in \mathcal{A}. \tag{2.4}$$

We close this section establishing a general result concerning algebras w.m.v. which will be used in our investigation on the homogenization of porous medium type equations in the last two sections of the present work. We first establish the following definition.

Definition 2.4. For a C^1 vector function $\psi = (\psi_1, \dots, \psi_N)$ whose components belong to an algebra w.m.v. $\mathcal{A}(\mathbb{R}^n)$, such that $\partial_{x_i}\psi_j \in \mathcal{A}(\mathbb{R}^n)$, $i = 1, \dots, n$, $j = 1, \dots, N$, we say that $\alpha \in \mathbb{R}^N$ is a *strongly regular* value of ψ if there exists $\delta_\alpha > 0$ such $|\psi(x) - \alpha|^2 + |D\psi(x)|^2 > \delta_\alpha$, for all $x \in \mathbb{R}^n$, where $|D\psi(x)|^2 = \sum_{i=1}^n \sum_{j=1}^N (\partial_{x_i}\psi_j(x))^2$.

Lemma 2.2. *Let $\mathcal{A}(\mathbb{R}^n)$ be an algebra w.m.v. and $\psi \in \mathcal{A}(\mathbb{R}^n)$ be such that $\partial_{x_i}\psi \in \mathcal{A}(\mathbb{R}^n)$, $i = 1, \dots, n$.*

(i) *If $\alpha \in \mathbb{R}$ is a strongly regular value of ψ, then*

$$\mathfrak{m}\left(\{z \in \mathcal{K} : \psi(z) = \alpha\}\right) = 0, \tag{2.5}$$

where $\mathcal{K}$ is the compact space given by Theorem 2.1 and $\mathfrak{m}$ is the associated invariant probability measure on $\mathcal{K}$.

(ii) *If also $\partial^2_{x_i x_j}\psi \in \mathcal{A}(\mathbb{R}^n)$, for $i, j \in \{1, \dots, n\}$, and 0 is a strongly regular value of $\nabla\psi$, then (2.5) holds for all $\alpha \in \mathbb{R}$.*

Proof. We first prove item (i). By the hypotheses and the properties of the algebras w.m.v., we have that $|\nabla\psi| \in \mathcal{A}(\mathbb{R}^n)$ and so it extends to a function in $C(\mathcal{K})$. Since α is a strongly regular value of ψ, the sets $A = \{z \in \mathcal{K} \,:\, \psi(z) = \alpha\}$ and $B = \{z \in \mathcal{K} \,:\, |\nabla\psi(z)| = 0\}$ are two disjoint compact subsets of $\mathcal{K}$. Hence, there exists $\delta_0 > 0$ such that $|\nabla\psi(z)| > \delta_0$ for $z \in A$. Now, given any $z \in A$, there exists $j \in \{1, \dots, n\}$ such that $|\partial_{x_j}\psi(z)| > \delta_0/n$.

We claim that

$$A \cap \mathbb{R}^n \subseteq \bigcup_{j=1}^{n} \bigcup_{k=1}^{\infty} S_k^j,$$

where each S_k^j is the graph of a C^1 function defined on an open subset of the space of the $n-1$ variables

$$(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n),$$

and, moreover, for each fixed $j \in \{1, \dots, n\}$, the sets S_k^j, $k \in \mathbb{Z}$, are separated from each other along the lines parallel to the x_j-axis by neighborhoods V_k^j defined by

$$V_k^j = \Big\{ x \in \mathbb{R}^n \,:\, x = x' + s\, e_j, \quad \text{for a unique pair } (x', s) \text{ with } x' \in S_k^j \text{ and } s \in (-\sigma_0, \sigma_0) \Big\},$$

for some $\sigma_0 > 0$, where e_j is the j-th unit vector of the canonical basis.

Indeed, let $A^j = \{x \in A \cap \mathbb{R}^n \,:\, |\partial_{x_j}\psi(x)| > \frac{\delta_0}{n}\}$. Clearly $A \cap \mathbb{R}^n = \cup_{j=1}^n A^j$. By the Implicit Function Theorem, we have that A^j is the union of a family of connected graphs of C^1 functions. Moreover, since $\partial_{x_j}\psi$ is uniformly continuous, there exists σ_0 such that $|x - y| < \sigma_0$ implies $|\partial_{x_j}\psi(x) - \partial_{x_j}\psi(y)| < \delta_0/2n$. Therefore, any two points $x, y \in A_j$ lying both in one line parallel to the x_j-axis must satisfy $|x - y| > 2\sigma_0$. In particular, the set of connected graphs in A^j is countable, since for each point in the hyperplane $\{x_j = 0\}$ with rational coordinates there corresponds at most a countable number of graphs whose projection in $\{x_j = 0\}$ contains that point. Now, given a connected graph contained in A^j, by Zorn's lemma, we can obtain a maximal family of connected graphs in A_j, containing the given graph, whose projections into the hyperplane $\{x_j = 0\}$ are disjoint from each other. We call this maximal family S_1^j. We then consider the family of connected graphs $A^j \setminus S_1^j$ and pick up a connected graph from it. Again by Zorn's lemma such graph belongs to a maximal family of connected graphs in $A^j \setminus S_1^j$ whose projections into the hyperplane $\{x_j = 0\}$ are pairwise disjoint. We call this maximal family S_2^j. We then consider the family of connected graphs $A^j \setminus (S_1^j \cup S_2^j)$ and, from it, we define a maximal family of connected graphs whose projections in the hyperplane $\{x_j = 0\}$ are pairwise disjoint, call this maximal family S_3^j, and so on. In this way, relabeling if necessary, we end up decomposing A^j into a disjoint union, $\cup_{k=1}^\infty S_k^j$, of maximal families of connected graphs whose projections in the hyperplane $\{x_j = 0\}$ do not intersect each other. Clearly, each such maximal family, S_k^j, maybe be viewed as a graph of a C^1 function defined on an open subset of the space of the variables $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n$. The assertion about the neighborhoods V_k^j is also clear.

For $0 < \varepsilon < (\delta_0\sigma_0)/2n$, let us define

$$\phi_\varepsilon(x) = \min\{|\psi(x) - \alpha|,\ \varepsilon\}, \qquad \psi_\varepsilon(x) = 1 - \varepsilon^{-1}\phi_\varepsilon.$$

We then have that $\operatorname{supp}\psi_\varepsilon \cap V_k^j \subseteq V_k^{j,\varepsilon}$, where

$$V_k^{j,\varepsilon} := \Big\{ x \in \mathbb{R}^n \,:\, x = x' + s\, e_j, \quad \text{for a unique pair } (x', s) \text{ with } x' \in S_k^j \text{ and } s \in (-\frac{2n\varepsilon}{\delta_0}, \frac{2n\varepsilon}{\delta_0}) \Big\}.$$

Therefore,

$$\mathfrak{m}(A) \le \mathfrak{m}(\{z \in \mathcal{K} \,:\, |\psi_\varepsilon(z)| > \frac{1}{2}\}) \le 2\int_{\mathcal{K}} \psi_\varepsilon(z)\, d\mathfrak{m}(z) = 2\lim_{R\to\infty} \frac{1}{|B(0;R)|}\int_{|x|<R} \psi_\varepsilon(x)\, dx. \tag{2.6}$$

Now, for each $j \in \{1, \dots, n\}$, $\#\{V_k^j : V_k^j \cap B(0;R) \neq \emptyset\} < \frac{R}{\sigma_0}$, and clearly

$$\frac{1}{R^{n-1}} \mathcal{H}^{n-1}\left(B(0;R) \cap S_k^j\right) \leq C,$$

for some $C > 0$ depending only on ψ. Hence, for any $R > 0$, we have

$$\begin{aligned}
\frac{1}{|B(0,R)|} \int_{|x|<R} \psi_\varepsilon(x)\, dx &= \sum_{j=1}^{n} \sum_{k \in \mathbb{Z}} \frac{1}{|B(0;R)|} \int_{B(0;R) \cap V_k^j} \psi_\varepsilon(x)\, dx \\
&\leq \sum_{j=1}^{n} \frac{R}{\sigma_0 |B(0;R)|} \max_k \int_{B(0;R) \cap V_k^j} \psi_\varepsilon(x)\, dx \\
&\leq \sum_{j=1}^{n} \frac{R}{\sigma_0 |B(0;R)|} \max_k \int_{B(0;R) \cap V_k^j \cap \operatorname{supp} \psi_\varepsilon} dx \\
&< C\varepsilon,
\end{aligned}$$

again for some $C > 0$ depending only on ψ. Thus, we get that $\mathfrak{m}(A) < C\varepsilon$, and, since $\varepsilon > 0$ is arbitrary, we arrive at the desired conclusion, concluding the proof of (i).

As for the proof of (ii), by (i) and the hypotheses on $\nabla\psi$, we see that

$$\mathfrak{m}\left(\{z \in \mathcal{K} :\ \nabla\psi(z) = 0\}\right) = 0,$$

and, so, we only need to prove that

$$\mathfrak{m}\left(\{z \in \mathcal{K} :\ \psi(z) = \alpha,\ |\nabla\psi(z)| > 0\}\right) = 0.$$

But, we may write

$$\{z \in \mathcal{K} :\ \psi(z) = \alpha,\ |\nabla\psi(z)| > 0\} = \bigcup_{l=1}^{\infty} B^l, \qquad B^l = \{z \in \mathcal{K} \,:\, \psi(z) = \alpha,\ |\nabla\psi(z)| > \frac{1}{l}\}.$$

Now, we claim that $\mathfrak{m}(B^l) = 0$, $l = 1, 2, \dots$. Indeed, the claim follows by arguments similar to those used in the proof of (i). The only nontrivial adaptation to be made, is that, instead of using the function ψ_ε defined above, we shall now use the function

$$\tilde{\psi}_\varepsilon(x) = \psi_\varepsilon(x)\theta_l(x), \qquad \theta_l(x) := 2l\left(\max\left\{\frac{1}{2l},\ \min\{\ |\nabla\psi(x)|,\ \frac{1}{l}\}\right\} - \frac{1}{2l}\right).$$

We then get an inequality similar to (2.6) with A replaced by B^l and ψ_ε replaced by $\tilde{\psi}_\varepsilon$. We also define the analogues of S_k^j and V_k^j and the remaining of the proof follows as in the proof of (i). This gives the desired conclusion.

□

Remark 2.3. In general, for any ψ in an algebra w.m.v. $\mathcal{A}(\mathbb{R}^n)$, we trivially have $\mathfrak{m}(\{z \in \mathcal{K} : \psi(z) = \alpha\}) = 0$, except for a countable set of α's. Nevertheless, in general we do not have any other information about the set of exceptional α's besides the fact that it is countable; in particular, it could be dense in $\mathbb{R}$. However, we can use Lemma 2.2 to provide examples where the set of exceptional α's is empty. For instance, if ψ_0 is a C^2 periodic function in $\mathbb{R}^n$ for which 0 is a regular value of $\nabla\psi_0$ in the usual sense, then, by the Implicit Function Theorem, we know that the set $\{x \in \mathbb{R}^n : |\nabla\psi_0(x)| = 0\}$ has n-dimensional Lebesgue measure zero, and since $\nabla\psi_0 = 0$ almost everywhere on the level sets $\{\psi_0 = \alpha\}$, we conclude that all these level sets have n-dimensional Lebesgue measure zero. Now, if $\mathcal{A}(\mathbb{R}^n)$ is an algebra w.m.v. containing such a periodic function ψ_0 and $\psi_1, \partial_{x_i}\psi_1, \partial^2_{x_i x_j}\psi_1 \in \mathcal{A}(\mathbb{R}^n)$, $i, j = 1, \dots, n$, then, for $\delta > 0$ sufficiently small, we have that $\psi = \psi_0 + \delta\psi_1$ satisfies the hypotheses of the item (ii) of Lemma 2.2, and so the conclusion of (ii) holds for ψ.

3. Regular algebras w.m.v. and the Fourier-Stieltjes space $\mathrm{FS}(\mathbb{R}^n)$.

In this section we introduce the concept of regular algebra w.m.v. and recall the definition and some basic properties of the Fourier-Stieltjes space introduced by the authors in [17], which is, to the best of our knowledge, the largest known example of a regular algebra w.m.v..

For any $f \in L^\infty(\mathbb{R}^n)$, let us denote by $\hat{f}$ the Fourier transform of f defined as the following distribution

$$\langle \hat{f}, \phi \rangle := \int f(x)\check{\phi}(x)\,dx, \qquad \text{for all } \phi \in C_c^\infty(\mathbb{R}^n),$$

where $\check{\phi}$ denotes the usual inverse Fourier transform of ϕ, i.e.,

$$\check{\phi}(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int \phi(y) e^{iy\cdot x}\,dx.$$

Given an algebra w.m.v. $\mathcal{A}$, let us denote by $V(\mathcal{A})$ the subspace formed by the elements $f \in \mathcal{A}$ such that $M(f) = 0$, namely,

$$V(\mathcal{A}) := \{ f \in \mathcal{A} \, : \, M(f) = 0 \}.$$

Also, let us denote by $Z(\mathcal{A})$ the subset of those $f \in \mathcal{A}$ such that the distribution $\hat{f}$ has compact support not containing the origin 0, that is,

$$Z(\mathcal{A}) := \{ f \in \mathcal{A} \, : \, \mathrm{supp}(\hat{f}) \text{ is compact and } 0 \notin \mathrm{supp}(\hat{f}) \}.$$

We collect in the following lemma some useful properties of the functions in $Z(\mathcal{A})$, whose proof is found in [19], p. 246.

Lemma 3.1 (cf. [19])**.** *Let $\mathcal{A}$ be an algebra w.m.v. in $\mathbb{R}^n$ and $f \in Z(\mathcal{A})$. Then:*

(i) *There exists $u \in C^\infty(\mathbb{R}^n) \cap Z(\mathcal{A})$ such that $\Delta u = f$, where Δ is the usual Laplace operator in $\mathbb{R}^n$; $u = f * \zeta$ for certain smooth function ζ, fast decaying together with all its derivatives, satisfying $\hat{\zeta} \in C_c^\infty(\mathbb{R}^n)$ and $0 \notin \mathrm{supp}(\hat{\zeta})$.*

(ii) *For any Borelian $Q \subseteq \mathbb{R}^n$, with $|Q| > 0$, we have*

$$\lim_{t\to\infty} \frac{1}{t^n |Q|} \int_{Q_t} f(x+y)\,dx = 0, \qquad \textit{uniformly in } y \in \mathbb{R}^n. \tag{3.1}$$

The fundamental result about ergodic algebras, proved by Zhikov and Krivenko [30], is the following.

Theorem 3.1 (cf. [30])**.** *If $\mathcal{A}$ is an ergodic algebra, then $Z(\mathcal{A})$ is dense in $V(\mathcal{A})$ in the topology of the corresponding space $\mathcal{B}^2$.*

The following immediate corollary of Theorem 3.1, established in [3], will be used in Section 6 concerning the homogenization of a porous medium type equation.

Lemma 3.2 (cf. [3])**.** *Let $\mathcal{A}$ be an ergodic algebra in $\mathrm{BUC}(\mathbb{R}^n)$ and $h \in \mathcal{B}^2$ such that $M(h\Delta f) = 0$ for all $f \in \mathcal{A}$ such that $\Delta f \in \mathcal{A}$. Then h is $\mathcal{B}^2$-equivalent to a constant.*

Theorem 3.1 also motivates the following definition.

Definition 3.1. An algebra w.m.v. $\mathcal{A}$ is said to be *regular* if $Z(\mathcal{A})$ is dense in $V(\mathcal{A})$ in the topology of the sup-norm.

We have the following important fact about regular algebras w.m.v..

Proposition 3.1. *If $\mathcal{A}$ is a regular algebra w.m.v., then $\mathcal{A}$ is ergodic.*

Proof. We are going to use the characterization of ergodic algebras provided by Lemma 2.1. Let $f \in \mathcal{A}$. Clearly, to prove (2.4), we may assume $M(f) = 0$. Now, since $\mathcal{A}$ is regular, given $\varepsilon > 0$, we may find $g \in Z(\mathcal{A})$ such that $\|f - g\|_\infty < \varepsilon$. Hence,

$$\limsup_{t\to\infty} M_y\left(\Big|\frac{1}{|B(0;t)|}\int_{B(0;t)} f(x+y)\,dx\Big|^2\right) \le 2\lim_{t\to\infty} M_y\left(\Big|\frac{1}{|B(0;t)|}\int_{B(0;t)} g(x+y)\,dx\Big|^2\right) + 2\varepsilon^2 = 2\varepsilon^2,$$

where we used Lemma 3.1(ii) for the last equality. This implies (2.4). □

We next state a property of regular algebras w.m.v. which will be used in our application to homogenization of porous medium type equations on bounded domains in the final part of this paper.

Lemma 3.3. *Let $\mathcal{A}$ be a regular algebra w.m.v. If $f \in V(\mathcal{A})$, then for any $\varepsilon > 0$ there exists a function $u_\varepsilon \in Z(\mathcal{A})$ satisfying the inequalities*

$$f - \varepsilon \le \Delta u_\varepsilon \le f + \varepsilon. \tag{3.2}$$

Proof. This follows immediately from Lemma 3.1(i) and Definition 3.1. □

The space $\mathrm{FS}(\mathbb{R}^n)$ introduced in [17] provides a very encompassing example of a regular algebra w.m.v..

Definition 3.2. The Fourier-Stieltjes space, denoted by $\mathrm{FS}(\mathbb{R}^n)$, is the completion relatively to the sup-norm of the space of functions $\mathrm{FS}_*(\mathbb{R}^n)$ defined by

$$\mathrm{FS}_*(\mathbb{R}^n) := \left\{ f : \mathbb{R}^n \to \mathbb{R} \,:\, f(x) = \int_{\mathbb{R}^n} e^{ix\cdot y}\,d\nu(y) \text{ for some } \nu \in \mathcal{M}_*(\mathbb{R}^n) \right\}, \tag{3.3}$$

where by $\mathcal{M}_*(\mathbb{R}^n)$ we denote the space of complex-valued measures μ with finite total variation, i.e., $|\mu|(\mathbb{R}^n) < \infty$.

Recall that a subalgebra $\mathcal{B} \subseteq \mathcal{A}$ is called an ideal of $\mathcal{A}$ if for any $f \in \mathcal{A}$ and $g \in \mathcal{B}$ we have $fg \in \mathcal{B}$. Let $C_0(\mathbb{R}^n)$ denote the closure of $C_c^\infty(\mathbb{R}^n)$ with respect to the sup norm. The following result was established in [17].

Proposition 3.2 (cf. [17])**.** $\mathrm{FS}(\mathbb{R}^n) \subseteq \mathrm{BUC}(\mathbb{R}^n)$ *and it is an algebra w.m.v. containing* $C_0(\mathbb{R}^n)$ *as an ideal. Moreover,* $\mathrm{FS}(\mathbb{R}^n)$ *is a regular algebra w.m.v. and the space* $\mathrm{PAP}(\mathbb{R}^n)$ *of the perturbed almost periodic functions, defined as*

$$\mathrm{PAP}(\mathbb{R}^n) := \{f \in \mathrm{BUC}(\mathbb{R}^n) \,:\, f = g + \psi,\ g \in \mathrm{AP}(\mathbb{R}^n),\ \psi \in C_0(\mathbb{R}^n)\},$$

is a closed strict subalgebra of $\mathrm{FS}(\mathbb{R}^n)$*.*

4. Two-scale Young Measures

In this section we recall the theorem giving the existence of two-scale Young measures established in [3]. We begin by recalling the concept of vector-valued algebra with mean value.

Given a Banach space E and an algebra w.m.v. $\mathcal{A}$, we denote by $\mathcal{A}(\mathbb{R}^n; E)$ the space of functions $f \in \mathrm{BUC}(\mathbb{R}^n; E)$ satisfying the following:

(i) $L_f := \langle L, f\rangle$ belongs to $\mathcal{A}$ for all $L \in E^*$;
(ii) The family $\{L_f \,:\, L \in E^*,\ \|L\| \le 1\}$ is relatively compact in $\mathcal{A}$.

Theorem 4.1 (cf. [3])**.** *Let E be a Banach space, $\mathcal{A}$ an algebra w.m.v. and $\mathcal{K}$ be the compact associated with $\mathcal{A}$. There is an isometric isomorphism between $\mathcal{A}(\mathbb{R}^n; E)$ and $C(\mathcal{K}; E)$. Denoting by $g \mapsto \underline{g}$ the canonical map from $\mathcal{A}$ to $C(\mathcal{K})$, the isomorphism associates to $f \in \mathcal{A}(\mathbb{R}^n; E)$ the map $\tilde{f} \in C(\mathcal{K}; E)$ satisfying*

$$\underline{\langle L, f\rangle} = \langle L, \tilde{f}\rangle \in C(\mathcal{K}) \qquad \forall L \in E^*. \tag{4.1}$$

In particular, for each $f \in \mathcal{A}(\mathbb{R}^n; E)$, $\|f\|_E \in \mathcal{A}$.

For $1 \le p < \infty$, we define the space $L^p(\mathcal{K}; E)$ as the completion of $C(\mathcal{K}; E)$ with respect to the norm $\|\cdot\|_p$, defined as usual,

$$\|f\|_p := \left(\int_{\mathcal{K}} \|f\|_E^p \, d\mathfrak{m} \right)^{1/p}.$$

As a standard procedure, we identify functions in L^p that coincide $\mathfrak{m}$-a.e. in $\mathcal{K}$.

Similarly, we define the space $\mathcal{B}^p(\mathbb{R}^n; E)$ as the completion of $\mathcal{A}(\mathbb{R}^n; E)$ with respect to the seminorm

$$|f|_p := \left(\fint_{\mathbb{R}^n} \|f\|_E^p \, dx \right)^{1/p},$$

identifying functions in the same equivalence class determined by the seminorm $|\cdot|_p$. Clearly, the isometric isomorphism given by Theorem 4.1 extends to an isometric isomorphism between $\mathcal{B}^p(\mathbb{R}^n; E)$ and $L^p(\mathcal{K}; E)$.

The next theorem gives the existence of two-scale Young measures associated with an algebra $\mathcal{A}$. For the proof, we again refer to [3].

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and $\{u_\varepsilon(x)\}_{\varepsilon>0}$ be a family of functions in $L^\infty(\Omega; K)$, for some compact metric space K.

Theorem 4.2. *Given any infinitesimal sequence $\{\varepsilon_i\}_{i\in\mathbb{N}}$ there exist a subnet $\{u_{\varepsilon_{i(d)}}\}_{d\in D}$, indexed by a certain directed set D, and a family of probability measures on K, $\{\nu_{z,x}\}_{z\in\mathcal{K},x\in\Omega}$, weakly measurable with respect to the product of the Borel σ-algebras in $\mathcal{K}$ and $\mathbb{R}^n$, such that*

$$\lim_D \int_\Omega \Phi(\frac{x}{\varepsilon_{i(d)}}, x, u_{\varepsilon_{i(d)}}(x))\, dx = \int_\Omega \int_{\mathcal{K}} \langle \nu_{z,x}, \underline{\Phi}(z,x,\cdot)\rangle \, d\mathfrak{m}(z)\, dx \qquad \forall \Phi \in \mathcal{A}\left(\mathbb{R}^n; C_0(\Omega\times K)\right). \tag{4.2}$$

Here $\underline{\Phi} \in C\left(\mathcal{K}; C_0(\Omega\times K)\right)$ denotes the unique extension of Φ. Moreover, equality (4.2) *still holds for functions Φ in the following function spaces:*

1. $\mathcal{B}^1(\mathbb{R}^n; C_0(\Omega\times K))$;
2. $\mathcal{B}^p(\mathbb{R}^n; C(\bar\Omega\times K))$ *with* $p>1$;
3. $L^1(\Omega; \mathcal{A}(\mathbb{R}^n; C(K)))$.

As in the classical theory of Young measures we have the following consequence of Theorem 4.2.

Theorem 4.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, let $\{u_\varepsilon\} \subseteq L^\infty(\Omega; \mathbb{R}^m)$ be uniformly bounded and let $\nu_{z,x}$ be a two-scale Young measure generated by a subnet $\{u_{\varepsilon(d)}\}_{d\in D}$, according to Theorem 4.2. Assume that U belongs either to $L^1(\Omega; \mathcal{A}(\mathbb{R}^n; \mathbb{R}^m))$ or to $\mathcal{B}^p(\mathbb{R}^n; C(\bar\Omega; \mathbb{R}^m))$ for some $p>1$. Then*

$$\nu_{z,x} = \delta_{\underline{U}(z,x)} \quad \text{if and only if} \quad \lim_D \|u_{\varepsilon(d)}(x) - U(\frac{x}{\varepsilon(d)}, x)\|_{L^1(\Omega)} = 0. \tag{4.3}$$

5. Some results about a porous medium type equation

In this section, we review some results about the Cauchy problem and an initial-boundary value problem for a porous medium type equation which will be used later. More specifically, we consider the Cauchy problem

$$\partial_t u - \Delta f(x,u) = 0, \qquad (x,t) \in \mathbb{R}_+^{n+1} := \mathbb{R}^n \times (0,+\infty), \tag{5.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n, \tag{5.2}$$

and, for $\Omega \subseteq \mathbb{R}^n$ open bounded with smooth boundary, we consider the initial-boundary value problem

$$\partial_t u - \Delta f(x,u) = 0, \qquad (x,t) \in Q := \Omega \times (0,+\infty), \tag{5.3}$$

$$u(x,0) = u_0(x),\ x \in \Omega, \qquad u(x,t) = 0,\ (x,t) \in \partial\Omega \times (0,\infty). \tag{5.4}$$

For the purposes of this paper, we consider two types of functions according to the following definitions.

Definition 5.1. We say that the function $f(x,u)$ is of *type 1* if the conditions below are satisfied, where I is an arbitrary compact interval of $\mathbb{R}$:

(f1.1) $f:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ ($f:\Omega\times\mathbb{R}\to\mathbb{R}$, resp.) is continuous and, for each $x\in\mathbb{R}^n$ ($x\in\Omega$, resp.), $f(x,\cdot):\mathbb{R}\to\mathbb{R}$ is increasing and locally Lipschitz continuous uniformly in x. Moreover, $\lim_{u\to\pm\infty} f(x,u)=\pm\infty$, uniformly in x.

(f1.2) $D_x^\alpha f(x,u)$ and $D_x^\alpha f_u(x,u)$, $|\alpha|\le 2$, are uniformly bounded for $(x,u)\in\mathbb{R}^n\times I$ (resp., $\Omega\times I$), and there exists a constant $\theta_0>0$ such that

$$-f_u(x,u)+\theta_0\sum_{i=1}^n |f_{u,x_i}(x,u)|\le 0, \tag{5.5}$$

for all $(x,u)\in\mathbb{R}^n\times I$ (resp., $(x,u)\in\Omega\times I$).

Observe that assumptions **(f1.1)**, **(f1.2)** are trivially satisfied by functions of the form $f(x,u)=a(x)u|u|^{\gamma(x)}+b(x)$, with γ,a,b smooth, bounded with bounded derivatives up to second order, $\gamma(x)>\gamma_0>0$ and $a(x)>a_0>0$, $x\in\mathbb{R}^n$ ($x\in\Omega$).

Specifically for the problem (5.3),(5.4), we add the following item to definition 5.1:

(f1.3) $f(x,0)=0$ for $x\in\partial\Omega$.

We will also consider the problems (5.1),(5.2) and (5.3),(5.4) when $f(x,u)$ is of the type described in the following definition.

Definition 5.2. We say that the function $f(x,u)$ is of *type 2* if $f(x,u)=h(x)F(u)+S(x)$, where:

(f2.1) $F:\mathbb{R}\to\mathbb{R}$ is locally Lipschitz continuous, nondecreasing, and $\lim_{u\to\pm\infty}F(u)=\pm\infty$. For definiteness, we assume that F is *not* strictly increasing.

(f2.2) $S,h:\mathbb{R}^n\to\mathbb{R}$ are functions in $C^2(\mathbb{R}^n)$, bounded with bounded derivatives up to the second order, and $h(x)\ge\delta_0>0$, for all $x\in\mathbb{R}^n$.

Observe that, for F satisfying **(f2.1)** we may define $G(r)=\min\{u\,:\,F(u)=r\}$, and we have $F(G(r))=r$, for all $r\in\mathbb{R}$, and $G(F(u))=u$ if $u\notin F^{-1}(E)$, where

$$E:=\{r\in\mathbb{R}\,:\,G\text{ is discontinuous at }r\}.$$

Again, specifically for the problem (5.3),(5.4), we add the following item to Definition 5.2

(f2.3) $h(x)F(0)+S(x)=0$, for $x\in\partial\Omega$.

Remark 5.1. We remark that all the results obtained in what follows for $f(x,u)$ of type 2 have identical versions for $f(x,u)$ of the form $f(x,u)=F(h(x)u)+S(x)$, with F,h,S satisfying the conditions in **(f2.1)**, **(f2.2)**, and **(f2.3)**, the latter specifically for the problem (5.3),(5.4), the proofs of which are easy adaptations of the proofs given herein for $f(x,u)$ of type 2, after the trivial change of variables $v=h(x)u$.

In the case where $f(x,u)$ is of type 1, since $f(x,\cdot)$ is (strictly) increasing, for each x, then the equation (5.1) is only mildly degenerate, in other words, it still belongs to the "non-degenerate" class, in the classification of [5]. Nevertheless, it is degenerate in the sense that $f_u(x,\cdot)$ can vanish on a set $\mathcal{N}\subseteq\mathbb{R}$, provided $\mathcal{N}$ does not contain a non-empty open interval. The simplest and prototypical example is the classical porous medium equation, for which $f(x,u)=u|u|^\gamma$, $\gamma>0$. We remark that for the latter, due to a comparison principle, we can always guarantee that $u(x,t)\ge 0$ if $u_0(x)\ge 0$, which is physically desirable. For this reason, we can view $f(u)=u^{\gamma+1}$, $u\ge 0$, as defined in $\mathbb{R}$, trivially extended as $u|u|^\gamma$. This motivates our choice of taking $f(x,\cdot)$ as defined in the whole $\mathbb{R}$, which is a matter of convenience. On the other hand, if $f(x,u)$ is of type 2, then the equation (5.1) falls into the degenerate class in the classification of [5].

Concerning the initial data, we assume

$$u_0\in L^\infty(\mathbb{R}^n)\qquad (u_0\in L^\infty(\Omega),\text{ resp.}).$$

The study of the well-posedness of the Cauchy problem for general quasilinear degenerate parabolic equations starts with Volpert and Hudjaev [28], for initial data in BV, where the L^1-stability was achieved completely only in the isotropic case, that is, for a diagonal viscosity matrix. The results in [28] were extended to the initial boundary value problem in [29]. Well-posedness in the isotropic case with initial data in L^∞ was established by Carrillo [5] in the homogeneous case where the coefficients do not explicitly depend on (x,t). A purely L^1 well-posedness theory for the homogeneous anisotropic case was established by Chen and Perthame in [9]. The latter was extended to the non-homogeneous anisotropic case in [8]. We refer to the bibliography in the cited papers for a more complete list of references on the subject.

Equation (5.1) is a particular case of a degenerate non-homegeneous isotropic equation and, as we said above, in the case where $f(x,u)$ is of type 1, its degeneration is of a mild type which makes its study a bit simpler than that of the general degenerate equation. On the other hand, in the case where $f(x,u)$ is of type 2, equation (5.1) is a particular case of a strongly degenerate parabolic equation. Here we will review the analysis of such equations for f belonging to both types in order to introduce some notations and some particular results that will be needed in our study of the homogenization of porous medium type equations in Section 6. For the stability results, in the type 1 case, we follow closely the analysis in [5] and show which adaptations of the results in [5] need to be made in order to handle the explicit dependence on x of f. Still for the stability results, in the type 2 case, we borrow as well some ideas from [20], which in turn is also based on the analysis of [5].

For the existence of solutions, which follows from the compactness of the sequence of solutions of regularized (nondegenerate) problems, we introduce here a method which is motivated by Kruzkhov [22]. We remark that recently Panov [26] has obtained a very general compactness result that, in particular, would imply the one proved here. However the techniques used in [26] are out of the scope of the present paper and we think it is appropriate here to provide a simple and direct proof of this compactness result.

Definition 5.3. A function $u\in L^\infty(\mathbb{R}_+^{n+1})$ is said to be a weak solution of the problem (5.1),(5.2) if the following hold:

(1) $f(x,u(x,t))\in L^2_{\rm loc}((0,\infty);H^1_{\rm loc}(\mathbb{R}^n))$;
(2) For any $\varphi\in C_c^\infty(\mathbb{R}^{n+1})$, we have

$$\int_{\mathbb{R}_+^{n+1}} u\varphi_t-\nabla f(x,u)\cdot\nabla\varphi\,dx\,dt+\int_{\mathbb{R}^n}u_0\varphi(x,0)\,dx=0. \tag{5.6}$$

Similarly, a function $u\in L^\infty(Q)$ is said to be a weak solution of the problem (5.3),(5.4) if the following hold:

(3) $f(x,u(x,t))\in L^2_{\rm loc}((0,+\infty);H^1_0(\Omega))$.
(4) Given $\varphi\in C_c^\infty(\Omega\times\mathbb{R})$, we have

$$\int_Q u\,\partial_t\varphi-\nabla f(x,u)\cdot\nabla\varphi\,dx\,dt+\int_\Omega u_0(x)\varphi(x,0)\,dx=0. \tag{5.7}$$

Let u be a weak solution of either (5.1),(5.2) or (5.3),(5.4). Denoting by $\langle\cdot,\cdot\rangle$ the usual pairing between $H^{-1}(U)$ and $H^1_0(U)$ when $U\subseteq\mathbb{R}^n$ is open, we can conclude from (5.6) (resp., from (5.7)) that

$$\partial_t u\in L^2_{\rm loc}(\mathbb{R}_+;H^{-1}_{\rm loc}(\mathbb{R}^n)),\qquad (\text{resp., } \partial_t u\in L^2_{\rm loc}(\mathbb{R}_+;H^{-1}_{\rm loc}(\Omega)))$$

so that the equality (5.6) is equivalent to

$$\int_0^\infty\langle\partial_t u,\varphi\rangle\,dt+\int_{\mathbb{R}_+^{n+1}}\nabla f(x,u)\cdot\nabla\varphi\,dx\,dt-\int_{\mathbb{R}^n}u_0\varphi(x,0)\,dx=0, \tag{5.8}$$

for all $\varphi\in C_c^\infty(\mathbb{R}^{n+1})$, while (5.7) is equivalent to

$$\int_0^\infty\langle\partial_t u,\varphi\rangle\,dt+\int_Q\nabla f(x,u)\cdot\nabla\varphi\,dx\,dt-\int_\Omega u_0\varphi(x,0)\,dx=0. \tag{5.9}$$

Let $H_\delta : \mathbb{R} \to \mathbb{R}$ be the approximation of the function sgn given by

$$H_\delta(s) := \begin{cases} 1, & \text{for } s > \delta, \\ \dfrac{s}{\delta}, & \text{for } |s| \le \delta, \\ -1, & \text{for } s < -\delta. \end{cases}$$

Given a nondecreasing Lipschitz continuous function $\vartheta : \mathbb{R} \to \mathbb{R}$ and $k \in \mathbb{R}$, we define

$$B_\vartheta^k(x,\lambda) := \begin{cases} \int_k^\lambda \vartheta(f(x,r))dr, & \text{if } f \text{ is of type 1}, \\ \int_k^\lambda \vartheta(F(r))dr, & \text{if } f \text{ is of type 2}. \end{cases}$$

Concerning the function B_ϑ^k, we will make use of the following lemma which is a version of a lemma in [5], whose proof remains essentially the same and for which, therefore, we refer to [5].

Lemma 5.1. *Let $u \in L^\infty(\mathbb{R}_+^{n+1})$ be a weak solution of (5.1),(5.2). Then, for a.e. $t \in (0,+\infty)$, we have*

$$\int_0^t \int_{\mathbb{R}^n} B_\vartheta^k(x,u)\varphi_s\,ds\,dx + \int_{\mathbb{R}^n} B_\vartheta^k(x,u_0)\varphi(x,0)\,dx - \int_{\mathbb{R}^n} B_\vartheta^k(x,u(t))\varphi(x,t)\,dx$$
$$= -\int_0^t \langle \partial_s u, \vartheta(f(x,u))\varphi\rangle\,ds$$

$\forall k \in \mathbb{R}$ and for all $0 \le \varphi \in C_c^\infty(\mathbb{R}^{n+1})$.

Similarly, let $u \in L^\infty(Q)$ be a weak solution of (5.3),(5.4). Then, for a.e. $t \in (0,+\infty)$, we have

$$\int_0^t \int_\Omega B_\vartheta^k(x,u)\varphi_s\,ds\,dx + \int_\Omega B_\vartheta^k(x,u(x,0))\varphi(x,0)\,dx - \int_\Omega B_\vartheta^k(x,u(t))\varphi(x,t)\,dx$$
$$= -\int_0^t \langle \partial_s u, \vartheta(f(x,u))\varphi\rangle\,ds,$$

$\forall k \in \mathbb{R}$ and $\forall\, 0 \le \varphi \in C_c^\infty(\Omega \times \mathbb{R})$.

Let us denote

$$\vartheta_\delta^1(\lambda;y) := H_\delta(\lambda - f(y,k)),\ B_{\vartheta_\delta^1}(x,\lambda;y) := B_{\vartheta_\delta^1(\cdot;y)}(x,\lambda) \quad \text{and} \quad \vartheta_\delta^2(\lambda) := H_\delta(\lambda - F(k)).$$

Next we state and prove a lemma which is also an adaptation of a similar result in [5].

Lemma 5.2 (Entropy production term: type 1 case)**.** *Let u be a weak solution of the Cauchy problem (5.1),(5.2), with $u_0 \in L^\infty(\mathbb{R}^n)$. If f is of type 1, Then*

$$\int_{\mathbb{R}_+^{n+1}} B_{\vartheta_\delta^1}^k(x,u;y)\varphi_t - H_\delta(f(x,u) - f(y,k))\nabla f(x,u)\cdot\nabla\varphi\,dx\,dt$$
$$= \int_{\mathbb{R}_+^{n+1}} |\nabla f(x,u)|^2 H_\delta'(f(x,u) - f(y,k))\varphi\,dx\,dt, \tag{5.10}$$

for all $k \in \mathbb{R}$ and all $0 \le \varphi \in C_c^\infty(\mathbb{R}_+^{n+1})$.

Similarly, let $u \in L^\infty(Q)$ be a weak solution of (5.3),(5.4). Then,

$$\int_Q B_{\vartheta_\delta^1}^k(x,u;y)\varphi_t - H_\delta(f(x,u) - f(y,k))\nabla f(x,u)\cdot\nabla\varphi\,dx\,dt$$
$$= \int_Q |\nabla f(x,u)|^2 H_\delta'(f(x,u) - f(y,k))\varphi\,dx\,dt \tag{5.11}$$

for all $k \in \mathbb{R}$ and all $0 \le \varphi \in C_c^\infty(Q)$.

Proof. By the Lemma 5.1, we have

$$-\int_0^{+\infty}\langle \partial_t u, H_\delta(f(x,u)-f(y,k))\varphi\rangle\,dt=\int_{\mathbb{R}^{n+1}_+}B^k_{\vartheta_\delta}(x,u;y)\varphi_t\,dx\,dt.$$

Since u is a weak solution and $H_\delta(f(x,u)-f(y,k))\varphi$ is a test function for each fixed y and k, we get

$$-\int_0^{+\infty}\langle \partial_t u, H_\delta(f(x,u)-f(y,k))\varphi\rangle\,dt-\int_{\mathbb{R}^{n+1}_+}\{\nabla f(x,u)\cdot\nabla(H_\delta(f(x,u)-f(y,k))\varphi)\}\,dx\,dt=0.$$

This equality with the previous one gives

$$\int_{\mathbb{R}^{n+1}_+}\{B^k_{\vartheta^1_\delta}(x,u;y)\varphi_t-\nabla f(x,u)\cdot\nabla(H_\delta(f(x,u)-f(y,k))\varphi)\}\,dx\,dt=0,$$

and this equality yields (5.10).

The proof of (5.11) follows similarly with obvious adaptations. □

Now, we establish a result which is the analogue of Lemma 5.2 for the case where f is of type 2.

Lemma 5.3 (Entropy production term: type 2 case)**.** *Let u be a weak solution of* (5.1),(5.2) *with $u_0\in L^\infty(\mathbb{R}^n)$. If f is of type* 2*, then*

$$\int_{\mathbb{R}^{n+1}_+}\big\{|u-k|\varphi_t-\nabla|f(x,u)-f(x,k)|\cdot\nabla\varphi-\operatorname{sgn}(u-k)\Delta f(x,k)\varphi\big\}\,dx\,dt \tag{5.12}$$
$$=\lim_{\delta\to0}\int_{\mathbb{R}^{n+1}_+}h(x)|\nabla F(u)|^2H'_\delta(F(u)-F(k))\varphi\,dx\,dt,$$

for all $k\in\mathbb{R}$ such that $F(k)\notin E$ and $0\le\varphi\in C^\infty_c(\mathbb{R}^{n+1}_+)$.

Proof. Similarly to what was done in lemma 2, we have

$$\int_{\mathbb{R}^{n+1}_+}\{B^k_{\vartheta^2_\delta}(u)\varphi_t-\nabla f(x,u)\cdot\nabla(H_\delta(F(u)-F(k))\varphi)\}\,dx\,dt=0,$$

which gives

$$\int_{\mathbb{R}^{n+1}_+}\left\{B^k_{\vartheta^2_\delta}(u)\varphi_t-H_\delta(F(u)-F(k))\nabla\left(f(x,u)-f(x,k)\right)\cdot\nabla\varphi+H_\delta(F(u)-F(k))\Delta f(x,k)\,\varphi\right\}dx\,dt$$
$$=\int_{\mathbb{R}^{n+1}_+}\left\{\,(F(u)-F(k))\,\nabla h(x)\cdot\nabla F(u)+h(x)|\nabla F(u)|^2\right\}H'_\delta(F(u)-F(k))\varphi\,dx\,dt.$$

Since $F(k)\notin E$, we obtain that $H_\delta(F(u)-F(k))\to\operatorname{sgn}(u-k)$ and $B^k_{\vartheta^2_\delta}(u)\to|u-k|$ as $\delta\to0$. So, in order to obtain 5.12, it suffices to show that the first integral on the right-hand side of the expression above goes to 0 as $\delta\to0$. For this, define

$$I_\delta:=\int_{\mathbb{R}^{n+1}_+}(F(u)-F(k))\,\nabla h(x)\cdot\nabla F(u)H'_\delta(F(u)-F(k))\varphi\,dx\,dt.$$

A simple computation shows that

$$I_\delta:=\int_{\mathbb{R}^{n+1}_+}\operatorname{div}\mathcal{F}_\delta(F(u))\varphi\,dx\,dt,$$

where

$$\mathcal{F}_\delta(z) := \nabla h(x) \int_{F(k)}^{z} (r - F(k)) H'_\delta(r - F(k))\, dr.$$

Since $\lim_{\delta\to 0} \mathcal{F}_\delta(z) = 0$ for all z, we have $\lim_{\delta\to 0} I_\delta = 0$. □

Definition 5.4. A function $u \in L^\infty(\mathbb{R}^{n+1}_+)$ is an entropy solution of the problem (5.1),(5.2) if u is a weak solution and satisfies, for all $k \in \mathbb{R}$,

$$\partial_t |u - k| - \Delta |f(x,u) - f(x,k)| + \operatorname{sgn}(u-k)\, \Delta f(x,k) \le 0,$$

in $\mathcal{D}'(\mathbb{R}^{n+1}_+)$.

As it will become clear from the proof of Theorem 5.1 below, when f is of type 1, all weak solutions are automatically also entropy solutions. On the other hand, since (5.12) in Lemma 5.3 only holds for $k \in \mathbb{R}$ such that $F(k) \notin E$, the notions of weak and entropy solutions do not coincide in general when f is of type 2.

The following theorem is a central tool in our analysis of the homogenization problem for porous medium type equation in Sections 6 and 7. Its proof follows from (5.10) (resp., (5.11)), by using doubling of variables, and the trick of completing the square in [5], theorem 13, p. 339. Of particular importance for our homogenization study in Section 6 will be the formula (5.14) below, which holds in the special case when one of the entropy solutions is stationary. We give the detailed proof here for the reader's convenience.

Theorem 5.1. *Let u_1, u_2 be entropy solutions of the Cauchy problem* (5.1),(5.2) *with initial data $u_{01}, u_{02} \in L^\infty(\mathbb{R}^n)$. Then we have the following:*

(i) *For all $0 \le \varphi \in C_c^\infty(\mathbb{R}^{n+1}_+)$, we have*

$$\int_{\mathbb{R}^{n+1}_+} |u_1(x,t) - u_2(x,t)|\varphi_t - \nabla |f(x,u_1(x,t)) - f(x,u_2(x,t))| \cdot \nabla\varphi \, dx\, dt \ge 0. \tag{5.13}$$

(ii) *If u_2 is a stationary solution, then*

$$\begin{aligned} &\int_{\mathbb{R}^{n+1}_+} |u_1(x,t) - u_2(x)|\varphi_t - \nabla |f(x,u_1(x,t)) - f(x,u_2(x))| \cdot \nabla\varphi \, dx\, dt \\ &\quad = \lim_{\delta\to 0} \int_{\mathbb{R}^{n+1}_+} |\nabla [f(x,u_1(x,t)) - f(x,u_2(x))]|^2 H'_\delta(f(x,u_1(x,t)) - f(x,u_2(x)))\varphi \, dx\, dt, \end{aligned} \tag{5.14}$$

for all $0 \le \varphi \in C_c^\infty(\mathbb{R}^{n+1}_+)$.

Similarly, if u_1, u_2 are weak solutions of the initial-boundary value problem (5.3),(5.4)*, with initial data $u_{01}, u_{02} \in L^\infty(\Omega)$, then we have the following:*

(iii) *For all $0 \le \varphi \in C_c^\infty(Q)$, we have*

$$\int_Q |u_1(x,t) - u_2(x,t)|\varphi_t - \nabla |f(x,u_1(x,t)) - f(x,u_2(x,t))| \cdot \nabla\varphi \, dx\, dt \ge 0. \tag{5.15}$$

(iv) *If u_2 is a stationary solution, then*

$$\begin{aligned} &\int_Q |u_1(x,t) - u_2(x)|\varphi_t - \nabla |f(x,u_1(x,t)) - f(x,u_2(x))| \cdot \nabla\varphi \, dx\, dt \\ &\quad = \lim_{\delta\to 0} \int_Q |\nabla [f(x,u_1(x,t)) - f(x,u_2(x))]|^2 H'_\delta(f(x,u_1(x,t)) - f(x,u_2(x)))\varphi \, dx\, dt, \end{aligned} \tag{5.16}$$

for all $0 \le \varphi \in C_c^\infty(Q)$.

Proof. 1. We first prove the assertions concerning the Cauchy problem (5.1),(5.2). In what follows, we use the abridged notation $u_1 = u_1(x,t)$ and $u_2 = u_2(y,s)$. We begin by proving (5.13) in the case where f is of type 1. For this, we apply (5.10) to u_1, to obtain

$$\int_{\mathbb{R}_+^{n+1}} \{B_{\vartheta_\delta^1}^k(x,u_1;y)\phi_t - H_\delta(f(x,u_1)-f(y,k))\nabla_x f(x,u_1)\cdot\nabla_x\phi\}\,dx\,dt$$
$$= \int_{\mathbb{R}_+^{n+1}} |\nabla_x f(x,u_1)|^2 H_\delta'(f(x,u_1)-f(y,k))\phi\,dx\,dt,$$

for all $k\in\mathbb{R}$ and for all $0\le\phi\in C_c^\infty((\mathbb{R}_+^{n+1})^2)$. Setting $k=u_2$ and integrating in y,s, we obtain

$$\int_{(\mathbb{R}_+^{n+1})^2} \{B_{\vartheta_\delta^1}^{u_2}(x,u_1;y)\phi_t - H_\delta(f(x,u_1)-f(y,u_2))\nabla_x f(x,u_1)\cdot\nabla_x\phi\,dx\,dt\,dy\,ds$$
$$= \int_{(\mathbb{R}_+^{n+1})^2} |\nabla_x f(x,u_1)|^2 H_\delta'(f(x,u_1)-f(y,u_2))\phi\,dx\,dt\,dy\,ds. \tag{5.17}$$

Now, applying (5.10) to u_2, taking $k=u_1$ and integrating in x,t, we obtain

$$\int_{(\mathbb{R}_+^{n+1})^2} \Big\{B_{\vartheta_\delta^1}^{u_1}(y,u_2;x)\phi_s + H_\delta(f(x,u_1)-f(y,u_2))\nabla_y f(y,u_2)\cdot\nabla_y\phi\Big\}\,dx\,dt\,dy\,ds$$
$$= \int_{(\mathbb{R}_+^{n+1})^2} |\nabla_y f(y,u_2)|^2 H_\delta'(f(x,u_1)-f(y,u_2))\phi\,dx\,dt\,dy\,ds \tag{5.18}$$

Now, note that

$$0 = \int_{\mathbb{R}_+^{n+1}} \nabla_y f(y,u_2)\cdot\nabla_x[H_\delta(f(x,u_1)-f(y,u_2))\phi]\,dx\,dt$$
$$= \int_{\mathbb{R}_+^{n+1}} \Big\{\nabla_y f(y,u_2)\cdot\nabla_x f(x,u_1)H_\delta'(f(x,u_1)-f(y,u_2))\phi + H_\delta(f(x,u_1)-f(y,u_2))\nabla_y f(y,u_2)\cdot\nabla_x\phi\Big\}\,dx\,dt$$

and so we have

$$\int_{(\mathbb{R}_+^{n+1})^2} H_\delta(f(x,u_1)-f(y,u_2))\nabla_y f(y,u_2)\cdot\nabla_x\phi\,dx\,dt\,dy\,ds$$
$$= -\int_{(\mathbb{R}_+^{n+1})^2} \nabla_y f(y,u_2)\cdot\nabla_x f(x,u_1)H_\delta'(f(x,u_1)-f(y,u_2))\phi\,dx\,dt\,dy\,ds \tag{5.19}$$

Analogously,

$$\int_{(\mathbb{R}_+^{n+1})^2} H_\delta(f(x,u_1)-f(y,u_2))\nabla_x f(x,u_1)\cdot\nabla_y\phi\,dx\,dt\,dy\,ds$$
$$= \int_{(\mathbb{R}_+^{n+1})^2} \nabla_y f(y,u_2)\cdot\nabla_x f(x,u_1)H_\delta'(f(x,u_1)-f(y,u_2))\phi\,dx\,dt\,dy\,ds \tag{5.20}$$

Making (5.17) minus (5.20) yields

$$\int_{(\mathbb{R}_+^{n+1})^2} \Big\{B_{\vartheta_\delta^1}^{u_2}(x,u_1;y)\phi_t - H_\delta(f(x,u_1)-f(y,u_2))\nabla_x f(x,u_1)\cdot(\nabla_x+\nabla_y)\phi\Big\}\,dx\,dt\,dy\,ds$$
$$= \int_{(\mathbb{R}_+^{n+1})^2} \Big\{|\nabla_x f(x,u_1)|^2 - \nabla_x f(x,u_1)\cdot\nabla_y f(y,u_2)\Big\}H_\delta'(f(x,u_1)-f(y,u_2))\phi\,dx\,dt\,dy\,ds \tag{5.21}$$

Further, adding (5.18) and (5.19) gives

$$\begin{aligned}&\int_{(\mathbb{R}^{n+1}_+)^2}\Big\{B^{u_1}_{\vartheta^1_\delta}(y,u_2;x)\phi_s+H_\delta(f(x,u_1)-f(y,u_2))\nabla_y f(y,u_2)\cdot(\nabla_x+\nabla_y)\phi\Big\}\,dx\,dt\,dy\,ds\\ &=\int_{(\mathbb{R}^{n+1}_+)^2}\Big\{|\nabla_y f(y,u_2)|^2-\nabla_x f(x,u_1)\cdot\nabla_y f(y,u_2)\Big\}H'_\delta(f(x,u_1)-f(y,u_2))\phi\,dx\,dt\,dy\,ds.\end{aligned}\tag{5.22}$$

Now, adding (5.21) and (5.22) we obtain

$$\begin{aligned}&\int_{(\mathbb{R}^{n+1}_+)^2}\Big\{B^{u_2}_{\vartheta^1_\delta}(x,u_1;y)\phi_t+B^{u_1}_{\vartheta^1_\delta}(y,u_2;x)\phi_s\\ &-H_\delta(f(x,u_1)-f(y,u_2))(\nabla_x+\nabla_y)(f(x,u_1)-f(y,u_2))\cdot(\nabla_x+\nabla_y)\phi\Big\}\,dx\,dt\,dy\,ds\\ &=+\int_{(\mathbb{R}^{n+1}_+)^2}|(\nabla_x+\nabla_y)(f(x,u_1)-f(y,u_2))|^2H'_\delta(f(x,u_1)-f(y,u_2))\phi\,dx\,dt\,dy\,ds.\end{aligned}\tag{5.23}$$

We then use test functions as $\phi(x,t,y,s):=\varphi(\frac{x+y}{2},\frac{t+s}{2})\rho_k(\frac{x-y}{2})\theta_l(\frac{t-s}{2})$, where $0\le\varphi\in C^\infty_c(\mathbb{R}^{n+1}_+)$, and ρ_k, θ_l are classical approximations of the identity in $\mathbb{R}^n$ and $\mathbb{R}$, respectively, as in the doubling of variables method. Hence, letting $k\to\infty$ first, later $\delta\to 0$ and then letting $l\to\infty$, we obtain (5.13) for f of type 1.

2. Now, assume that the function f is of type 2 and define the sets

$$E_1:=\big\{(x,t)\in\mathbb{R}^{n+1}_+ : F(u_1(x,t))\in E\big\}\ \text{ and }\ E_2:=\big\{(y,s)\in\mathbb{R}^{n+1}_+ : F(u_2(y,s))\in E\big\}.$$

Observe that

$$\operatorname{sgn}(u_1-u_2)=\operatorname{sgn}(F(u_1)-F(u_2)),\tag{5.24}$$

for all $(x,t,y,s)\in\big\{\big(\mathbb{R}^{n+1}_+\setminus E_1\big)\times\mathbb{R}^{n+1}_+\big\}\cup\big\{\mathbb{R}^{n+1}_+\times\big(\mathbb{R}^{n+1}_+\setminus E_2\big)\big\}$. Moreover,

$$\nabla_x F(u_1)=0,\qquad \text{a.e. in } E_1,\tag{5.25}$$

$$\nabla_y F(u_2)=0,\qquad \text{a.e. in } E_2.\tag{5.26}$$

Let ϕ be as in step 1. Using the definition 5.4, taking $k=u_2$ and integrating over E_2, we get

$$\int_{\mathbb{R}^{n+1}_+\times E_2}\big\{|u_1-u_2|\phi_t-\nabla_x|f(x,u_1)-f(x,u_2)|\cdot\nabla_x\phi-\operatorname{sgn}(u_1-u_2)(\Delta f)(x,u_2)\phi\big\}\,dx\,dt\ge 0,\tag{5.27}$$

where $(\Delta f)(x,u):=\sum_{i=1}^n f_{x_ix_i}(x,u)$. Now, by applying Lemma 5.3 for u_1, taking $k=u_2(y,s)$ such that $(y,s)\notin E_2$, integrating over $\mathbb{R}^{n+1}_+\setminus E_2$ and adding to (5.27), we have

$$\begin{aligned}&\int_{(\mathbb{R}^{n+1}_+)^2}\Big\{|u_1-u_2|\phi_t-|F(u_1)-F(u_2)|(\nabla h)(x)\cdot\nabla_x\phi-h(x)\nabla_x|F(u_1)-F(u_2)|\cdot\nabla_x\phi\\ &\qquad-\operatorname{sgn}(u_1-u_2)(\Delta f)(x,u_2)\phi\Big\}\,dx\,dt\,dy\,ds\\ &\ge\lim_{\delta\to 0}\int_{\left(\mathbb{R}^{n+1}_+\setminus E_1\right)\times\left(\mathbb{R}^{n+1}_+\setminus E_2\right)}h(x)|\nabla_x F(u_1)|^2H'_\delta(F(u_1)-F(u_2))\phi\,dx\,dt\,dy\,ds.\end{aligned}\tag{5.28}$$

By arguing in a similar way for u_2 we can prove that

$$
\begin{aligned}
&\int_{(\mathbb{R}^{n+1}_+)^2} \Big\{ |u_1-u_2|\phi_s - |F(u_1)-F(u_2)|(\nabla h)(y)\cdot\nabla_y\phi - h(y)\nabla_y|F(u_1)-F(u_2)|\cdot\nabla_y\phi \\
&\qquad + \operatorname{sgn}(u_1-u_2)(\Delta f)(y,u_1)\phi \Big\}\,dx\,dt\,dy\,ds \\
&\geq \lim_{\delta\to 0}\int_{\left(\mathbb{R}^{n+1}_+\setminus E_1\right)\times\left(\mathbb{R}^{n+1}_+\setminus E_2\right)} h(y)|\nabla_y F(u_2)|^2 H'_\delta(F(u_1)-F(u_2))\phi\,dx\,dt\,dy\,ds.
\end{aligned}\tag{5.29}
$$

3. Since

$$
0=\int_{\mathbb{R}^{n+1}_+} h(y)\nabla_y F(u_2)\cdot\nabla_x\left(H_\delta(F(u_1)-F(u_2))\phi\right)\,dx\,dt,
$$

we obtain, taking into account (5.24)–(5.26)

$$
\begin{aligned}
&\int_{(\mathbb{R}^{n+1}_+)^2} h(y)\nabla_y|F(u_1)-F(u_2)|\cdot\nabla_x\phi\,dx\,dt\,dy\,ds \\
&= \lim_{\delta\to 0}\int_{\left(\mathbb{R}^{n+1}_+\setminus E_1\right)\times\left(\mathbb{R}^{n+1}_+\setminus E_2\right)} h(y)\nabla_y F(u_2)\cdot\nabla_x F(u_2) H'_\delta(F(u_1)-F(u_2))\phi\,dx\,dt\,dy\,ds.
\end{aligned}\tag{5.30}
$$

Analogously,

$$
\begin{aligned}
&\int_{(\mathbb{R}^{n+1}_+)^2} h(x)\nabla_x|F(u_1)-F(u_2)|\cdot\nabla_y\phi\,dx\,dt\,dy\,ds \\
&= \lim_{\delta\to 0}\int_{\left(\mathbb{R}^{n+1}_+\setminus E_1\right)\times\left(\mathbb{R}^{n+1}_+\setminus E_2\right)} h(x)\nabla_y F(u_2)\cdot\nabla_x F(u_2) H'_\delta(F(u_1)-F(u_2))\phi\,dx\,dt\,dy\,ds.
\end{aligned}\tag{5.31}
$$

4. Multiplying (5.31) by -1 and adding to (5.28), we get

$$
\begin{aligned}
&\int_{(\mathbb{R}^{n+1}_+)^2} \Big\{ |u_1-u_2|\phi_t - |F(u_1)-F(u_2)|(\nabla h)(x)\cdot\nabla_x\phi - h(x)\nabla_x|F(u_1)-F(u_2)|\cdot\left(\nabla_x+\nabla_y\right)\phi \\
&\qquad - \operatorname{sgn}(u_1-u_2)(\Delta f)(x,u_2)\phi \Big\}\,dx\,dt\,dy\,ds \\
&\geq \lim_{\delta\to 0}\int_{\left(\mathbb{R}^{n+1}_+\setminus E_1\right)\times\left(\mathbb{R}^{n+1}_+\setminus E_2\right)} \Big\{ h(x)|\nabla_x F(u_1)|^2 \\
&\qquad - h(x)\nabla_x F(u_1)\cdot\nabla_y F(u_2)\Big\} H'_\delta(F(u_1)-F(u_2))\phi\,dx\,dt\,dy\,ds.
\end{aligned}\tag{5.32}
$$

Similarly with respect to (5.30) and (5.29),

$$\begin{aligned}\int_{(\mathbb{R}_+^{n+1})^2} \Big\{ &|u_1-u_2|\phi_s - |F(u_1)-F(u_2)|(\nabla h)(y)\cdot\nabla_y\phi - h(y)\nabla_y|F(u_1)-F(u_2)|\cdot(\nabla_x+\nabla_y)\,\phi \\ &+ \operatorname{sgn}(u_1-u_2)(\Delta f)(y,u_1)\phi\Big\}\,dx\,dt\,dy\,ds \\ &\geq \lim_{\delta\to 0}\int_{(\mathbb{R}_+^{n+1}\setminus E_1)\times(\mathbb{R}_+^{n+1}\setminus E_2)}\Big\{h(y)|\nabla_y F(u_2)|^2 \\ &\quad - h(y)\nabla_x F(u_1)\cdot\nabla_y F(u_2)\Big\}H'_\delta(F(u_1)-F(u_2))\phi\,dx\,dt\,dy\,ds. \end{aligned} \tag{5.33}$$

Finally, adding the last two inequalities yields

$$\begin{aligned}\int_{(\mathbb{R}_+^{n+1})^2} \Big\{ &|u_1-u_2|\,(\phi_t+\phi_s) - |F(u_1)-F(u_2)|\Big((\nabla h)(x)\cdot\nabla_x\phi + (\nabla h)(y)\cdot\nabla_y\phi\Big) \\ &- \Big(h(x)\,\nabla_x|F(u_1)-F(u_2)| + h(y)\,\nabla_y|F(u_1)-F(u_2)|\Big)\cdot\big(\nabla_x+\nabla_y\big)\phi \\ &- \operatorname{sgn}(u_1-u_2)\Big((\Delta f)(x,u_2)-(\Delta f)(y,u_1)\Big)\phi\Big\}\,dx\,dt\,dy\,ds \\ &\geq \lim_{\delta\to 0}\int_{(\mathbb{R}_+^{n+1}\setminus E_1)\times(\mathbb{R}_+^{n+1}\setminus E_2)}\Big\{\big|h(x)\,\nabla_x F(u_1) - h(y)\,\nabla_y F(u_2)\big|^2 \\ &\quad + \big(h(x)-h(y)\big)^2\,\nabla_x F(u_1)\cdot\nabla_y F(u_2)\Big\}H'_\delta(F(u_1)-F(u_2))\phi\,dx\,dt\,dy\,ds, \end{aligned}$$

which is equivalent to

$$\begin{aligned}\int_{(\mathbb{R}_+^{n+1})^2} \Big\{ &|u_1-u_2|\,(\phi_t+\phi_s) - \big(\nabla_x+\nabla_y\big)|f(y,u_1)-f(y,u_2)|\cdot\big(\nabla_x+\nabla_y\big)\phi \\ &- \operatorname{sgn}(u_1-u_2)\Big((\Delta f)(x,u_2)-(\Delta f)(y,u_1)\Big)\phi\Big\}\,dx\,dt\,dy\,ds \\ \geq \lim_{\delta\to 0}\int_{(\mathbb{R}_+^{n+1})^2} &\Big\{\big(h(x)-h(y)\big)^2\,\nabla_x F(u_1)\cdot\nabla_y F(u_2) H'_\delta(F(u_1)-F(u_2))\phi \\ &- \big(h(x)-h(y)\big)\,\nabla_x|F(u_1)-F(u_2)|\cdot\big(\nabla_x+\nabla_y\big)\phi \\ &+ |F(u_1)-F(u_2)|\,\Big((\nabla h)(x)-(\nabla h)(y)\Big)\cdot\nabla_x\phi\Big\}\,dx\,dt\,dy\,ds \\ = \lim_{\delta\to 0}&\big(I_1^\delta + I_2 + I_3\big). \end{aligned} \tag{5.34}$$

Now, observe that

$$
\begin{aligned}
I_1^\delta &= \int_{(\mathbb{R}_+^{n+1})^2} \big(h(x)-h(y)\big)^2 \nabla_x F(u_1)\cdot\nabla_y F(u_2) H_\delta'(F(u_1)-F(u_2))\phi\,dx\,dt\,dy\,ds\\
&= \int_{(\mathbb{R}_+^{n+1})^2} \big(h(x)-h(y)\big)^2 \nabla_y F(u_2)\cdot\nabla_x\big(H_\delta(F(u_1)-F(u_2))\big)\phi\,dx\,dt\,dy\,ds\\
&= -\int_{(\mathbb{R}_+^{n+1})^2} H_\delta(F(u_1)-F(u_2))\nabla_y F(u_2)\cdot\Big(\nabla_x\phi\,\big(h(x)-h(y)\big)^2 + 2(\nabla h)(x)\,\big(h(x)-h(y)\big)\phi\Big)\,dx\,dt\,dy\,ds\\
&\le C\int_{(\mathbb{R}_+^{n+1})^2} |\nabla_y F(u_2)|\,|x-y|\Big(|x-y|\,|\nabla_x\phi| + 2|(\nabla h)(x)|\phi\Big)\,dx\,dt\,dy\,ds.
\end{aligned}
$$

Taking $\phi(x,t,y,s) := \varphi(\frac{x+y}{2},\frac{t+s}{2})\rho_k(\frac{x-y}{2})\theta_l(\frac{t-s}{2})$ as in the step 1, the previous inequality shows that $I_1^\delta \to 0$ when $k\to\infty$ uniformly in δ. Similarly, we can prove that $I_2\to 0$ as $k\to\infty$. Moreover,

$$
I_3 = -\int_{(\mathbb{R}_+^{n+1})^2} \Big\{\nabla_x|F(u_1)-F(u_2)|\cdot\big((\nabla h)(x)-(\nabla h)(y)\big)\phi + |F(u_1)-F(u_2)|\,(\Delta h)(x)\phi\Big\}\,dx\,dt\,dy\,ds,
$$

where, like above, the first integral goes to 0 as $k\to\infty$ and it is easy to check that the second one goes to

$$
-\int_{\mathbb{R}_+^{n+1}} \operatorname{sgn}(u_1-u_2)\Big((\Delta f)(x,u_2(x,t)) - (\Delta f)(x,u_1(x,t))\Big)\varphi(x,t)\,dx\,dt,
$$

as $k,l\to\infty$. Finally, using this facts and taking $k,l\to\infty$ in (5.34), we obtain (5.13) for f of type 2.

4. To obtain (5.14), we observe that if u_2 is stationary solution then $B_{\vartheta_\delta}^{u_1}(y,u_2;x)$ and $B_{\vartheta_\delta}^{u_2}(x,u_1;y)$ are independent of s and so, we can write the trivial equality where both members are null

$$
\int_{(\mathbb{R}_+^{n+1})^2} B_{\vartheta_\delta}^{u_1}(y,u_2;x)\phi_s\,dx\,dt\,dy\,ds = \int_{(\mathbb{R}_+^{n+1})^2} B_{\vartheta_\delta}^{u_2}(x,u_1;y)\phi_s\,dx\,dt\,dy\,ds
$$

Combining the previous equality in (5.23), we have

$$
\begin{aligned}
&\int_{(\mathbb{R}_+^{n+1})^2} \Big\{B_{\vartheta_\delta}^{u_2}(x,u_1;y)(\phi_t+\phi_s)\\
&- H_\delta(f(x,u_1)-f(y,u_2))(\nabla_x+\nabla_y)(f(x,u_1)-f(y,u_2))\cdot(\nabla_x+\nabla_y)\phi\Big\}\,dx\,dy\,dt\,ds\\
&= \int_{(\mathbb{R}_+^{n+1})^2} |(\nabla_x+\nabla_y)(f(x,u_1)-f(y,u_2))|^2 H_\delta'(f(x,u_1)-f(y,u_2))\phi\,dx\,dt\,dy\,ds.
\end{aligned}
$$

Now, using test functions as above and letting $k,l\to\infty$, we get (5.14).

The relations (5.15) and (5.16) concerning problem (5.3),(5.4) are proved in an entirely similar way.

□

Remark 5.2. As usual, we denote $(s)_\pm := \max\{\pm s, 0\}$. The same arguments in the above proof lead to an inequality similar to (5.13) (resp., (5.15)) with $|u_1-u_2|$, $|f(x,u_1)-f(x,u_2)|$ replaced by $(u_1-u_2)_\pm$, $(f(x,u_1)-f(x,u_2))_\pm$, respectively, just by using $B^k_{(\vartheta_\delta)_\pm}$, $(H_\delta)_\pm$, instead of $B^k_{\vartheta_\delta}$, H_δ, respectively. We thus obtain

$$
\int_{\mathbb{R}_+^{n+1}} (u_1(x,t)-u_2(x,t))_\pm\varphi_t - \nabla(f(x,u_1(x,t))-f(x,u_2(x,t)))_\pm\cdot\nabla\varphi\,dx\,dt \ge 0. \tag{5.35}
$$

in the case of problem (5.1),(5.2), and

$$
\int_Q (u_1(x,t)-u_2(x,t))_\pm\varphi_t - \nabla(f(x,u_1(x,t))-f(x,u_2(x,t)))_\pm\cdot\nabla\varphi\,dx\,dt \ge 0, \tag{5.36}
$$

in the case of problem (5.3),(5.4), where we mean one inequality holding with $(\cdot)_+$ and another holding for $(\cdot)_-$. Moreover, in the latter case, to obtain (5.15) and (5.36) we only need that $u_i \in L^\infty(Q)$ satisfies (5.7) and $f(x,u_i(x,t)) \in L^2_{\rm loc}((0,\infty);H^1(\Omega))$ instead of $f(x,u_i(x,t)) \in L^2_{\rm loc}((0,\infty);H^1_0(\Omega))$, $i=1,2$, as can be easily checked.

Concerning the Cauchy problem (5.1),(5.2), we now consider the following weight function $\Lambda:\mathbb{R}^n\to\mathbb{R}$ defined by

$$\Lambda(x) := e^{-\sqrt{1+|x|^2}}. \tag{5.37}$$

The relevance of the weight function Λ for our purposes is that

$$|D_i\Lambda(x)|\le \Lambda(x), \quad \text{for } i=1,\dots,n, \quad \text{and} \quad |\Delta\Lambda(x)|\le (n+1)\Lambda(x), \quad \text{for } x\in\mathbb{R}^n. \tag{5.38}$$

Concerning the initial-boundary value problem (5.3),(5.4), we let $\xi\in H^1_0(\Omega)$ be the eigenfunction of $-\Delta$ associated with the eigenvalue $\lambda_1>0$ such that $\xi>0$ in Ω (see, e.g., [16]).

Theorem 5.2 (Uniqueness)**.** *Let u_1,u_2 be entropy solutions of the Cauchy problem* (5.1),(5.2) *with initial data $u_{01},u_{02}\in L^\infty(\mathbb{R}^n)$. Then, there exists $C>0$ such that for a.e. $t>0$, we have*

$$\int_{\mathbb{R}^n}|u_1(t)-u_2(t)|\Lambda(x)\,dx \le e^{Ct}\int_{\mathbb{R}^n}|u_{01}(x)-u_{02}(x)|\Lambda(x)\,dx. \tag{5.39}$$

Similarly, let u_1,u_2 be entropy solutions of the initial-boundary value problem (5.3),(5.4) *with initial data $u_{01},u_{02}\in L^\infty(\Omega)$. Then, there exists $C>0$ such that for a.e. $t>0$, we have*

$$\int_{\Omega}|u_1(t)-u_2(t)|\xi(x)\,dx \le e^{Ct}\int_{\Omega}|u_{01}(x)-u_{02}(x)|\xi(x)\,dx. \tag{5.40}$$

Proof. Taking $\varphi(x,t)=\delta_h(t)\Lambda(x)$, with $0\le\delta_h\in C_c^\infty((0,+\infty))$ in (i) of Theorem (5.1), we obtain

$$\int_{\mathbb{R}^{n+1}_+}\Big\{-|u_1-u_2|\delta_h'(t)\Lambda(x)-|f(x,u_1)-f(x,u_2)|\delta_h(t)\Delta\Lambda\Big\}\,dx\,dt\le 0.$$

Observe that

$$\begin{aligned}-\int_{\mathbb{R}^{n+1}_+}|u_1-u_2|\delta_h'(t)\Lambda(x)\,dx\,dt &\le \int_{\mathbb{R}^{n+1}_+}\Big\{|f(x,u_1)-f(x,u_2)|\delta_h(t)|\Delta\Lambda|\Big\}\,dx\,dt\\ &\le C\int_{\mathbb{R}^{n+1}_+}|u_1-u_2|\delta_h(t)\Lambda(x)\,dx\,dt,\end{aligned}$$

where we use that $|\Delta\Lambda|\le(n+1)\Lambda$ and the Lipschitz condition on $f(x,u)$. We define

$$\beta(s):=\int_{\mathbb{R}^n}|u_1(x,s)-u_2(x,s)|\Lambda(x)\,dx.$$

Then, using a suitable sequence of functions δ_h and letting $h\to0$, we arrive at

$$\beta(t)\le\int_{\mathbb{R}^n}|u_{01}(x)-u_{02}(x)|\Lambda(x)\,dx + C\int_0^t\beta(s)\,ds.$$

Hence, we may apply Gronwall's lemma to conclude the proof of (5.39).

The proof of (5.40) is entirely similar starting now by taking $\varphi(x,t)=\delta_h(t)\xi(x)$ in (iii) of Theorem 5.1. □

Remark 5.3. Noting that $(f(x,u_1)-f(x,u_2))_\pm\le C(u_1-u_2)_\pm$, respectively, and using Remark 5.2 we see that the same arguments show that

$$\int_{\mathbb{R}^n}(u_1(t)-u_2(t))_\pm\Lambda(x)\,dx\le e^{Ct}\int_{\mathbb{R}^n}(u_{01}(x)-u_{02}(x))_\pm\Lambda\,dx \tag{5.41}$$

and

$$\int_\Omega (u_1(t)-u_2(t))_\pm \xi(x)\,dx \le e^{Ct}\int_\Omega (u_{01}(x)-u_{02}(x))_\pm \xi(x)\,dx \tag{5.42}$$

for a.e. $t>0$ for weak solutions of problems (5.1),(5.2) and (5.3),(5.4), respectively. Moreover, as a consequence of Remark 5.2, for the problem (5.3),(5.4), to obtain (5.42) we only need that $u_i\in L^\infty(Q)$ satisfies (5.7) and $f(x,u_i(x,t))\in L^2_{\text{loc}}((0,\infty);H^1(\Omega))$ instead of $f(x,u_i(x,t))\in L^2_{\text{loc}}((0,\infty);H_0^1(\Omega))$, $i=1,2$, provided

$$(f(x,u_1(x,t))-f(x,u_2(x,t)))_\pm|\partial\Omega\equiv 0, \qquad a.e.\ t\in(0,\infty), \quad \text{respectively},$$

the latter meaning the trace for functions in $H^1(\Omega)$.

The above remark immediately implies the following result.

Corollary 5.1 (Monotonicity). *Let u_1,u_2 to be as in the Theorem 5.2. Suppose that $u_{01}\le u_{02}$ a.e. in $\mathbb{R}^n$ (resp., in Ω). Then,*

$$u_1\le u_2, \qquad a.e.\ in\ \mathbb{R}^n\ (resp.,\ a.e.\ in\ \Omega).$$

Remark 5.4. We remark that so far we have only used that $f(x,u)$ satisfies **(f1.1)**, when it is of type 1, or **(f2.1)**, when it is of type 2. In particular, for the stability and monotonicity results it suffices **(f1.1)**, for f of type 1, and **(f2.1)**, for f of type 2. The assumptions **(f1.2)** and **(f1.3)**, the latter only for bounded domains, for f of type 1, or **(f2.2)** and **(f2.3)**, for f of type 2, will only be needed for the subsequent discussion on the existence of solutions.

Our next goal is to prove the existence of an entropy solution for (5.1),(5.2) and for (5.3),(5.4).

Before we begin properly the discussion about the existence question, we state a well known result on the compactness in the space L^1, which will be needed. The proof, which we omit here, follows in a standard way by mollification and application of Arzela-Ascoli theorem.

Lemma 5.4. *Let $U\subseteq\mathbb{R}^n$ be an open set and $\mathbb{F}\subseteq L^1_{loc}(U)$ be a family uniformly bounded in $L^1(B)$, for any closed ball $B\subseteq U$. Suppose that for any $\sigma>0$ there exists $\delta>0$ such that for $|y|<\delta$ we have*

$$\int_B |u(x+y)-u(x)|\,dx<\sigma, \quad \forall\, u\in\mathbb{F}.$$

Then, $\mathbb{F}$ is relatively compact in $L^1_{loc}(U)$.

We consider the following regularized version of (5.1),(5.2),

$$\partial_t u-\Delta f^\sigma(x,u)=0, \qquad (x,t)\in\mathbb{R}^{n+1}_+, \tag{5.43}$$

$$u(x,0)=u_0(x), \qquad x\in\mathbb{R}^n, \tag{5.44}$$

where $f^\sigma(x,u):=f(x,u)+\sigma u$ and, for the moment, we assume

$$u_0\in W^{2,\infty}(\mathbb{R}^n). \tag{5.45}$$

Similarly, we also consider the regularized version of (5.3),(5.4),

$$\partial_t u-\Delta f^\sigma(x,u)=0, \qquad (x,t)\in Q, \tag{5.46}$$

$$u(x,0)=u_0(x), \qquad x\in\Omega, \qquad u|\partial\Omega\times(0,\infty)=0, \tag{5.47}$$

where f^σ is as above and, again, for the moment, we assume

$$u_0\in W_0^{2,\infty}(\Omega). \tag{5.48}$$

The existence and uniqueness of a classical solution of (5.43),(5.44) (resp., (5.46),(5.47)) for $\sigma>0$ having bounded derivatives is proved, for example, in [24].

Motivated by [22] we now establish the following result.

Theorem 5.3. *Let u_σ be the solution of the regularized problem* (5.43),(5.44). *Then, for $|y| < \delta < 1$ and $t \in [0,T]$, we have*

$$\int_{\mathbb{R}^n} |u_\sigma(x+y,t) - u_\sigma(x,t)|\Lambda(x)\,dx \le c_0\delta, \tag{5.49}$$

where the constant c_0 is independent of σ. Moreover, for some constant $M > 0$ independent of σ, we have

$$\int_{\mathbb{R}^n} |u_\sigma(x,t+s) - u_\sigma(x,t)|\Lambda(x)\,dx \le \min_{0<\delta<1}\left\{(2c_0 + \|\nabla\Lambda\|_1)\delta + s\,M\left(\frac{1}{\delta^2} + \frac{2}{\delta} + 1\right)\|\Lambda\|_1\right\} \overset{s\to 0}{\longrightarrow} 0. \tag{5.50}$$

Similarly, if u_σ is the solution of (5.46),(5.47), *then for any $0 \le \varphi \in C_c^\infty(\Omega)$ and $|y| < \delta$, with δ sufficiently small,*

$$\int_{\Omega} |u_\sigma(x+y,t) - u_\sigma(x,t)|\varphi(x)\,dx \le c_1\delta, \tag{5.51}$$

where the constant c_1 is independent of σ. Moreover, for some constant $M > 0$ independent of σ, we have

$$\int_{\Omega} |u_\sigma(x,t+s) - u_\sigma(x,t)|\varphi(x)\,dx \le \min_{0<\delta<1}\left\{(2c_1 + \|\nabla\Lambda\|_1)\delta + s\,M\left(\frac{1}{\delta^2} + \frac{2}{\delta} + 1\right)\|\varphi\|_1\right\} \overset{s\to 0}{\longrightarrow} 0. \tag{5.52}$$

Proof. 1. To deduce (5.49), for each $k = 1,\cdots,n$ define $v^k := \partial_{x_k}u_\sigma$ and observe that

$$\partial_t v^k - \Delta(f_u^\sigma(x,u)v^k) - \nabla\cdot(f_{x_k u}^\sigma(x,u)\nabla u) - \big(\nabla f_{x_k u}^\sigma\big)(x,u)\cdot\nabla u = -\big(\Delta f^\sigma\big)(x,u), \tag{5.53}$$

where, for simplicity of notation, we denote u_σ by u, $\big(f_{x_1 x_k u}^\sigma(x,u),\cdots,f_{x_n x_k u}^\sigma(x,u)\big)$ by $\big(\nabla f_{x_k u}^\sigma\big)(x,u)$ and $\sum_{i=1}^n f_{x_i x_i}^\sigma(x,u)$ by $\big(\Delta f^\sigma\big)(x,u)$.

We fix a number $T > 0$ and let $g^k \in C^\infty([0,T]\times\mathbb{R}^n)$ be such that $g^k(t) \in C_c^\infty(\mathbb{R}^n)$ for all $t \in [0,T]$. Now, taking $0 < t_0 \le T$, multiplying the equation (5.53) by g^k, integrating by parts and summing over k from 1 to n, we get

$$\begin{aligned}
&\int_0^{t_0}\int_{\mathbb{R}^n} -\sum_{k=1}^n\left\{\partial_t g^k + f_u^\sigma(x,u)\Delta g^k - \sum_{i=1}^n\big(f_{x_i u}^\sigma(x,u)g_{x_k}^i - f_{x_i x_k u}^\sigma(x,u)g^i\big)\right\}v^k\,dx\,dt\\
&\quad + \int_{\mathbb{R}^n}\sum_{k=1}^n v^k(t_0)g^k(t_0)\,dx = \int_{\mathbb{R}^n}\sum_{k=1}^n\left\{v^k(0)g^k(0) - (\Delta f)(x,u)g^k(t_0)\right\}dx.
\end{aligned} \tag{5.54}$$

For $k = 1,\cdots,n$ and $g = (g^1,\cdots,g^n)$, we define

$$\mathcal{L}_k(g) := \partial_t g^k + f_u^\sigma(x,u)\Delta g^k - \sum_{i=1}^n\big(g_{x_k}^i f_{x_i u}^\sigma(x,u) - f_{x_i x_k u}^\sigma(x,u)g^i\big). \tag{5.55}$$

Let φ_h^k, $k = 1,\cdots,n$ be the solution of the Cauchy problem

$$\begin{cases}\mathcal{L}_k(\varphi_h) = 0, & (x,t) \in \mathbb{R}^n\times(0,t_0),\\ \varphi_h^k(t_0) = \operatorname{sgn}(v^k(t_0)) * \rho_h\,e^{-|x|}, & x \in \mathbb{R}^n,\end{cases} \tag{5.56}$$

where $\rho_h = h^{-n}\rho(h^{-1}x)$, and $0 \le \rho \in C_c(\mathbb{R}^n)$ is a standard symmetric mollifier satisfying $\operatorname{supp}\rho \subseteq \{x : |x| \le 1\}$ and $\int_{\mathbb{R}^n}\rho\,dx = 1$.

Now, observe that

$$\begin{aligned}
0 = 2\mathcal{L}_k(\varphi_h)\varphi_h^k &= \partial_t(\varphi_h^k)^2 + f_u^\sigma(x,u)\Delta(\varphi_h^k)^2 - 2f_u^\sigma(x,u)|\nabla\varphi_h^k|^2\\
&\quad - 2\sum_{i=1}^n f_{x_i u}^\sigma(x,u)\varphi_{h,x_k}^i\varphi_h^k + 2\sum_{i=i}^n f_{x_i x_k u}^\sigma(x,u)\varphi_h^i\varphi_h^k
\end{aligned}$$

Summing over k, using the Cauchy inequality with δ, the fact that $f^\sigma_{x_j u}(x,u)=f_{x_j u}(x,u)$ and (5.5), we have

$$
\begin{aligned}
0 &\le \partial_t|\varphi_h|^2+f^\sigma_u(x,u)\Delta|\varphi_h|^2+2\Big(-f_u(x,u)+\theta_0\sum_{i=1}^n|f_{x_i u}(x,u)|\Big)\sum_{k=1}^n|\nabla\varphi_h^k|^2+c(\theta_0)|\varphi_h|^2 \\
&\le \partial_t|\varphi_h|^2+f^\sigma_u(x,u)\Delta|\varphi_h|^2+c|\varphi_h|^2.
\end{aligned}
\tag{5.57}
$$

2. In this step, we prove that

$$|\varphi_h|^2\le c(\theta_0,T)\,e^{-\frac{|x|}{M}},$$

for all $(x,t)\in\mathbb{R}^n\times[0,t_0]$.

We begin by defining $\mathcal{L}(v):=\partial_t v+f^\sigma_u(x,u)\Delta v$, $w:=e^{ct}|\varphi_h|^2$, and observing that (5.57) implies $\mathcal{L}(w)\ge 0$. From the latter, it follows by the maximum principle that $|\varphi_h(x,t)|\le 1$ for all $(x,t)\in\mathbb{R}^n\times[0,t_0]$. In particular, given $q_0>n\,e^{2cT}$, we obtain that $|w|\le q_0$ for all $(x,t)\in\mathbb{R}^n\times[0,t_0]$.

Now, set

$$q(x,t):=q_0\,e^{\frac{1}{M}\left(t_0-t-|x|\right)},$$

with $M>\sup_{\mathbb{R}^n\times I}f_u(x,u)$, $I\supset[-\|u_\sigma\|_\infty,\|u_\sigma\|_\infty]$ for $0<\sigma<1$, and note that

$$\mathcal{L}(q)=-q\Big\{\frac{1}{M}\Big(1-\frac{f^\sigma_u(x,u)}{M}\Big)+\frac{f^\sigma_u(x,u)}{M}\frac{n-1}{|x|}\Big\}\le 0,$$

which yields $\mathcal{L}(w-q)\ge 0$. It is easily seen that

$$w-q|_{\{0\le t\le t_0;\,|x|=t_0-t\}}=w-q_0\le 0,\qquad w(x,t_0)-q(x,t_0)\le 0.$$

Then, the claim follows by the maximum principle (cf., e.g., [27]).

3. Let $0\le\rho\in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\rho\subseteq[-1,1]$ and $\int_{\mathbb{R}}\rho\,dx=1$. Set

$$\eta_m(\lambda):=1-\int_{-\infty}^{\lambda}\rho(s-m)\,ds,$$

for $m\in\mathbb{N}$, and take

$$g^k(x,t):=\varphi_h^k(x,t)\,\eta_m(|x|)$$

as a test function in (5.54). Hence

$$
\begin{aligned}
\int_{\mathbb{R}^n}\sum_{k=1}^n v^k(t_0)\operatorname{sgn}(v^k(t_0))*\rho_h\,e^{-|x|}\eta_m(|x|)\,dx &=\sum_{k=1}^n\int_0^{t_0}\int_{\mathbb{R}^n}\Big\{2f^\sigma(x,u)\nabla\varphi_h^k\cdot\nabla\eta_m(|x|) \\
&\quad+f^\sigma(x,u)\varphi_h^k\Delta\eta_m(|x|)-\sum_{i=1}^n f^\sigma_{x_i u}(x,u)\partial_{x_k}\eta_m(|x|)\varphi_h^k\Big\}\,dx\,dt \\
&\quad+\int_{\mathbb{R}^n}\sum_{k=1}^n\Big\{v^k(0)\varphi_h^k(x,0)+-(\Delta f)(x,u)\varphi_h^k(x,t_0)\Big\}\eta_m(|x|)\,dx.
\end{aligned}
\tag{5.58}
$$

Thus, letting $m\to\infty$ first and then letting $h\to 0$, we obtain an estimate of the form

$$\int_{\mathbb{R}^n}\sum_{k=1}^n|v^k(t_0)|\,e^{-|x|}\,dx\le c(T,\theta_0,\|\nabla u_0\|_\infty)<\infty,$$

for all $t_0\in[0,T]$, where, in particular, the right-hand side does not depend on σ. Consequently

$$\int_{\mathbb{R}^n}|u_\sigma(x+y,t)-u_\sigma(x,t)|\Lambda(x)\,dx\le c_0|y|,$$

for some c_0 independent of σ, which gives (5.49).

4. To deduce (5.50), we first note that from the maximum principle and the hypotheses on f, we know that there exists $M > 0$ such that $|f^\sigma(x, u_\sigma(x,t))| \le M$ for all $(x,t) \in \mathbb{R}^{n+1}_+$ and for all $\sigma > 0$. Now, fix t, s, σ and set $w(x) := u_\sigma(x, t+s) - u_\sigma(x,t)$. Given $\varphi \in W^{2,\infty}(\mathbb{R}^n)$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} w(x)\varphi(x)\Lambda(x)\,dx &= \int_{\mathbb{R}^n}\int_t^{t+s} \partial_t u_\sigma(x,\tau)\varphi\Lambda\,d\tau\,dx = \int_{\mathbb{R}^n}\int_t^{t+s} \Delta f^\sigma(x,u_\sigma)\varphi\Lambda\,d\tau\,dx \\ &= \int_{\mathbb{R}^n}\int_t^{t+s} f^\sigma(x,u_\sigma)\Delta(\varphi\Lambda)\,d\tau\,dx \\ &= \int_{\mathbb{R}^n}\int_t^{t+s} \Big\{ f^\sigma(x,u_\sigma)\Delta\varphi\Lambda + 2f^\sigma(x,u_\sigma)\nabla\varphi\cdot\nabla\Lambda + f^\sigma(x,u_\sigma)\varphi\Delta\Lambda\Big\}\,d\tau\,dx,\end{aligned}$$

and this implies

$$\left|\int_{\mathbb{R}^n} w(x)\varphi(x)\Lambda(x)\,dx\right| \le M\Big\{\|\Delta\varphi\|_\infty + 2\|\nabla\varphi\|_\infty + \|\varphi\|_\infty\Big\}\|\Lambda\|_1 s. \tag{5.59}$$

Taking $\varphi = (\mathrm{sgn}\; w) * \rho_\delta$ and observing that $\|\nabla\varphi\|_\infty \le \frac{c}{\delta}, \|\Delta\varphi\|_\infty \le \frac{c}{\delta^2}$ and $\|\varphi\|_\infty \le 1$, where c only depends on the dimension, we get

$$\begin{aligned}\int_{\mathbb{R}^n} |w(x)|\Lambda(x)\,dx &= \left(\int_{\mathbb{R}^n} w(x)\;\mathrm{sgn}(w(x))\,\Lambda(x)\,dx\right)\int_{\mathbb{R}^n}\rho(y)\,dy \\ &= \int_{\mathbb{R}^n\times\mathbb{R}^n} w(x-\delta y)\;\mathrm{sgn}(w(x-\delta y))\,\Lambda(x-\delta y)\rho(y)\,dx\,dy\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n} w(x)\varphi(x)\Lambda(x)\,dx &= \int_{\mathbb{R}^n} w(x)\Lambda(x)\left(\int_{\mathbb{R}^n}\mathrm{sgn}(w(y))\,\rho_\delta(x-y)\,dy\right)dx \\ &= \int_{\mathbb{R}^n} w(x)\Lambda(x)\left(\int_{\mathbb{R}^n}\mathrm{sgn}(w(x-\delta y))\rho(y)\,dy\right)dx \\ &= \int_{\mathbb{R}^n\times\mathbb{R}^n} w(x)\,\Lambda(x)\;\mathrm{sgn}(w(x-\delta y))\,\rho(y)\,dx\,dy.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_{\mathbb{R}^n} |w(x)|\Lambda(x)\,dx - \int_{\mathbb{R}^n} w(x)\varphi(x)\Lambda(x)\,dx \\ &= \int_{\mathbb{R}^n\times\mathbb{R}^n}\Big\{w(x-\delta y)\;\mathrm{sgn}(w(x-\delta y))\,\Lambda(x-\delta y) - w(x)\,\Lambda(x)\;\mathrm{sgn}(w(x-\delta y))\Big\}\rho(y)\,dx\,dy \\ &= \int_{\mathbb{R}^n\times\mathbb{R}^n}\Big\{\big[w(x-\delta y)-w(x)\big]\;\mathrm{sgn}(w(x-\delta y))\Lambda(x) + \big[\Lambda(x-\delta y)-\Lambda(x)\big]\;\mathrm{sgn}(w(x-\delta y))\,w(x-\delta y)\Big\}\rho(y)\,dx\,dy.\end{aligned}$$

Therefore,

$$\left|\int_{\mathbb{R}^n} |w(x)|\Lambda(x)\,dx - \int_{\mathbb{R}^n} w(x)\varphi(x)\Lambda(x)\,dx\right| \le (2c_0 + \|\nabla\Lambda\|_1)\delta. \tag{5.60}$$

Thus, we conclude from (5.60) and from (5.59) that

$$\int_{\mathbb{R}^n} |w(x)|\Lambda(x)\,dx \le \,(2c_0+\|\nabla\Lambda\|_1)\delta + s\,M\Big\{\frac{1}{\delta^2}+\frac{2}{\delta}+1\Big\}\|\Lambda\|_1,$$

for all $0<\delta<1$, which completes the proof of the assertions concerning the Cauchy problem (5.43),(5.44).

5. As to the proof of the assertions concerning the initial-boundary value problem (5.46),(5.47) we have the following. The proof of (5.51) is achieved following the same lines as the proof of (5.49) with the following small adaptations. We first get the analogue of equation (5.54),

$$\begin{aligned}&\int_0^{t_0}\int_\Omega -\sum_{k=1}^n\Big\{\partial_t g^k + f_u^\sigma(x,u)\Delta g^k - \sum_{i=1}^n\big(f_{x_iu}^\sigma(x,u)g_{x_k}^i - f_{x_ix_ku}^\sigma(x,u)g^i\big)\Big\}v^k\,dx\,dt\\ &\quad+\int_{\mathbb{R}^n}\sum_{k=1}^n v^k(t_0)g^k(t_0)\,dx = \int_\Omega\sum_{k=1}^n\Big\{v^k(0)g^k(0)-(\Delta f)(x,u)g^k(t_0)\Big\}\,dx.\end{aligned}\tag{5.61}$$

Now we define φ_h^k, $k=1,\cdots,n$, as the solution of the initial-boundary value problem

$$\begin{cases}\mathcal{L}_k(\varphi_h)=0, & (x,t)\in\Omega\times(0,t_0),\\ \varphi_h^k(t_0)=\big(\operatorname{sgn}(v^k(t_0))\chi_{\Omega_{2h}}\big)*\rho_h, & x\in\Omega,\\ \varphi_h^k(x,t)=0, & (x,t)\in\partial\Omega\times(0,t_0),\end{cases}\tag{5.62}$$

where χ_A denotes, as usual, the indicator function of the set A, and $\Omega_{2h}:=\{\,x\in\Omega\,:\,\operatorname{dist}(x,\partial\Omega)>2h\,\}$. Again we have inequality (5.57) from which we deduce by the maximum principle that $|\varphi_h|\le 1$. Since now we are in a bounded domain Ω, the latter inequality for $|\varphi_h|$ is enough and we may skip step 2.

In (5.61), we now take $g^k(x,t)=\eta_\delta(x)\varphi_h^k(x,t)$ where $\eta_\delta\in C_c^\infty(\Omega)$, $\|\eta_\delta\|_\infty\le 1$ and $\eta_\delta\to\chi_\Omega$ pointwise as $\delta\to 0$. We obtain the analogue of equation (5.58)

$$\begin{aligned}\int_\Omega\sum_{k=1}^n v^k(t_0)\big(\operatorname{sgn}(v^k(t_0))\chi_{\Omega_h}\big)*\rho_h\eta_\delta(x)\,dx &= \sum_{k=1}^n\int_0^{t_0}\int_\Omega\Big\{2f^\sigma(x,u)\nabla\varphi_h^k\cdot\nabla\eta_\delta(x)\\ &\quad+f^\sigma(x,u)\varphi_h^k\Delta\eta_\delta(x)-\sum_{i=1}^n f_{x_iu}^\sigma(x,u)\partial_{x_k}\eta_\delta(x)\varphi_h^k\Big\}\,dx\,dt\\ &\quad+\int_\Omega\sum_{k=1}^n\Big\{v^k(0)\varphi_h^k(x,0)-(\Delta f)(x,u)\varphi_h^k(x,t_0)\Big\}\eta_\delta(|x|)\,dx.\end{aligned}\tag{5.63}$$

We use integration by parts to move the derivatives from η_δ to the product of remaining functions in the integrals of the first three terms inside the integral sign on the right-hand member of (5.63). We then make $\delta\to 0$, use Gauss-Green (divergence) theorem and the fact that $\varphi_h(x,t)$ and $f^\sigma(x,u_\sigma(x,t))$ vanish for $x\in\partial\Omega$ to conclude that those three integrals converge to 0 as $\delta\to 0$. We then make $h\to 0$, and the remaining of the proof of (5.51) is entirely similar to the corresponding part of the proof of (5.49).

The proof of (5.52) is totally similar to the one of (5.50) given above.

□

Theorem 5.4 (Existence)**.** *Let u_σ be the unique solution of* (5.43),(5.44)*. There exists $u\in L^\infty(\mathbb{R}_+^{n+1})$ such that, passing to a suitable subsequence if necessary, $u_\sigma\to u$ a.e. in $\mathbb{R}_+^{n+1}$ as $\sigma\to 0$. Moreover, u is the unique entropy solution of* (5.1),(5.2)*. Finally, we may relax* (5.45)*, take $u_0\in L^\infty(\mathbb{R}^n)$, and still obtain a weak solution for* (5.1),(5.2)*, which is unique.*

Similarly, if u_σ be the unique solution of (5.46),(5.47)*, there exists $u\in L^\infty(Q)$ such that, passing to a suitable subsequence if necessary, $u_\sigma\to u$ a.e. in Q as $\sigma\to 0$. Moreover, u is the unique entropy solution of* (5.3),(5.4)*. Finally, we may relax* (5.48)*, take $u_0\in L^\infty(\Omega)$, and still obtain a weak solution for* (5.3),(5.4)*, which is unique.*

Proof. We only prove the part concerning the Cauchy problem (5.1),(5.2). The assertions concerning the problem (5.3),(5.4) are proved in an entirely similar (even easier) way.

1. Let us first prove the case where (5.45) holds. By Theorem 5.3, $\{u_\sigma\}_{\sigma>0}$ satisfies the hypotheses of Lemma 5.4. Therefore, there exists $u \in L^\infty(\mathbb{R}_+^{n+1})$ such that, passing to a subsequence if necessary, $u_\sigma \to u$ in $L^1_{loc}(\mathbb{R}_+^{n+1})$.

2. Multiplying (5.43) by $f^\sigma(x,u_\sigma)\Lambda$ and integrating by parts, we get

$$\int_0^T \int_{\mathbb{R}^n} \Big\{ \partial_t u_\sigma f^\sigma(x,u_\sigma)\Lambda + \nabla f^\sigma(x,u_\sigma)\cdot \nabla (f^\sigma(x,u_\sigma)\Lambda) \Big\}\, dx\, dt = 0.$$

This yields the equality

$$\int_0^T \int_{\mathbb{R}^n} \Lambda\, \partial_t \Big[\int_0^{u_\sigma} f^\sigma(x,s) ds \Big]\, dx\, dt + \int_0^T \int_{\mathbb{R}^n} |\nabla f^\sigma(x,u_\sigma)|^2 \Lambda + \nabla f^\sigma(x,u_\sigma)\cdot \nabla \Lambda f^\sigma(x,u_\sigma) \Big\}\, dx\, dt = 0,$$

which gives

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^n} |\nabla f^\sigma(x,u_\sigma)|^2 \Lambda\, dx\, dt &= -\int_0^T \int_{\mathbb{R}^n} \nabla f^\sigma(x,u_\sigma)\cdot \nabla\Lambda f^\sigma(x,u_\sigma)\, dt\, dx + \int_{\mathbb{R}^n} \Lambda(x) \int_{u_0}^{u_\sigma(T)} f^\sigma(x,s)\, ds\, dx \\ &\le n \int_0^T \int_{\mathbb{R}^n} |\nabla f^\sigma(x,u_\sigma)||f^\sigma(x,u_\sigma)|\Lambda\, dx\, dt + \int_{\mathbb{R}^n} \Lambda(x) \Big| \int_{u_0}^{u_\sigma(T)} f^\sigma(x,s)\, ds \Big|\, dx,\end{aligned}$$

where we have used that $|\nabla \Lambda| \le n\Lambda$. The Cauchy inequality with δ gives

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^n} |\nabla f^\sigma(x,u_\sigma)|^2 \Lambda\, dx\, dt \\ &\qquad \le \int_0^T \int_{\mathbb{R}^n} \Big\{ 2n\delta |\nabla f^\sigma(x,u_\sigma)|^2 \Lambda + \frac{n}{4\delta} (f^\sigma)^2(x,u_\sigma)\Lambda \Big\}\, dx\, dt + \int_{\mathbb{R}^n} \Lambda(x) \Big| \int_{u_0(x)}^{u_\sigma(T)} f^\sigma(x,s)\, ds \Big|\, dx.\end{aligned}$$

taking $\delta = \dfrac{1}{4n}$ in the previous inequality, we obtain

$$\frac{1}{2}\int_0^T \int_{\mathbb{R}^n} |\nabla f^\sigma(x,u_\sigma)|^2 \Lambda\, dx\, dt \le n^2 \int_0^T \int_{\mathbb{R}^n} (f^\sigma)^2(x,u_\sigma)\Lambda\, dx\, dt + \int_{\mathbb{R}^n} \Lambda(x) \Big| \int_{u_0(x)}^{u_\sigma(T)} f^\sigma(x,s)\, ds \Big|\, dx.$$

Therefore,

$$\int_0^T \int_{\mathbb{R}^n} |\nabla f^\sigma(x,u_\sigma)|^2 \Lambda\, dx\, dt \le c(\|u_0\|_\infty, T) \int_{\mathbb{R}^n} \Lambda(x)\, dx,$$

for all $0 < \sigma < 1$. Given $R > 0$, it follows that

$$\Lambda(R) \int_0^T \int_{B_R} |\nabla f^\sigma(x,u_\sigma)|^2\, dx\, dt \le \int_0^T \int_{B_R} |\nabla f^\sigma(x,u_\sigma)|^2 \Lambda(x)\, dx\, dt \le c \int_{\mathbb{R}^n} \Lambda(x)\, dx$$

and so

$$\int_0^T \int_{B_R} |\nabla f^\sigma(x,u_\sigma)|^2\, dx\, dt \le \frac{c}{\Lambda(R)} \int_{\mathbb{R}^n} \Lambda(x)\, dx,$$

for any $0 < \sigma < 1$.

Thus,

$$\|f^\sigma(x,u_\sigma)\|_{L^2(0,T;H^1(B_R))} \le c(R,T,\|u_0\|_\infty),$$

uniformly in σ. Hence, there exists $v \in L^2_{\rm loc}(\mathbb{R}_+; H^1_{\rm loc}(\mathbb{R}^n))$ such that $f^\sigma(x,u_\sigma) \to v$ weakly. Since $f^\sigma(x,u_\sigma) \to f(x,u)$ a.e., then $v = f(x,u)$ and for this reason we conclude that $f(x,u) \in L^2_{\rm loc}(\mathbb{R}_+; H^1_{\rm loc}(\mathbb{R}^n))$.

3. Finally, when $u_0 \in L^\infty(\mathbb{R}^n)$, we may approximate u_0 in $L^1_{\text{loc}}(\mathbb{R}^n)$ by a sequence $u_{0k} \in W^{2,\infty}(\mathbb{R}^n)$ obtaining a sequence u_k of weak solutions of (5.1),(5.2), with initial data $u_0 = u_{0k}$, and then use the stability Theorem 5.2 to deduce that u_k is a Cauchy sequence in $L^1_{\text{loc}}(\mathbb{R}^{n+1}_+)$. We then easily conclude that the limit $u \in L^\infty(\mathbb{R}^{n+1}_+)$ of the sequence u_k is a weak solution of (5.1),(5.2).

□

We close this section by establishing an elementary result which will be needed in the following sections.

Lemma 5.5. *Let $f(x,u)$ be either of type 1 or type 2, and let $g(x,v)$ be the function left-continuous on v determined by the relation $f(x,g(x,v)) = v$, for all $x \in \mathbb{R}^n$, $v \in \mathbb{R}$. Then, $\lim_{v\to\pm\infty} g(x,v) = \pm\infty$, uniformly in x.*

Proof. Let us prove that $\lim_{v\to+\infty} g(x,v) = +\infty$ uniformly in x. From **(f1.1)**, when f of type 1, or **(f2.1)**, when f is of type 2, there exist $u_1 < 0 < u_2$ such that $f(x,u_1) \le 0 \le f(x,u_2)$, for all x. Now, given any $M > 0$, if $M' := \max\{|u_1|, u_2, M\}$, then, for $M' \le u \le 2M'$, we have $0 \le f(x,u) \le f(x,u) - f(x,u_2) \le 3CM'$, where $C > 0$ is the uniform in x Lipschitz constant of $f(x,\cdot)$ on $[-M', 2M']$. Hence, $g(x, 3CM') \ge u \ge M$, for all x, and, since $g(x,\cdot)$ is increasing, we have $g(x,v) > M$ for all $v > 3CM'$, uniformly in x. This concludes the proof that $\lim_{v\to+\infty} g(x,v) = +\infty$ uniformly in x; the proof that $\lim_{v\to-\infty} g(x,v) = -\infty$, uniformly in x, is completely similar. □

6. Homogenization of Porous Medium Type Equations: The Cauchy problem

In this section, we consider the following homogenization problem

$$
\begin{cases} \partial_t u = \Delta f(\frac{x}{\varepsilon}, u), & (x,t) \in \mathbb{R}^{n+1}_+, \\ u(x,0) = u_0(\frac{x}{\varepsilon}, x), & x \in \mathbb{R}^n, \end{cases} \tag{6.1}
$$

where $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ can be of type 1 or 2 as in Definition 5.1 and Definition 5.2, respectively. We observe that here we consider f depending explicitly only on the "fast variable" $y = x/\varepsilon$, that is, $f = f(y,u)$, and *not* also on the "slow variable" x, in which case $f = f(x,y,u)$. The reason for this restriction will be indicated in the proof of the Theorem 6.1 below.

(h1.1) In the type 1 case, we assume that, for each $u \in \mathbb{R}$, $f(\cdot,u) : \mathbb{R}^n \to \mathbb{R}$ belongs to a given fixed ergodic algebra $\mathcal{A}(\mathbb{R}^n)$. Further, if $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is the function determined by $g(z, f(z,u)) = u$ and $f(z, g(z,v)) = v$, for all $u, v \in \mathbb{R}$, for each $z \in \mathbb{R}^n$, then we assume that g is continuous in $\mathbb{R}^n \times \mathbb{R}$ and for each $v \in \mathbb{R}$, $g(\cdot, v) \in \mathcal{A}(\mathbb{R}^n)$.

We define the function $\bar f : \mathbb{R} \to \mathbb{R}$ by

$$
p = \fint g(z, \bar f(p))\, dz, \tag{6.2}
$$

and we easily deduce that $\bar f$ is monotone increasing and locally Lipschitz continuous. Indeed, that $\bar f$ is increasing follows immediately by the definition and the fact that g is increasing. As for the local Lipschitz continuity of $\bar f$, given any $M > 0$, let $K > 0$ be such that $f(x,-K) \le \bar f(-M)$ and $f(x,K) \ge \bar f(M)$, for all x, and let $C > 0$ be the uniform Lipschitz constant of $f(x,\cdot)$ on the interval $[-K,K]$. We have

$$
|g(z,v_1) - g(z,v_2)| \ge C^{-1}|v_1 - v_2|, \qquad \text{for } v_1, v_2 \in [\bar f(-M), \bar f(M)].
$$

Therefore, for $p_1, p_2 \in [-M, M]$, using the monotonicity of $g(z,\cdot)$, we get

$$
|p_1 - p_2| = \fint |g(z, \bar f(p_1)) - g(z, \bar f(p_2))|\, dz \ge C^{-1}|\bar f(p_1) - \bar f(p_2)|, \tag{6.3}
$$

which proves the assertion.

(h2.1) In the case where f is of type 2, we assume that h, S in Definition 5.2 are functions in $C^2(\mathbb{R}^n)$ which belong to an ergodic algebra $\mathcal{A}(\mathbb{R}^n)$.

For f of type 2, we define $g(z,v) := G\left(\frac{v-S(z)}{h(z)}\right)$, where G is as in Definition 5.2. Instead of defining $\bar f$ directly by (6.2), as in the type 1 case, which in the type 2 case is no longer possible, one proceeds differently. We first define the function $\bar g$ on the set $D \subseteq \mathbb{R}$, of all $q \in \mathbb{R}$ such that, for all functions $\psi_\alpha(z) := \alpha h(z) + S(z)$, with $\alpha \in E$, where E is as in Definition 5.2, we have $\mathfrak{m}\left(\{z \in \mathcal{K} : \psi_\alpha(z) = q\}\right) = 0$. Clearly, since E is countable, $\mathbb{R} \setminus D$ is a countable set.

We set

$$\bar g(q) = \fint g(z,q)\, dz, \qquad \text{for } q \in D. \tag{6.4}$$

We notice that $\bar g : D \to \mathbb{R}$ is (strictly) increasing and so the limits

$$\bar g(q \pm 0) := \lim_{\substack{s \to q \pm 0 \\ s \in D}} \bar g(s)$$

exist for all $q \in \mathbb{R}$, and coincide except on a countable set $\mathcal{D}_0 \subseteq \mathbb{R}$. Moreover, by the fact that $\lim_{v\to\pm\infty} g(z,v) = \pm\infty$, uniformly in z, we have that $\lim_{s\to\pm\infty} \bar g(s) = \pm\infty$. Therefore, we may extend $\bar g$ to $\mathbb{R}$, as a strictly increasing function, by setting

$$\bar g(q) = \bar g(q-0), \qquad \text{for all } q \in \mathbb{R}.$$

We observe that $\bar g$ is continuous except on the countable set $\mathcal{D}_0$. We then set $\bar f(p) = \sigma$ for $p \in [\bar g(\sigma - 0), \bar g(\sigma + 0)]$ and this defines $\bar f$ everywhere in $\mathbb{R}$.

Now, we claim that (6.4), the monotonicity of $\bar g$, and the density of D in $\mathbb{R}$, imply that $D \subseteq \mathbb{R} \setminus \mathcal{D}_0$. Indeed, by the definition of D, we see that $g(z,q)$ is continuous in q for $\mathfrak{m}$-a.e. z, for all $q \in D$. Thus the claim follows from the dominated convergence theorem. Therefore, we see that for each $q \in D$ there is one only $p \in \mathbb{R}$ such that $q = \bar f(p)$. In particular, formula (6.2) holds for $p \in \bar g(D)$. We have that

$$\bar g(\mathbb{R}) = \mathbb{R} \setminus \bigcup_{\sigma \in \mathcal{D}_0} (\bar g(\sigma - 0), \bar g(\sigma + 0)],$$

and we also observe that $\bar g(D)$ is a dense subset of $\bar g(\mathbb{R})$, which follows directly from the density of D in $\mathbb{R}$. Thus, (6.3) holds if $p_1, p_2 \in \bar g(D)$, and so also for $p_1, p_2 \in \bar g(\mathbb{R})$. More generally, we easily see that, if $p_1 \in [\bar g(\sigma_1 - 0), \bar g(\sigma_1 + 0)]$ and $p_2 \in [\bar g(\sigma_2 - 0), \bar g(\sigma_2 + 0)]$, with $p_1 < p_2$, for some $\sigma_1 < \sigma_2$, then we have

$$p_2 - p_1 \geq \bar g(\sigma_2 - 0) - \bar g(\sigma_1 + 0) \geq C^{-1}(\sigma_2 - \sigma_1) = C^{-1}(\bar f(p_2) - \bar f(p_1)),$$

which easily follows by approximation. Hence, $\bar f : \mathbb{R} \to \mathbb{R}$ is a non-decreasing Lipschitz continuos function and formula (6.2) holds for $p \in \bar g(\mathbb{R})$.

As to the initial data, in the where f is of type 1, we assume

(h1.2) $u_0(z,x) = g(z, \phi_0(x))$ with $\phi_0 \in L^\infty(\mathbb{R}^n)$,

while in the case where f is of type 2, we assume

(h2.2) $u_0(z,x) = G\left((\phi_0(x) - S(z))/h(z)\right)$, with $\phi_0 \in L^\infty(\mathbb{R}^n)$, and $\phi_0(x) \in I \subseteq D$, for a.e. $x \in \mathbb{R}^n$, where I is a closed interval and D is as in the above discussion on the definition of $\bar f$ when f is of type 2.

The particular form of the initial data prescribed in **(h1.2)** and **(h2.2)** is sometimes summarized by saying that the initial data are "well-prepared". The additional restriction on $u_0(z,x)$ in **(h2.2)**, when f is of type 2, namely, that $\phi_0(x) \in I \subseteq D$, where I is a closed interval, requires, in order to allow ϕ_0's that are not equivalent to a constant, that D contains a non-degenerate interval. This motivates the search of conditions on the functions $\psi_\alpha(z) = \alpha h(z) + S(z)$, with $\alpha \in E$ (see the definition of D above), under which one can infer that D contains such a non-degenerate interval. This investigation in turn is the motivation behind Lemma 2.2 and Remark 2.3. The latter, in particular, provides examples where $D = \mathbb{R}$, for ψ_α, $\alpha \in E$, of the form $\psi_{0,\alpha} + \delta\psi_{1,\alpha}$, with $\psi_{0,\alpha}$ periodic, $\psi_{1,\alpha} \in \mathcal{A}(\mathbb{R}^n)$, $\delta > 0$ sufficiently small, in the case where

E is finite and $\mathcal{A}(\mathbb{R}^n)$ is an algebra w.m.v. containing periodic functions $\psi_{0,\alpha}$ for which 0 is a regular value of $\nabla\psi_{0,\alpha}$.

We define

$$\bar u_0(x) = \fint u_0(x,z)\,dz. \tag{6.5}$$

Observe that, by the hypotheses **(f1.2)**, when f is of type 1, or **(f2.2)**, when f is of type 2, (6.5) is equivalent to $\bar u_0(x) = \bar g(\phi_0(x))$.

For each $\alpha\in\mathbb{R}$, we define

$$\Phi_\alpha(\cdot) := g(\cdot,\alpha).$$

We observe that Φ_α is a steady weak solution of (6.1). We recall that, for f of type 2, $g(\cdot,\alpha) = G\left((\alpha - S(\cdot))/h(\cdot)\right)$.

Theorem 6.1. *Let $u_\varepsilon(x,t)$ be the entropy solution of (6.1). For f of type 1, assume that* **(h1.1)**, **(h1.2)** *hold; for f is of type 2, assume that* **(h2.1)** *and* **(h2.2)** *hold. Then u_ε weak star converge in $L^\infty(\mathbb{R}^{n+1}_+)$ to $\bar u(x,t)$, where the latter is the entropy solution to the Cauchy problem*

$$\begin{cases}\partial_t\bar u = \Delta\bar f(\bar u), & (x,t)\in\mathbb{R}^{n+1}_+,\\ \bar u(x,0) = \bar u_0(x), & x\in\mathbb{R}^n.\end{cases} \tag{6.6}$$

Moreover, we have

$$u_\varepsilon(x,t) - g\big(\frac{x}{\varepsilon},\bar f(\bar u(x,t))\big)\to 0, \qquad \text{as } \varepsilon\to 0 \text{ in } L^1_{loc}(\mathbb{R}^{n+1}_+). \tag{6.7}$$

Proof. Since the proof is similar to that of Theorem 7.1 in [3] with slight modifications, we will only outline the main steps of it. The case where f is of type 2 will demand some specific considerations which are fully provided here. For details omitted here, we refer the reader to [3].

1. First, we observe that the weak solutions u_ε, $\varepsilon>0$, of (6.1) are bounded uniformly with respect to ε in $L^\infty(\mathbb{R}^{n+1}_+)$. For this, we note that if α_1,α_2 are such that $\alpha_1\le\varphi_0(x)\le\alpha_2$ for $x\in\mathbb{R}$, we have

$$g(\frac{x}{\varepsilon},\alpha_1)\le u_0(\frac{x}{\varepsilon},x)\le g(\frac{x}{\varepsilon},\alpha_2) \qquad \text{for all } x\in\mathbb{R}^n. \tag{6.8}$$

We also note that, when f is of type 1, $g(\frac{x}{\varepsilon},\alpha)$ is a stationary entropy solution of (6.1), for all $\alpha\in\mathbb{R}$. When f is of type 2, this is also true for $\alpha\in D$, where D is as in the above discussion about the definition of $\bar f$ in that case. Indeed, if $f_\delta(z,u) := f(z,u)+\delta u$ and $g_\delta(x,v)$ is such that $f_\delta(z,g_\delta(z,v)) = v$ and $g_\delta(z,f_\delta(z,u)) = u$, then $g_\delta(\frac{x}{\varepsilon},\alpha)$ is a stationary entropy solution of the regularized equation obtained from (6.1) replacing f by f_δ. Now, for $\alpha\in D$, $g_\delta(\frac{x}{\varepsilon},\alpha)\to g(\frac{x}{\varepsilon},\alpha)$ as $\delta\to 0$, for all $x\in\mathbb{R}^n$. This can be seen by the fact that, for each $x\in\mathbb{R}^n$, $g_\delta(\frac{x}{\varepsilon},\alpha)$ is a bounded sequence, and for any converging subsequence $g_{\delta_i}(\frac{x}{\varepsilon},\alpha)\to\beta_x$, we have $\alpha = \lim_{i\to\infty} f_{\delta_i}(\frac{x}{\varepsilon}, g_{\delta_i}(\frac{x}{\varepsilon},\alpha)) = f(\frac{x}{\varepsilon},\beta_x)$, and so $\beta_x = g(\frac{x}{\varepsilon},\alpha)$, which proves the assertion. Let us agree to set $D=\mathbb{R}$ if f is of type 1. So, choosing $\alpha_1,\alpha_2\in D$ in (6.8), and using the monotonicity of the solution operator of (6.1) (see Corollary 5.1), we get

$$g(\frac{x}{\varepsilon},\alpha_1)\le u_\varepsilon(x,t)\le g(\frac{x}{\varepsilon},\alpha_2) \qquad \text{for a.e. } (x,t)\in\mathbb{R}^{n+1}_+.$$

Choosing $A_1,A_2\in\mathbb{R}$ such that $f(y,A_1)\le\alpha_1$ and $f(y,A_2)\ge\alpha_2$ for all $y\in\mathbb{R}^n$, since then $A_1\le g(\frac{x}{\varepsilon},\alpha_1)$ and $A_2\ge g(\frac{x}{\varepsilon},\alpha_2)$, for all $x\in\mathbb{R}^n$, we obtain a compact interval $K=[A_1,A_2]$ in which $u_\varepsilon(x,t)$ takes its values for a.e. $(x,t)\in\mathbb{R}^n\times(0,\infty)$.

Let $\nu_{z,x,t}\in\mathcal{M}(K)$, with $(z,x,t)\in\mathcal{K}\times\mathbb{R}^{n+1}_+$, be the two-scale space-time Young measures associated with a subnet of $\{u_\varepsilon\}_{\varepsilon>0}$ with test functions oscillating only on the space variable. Following [14], [2] and [3], the theorem is proved by adapting DiPerna's method in [12], that is, by showing that $\nu_{z,x,t}$ is a Dirac measure for almost all $(z,x,t)\in\mathcal{K}\times\mathbb{R}^{n+1}_+$. Since we are going to show that $\nu_{z,x,t}$ does not depend on the chosen subnet (so that, a posteriori, a full limit as $\varepsilon\to 0$ occurs), in order to simplify our notation we will use the notation $\lim_{\varepsilon\to 0}$, with no reference to the subnet.

Observe that, for every $\alpha \in D$, the entropy solutions u_ε and $\Phi_\alpha(\frac{x}{\varepsilon}) := g(\frac{x}{\varepsilon}, \alpha)$ satisfy (see Theorem 5.1)

$$\int_{\mathbb{R}^{n+1}_+} |u_\varepsilon(x,t) - \Phi_\alpha(\frac{x}{\varepsilon})|\phi_t + |f(\frac{x}{\varepsilon}, u_\varepsilon(x,t)) - f(\frac{x}{\varepsilon}, \Phi_\alpha(\frac{x}{\varepsilon}))|\Delta\phi\,dx\,dt + \int_{\mathbb{R}^n} |u_0(\frac{x}{\varepsilon}, x) - \Phi_\alpha(\frac{x}{\varepsilon})|\phi(x,0)\,dx \geq 0, \tag{6.9}$$

for all $0 \leq \phi \in C_c^\infty(\mathbb{R}^{n+1})$. In (6.9), we take $\phi(x,t) = \varepsilon^2\varphi(\frac{x}{\varepsilon})\psi(x,t)$ with $0 \leq \psi \in C_c^\infty(\mathbb{R}^{n+1}_+)$, φ, $\Delta\varphi \in \mathcal{A}(\mathbb{R}^n)$ and $\varphi \geq 0$. Observe that

$$\Delta\phi = \Delta\varphi(\frac{x}{\varepsilon})\psi(x,t) + 2\varepsilon\nabla\varphi(\frac{x}{\varepsilon})\cdot\nabla\psi(x,t) + \varepsilon^2\varphi(\frac{x}{\varepsilon})\Delta\psi(x,t).$$

Letting $\varepsilon \to 0$ and using Theorem 4.2, we get

$$\int_{\mathbb{R}^{n+1}_+}\int_{\mathcal{K}} \psi(x,t)\langle\nu_{z,x,t}, |f(z,\cdot) - f(z,\psi_\alpha(z)|\rangle\underline{\Delta\varphi}(z)\,d\mathfrak{m}(z)\,dx\,dt \geq 0.$$

Now apply the inequality above to $\|\varphi\|_\infty \pm \varphi$ to obtain

$$\int_{\mathbb{R}^{n+1}_+}\int_{\mathcal{K}} \psi(x,t)\langle\nu_{z,x,t}, |f(z,\cdot) - \alpha|\rangle\underline{\Delta\varphi}(z)\,d\mathfrak{m}(z)\,dx\,dt = 0, \tag{6.10}$$

for all φ such that $\varphi, \Delta\varphi \in \mathcal{A}(\mathbb{R}^n)$ and all $0 \leq \psi \in C_c^\infty(\mathbb{R}^{n+1}_+)$. This equality, holding in principle for $\alpha \in D$, easily extends to all $\alpha \in \mathbb{R}$. Moreover, equality (6.10) also holds if we replace $|f(z,\cdot) - \alpha|$ by $f(z,\cdot) - \alpha$, which is achieved in the same way by using the integral equality in the definition of weak solution instead of the entropy inequality. Therefore, we obtain

$$\int_{\mathbb{R}^{n+1}_+}\int_{\mathcal{K}} \psi(x,t)\langle\nu_{z,x,t}, \theta(f(z,\cdot))\rangle\underline{\Delta\varphi}(z)\,d\mathfrak{m}(z)\,dx\,dt = 0, \tag{6.11}$$

for any affine function θ, and, by approximation, we get that (6.11) holds for any $\theta \in C(K')$, where K' is a compact interval such that $f(x,K) \subseteq K'$, for all $x \in \mathbb{R}^n$.

2. Define a new family of parametrized measures $\mu_{z,x,t}$ given by

$$\langle\mu_{z,x,t}, \theta\rangle := \langle\nu_{z,x,t}, \theta(f(z,\cdot))\rangle, \qquad \theta \in C(K'). \tag{6.12}$$

By (6.11), we have

$$\Delta_z\langle\mu_{z,x,t}, \theta\rangle = 0, \qquad \text{in the sense of } \mathcal{B}^2. \tag{6.13}$$

Therefore, by the ergodicity of $\mathcal{A}(\mathbb{R}^n)$, using Lemma 3.2, we have that (6.13) implies that $\mu_{z,x,t}$ does not depend of z, that is, $\langle\mu_{z,x,y}, \theta(\cdot)\rangle = \langle\mu_{x,t}, \theta(\cdot)\rangle := \int_{\mathcal{K}}\langle\mu_{z,x,t}, \theta(\cdot)\rangle\,d\mathfrak{m}(z)$, for any $\theta \in C(K')$, a.e. $(x,t) \in \mathbb{R}^n \times (0,\infty)$.

3. The central strategy of the proof is then to show that $\mu_{x,t} = \delta_{\xi(x,t)}$, with $\xi(x,t) := \bar{f}(\bar{u}(x,t))$, where $\bar{u}(x,t)$ is the entropy solution of (6.6). In order to achieve this, a major step is to obtain the inequality

$$\int_{\mathbb{R}^{n+1}_+} \langle\mu_{x,t}, I\left(\cdot, \bar{f}(\bar{u}(x,t))\right)\rangle\varphi_t + \langle\mu_{x,t}, G\left(\cdot, \bar{f}(\bar{u}(x,t))\right)\rangle\Delta\varphi\,dx\,dt \geq 0, \tag{6.14}$$

for all $0 \leq \varphi \in C_c^\infty(\mathbb{R}^{n+1}_+)$, where

$$I(\rho,\alpha) := \int_{\mathcal{K}} |g(z,\rho) - g(z,\alpha)|\,d\mathfrak{m}(z), \tag{6.15}$$

$$G(\rho,\alpha) := |\rho - \alpha|. \tag{6.16}$$

The inequality (6.14) is obtained as follows. We first use (5.14), in item (ii) of Theorem 5.1, making $u_1(x,t)=u_\varepsilon(x,t)$ and $u_2(x)=\Phi_\alpha(\frac{x}{\varepsilon})$. Then, we set $\alpha=\bar f(\bar u(y,s))$, integrate in $(y,s)\in\mathbb{R}^n\times(0,\infty)$, and make $\varepsilon\to0$ to obtain, after some manipulations,

$$
\begin{aligned}
(6.17)\qquad &\int_{(\mathbb{R}_+^{n+1})^2}\langle\mu_{x,t},I(\cdot,\xi)\rangle\phi_t+\langle\mu_{x,t},G(\cdot,\xi)\rangle\big(\Delta_x\phi+\operatorname{div}{}_y\nabla_x\phi\big)\,dx\,dt\,dy\,ds\\
&=\lim_{\varepsilon\to0}\lim_{\delta\to0}\int_{(\mathbb{R}_+^{n+1})^2}\Big\{|\nabla_x[f(\frac{x}{\varepsilon},u_\varepsilon)-f(\frac{x}{\varepsilon},\Phi_\xi(\frac{x}{\varepsilon}))]|^2\\
&\quad+\nabla_y[f(\frac{x}{\varepsilon},u_\varepsilon)-f(\frac{x}{\varepsilon},\Phi_\xi(\frac{x}{\varepsilon}))]\cdot\nabla_x[f(\frac{x}{\varepsilon},u_\varepsilon)-f(\frac{x}{\varepsilon},\Phi_\xi(\frac{x}{\varepsilon}))]\Big\}\\
&\quad\times H_\delta'(f(\frac{x}{\varepsilon},u_\varepsilon)-f(\frac{x}{\varepsilon},\Phi_\xi(\frac{x}{\varepsilon})))\phi\,dx\,dt\,dy\,ds,
\end{aligned}
$$

where $u_\varepsilon=u_\varepsilon(x,t)$, $\xi=\xi(y,s)$.

Next we use again (5.14), in item (ii) of Theorem 5.1, now making $u_1(x,t)=\bar u(x,t)$ and $u_2(x)=l$, where $l\in\mathbb{R}$ is a constant, to obtain

$$
\begin{aligned}
(6.18)\qquad &\int_{\mathbb{R}_+^{n+1}}|l-\bar u(y,s)|\phi_s+\operatorname{sgn}(\bar f(l)-\bar f(\bar u(y,s)))\nabla_y\bar f(\bar u)\cdot\nabla_y\phi\,dy\,ds\\
&=\lim_{\delta\to0}\int_{\mathbb{R}_+^{n+1}}|\nabla_y\bar f(\bar u)|^2H_\delta'(\bar f(l)-\bar f(\bar u(y,s)))\phi\,dy\,ds,\qquad\text{for all } l\in\mathbb{R}.
\end{aligned}
$$

Precisely at this point we will need the additional restriction in **(h2.2)**, in the case where f is of type 2. Namely, we need the validity of the formula

$$
(6.19)\qquad \bar u(y,s)=\int_{\mathcal{K}}g(z,\xi(y,s))\,d\mathfrak{m}(z),
$$

which holds under the additional assumption that $\phi_0(x)\in I\subseteq D$, implying that $\bar u_0(x)=\bar g(\phi_0(x))\in\bar g(I)\subseteq\bar g(D)$, for some closed interval I, because then, by the maximum principle for (6.6), we also have $\bar u(x,t)\in\bar g(D)$, and so equation (6.19) holds. We can then extend $\bar f$ out of $\bar g(I)$ so as to have $\bar f$ strictly increasing in $\mathbb{R}$ and thus, for all $l\in\mathbb{R}$, we have

$$
l=\int_{\mathcal{K}}g(z,\bar f(l))\,d\mathfrak{m}(z).
$$

Also, to obtain (6.18), we used the fact that $\bar f$ does not depend explicitly on x, in order to have that a constant l is a stationary solution of the homogenized equation (6.6), which follows from our taking f not depending explicitly on the “slow variable” x but just on the fast variable $y=x/\varepsilon$.

Thus, making $\bar f(l)=k$ in (6.18), using the definition of I and G, the fact that, since $\nabla_y\xi(y,s)=\nabla_y[f(\frac{x}{\varepsilon},\Phi_{\xi(y,s)}(\frac{x}{\varepsilon}))]$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}_+^{n+1}}|\nabla_y\bar f(\bar u)|^2H_\delta'(\bar f(l)-\bar f(\bar u(y,s)))\phi\,dy\,ds=\int_{\mathbb{R}_+^{n+1}}|\nabla_y\xi(y,s)|^2H_\delta'(k-\xi(y,s))\phi\,dy\,ds\\
&=\int_{\mathbb{R}_+^{n+1}}|\nabla_yf(\frac{x}{\varepsilon},\Phi_{\xi(y,s)}(\frac{x}{\varepsilon}))|^2H_\delta'(k-\xi(y,s))\phi\,dy\,ds,
\end{aligned}
$$

we arrive at

$$
\int_{\mathbb{R}_+^{n+1}}I(k,\xi(y,s))\phi_s+G(k,\xi(y,s))\Delta_y\phi\,dy\,ds=\lim_{\delta\to0}\int_{\mathbb{R}_+^{n+1}}|\nabla_yf(\frac{x}{\varepsilon},\Phi_{\xi(y,s)}(\frac{x}{\varepsilon}))|^2H_\delta'(k-\xi(y,s))\phi\,dy\,ds.
$$

for all $k\in\mathbb{R}$ and all $0\le\phi\in C_c^\infty((\mathbb{R}_+^{n+1})^2)$.

We then take $k = f(\frac{x}{\varepsilon}, u_\varepsilon(x,t))$, integrate in (x,t), make $\varepsilon \to 0$, using Theorem 4.2, use the definition of μ, and after some manipulations we obtain

$$\begin{aligned}
&\int_{(\mathbb{R}_+^{n+1})^2} \langle \mu_{x,t}, I(\cdot,\xi)\rangle \phi_s + \langle \mu_{x,t}, G(\cdot,\xi)\rangle \big(\Delta_y \phi + \operatorname{div}_x \nabla_y \phi\big)\, dx\, dt\, dy\, ds \\
&= \lim_{\varepsilon\to 0}\lim_{\delta\to 0} \int_{(\mathbb{R}_+^{n+1})^2} \Big\{ |\nabla_y [f(\frac{x}{\varepsilon}, u_\varepsilon) - f(\frac{x}{\varepsilon}, \Phi_\xi(\frac{x}{\varepsilon}))]|^2 \\
&+ \nabla_y [f(\frac{x}{\varepsilon}, u_\varepsilon) - f(\frac{x}{\varepsilon}, \Phi_\xi(\frac{x}{\varepsilon}))] \cdot \nabla_x [f(\frac{x}{\varepsilon}, u_\varepsilon) - f(\frac{x}{\varepsilon}, \Phi_\xi(\frac{x}{\varepsilon}))] \Big\} \\
&\qquad \times H_\delta'(f(\frac{x}{\varepsilon}, u_\varepsilon) - f(\frac{x}{\varepsilon}, \Phi_\xi(\frac{x}{\varepsilon}))\phi\, dx\, dt\, dy\, ds.
\end{aligned} \tag{6.20}$$

Finally we add (6.17) with (6.20), use suitable test functions as in Kruzkov's doubling variables method (cf. [22]), to conclude the proof of (6.14).

4. From (6.14) it follows that $\mu_{x,t} = \delta_{\xi(x,t)}$, with $\xi(x,t) = \bar f(\bar u(x,t))$, as asserted. This is achieved in a standard way, where an essential point is to show that

$$\lim_{\tau\to 0} \frac{1}{\tau} \int_0^\tau \int_{B_R} \langle \mu_{x,t}, I(\cdot, \bar f(\bar u_0(x)))\rangle\, dx\, dt = 0, \qquad \text{for all } R > 0, \tag{6.21}$$

where B_R is the open ball centered at the origin with radius R. It is in the proof of (6.21) that we need to use the fact that $u_0(z,x)$ has the for $u_0(z,x) = g(z, \phi_0(x))$ in hypotheses **(h1.2)**, if f is of type 1, or **(h2.2)**, if f is of type 2. Indeed, (6.21) follows from the relation

$$\begin{aligned}
&\int_{\mathbb{R}_+^{n+1}} \int_{\mathcal{K}} \langle \nu_{z,x,t}, |\cdot - \Phi_\alpha(z)|\rangle \varphi_t + \langle \nu_{z,x,t}, |f(z,\cdot) - f(z, \Phi_\alpha(z))|\rangle \Delta\varphi(z)\, d\mathfrak{m}(z)\, dx\, dt \\
&\qquad + \int_{\mathbb{R}^n} \int_{\mathcal{K}} |u_0(z,x) - \Phi_\alpha(z)| \varphi(x,0)\, d\mathfrak{m}(z)\, dx \ge 0
\end{aligned} \tag{6.22}$$

for all $\alpha \in \mathbb{R}$ and for all $0 \le \varphi \in C_c^\infty(\mathbb{R}^{n+1})$, obtained from (6.9) by sending $\varepsilon \to 0$ and using Theorem 4.2. From (6.22), using the definition of $\mu_{x,t}$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}_+^{n+1}} \langle \mu_{x,t}, I(\cdot,\alpha)\rangle \varphi_t + \langle \mu_{x,t}, G(\cdot,\alpha)\rangle \Delta\varphi\, dx\, dt \\
&\qquad + \int_{\mathbb{R}^n} \int_{\mathcal{K}} |u_0(z,x) - \Phi_\alpha(z)| \varphi(x,0)\, d\mathfrak{m}(z)\, dx \ge 0,
\end{aligned} \tag{6.23}$$

for all $0 \le \varphi \in C_c^\infty(\mathbb{R}^{n+1})$ and all $\alpha \in \mathbb{R}$. Now, from (6.23) in a standard way, we obtain

$$\overline{\lim_{h\to 0}} \frac{1}{h} \int_0^h \int_{\mathbb{R}^n} \langle \mu_{x,t}, I(\cdot,\alpha)\rangle \phi(x)\, dx\, dt \le \int_{\mathbb{R}^n} \int_{\mathcal{K}} |u_0(z,x) - \Phi_\alpha(z)| \phi(x)\, d\mathfrak{m}(z)\, dx, \tag{6.24}$$

for all $0 \le \phi \in C_0^\infty(\mathbb{R}^n)$. Using the flexibility provided by the presence of the test function ϕ in (6.24), we get to replace α by $\phi_0(x)$ in (6.24), then getting (6.21).

5. Therefore, using the definition of $\mu_{x,t}$, we deduce that $\nu_{z,x,t} = \delta_{g(z, \bar f(\bar u(x,t)))}$, and so by Theorem 4.2

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}_+^{n+1}} u_\varepsilon(x,t)\phi(x,t)\, dx\, dt = \int_{\mathbb{R}_0^{n+1}} \int_{\mathcal{K}} g(z, \bar f(\bar u(x,t)))\phi(x,t)\, d\mathfrak{m}(z)\, dx\, dt = \int_{\mathbb{R}_+^{n+1}} \bar u(x,t)\phi(x,t)\, dx\, dt,$$

for all $\phi \in C_0(\mathbb{R}_+^{n+1})$, while Theorem 4.3 gives (6.7), concluding the proof.

□

7. Homogenization of Porous medium type equations on bounded domains

In this section we consider a homogenization problem for a porous medium type equation, similar to the one analyzed in the previous section, but now in a bounded domain. Because of boundary constraints, we must consider a flux function of the form $f(x, \frac{x}{\varepsilon}, u)$, depending also on the "slow variable" x, instead of simply $f(\frac{x}{\varepsilon}, u)$. As we have seen, the technique used in the last section for the Cauchy problem takes advantage from the fact that the flux function of the homogenized equation does not depend explicitly on x, in which case constants are stationary solutions, which is no longer the case when f depends explicitly on the "slow variable" x. That technique also requires the initial data to be well-prepared, that is, of the particular form prescribed in **(h1.2)** and **(h2.2)**. Another difficulty in using the method of the last section, also due to the fact that here f depends explicitly on x and so the flux function of the homogenized equation, $\bar{f}(x, u)$, also depends explicitly on x, is that, even when hypotheses on the homogenized initial data guarantee that the flux function of the homogenized equation is strictly increasing, it is not at all clear how to verify the condition **(f1.2)** for $\bar{f}(x, u)$, which is required for our existence result, Theorem 5.4. In this connection, however, we recall that, if $\bar{f}(x, u)$ is strictly increasing, then the recent compactness result in [26] could be invoked in order to guarantee the existence of a weak solution with no need of **(f1.2)**.

Here, we use a completely different approach to address the homogenization problem on bounded domains, whose important advantage is that it allows us to consider more general initial data, namely, initial data that are not necessarily "well prepared" as prescribed in **(h1.2)** and **(h2.2)**. On the other hand, our approach here will require that we restrict ourselves to the case where $\mathcal{A}(\mathbb{R}^n)$ is a regular algebra w.m.v. instead of a general ergodic algebra. As it was shown in Section 3, $\mathrm{FS}(\mathbb{R}^n)$ provides a very encompassing example of regular algebra w.m.v., and it is not even known so far whether there are ergodic algebras that are not regular algebras w.m.v., neither whether there are ergodic algebras that are not subalgebras of $\mathrm{FS}(\mathbb{R}^n)$. Another loss, is that, here, for our result on the existence of oscillatory profiles correcting the weak convergence into a strong convergence, we need to ask the flux function to be convex, which was not necessary for the corresponding result in Theorem 6.1 and excludes completely flux functions of type 2. The discussion in this section largely extends the corresponding one in [17] concerning the homogenization of a particular type of the general equation considered here, in the nondegenerate case.

So, let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary and $Q = \Omega \times (0, \infty)$. We consider the initial-boundary value problem

$$\begin{cases} \partial_t u = \Delta f(x, \frac{x}{\varepsilon}, u), & (x,t) \in Q, \\ u(x,0) = u_0(x, \frac{x}{\varepsilon}), & x \in \Omega, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0, \infty). \end{cases} \tag{7.1}$$

Set $f_\varepsilon(x, u) := f(x, \frac{x}{\varepsilon}, u)$, for each fixed $\varepsilon > 0$. Then we will impose hypotheses on $f(x, y, u)$ considering separately the cases where $f_\varepsilon(x, u)$ is of type 1 and of type 2, according to Definitions 5.1 and 5.2, respectively.

(h1.3) For each $\varepsilon > 0$, $f_\varepsilon(x, u)$ is of type 1, $f : \Omega \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is continuous and, for each $(x, u) \in \Omega \times \mathbb{R}$ fixed, $f(x, \cdot, u)$ belongs to a given regular algebra w.m.v. $\mathcal{A}(\mathbb{R}^n)$. We denote by $g(x, y, v)$ the function satisfying $g(x, y, f(x, y, u)) = u$, $f(x, y, g(x, y, v)) = v$, and we assume that g is continuous over $\Omega \times \mathbb{R}^n \times \mathbb{R}$ and, for each fixed $(x, v) \in \Omega \times \mathbb{R}$, $g(x, \cdot, v) \in \mathcal{A}(\mathbb{R}^n)$. Also, $f(x, y, 0) = 0$ for all $(x, y) \in \partial\Omega \times \mathbb{R}^n$ (cf. **(f1.3)** in Section 5).

Define $\bar{f} : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$p = ⨍_{\mathbb{R}^n} g(x, z, \bar{f}(x, p))\, dz. \tag{7.2}$$

As in the last section, we can prove that $\bar{f}(x, u)$ satisfies **(f1.1)**, and it is easy to see that it also satisfies **(f1.3)**. Nevertheless, in general, it is not at all clear whether **(f1.2)** holds. Therefore, we cannot use the

Theorem 5.4 to assert the existence of a weak solution for the initial-boundary value problem

$$\begin{cases} \partial_t u = \Delta \bar f(x,u), & (x,t)\in Q,\\ u(x,0)=\bar u_0(x) := \fint_{\mathbb{R}^n} u_0(x,z)\,dz, & x\in\Omega,\\ u(x,t)=0, & (x,t)\in\partial\Omega\times[0,\infty). \end{cases} \tag{7.3}$$

That is the reason why, in Theorem 7.1 below, we characterize the homogenized limit $\bar u$ in a different way. However, since the existence of a weak solution to (7.3) follows from the general compactness result in [26], given that **(f1.1)** and **(f1.3)** hold, and uniqueness is also guaranteed by **(f1.1)** and **(f1.3)**, for f satisfying **(h1.3)**, we actually could characterize $\bar u$ as the unique weak solution of (7.3).

We now state our assumptions in case $f(x,\frac{x}{\varepsilon},u)$ is of type 2.

(h2.3) For each $\varepsilon>0$, $f_\varepsilon(x,u)$ is of type 2. More specifically, $f(x,y,u)=h(x,y)F(u)+S(x,y)$, with F satisfying the conditions in **(f2.1)**, and for each $\varepsilon>0$ fixed, $h(x,\frac{x}{\varepsilon})$ and $S(x,\frac{x}{\varepsilon})$ satisfy the conditions in **(f2.2)**. Moreover, $h,S:\bar\Omega\times\mathbb{R}^n\to\mathbb{R}$ are continuous and, for each $x\in\bar\Omega$, $h(x,\cdot)$ and $S(x,\cdot)$ belong to a given regular algebra w.m.v. $\mathcal{A}(\mathbb{R}^n)$, and $h(x,y)F(0)+S(x,y)=0$ for all $(x,y)\in\partial\Omega\times\mathbb{R}^n$ (cf. **(f2.3)**). If G is as in Definition 5.2, we denote $g(x,y,v):=G((v-S(x,y))/h(x,y))$.

Again, as in the last section, when $f_\varepsilon(x,u)$ is of type 2, we cannot define $\bar f(x,u)$ directly by (7.2). For each fixed $x\in\bar\Omega$, we argue as in the last section in order to define first $\bar g(x,v)$ by

$$\bar g(x,v) = \fint_{\mathbb{R}^n} g(x,y,v)\,dy, \tag{7.4}$$

for $v\in D_x$, where D_x is the set of $v\in\mathbb{R}$ such that $\mathfrak{m}\left(\{z\in\mathcal{K} : \alpha h(x,z)+S(x,z)=v\}\right)=0$, for all $\alpha\in E$, where E is the set of discontinuities of G (see Definition 5.2), and then using the monotonicity of $\bar g(x,\cdot)$ and the density of D_x to extend $\bar g(x,\cdot)$ to $\mathbb{R}$ as a left-continuous function. We then define $\bar f(x,u)$ such that $\bar f(x,\bar g(x,v))=v$, for all $(x,v)\in\Omega\times\mathbb{R}$, and prove that $\bar f(x,\cdot)$ is Lipschitz continuous, as in the last section. Now, for f_ε of type 2, the explicit dependence of $\bar f$ on x imposes the need of conditions to guarantee the continuity of $\bar f(x,u)$ also in the variable x. In case we have that there exists a closed interval $J\subseteq\mathbb{R}$ such that $J\subseteq D_x$ for all $x\in\bar\Omega$, then, by dominated convergence, it follows that $\bar g$ is continuous on $\bar\Omega\times J$. Therefore, if there is a closed interval $I\subseteq\mathbb{R}$ such that $I\subseteq\bar g(x,J)$, for all $x\in\bar\Omega$, then $\bar f$ is continuous on $\bar\Omega\times I$.

Concerning the initial data we assume

(h1.4) $u_0\in L^\infty(\Omega;\mathcal{A}(\mathbb{R}^n))$;

if $f_\varepsilon(x,u)$ is of type 1, and

(h2.4) $u_0\in L^\infty(\Omega;\mathcal{A}(\mathbb{R}^n))$. There exist $\alpha_1<0<\alpha_2$, with $\alpha_1,\alpha_2\in D_x$ for all $x\in\bar\Omega$, such that $g(x,y,\alpha_1)\le u_0(x,y)\le g(x,y,\alpha_2)$, for all $(x,y)\in\bar\Omega\times\mathbb{R}^n$. Further, if $J_1:=[\alpha_1,\alpha_2]$, there are closed intervals $J_2,I\subseteq\mathbb{R}$, such that $J_1\subseteq J_2\subseteq D_x$ for all $x\in\bar\Omega$, and $\bar g(x,J_1)\subseteq I\subseteq\bar g(x,J_2)$, for all $x\in\bar\Omega$;

if f is of type 2.

A few comments on the additional assumptions in **(h2.4)** are in order. The first additional requirement, namely, there exist $\alpha_1<0<\alpha_2$, with $\alpha_1,\alpha_2\in D_x$ for all $x\in\bar\Omega$, such that $g(x,y,\alpha_1)\le u_0(x,y)\le g(x,y,\alpha_2)$, for all $(x,y)\in\bar\Omega\times\mathbb{R}^n$, is needed here because now the set of "good values" D_x depends on x and so it is not possible to guarantee the existence of such α_1,α_2 that are good for all $x\in\bar\Omega$. The other additional requirement, namely, if $J_1:=[\alpha_1,\alpha_2]$, there are closed intervals $J_2,I\subseteq\mathbb{R}$, such that $J_1\subseteq J_2\subseteq D_x$ for all $x\in\bar\Omega$, and $\bar g(x,J_1)\subseteq I\subseteq\bar g(x,J_2)$, for all $x\in\bar\Omega$, is needed in order to ensure the continuity of $\bar f(x,u)$ on $\bar\Omega\times I$ and that the entropy solutions of (7.1) assume values on I. Here also we refer to Lemma 2.2, and in particular to the item (ii) of its statement, for conditions that can guarantee nice properties for the sets D_x, for instance, that $D_x=\mathbb{R}$.

We next state and prove the main result of this section. We will use the concept and some basic facts about viscosity solutions of fully nonlinear parabolic equations. We refer to [10] for a general exposition of the theory of viscosity solutions of fully-nonlinear elliptic and parabolic equations.

Before stating the theorem, let us introduce the following notation. Given a function $h \in L^\infty(\Omega)$, we denote by $\Delta^{-1}h$ the solution of the boundary value problem

$$\begin{cases} \Delta v(x) = h(x), & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega. \end{cases} \tag{7.5}$$

Theorem 7.1. *Assume that, for each fixed $\varepsilon > 0$, $f_\varepsilon(x,u) = f(x, \frac{x}{\varepsilon}, u)$ is either of type 1 and satisfies* **(h1.3)***, while u_0 satisfies* **(h1.4)***, or it is of type 2 and satisfies* **(h2.3)***, while u_0 satisfies* **(h2.4)***. Let $u_\varepsilon(x,t)$ be the entropy solution of (7.1). Then, as $\varepsilon \to 0$, u_ε weak star converges in $L^\infty(\Omega \times [0,\infty))$ to $\bar u(x,t)$, which is uniquely defined as follows. Let $\bar U$ be the viscosity solution of*

$$\begin{cases} \partial_t U = \bar f(x, \Delta U), & (x,t) \in Q, \\ U(x,0) = \bar U_0(x) := \Delta^{-1}\left\{ \fint u_0(z,x) \right\}, & x \in \Omega, \\ U(x,t) = 0, & (x,t) \in \partial\Omega \times [0,\infty). \end{cases} \tag{7.6}$$

Then $\bar U \in L^\infty((0,\infty); W^{2,p}(\Omega))$, for any $1 < p < \infty$, and

$$\bar u(x,t) := \Delta \bar U, \qquad \text{a.e. } (x,t) \in \Omega \times (0,\infty).$$

Moreover, assuming the existence of an entropy solution to (7.3), $\bar u$ is the entropy solution of (7.3), and if $f(x,z,\cdot)$ is strictly convex, for all $(x,z) \in \Omega \times \mathbb{R}^n$, then

$$u_\varepsilon(x,t) - g\big(x, \frac{x}{\varepsilon}, \bar f(x, \bar u(x,t))\big) \to 0 \qquad \text{as } \varepsilon \to 0 \text{ in } L^1_{loc}(\Omega \times [0,\infty)). \tag{7.7}$$

Proof. 1. The fact that the solutions of (7.1) form a uniformly bounded sequence in $L^\infty(Q)$ follows from the last part of Remark 5.3. To see this, let us denote $\Phi_\alpha(x,y) := g(x,y,\alpha)$. Take $\alpha_1 < 0 < \alpha_2$ such that $\Phi_{\alpha_1}(x, \frac{x}{\varepsilon}) \le u_0(x, \frac{x}{\varepsilon}) \le \Phi_{\alpha_2}(x, \frac{x}{\varepsilon})$, where we use **(h2.4)** in case f is of type 2. Hence, using (5.42), once for $(u_\varepsilon - \Phi_{\alpha_1})_-$ and again for $(u_\varepsilon - \Phi_{\alpha_2})_+$, we obtain

$$\Phi_{\alpha_1}(x, \frac{x}{\varepsilon}) \le u_\varepsilon(x,t) \le \Phi_{\alpha_2}(x, \frac{x}{\varepsilon}),$$

which proves the uniform boundedness of the family $\{u_\varepsilon\}$.

2. Now, let us make a general observation concerning problem (5.3)–(5.4), under assumptions **(f1.1)**–**(f1.3)**, or **(f2.1)-(f2.3)**, for f of type 1 or type 2, respectively. So, let u be the weak solution of (5.3)–(5.4) and, for each $t \in [0,\infty)$, let $U(\cdot,t) := \Delta^{-1}u(\cdot,t)$. We claim that U is the viscosity solution of

$$\begin{cases} \partial_t U - f(x, \Delta U) = 0, & (x,t) \in Q, \\ U(x,0) = U_0(x), & x \in \Omega, \\ U(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \end{cases} \tag{7.8}$$

where $U_0 = \Delta^{-1}u_0$. Indeed, let u_σ be the smooth solution of the corresponding regularized problem (5.46)–(5.47). For each $t \in [0,\infty)$, let $U_\sigma(\cdot,t) := \Delta^{-1}u_\sigma(\cdot,t)$. Since u_σ and U_σ are smooth, it is clear that the latter is the (viscosity) solution of

$$\begin{cases} \partial_t U_\sigma - f^\sigma(x, \Delta U_\sigma) = 0, & (x,t) \in Q, \\ U_\sigma(x,0) = U_0(x), & x \in \Omega, \\ U_\sigma(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty). \end{cases} \tag{7.9}$$

Since $\{u_\sigma(x,t)\}_{0<\sigma<1}$ is uniformly bounded in $L^\infty(\Omega\times[0,\infty))$, we easily see that the $U_\sigma(x,t)$ form a uniformly bounded sequence in $L^\infty([0,\infty); W^{2,p}(\Omega))$ for all $p \in (1,\infty)$. On the other hand, from (7.9) we easily deduce that $|U_\sigma(x,t) - U_\sigma(x,s)| \le C|t-s|$ for all $x \in \Omega$ for some constant $C > 0$, independent of σ. Hence, we see that U_σ is uniformly bounded in $W^{1,\infty}(\bar Q)$. In particular, there is a subsequence U_{σ_i} of U_σ converging locally uniformly in $\bar Q$ to a function $U \in W^{1,\infty}(\bar Q)$ which satisfies $U = \Delta^{-1}u$.

It follows in a standard way that U is the viscosity solution of (7.8). Indeed, given any $(x_0,t_0)\in Q$, we consider $\varphi\in C^2(Q)$ such that $U-\varphi$ has a strict local maximum at (x_0,t_0). Since $U_{\sigma_i}-\varphi$ converges locally uniformly in $\bar Q$ to the function $U-\varphi$, we may obtain a sequence $(x_i,t_i)\in Q$ such that (x_i,t_i) is a point of local maximum of $U_{\sigma_i}-\varphi$ and $(x_i,t_i)\to(x_0,t_0)$ as $i\to\infty$. Thus, we have

$$\partial_t\varphi(x_i,t_i)-f^{\sigma_i}(x_i,\Delta\varphi(x_i,t_i))\le 0,$$

from which follows, as $i\to\infty$,

$$\partial_t\varphi(x_0,t_0)-f(x_0,\Delta\varphi(x_0,t_0))\le 0. \tag{7.10}$$

To relax the assumption of a strict local maximum to just a local maximum we proceed as usual replacing φ by, say, $\tilde\varphi(x,t):=\varphi(x,t)+\delta(|x-x_0|^2+(t-t_0)^2)$ obtaining (7.10) with $\tilde\varphi$ instead of φ and from that we obtain again (7.10) for φ passing to the limit when $\delta\to 0$. In an entirely similar way we prove the reverse inequality when $U-\varphi$ has a local minimum at (x_0,t_0), so proving that U is a viscosity solution of (7.8).

3. In the next steps we shall study the homogenization of (7.11) using a method motivated by [18]. As we will see, the ε-Laplacian property in Lemma 3.3 plays a decisive role at this point, and this explains our assumption that $\mathcal{A}(\mathbb{R}^n)$ is a regular algebra w.m.v. We define $U_\varepsilon(x,t)$ in $\Omega\times[0,\infty)$ by $U_\varepsilon:=\Delta^{-1}u_\varepsilon$ where u_ε is the weak solution of (7.1). By step 2, we have that U_ε is the viscosity solution of

$$\begin{cases}\partial_t U_\varepsilon-f(x,\frac{x}{\varepsilon},\Delta U_\varepsilon)=0, & (x,t)\in Q,\\ U_\varepsilon(x,0)=U_{0,\varepsilon}(x), & x\in\Omega,\\ U_\varepsilon(x,t)=0, & (x,t)\in\partial\Omega\times(0,\infty).\end{cases} \tag{7.11}$$

where $U_{0,\varepsilon}=\Delta^{-1}u_{0,\varepsilon}$, with $u_{0,\varepsilon}(x)=u_0(\frac{x}{\varepsilon},x)$. The same argument used in the previous step shows that

$$U_\varepsilon\in L^\infty((0,\infty);W^{2,p}(\Omega))\bigcap \mathrm{Lip}((0,\infty);L^\infty(\Omega)),$$

and so there is a subsequence U_{ε_i} of U_ε converging locally uniformly in $\bar Q$ to a function

$$\bar U\in L^\infty((0,\infty);W^{2,p}(\Omega))\bigcap \mathrm{Lip}((0,\infty);L^\infty(\Omega)),$$

in particular, $\bar U\in W^{1,\infty}(\bar Q)$.

4. We claim that $\bar U(x,t)$ is the viscosity solution of the initial-boundary value problem

$$\begin{cases}\partial_t U-\bar f(x,\Delta U)=0, & (x,t)\in Q,\\ U(x,0)=\bar U_0(x), & x\in\Omega,\\ U(x,t)=0, & (x,t)\in\partial\Omega\times(0,\infty).\end{cases} \tag{7.12}$$

where

$$\bar U_0:=\Delta^{-1}⨍ u_0(z,x)\,dz.$$

5. Indeed, let $(\hat x,\hat t)\in Q$ and let $\varphi\in C^2(Q)$ be such $\bar U-\varphi$ has a local maximum at $(\hat x,\hat t)$. Also, let $v_{\sigma,\delta}\in\mathcal{A}(\mathbb{R}^n)$ be a smooth function satisfying

$$g_\sigma\left(\hat x,z,\bar f_\sigma(\hat x,p)\right)-p-\delta\le\Delta_z v_{\sigma,\delta}\le g_\sigma\left(\hat x,z,\bar f_\sigma(\hat x,p)\right)-p+\delta, \tag{7.13}$$

with $p=\Delta\varphi(\hat x,\hat t)$, whose existence is asserted by Lemma 3.3, where $g_\sigma(x,y,\cdot)$ is the inverse of $f_\sigma(x,y,\cdot)=f(x,y,\cdot)+\sigma\cdot$, and $\bar f_\sigma$ is given by (7.2) with g_σ replacing g. In particular, given any $\delta'>0$ we can find $\delta>0$ sufficiently small such that

$$\bar f_\sigma(\hat x,\Delta\varphi(\hat x,\hat t))-\delta'\le f_\sigma\left(\hat x,z,\Delta\varphi(\hat x,\hat t)+\Delta v_{\sigma,\delta}(z)\right)\le\bar f_\sigma(\hat x,\Delta\varphi(\hat x,\hat t))+\delta',$$

from which it follows

$$\bar f(\hat x,\Delta\varphi(\hat x,\hat t))-2\delta'\le f\left(\hat x,z,\Delta\varphi(\hat x,\hat t)+\Delta v_\delta(z)\right)\le\bar f(\hat x,\Delta\varphi(\hat x,\hat t))+2\delta',$$

for $\sigma > 0$ sufficiently small, since $\bar f_\sigma$ converges pointwise to $\bar f$, where we set $v_\delta := v_{\sigma,\delta}$. Here we use **(h2.4)**, in case f is of type 2, which implies that $\bar f$ is strictly increasing in $\bar\Omega \times I$ and $\bar u(x,t)$ assumes values in I, so that we may assume that $\bar f(x,\cdot)$ is strictly increasing in $\mathbb{R}$, and so $\bar f_\sigma$ converges everywhere to $\bar f$.

Take $\rho > 0$ and let $x_j \in \Omega$ be a point of maximum of

$$U_j(x,\hat t) - \varphi(x,\hat t) - \varepsilon_j^2 v_\delta(\frac{x}{\varepsilon_j}) - \rho|x-\hat x|^2 + \rho,$$

where we denote $U_j = U_{\varepsilon_j}$, such that $x_j \to \hat x$ as $j \to \infty$. Such sequence (x_j) exists since U_j converges locally uniformly to $\bar U$ and v_δ is bounded. We have

$$\varphi_t(x_j,\hat t) - f\left(x_j, \frac{x_j}{\varepsilon_j}, \Delta\varphi(x_j,\hat t) + \Delta v_\delta(\frac{x_j}{\varepsilon_j}) + \rho\right) \le 0,$$

and

$$f\left(\hat x, \frac{x_j}{\varepsilon_j}, \Delta\varphi(\hat x,\hat t) + \Delta v_\delta(\frac{x_j}{\varepsilon_j})\right) \le \bar f(\hat x, \Delta\varphi(\hat x,\hat t)) + 2\delta',$$

which, after addition, gives

$$\varphi_t(x_j,\hat t) - \bar f(\hat x, \Delta\varphi(\hat x,\hat t)) \le O(|x_j - \hat x|) + O(\rho) + 2\delta'.$$

Hence, letting $j \to \infty$ first, and then letting $\rho, \delta' \to 0$, we obtain

$$\varphi_t(\hat x,\hat t) - \bar f(\hat x, \Delta\varphi(\hat x,\hat t)) \le 0.$$

The reverse inequality, when $\bar U - \varphi$ has a local minimum at $(\hat x,\hat t)$, follows in an entirely similar way, which concludes the proof of the claim.

6. By the uniqueness of the viscosity solution of (7.12) (see for instance [10], Theorem 8.2), we conclude that the whole sequence $U_\varepsilon(x,t)$ converges uniformly to $\bar U(x,t)$. Let $\bar u := \Delta \bar U$. Given any $\varphi \in C_c^\infty(Q)$, we have

$$\int_Q u_\varepsilon(x,t)\varphi(x,t)\,dx\,dt = \int_Q \Delta U_\varepsilon \varphi\,dx\,dt = \int_Q U_\varepsilon \Delta\varphi\,dx\,dt \overset{\varepsilon\to0}{\longrightarrow}$$
$$\int_Q \bar U \Delta\varphi\,dx\,dt = \int_Q \bar u\varphi\,dx\,dt.$$

Consequently, $u_\varepsilon(x,t)$ converges in the weak-$*$ topology of $L^\infty(Q)$ to $\bar u = \Delta\bar U(x,t)$, which concludes the proof of the first part of the theorem.

7. Now, let us assume the existence of a weak solution $\tilde u$ of (7.3), which actually follows from the compactness result in [26]. Let $\tilde U := \Delta^{-1}\tilde u$. As it was done above, we easily prove that $\tilde U$ is the viscosity solution of (7.12). Therefore, $\tilde U \equiv \bar U$, and so $\tilde u = \bar u$.

8. We are going to prove (7.7) under the additional assumption that $f(x,z,\cdot)$ is strictly convex for all $(x,z) \in \Omega \times \mathbb{R}^n$. In particular, f is of type 1. We first observe that the identity

$$\partial_t U_\varepsilon - f(x, \frac{x}{\varepsilon}, \Delta U_\varepsilon) = 0,$$

holds in the sense of distributions in Q. Indeed, for any $\varphi \in C_0^\infty((0,\infty); H_0^1(\Omega))$, we have

$$\int_Q u_\varepsilon \varphi_t - \nabla f(x, \frac{x}{\varepsilon}, u_\varepsilon)\cdot\nabla\varphi\,dx\,dt = 0. \tag{7.14}$$

Given $\phi \in C_0^\infty(Q)$, we take $\varphi = \Delta^{-1}\phi$ in (7.14), use $u_\varepsilon = \Delta U_\varepsilon$ and integration by parts, to obtain that

$$\int_Q U_\varepsilon \phi_t + f(x, \frac{x}{\varepsilon}, \Delta U_\varepsilon)\phi\,dx\,dt = 0, \tag{7.15}$$

holds for any $\phi \in C_0^\infty(Q)$. Similarly, since $\bar u$ is the weak solution of (7.3), we have

$$\int_Q \bar U \phi_t + \bar f(x, \Delta \bar U)\phi \, dx\, dt = 0, \tag{7.16}$$

for any $\phi \in C_0^\infty(Q)$.

Using $\phi(x,t)\varphi(\frac{x}{\varepsilon})$ with $\phi \in C_0^\infty(Q)$ and $\varphi \in \mathcal{A}(\mathbb{R}^n)$, as the test function ϕ in (7.15), which is clearly possible, and taking the limit along a suitable subnet $\varepsilon(d)$, $d \in D$, we obtain by Theorem 4.2

$$\int_0^\infty \int_\Omega \int_{\mathcal{K}} \{\langle \nu_{x,t,z}, f(x,z,\cdot)\rangle - \bar f(x, \Delta \bar U)\}\phi(x,t)\varphi(z)\, d\mathfrak{m}(z)\, dx\, dt = 0,$$

where $\mathcal{K}$ is the compactification of $\mathbb{R}^n$ associated with $\mathcal{A}(\mathbb{R}^n)$, and we have used (7.16). Since ϕ and φ are arbitrary, we have

$$\langle \nu_{x,t,z}, f(x,z,\cdot)\rangle = \bar f(x, \Delta \bar U) = f\big(x, z, g(x,z,\bar f(x,\Delta \bar U))\big), \quad \text{for a.e. } (x,t,z) \in Q \times \mathcal{K}.$$

Since $f(x,z,\cdot)$ is strictly convex for all $(x,z) \in \Omega \times \mathbb{R}^n$, we conclude that

$$\nu_{x,t,z} = \delta_{g(x,z,\bar f(x,\Delta \bar U))}, \quad \text{for a.e. } (x,t,z) \in Q \times \mathcal{K}.$$

and this implies through Theorem 4.3 that (7.7) holds. □

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