A non-type (D) operator in c_0

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Dedicated to Professor J. M. Borwein on the occasion of his 60th birthday

Abstract

Previous examples of non-type (D) maximal monotone operators were restricted to ℓ^1, L^1, and Banach spaces containing isometric copies of these spaces. This fact led to the conjecture that non-type (D) operators were restricted to this class of Banach spaces. We present a linear non-type (D) operator in c_0.

keywords: maximal monotone, type (D), Banach space, extension, bidual.

1 Introduction

Let U, V arbitrary sets. A *point-to-set* (or multivalued) operator $T : U \rightrightarrows V$ is a map $T : U \to \mathcal{P}(V)$, where $\mathcal{P}(V)$ is the power set of V. Given $T : U \rightrightarrows V$, the *graph* of T is the set

$$\mathrm{Gr}(T) := \{(u, v) \in U \times V \mid v \in T(u)\},$$

the *domain* and the *range* of T are, respectively,

$$\mathrm{dom}(T) := \{u \in U \mid T(u) \neq \emptyset\}, \qquad \mathrm{R}(T) := \{v \in V \mid \exists u \in U,\ v \in T(u)\}$$

and the *inverse* of T is the point-to-set operator $T^{-1} : V \rightrightarrows U$,

$$T^{-1}(v) = \{u \in U \mid v \in T(u)\}.$$

A point-to-set operator $T : U \rightrightarrows V$ is called *point-to-point* if for every $u \in \mathrm{dom}(T)$, $T(u)$ has only one element. Trivially, a point-to-point operator is injective if, and only if, its inverse is also point-to-point.

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Let X be a real Banach space. We use the notation X^* for the topological dual of X. From now on X is identified with its canonical injection into $X^{**} = (X^*)^*$ and the duality product in $X \times X^*$ will be denoted by $\langle \cdot, \cdot \rangle$,

$$\langle x, x^* \rangle = \langle x^*, x \rangle = x^*(x), \qquad x \in X, x^* \in X^*.$$

A point-to-set operator $T : X \rightrightarrows X^*$ (respectively $T : X^{**} \rightrightarrows X^*$) is *monotone*, if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (x, x^*), (y, y^*) \in \mathrm{Gr}(T),$$

(resp. $\langle x^* - y^*, x^{**} - y^{**} \rangle \geq 0$, $\forall (x^{**}, x^*), (y^{**}, y^*) \in \mathrm{Gr}(T)$), and it is *maximal monotone* if it is monotone and maximal in the family of monotone operators in $X \times X^*$ (resp. $X^{**} \times X^*$) with respect to the order of inclusion of the graphs.

We denote c_0 as the space of real sequences converging to 0 and ℓ^∞ as the space of real bounded sequences, both endowed with the sup-norm

$$\|(x_k)_k\|_\infty = \sup_{k \in \mathbb{N}} |x_k|,$$

and ℓ^1 as the space of absolutely summable real sequences, endowed with the 1-norm,

$$\|(x_k)_k\|_1 = \sum_{k=1}^{\infty} |x_k|.$$

The dual of c_0 is identified with ℓ^1 in the following sense: for $y \in \ell^1$

$$y(x) = \langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i, \qquad \forall x \in c_0.$$

Likewise, the dual of ℓ^1 is identified with ℓ^∞. It is well known that c_0 (as well as ℓ^1, ℓ^∞, etc.) is a non-reflexive Banach space.

Let X be a *non-reflexive* real Banach space and $T : X \rightrightarrows X^*$ be maximal monotone. Since $X \subset X^{**}$, the point-to-set operator T can also be regarded as an operator from X^{**} to X^*. We denote $\widehat{T} : X^{**} \rightrightarrows X^*$ as the operator such that

$$\mathrm{Gr}(\widehat{T}) = \mathrm{Gr}(T).$$

If $T : X \rightrightarrows X^*$ is maximal monotone then $\widehat{T}$ is (still) trivially monotone but, in general, not maximal monotone. Direct use of the Zorn's Lemma shows that $\widehat{T}$ has a maximal monotone extension. So it is natural to ask if such maximal monotone extension to the bidual is unique. Gossez [5, 6, 7, 8] gave a sufficient condition for uniqueness of such an extension.

Definition 1.1 ([5]). *Gossez's* monotone closure *(with respect to $X^{**} \times X^*$) of a maximal monotone operator $T : X \rightrightarrows X^*$, is the point-to-set operator $\widetilde{T} : X^{**} \rightrightarrows X^*$ whose graph* $\mathrm{Gr}\left(\widetilde{T}\right)$ *is given by*

$$\mathrm{Gr}\left(\widetilde{T}\right) = \{(x^{**}, x^*) \in X^{**} \times X^* \mid \langle x^* - y^*, x^{**} - y \rangle \geq 0, \ \forall (y, y^*) \in T\}.$$

A maximal monotone operator $T : X \rightrightarrows X^*$, *is of* Gossez type (D) *if for any* $(x^{**}, x^*) \in \mathrm{Gr}\left(\widetilde{T}\right)$ *, there exists a bounded net* $\left((x_i, x_i^*)\right)_{i \in I}$ *in* $\mathrm{Gr}(T)$ *which converges to* (x^{**}, x^*) *in the* $\sigma(X^{**}, X^*) \times$*strong topology of* $X^{**} \times X^*$.

Gossez proved [8] that a maximal monotone operator $T : X \rightrightarrows X^*$ of of type (D) has unique maximal monotone extension to the bidual, namely, its Gossez's monotone closure $\widetilde{T} : X^{**} \rightrightarrows X^*$. Beside this fact, maximal monotone operators of type (D) share many properties with maximal monotone operators defined in *reflexive* Banach spaces as, for example, convexity of the closure of the domain and convexity of the closure of the range [5].

Gossez gave an example of a non-type (D) operator on ℓ^1 [6]. Later, Fitzpatrick and Phelps gave an example of a non-type (D) on $L^1[0, 1]$ [4]. In [1], Profs. Borwein and Bauschke proved that if a monotone continuous linear operator in a Banach space has a monotone conjugate, then this operator is of type (D), and defined *conjugate monotone spaces* as those Banach spaces X such that the conjugate of any continuous monotone linear operator form X to X^* is monotone as well. Still in [1] it is observed that c_0, c (convergent real sequences) ℓ^∞ and $L^\infty[0, 1]$ are conjugate monotone spaces while ℓ^1, $L^1[0, 1]$, $(\ell^\infty)^*$ and $(L^\infty[0, 1])^*$ are not conjugate monotone spaces. These facts led Professor J. M. Borwein to define *Banach spaces of type (D)* as those Banach spaces where every maximal monotone operator is of type (D), and to formulate the following most interesting conjecture [2, §4, question 3]:

- Are any non-reflexive spaces X of type (D)? That is, are there non reflexive spaces on which all maximal monotones on X are type (D). I conjecture 'weakly' that if X contains no copy of $\ell^1(\mathbb{N})$ then X is type (D) as would hold in $X = c_0$.

In this work, we answer negatively such conjecture by giving an example of a non-type (D) operator on c_0 and proving that, for every space which contains a norm-isomorphic (in particular, isometric) copy of c_0, a non-type (D) operator can be defined.

2 A non-type (D) operator on c_0

Gossez's operator [6] $G : \ell^1 \to \ell^\infty$ is defined as

$$G(y) = x, \qquad x_n = \sum_{i=n+1}^{\infty} y_i - \sum_{i=1}^{n-1} y_i, \tag{1}$$

which is linear, continuous, anti-symmetric and, therefore, maximal monotone. This operator will be used to define a non-type (D) maximal monotone operator in c_0, which was previously defined in [11].

Lemma 2.1. *The operator*

$$T : c_0 \rightrightarrows \ell^1, \qquad T(x) = \{y \in \ell^1 \mid -G(y) = x\} \tag{2}$$

is point-to-point in its domain, is maximal monotone and its range is

$$R(T) = \left\{ y \in \ell^1 \;\middle|\; \sum_{i=1}^{\infty} y_i = 0 \right\} . \tag{3}$$

Proof. Gossez's operator is injective (see the proof of [3, Proposition 3.2]). Hence, by Definition (2), T is point-to-point in its domain. Moreover, direct use of (1) shows that G is linear and $\langle y, G(y) \rangle = 0$ for any $y \in \ell^1$. In particular, T is monotone.

Note that using (1) we have for any $y \in \ell^1$

$$\lim_{n\to\infty} (G(y))_n = \sum_{i=1}^{\infty} y_i.$$

Hence $x = G(y) \in c_0$ if and only if $\sum y_i = 0$ which, in view of definition (2), proves (3).

Suppose that $x \in c_0$, $y \in \ell^1$ and

$$\langle x - x', y - y' \rangle \geq 0, \qquad \forall (x', y') \in \mathrm{Gr}(T). \tag{4}$$

Define, $u^1 = (-1, 1, 0, 0, \dots)$, $u^2 = (0, -1, 1, 0, 0), \dots$, that is

$$(u^m)_i = \begin{cases} -1, & i = m \\ 1, & i = m+1 \\ 0, & \text{otherwise} \end{cases} \qquad i = 1, 2, \dots \tag{5}$$

and let

$$v^m = G(u^m), \qquad (v^m)_i = \begin{cases} 1, & i = m \text{ or } i = m+1 \\ 0, & \text{otherwise} \end{cases} \qquad i = 1, 2, \dots \tag{6}$$

where the expression for $(v^m)_i$ follows from (1) and (5).

Direct use of (5), (6) and (2) shows that $T(-\lambda v^m) = \lambda u^m$ for $\lambda \in \mathbb{R}$ and $m = 1, 2, \dots$. Therefore, for any $\lambda \in \mathbb{R}$, $m = 1, 2, \dots$

$$\langle x + \lambda v^m, y - \lambda u^m \rangle \geq 0$$

which is equivalent to

$$\langle x, y \rangle + \lambda [\langle v^m, y \rangle - \langle x, u^m \rangle] \geq 0.$$

Since the above inequality holds for any λ,

$$\langle x, u^m \rangle = \langle v^m, y \rangle, \qquad m = 1, 2, \dots$$

which, in view of (5), (6) is equivalent to

$$x_{m+1} - x_m = y_{m+1} + y_m, \qquad m = 1, 2, \dots \tag{7}$$

Adding the above equality for $m = i, i+1, \dots, j$ we conclude that

$$x_{j+1} = x_i + y_i + 2\sum_{k=i+1}^{j} y_k + y_{j+1}, \qquad i < j.$$

Using the assumptions $x \in c_0$, $y \in \ell^1$ and taking the limit $j \to \infty$ in the above equation, we conclude that

$$x_i = -\left[y_i + 2\sum_{k=i+1}^{\infty} y_k\right] = -\left[G(y)_i + \sum_{k=1}^{\infty} y_k\right]. \tag{8}$$

From (4) with $x' = 0$ and $y' = 0$, $\sum_{i=1}^{\infty} x_i y_i \geq 0$. Substituting into this the expression for x_i obtained in (8), we obtain

$$-\left[\sum_{k=1}^{\infty} y_k\right]^2 \geq 0. \tag{9}$$

Combining (8) and (9) we conclude that $x = -G(y)$. Hence $(x, y) \in \mathrm{Gr}(T)$, which proves the maximal monotonicity of T. □

Proposition 2.2. *The operator $T : c_0 \rightrightarrows \ell^1$ defined in Lemma 2.1 has infinitely many maximal monotone extensions to $\ell^\infty \rightrightarrows \ell^1$. In particular, T is non-type (D).*

Proof. Let

$$e = (1, 1, 1, \dots)$$

We claim that

$$\langle -G(y) + \alpha e - x', y - y'\rangle = \alpha\langle y, e\rangle, \qquad \forall (x', y') \in \mathrm{Gr}(T), y \in \ell^1, \alpha \in \mathbb{R}. \tag{10}$$

To prove this claim, first use (2) and (1) to conclude that $x' = -G(y')$ and

$$\langle -G(y) - x', y - y'\rangle = \langle G(y' - y), y - y'\rangle = 0$$

As $y' \in R(T)$, using (3) we have $\langle e, y'\rangle = 0$, which combined with the above equation yields (10).

Take $\widetilde{y} \in \ell^1$ such that $\langle \widetilde{y}, e\rangle > 0$ and define

$$x^\tau = -G(\tau\widetilde{y}) + \frac{1}{\tau}e, \qquad 0 < \tau < \infty.$$

In view of (10),

$$\left(x^\tau, \tau\widetilde{y}\right) \in \mathrm{Gr}\left(\widetilde{T}\right), \qquad 0 < \tau < \infty.$$

Therefore, for each $\tau \in (0, \infty)$ there exists a maximal monotone extension $T_\tau : \ell^\infty \rightrightarrows \ell^1$ of T such that

$$(x^\tau, \tau\widetilde{y}) \in G(T_\tau).$$

However, these extensions are distinct because if $\tau, \tau' \in (0, \infty)$ and $\tau \neq \tau'$ then

$$\langle x^\tau - x^{\tau'}, \tau\widetilde{y} - \tau'\widetilde{y}\rangle = (\tau - \tau')(1/\tau - 1/\tau')\langle \widetilde{y}, e\rangle < 0.$$

□

3 Other spaces with non-type (D) operators

In this section we will prove that if a Banach spaces contains a closed subspace which is norm-isomorphic to c_0, then there exists non-type (D) maximal monotone operators in this space. We begin with an auxiliary result.

Lemma 3.1. *Let X, Ω be real Banach space, and suppose that $A : X \to \Omega$ is a linear norm-isomorphism from X onto a closed subspace of Ω. For $T : X \rightrightarrows X^*$ define*

$$T_A : \Omega \rightrightarrows \Omega^*, \quad \mathrm{Gr}(T_A) = \{(w, w^*) \in \Omega\times\Omega^* \mid \exists (x, x^*) \in \mathrm{Gr}(T), \;\; w = A(x), \; x^* = A'(w^*)\}, \tag{11}$$

where $A' : \Omega^* \to X^*$ is the adjoint (or conjugate) of A, that is $A'(w^*) = w^* \circ A$. Then the application

$$\mathrm{Gr}(T_A) \to \mathrm{Gr}(T), \qquad (w, w^*) \mapsto (A^{-1}(w), A'(w^*)) \tag{12}$$

maps $\mathrm{Gr}(T_A)$ onto $\mathrm{Gr}(T)$ and, if T is maximal monotone, then T_A is also maximal monotone.

Proof. We claim that for any $x^* \in X^*$, there exists $w^* \in \Omega^*$ such that $A'(w^*) = x^*$, that is

$$(A')^{-1}(\{x^*\}) \neq \emptyset \qquad \forall x^* \in X^*. \tag{13}$$

For proving this claim, note that $A^{-1} : R(A) \to X$ is a continuous linear map. Therefore, $\xi^* = x^* \circ A^{-1}$ is a continuous linear functional defined in $R(A)$. Using the Hahn-Banach Theorem we conclude that there exists $w^* \in \Omega^*$ which extends ξ^*. To end the proof of the claim, note that as w^* and ξ^* coincides in $R(A)$, $w^* \circ A = \xi^* \circ A = x^*$ and so, $A'(w^*) = x^*$.

Direct use of (11) and (13) shows that the application defined in (12) maps $\mathrm{Gr}(T_A)$ onto $\mathrm{Gr}(T)$.

Now assume that T is maximal monotone. To prove that T_A is monotone, note that if $w_1^* \in T_A(w_1)$ and $w_2^* \in T_A(w_2)$ then, by definition (11), there exist $x_1, x_2 \in X$ and $x_1^*, x_2^* \in X^*$ such that

$$w_i = A(x_i), \; A'(w_i^*) = w_i^* \circ A = x_i^* \in T(x_i), \qquad i = 1, 2.$$

Therefore

$$\begin{aligned} \langle w_1 - w_2, w_1^* - w_2^*\rangle &= \langle A(x_1) - A(x_2), w_1^* - w_2^*\rangle \\ &= \langle A(x_1 - x_2), w_1^* - w_2^*\rangle \\ &= \langle x_1 - x_2, A'(w_1^* - w_2^*)\rangle = \langle x_1 - x_2, x_1^* - x_2^*\rangle \geq 0, \end{aligned}$$

where the last inequality follows from the monotonicity of T. Hence, T_A is monotone.

To prove that T_A is maximal monotone, suppose that $(w_0, w_0^*) \in \Omega \times \Omega^*$ is in monotone relation with any point in $\mathrm{Gr}(T_A)$, that is,

$$\langle w_0 - w, w_0^* - w^*\rangle \geq 0, \qquad \forall (w, w^*) \in \mathrm{Gr}(T_A) \tag{14}$$

Suppose that $w_0 \notin R(A)$. Take $(\bar{w}, \bar{w}^*) \in \mathrm{Gr}(T_A)$ and let $(\bar{x}, \bar{x}^*) = (A^{-1}(\bar{w}), A'(\bar{w}^*)) \in \mathrm{Gr}(T)$. Since $\bar{w} \in R(A)$, $w_0 - \bar{w} \notin R(A)$ and using the Hahn-Banach theorem (and the assumption of $R(A)$ being closed) we conclude that there exists $\bar{u}^* \in \Omega^*$ such that

$$\langle w, \bar{u}^* \rangle = 0, \quad \forall w \in R(A),$$
$$\langle w_0 - \bar{w}, \bar{u}^* \rangle > \langle w_0 - \bar{w}, w_0^* - \bar{w}^* \rangle.$$

Therefore, $\bar{u}^* \in \ker(A')$,

$$A'(\bar{w}^* + \bar{u}^*) = A'(\bar{w}^*) = \bar{x}^* \in T(\bar{x}),$$

so $\bar{w}^* + \bar{u}^* \in T_A(A(\bar{x})) = T_A(\bar{w})$, implying

$$\langle w_0 - \bar{w}, w_0^* - (\bar{w}^* + \bar{u}^*) \rangle < 0$$

in contradiction with (14). Therefore, $w_0 \in R(A)$. Now define

$$x_0 = A^{-1}(w_0), \qquad x_0^* = A'(w_0^*) = w_0^* \circ A. \tag{15}$$

If $(x, x^*) \in \mathrm{Gr}(T)$, then there exists $w^* \in (A')^{-1}(x^*)$ and, by the definition of T_A, $(Ax, w^*) \in \mathrm{Gr}(T_A)$. Therefore, by (14)

$$\begin{aligned} 0 &\le \langle w_0 - A(x), w_0^* - w^* \rangle \\ &= \langle A(x_0) - A(x), w_0^* - w^* \rangle \\ &= \langle x_0 - x, A'(w_0^*) - A'(w^*) \rangle = \langle x_0 - x, x_0^* - x^* \rangle. \end{aligned}$$

Hence, using the maximal monotonicity of T, we conclude that

$$(x_0, A'(w_0^*)) = (x_0, x_0^*) \in \mathrm{Gr}(T),$$

which, in view of (15) and the definition of T_A, shows that $(w_0, w_0^*) \in \mathrm{Gr}(T_A)$. Altogether, we proved that T_A is maximal monotone. □

Lemma 3.2. *Let X, Ω be real Banach space, and suppose that $A : X \to \Omega$ is a linear norm-isomorphism from X onto a closed subspace of Ω. Let $T : X \rightrightarrows X^*$ be a maximal monotone operator and define $T_A : \Omega \rightrightarrows \Omega^*$ as in Lemma 3.1, that is,*

$$\mathrm{Gr}(T_A) = \{(w, w^*) \in \Omega \times \Omega^* \mid \exists (x, x^*) \in \mathrm{Gr}(T), \;\; w = A(x), \; x^* = A'(w^*)\}.$$

If T_A is of type (D) on $\Omega \times \Omega^*$ then T is of type (D) on $X \times X^*$.

Proof. Suppose that $(\widehat{x}^*, \widehat{x}^{**}) \in X^* \times X^{**}$. Using (13) we can find $\widehat{w}^* \in (A')^{-1}(\widehat{x}^*)$. Using the first part of Lemma 3.1 we have

$$\begin{aligned} \inf_{(x,x^*)\in \mathrm{Gr}(T)} \langle \widehat{x}^* - x^*, \widehat{x}^{**} - x \rangle &= \inf_{(w,w^*)\in \mathrm{Gr}(T_A)} \langle \widehat{x}^* - A'(w^*), \widehat{x}^{**} - A^{-1}(w) \rangle \\ &= \inf_{(w,w^*)\in \mathrm{Gr}(T_A)} \langle A'(\widehat{w}^*) - A'(w^*), \widehat{x}^{**} - A^{-1}(w) \rangle \\ &= \inf_{(w,w^*)\in \mathrm{Gr}(T_A)} \langle \widehat{w}^* - w^*, A''(\widehat{x}^{**}) - A(A^{-1}(w)) \rangle \\ &= \inf_{(w,w^*)\in \mathrm{Gr}(T_A)} \langle \widehat{w}^* - w^*, A''(\widehat{x}^{**}) - w \rangle \le 0 \end{aligned}$$

where the last inequality follows from the assumption of T_A being of type (D) and [9]. Since $(\widehat{x}^*, \widehat{x}^{**})$ is a generic element of $X^* \times X^{**}$, in view of the above result and [10, eq. (5) and Theorem 4.4, item 1] we conclude that T is also of type (D). $\square$

Using Lemmas 3.1 and 3.2, we obtain the following Theorem.

Theorem 3.3. *Let X be a Banach space such that there exists a non-type (D) maximal monotone operator $T : X \rightrightarrows X^*$, and let Ω be another Banach space. If there exists a linear map $A : X \to \Omega$ such that A is a norm-isomorphism from X onto a closed subspace of Ω, then there exists a non-type (D) maximal monotone operator $S : \Omega \rightrightarrows \Omega^*$.*

Proof. Define $T_A : \Omega \rightrightarrows \Omega^*$ as in lemmas 3.1, 3.2. Using Lemma 3.1 we conclude that T_A is maximal monotone. It T_A is of type (D), then by Lemma 3.2 T is also of type (D), in contradiction with the assumptions of the theorem. Therefore T_A is a maximal monotone non-type (D) operator. $\square$

Using Proposition 2.2 and Theorem 3.3, we have the following Corollary.

Corollary 3.4. *Any real Banach space Ω which contains an norm-isomorphic copy of c_0 has a non-type (D) maximal monotone operator.*

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