Fixed point methods for a certain class of operators

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Abstract

We introduce in this paper a new class of nonlinear operators which contains, among others, the class of operators with semimonotone additive inverse and also the class of nonexpansive mappings. We study this class and discuss some of its properties. Then we present iterative procedures for computing fixed points of operators in this class, which allow for inexact solutions of the subproblems and relative error criteria. We prove weak convergence of the generated sequences in the context of Hilbert spaces. Strong convergence is also discussed.

keywords: fixed points, approximate solutions, hypomonotone operators, semimonotone operators, proximal point algorithms.

1 Introduction

If $T : C \to C$ is a contraction on the closed subset C of a Banach space ($\|T(x) - T(y)\| \le \theta\|x - y\|$ for all $x, y \in C$ and some $0 \le \theta < 1$), then Banach's fixed point Theorem asserts that the sequence of successive approximations $x^{k+1} = T(x^k)$, with any initial guess $x^0 \in C$, is strongly convergent to the unique fixed point of T. For the wider class of nonexpansive mappings ($\|T(x)-T(y)\| \le \|x-y\|$ for all $x, y \in C$), it is necessary to use some regularization techniques, like e.g. in Mann's method, introduced in [5] and further studied by other authors (see, e.g., [8]): given $x^0 \in C$, define $\{x^k\}$ by

$$x^{k+1} = (1 - \beta_k)x^k + \beta_k T(x^k) = [I - \beta_k(I - T)](x^k) \tag{1}$$

with $\beta_k \in (0, 1]$. This algorithm generates weakly convergent sequences, assuming existence of fixed points of T, when the set C is closed and convex, but the convergence may be just weak in

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infinite dimensional Hilbert spaces (see [1, 3]). Many attempts to modify Mann's iteration in order to get strong convergence have been made. We mention the works of Nakajo and Takahashi [7] and Moudafi [6]. In [7] the authors added an extra step, projecting the initial iterate onto a certain set, which is constructed using the information of Mann's iterate: given $x^0 \in C$, take

$$y^k = (1-\beta_k)x^k + \beta_k T(x^k) \quad \text{and} \quad x^{k+1} = P_{R_k}(x_0), \tag{2}$$

where P_K denotes the orthogonal projection onto the set K in a Hilbert space, and $R_k = C_k \cap Q_k$, with

$$C_k = \left\{ z \in C : \left\| y^k - z \right\| \leq \left\| x^k - z \right\| \right\}, \;\; Q_k = \{z \in C : \langle x^k - z, x_0 - x^k \rangle \geq 0\}.$$

This modification generates a sequence $\{x^k\}$ that converges strongly to $P_{F(T)}(x_0)$, where $F(T) = \{x \in C : T(x) = x\}$ is the set of fixed points of F.

Concerning the work in [6], we focus on the implicit iteration proposed there: for each k, find $x^k \in C$ such that

$$x^k = (1-\beta_k)T(x^k) + \beta_k \pi(x^k), \tag{3}$$

where $\beta_k = \varepsilon_k/(1+\varepsilon_k)$, $\{\varepsilon_k\}$ is a sequence of positive real numbers converging to zero and $\pi : H \to C$ is a contraction, with constant $\theta < 1$, from the real Hilbert space H to the closed convex set C. The generated sequence $\{x^k\}$ converges strongly to the unique fixed point of $P_{F(T)} \circ \pi$. It is worthwhile to mention that iteration (3) is equivalent to

$$x^k = \left[(I-T) + \varepsilon_k(I-\pi)\right]^{-1}(0).$$

Moreover, the operator $I-T$ is monotone because T is nonexpansive, and $I-\pi$ is strongly monotone with parameter $1-\theta$ because π is a contraction with parameter θ. Thus, Moudafi's iteration has the structure of an approximate proximal iteration in $x_0 = 0$, taking into account the convergence to zero of $\{\varepsilon_k\}$; hence, it can also be seen as a sequence of projections of the initial iterate onto certain separation hyperplanes (cf. the algorithm in [7], i.e. the iteration given by (2)).

In our work we assume that $C = H$, where H is a real Hilbert space, and we consider a set-valued operator $T : H \to \mathcal{P}(H)$. In this context, $p \in H$ is a fixed point of T, that is $p \in F(T)$, if and only if $p \in T(p)$. The basic scheme behind the algorithms in this paper is the following iteration: given $x^0 \in H$, define the sequence $\{x^k\}$ as

$$x^{k+1} = \left[I + \beta_k(I-T)\right]^{-1}(x^k), \tag{4}$$

with $\beta_k > 0$, namely the proximal point iteration for the operator $I-T$, but here $I-T$ is not necessary monotone, i.e., T may fail to be nonexpansive.

The novelty here is not the procedure given by (4), but rather the class of operators for which we will establish convergence, which we call class (QL) (see Section 2). Differently from the above mentioned methods, which require at least nonexpansiveness of the operator, this class contains nonexpansive operators as well as expansive ones. Moreover, we allow for approximate solution of the subproblems with relative error criteria, which leads to iteration formulae much more involved than (4) (see Algorithms 1 and 2 in Section 3).

We also mention that, as in Mann's iteration (where the iteration operator is $I - \beta_k(I - T)$, see (1)), our basic iteration operator $I + \beta_k(I - T)$ is in general better conditioned than $I - T$, as we explain next. Assume, for the sake of simplicity, that T is a finite dimensional linear operator. Observe that finding a fixed point of T is equivalent to solving the possibly ill-conditioned system $(I - T)(x) = 0$, since, even when $I - T$ is non-singular, the condition number $\mu(I - T)$ of $I - T$, defined as $\mu(I - T) = \|I - T\| \left\|(I - T)^{-1}\right\|$, can be much larger than one. In our procedure, on the other hand, we solve at each iteration the system in (4), namely

$$[I + \beta(I - T)](x) = x^0,$$

with $\beta > 0$, so that, for β small enough, $\|\beta(T - I)\| = \beta \|T - I\| < 1$, and hence $T_\beta = \beta(I - T) + I$ is non-singular and

$$\left\|T_\beta^{-1}\right\| \leq \frac{1}{1 - \|\beta(T - I)\|}.$$

Thus, the condition number $\mu(T_\beta)$ of T_β, satisfies

$$\begin{aligned} \mu(T_\beta) &= \|T_\beta\| \left\|T_\beta^{-1}\right\| \leq \frac{1}{1 - \|\beta(T - I)\|} \|\beta(I - T) + I\| \\ &\leq \frac{1 + \beta \|T - I\|}{1 - \beta \|T - I\|} = \frac{\left\|(T - I)^{-1}\right\| + \beta\mu(T - I)}{\|(T - I)^{-1}\| - \beta\mu(T - I)}. \end{aligned}$$

Thus, $\mu(T_\beta)$ approaches 1 as β decreases to zero. Summarizing, with a proper choice of β we can improve as much as we want the condition number of the subproblems in (4).

Next we recall the concepts of monotonicity, strong monotonicity and hypomonotonicity for a set valued operator $A : H \to \mathcal{P}(H)$ with effective domain $D(A) = \{x \in H;\ A(x) \neq \emptyset\}$, and graph $G(A) = \{(x, u) \in H \times H;\ x \in D(A),\ u \in A(x)\}$.

Definition 1. *The operator A is monotone when*

$$\langle x - y, u - v\rangle \geq 0 \quad \forall (x, u), (y, v) \in G(A).$$

If A is monotone and additionally $G(A) = G(A')$ for all monotone operator $A' : H \to \mathcal{P}(H)$ such that $G(A) \subset G(A')$, then A is said to be maximal monotone.

Definition 2. *The operator A is strongly monotone with parameter $\delta > 0$, or δ-strongly monotone, when*

$$\langle x - y, u - v\rangle \geq \delta \|x - y\|^2 \quad \forall (x, u), (y, v) \in G(A).$$

If A is δ-strongly monotone and additionally $G(A) = G(A')$ for all δ-strongly monotone operator $A' : H \to \mathcal{P}(H)$ such that $G(A) \subset G(A')$, then A is said to be maximal δ-strongly monotone.

Definition 3. *A is said to be ρ-hypomonotone, with $\rho > 0$, when*

$$\langle x-y, u-v\rangle \geq -\rho\|x-y\|^2 \quad \forall (x,u),(y,v) \in G(A).$$

A is said to be maximal ρ-hypomonotone if it is ρ-hypomonotone and additionally $G(A) = G(A')$ for all ρ-hypomonotone operator $A' : H \to \mathcal{P}(H)$ such that $G(A) \subset G(A')$.

Observe that ρ-hypomonotonicity of A^{-1} means $\langle x-y, u-v\rangle \geq -\rho\|u-v\|^2$ for all $(x,u),(y,v) \in G(A)$, or, equivalently, monotonicity of $A^{-1} + \rho I$.

2 (QL) type operators.

Let H be a real Hilbert space with inner product denoted by $\langle \cdot, \cdot \rangle$, $T : H \to \mathcal{P}(H)$, a set-valued operator and $G(T)$ its graph. We introduce next the class of operators we are concerned with in this paper.

Definition 4. *An operator $T : H \to \mathcal{P}(H)$ is of type (QL) with parameter $\tau \in (0, 1/2)$, or simply τ-(QL), iff*

$$\langle x-y, u-v\rangle \leq (1-\tau)\|x-y\|^2 + \tau\|u-v\|^2 \quad \forall (x,u),(y,v) \in G(T). \tag{5}$$

Moreover, T is said to be maximal τ-(QL) if it is τ-(QL) and additionally $G(T) = G(T')$ for all τ-(QL) operator $T' : H \to \mathcal{P}(H)$ such that $G(T) \subset G(T')$

Contractions and, more generally, nonexpansive mappings are of type (QL) for all $\tau \in (0, 1/2)$, as we prove next.

Lemma 1. *If $F : H \to H$ is nonexpansive then F satisfies property (QL) for any $\tau \in (0, 1/2)$.*

Proof. Assume that $\|F(x) - F(y)\| \leq \|x-y\|$ for all $x, y \in H$, and fix any $\tau \in (0, 1/2)$. In view of Cauchy-Schwartz inequality, it suffices to prove that, for any $x \neq y$,

$$\|F(x) - F(y)\|\,\|x-y\| \leq (1-\tau)\|x-y\|^2 + \tau\|F(x) - F(y)\|^2,$$

or equivalently

$$t := \frac{\|F(x) - F(y)\|}{\|x-y\|} \leq (1-\tau) + \tau\left(\frac{\|F(x) - F(y)\|}{\|x-y\|}\right)^2 = (1-\tau) + \tau t^2. \tag{6}$$

which reduces to

$$\tau t^2 - t + (1-\tau) \geq 0. \tag{7}$$

Note that the quadratic in the left hand side of (7) has its roots at $t_1 = 1$ and $t_2 = 1/\tau - 1$, and that (7) holds if and only if t is outside the open interval between these two roots. Since F is nonexpansive, we have $t = \|F(x) - F(y)\| / \|x-y\| \leq 1$ for all $x \neq y$, completing the proof. $\square$

Similarly, we show next that expansive mappings are also of type (QL), for an appropriate τ.

Lemma 2. *If $F : H \to \mathcal{P}(H)$ is η-expansive, then F is τ-(QL) for all $\tau \in (0, 1/2)$ such that $\eta \geq 1/\tau - 1$.*

Proof. By definition of expansivity, $\|u - v\| \geq \eta \|x - y\|$ for all $(x, u), (y, v) \in G(F)$. Then, taking $\tau \in (0, 1/2)$ such that $\eta \geq 1/\tau - 1$, we obtain, whenever $x \neq y$, that $t = \|u - v\| / \|x - y\| \geq t_2$, where $t_2 = 1/\tau - 1$ is the second root of the quadratic $\tau t^2 - t + (1 - \tau)$. Thus, as in the proof of Lemma 1, we can conclude that

$$\|u - v\| \|x - y\| \leq (1 - \tau) \|x - y\|^2 + \tau \|u - v\|^2$$

for all $(x, u), (y, v) \in G(F)$ with $x \neq y$, which in turn implies

$$\langle u - v, x - y \rangle \leq (1 - \tau) \|x - y\|^2 + \tau \|u - v\|^2$$

for all $(x, u), (y, v) \in G(F)$, as required. ☐

Remark 1. *The choice of QL as the name of the class originates in "Quadratic Lipschitz", which is not a fully adequate description, and hence we keep only the acronym. The idea is that, as suggested by* (6) *and* (7) *above, operators which satisfy a "quadratic Lipschitz bound" belong to this class, while usual Lipschitz continuous operators with constant L satisfy a "constant Lipschitz bound" ($t \leq L$ with t as in* (6)*, instead of the quadratic bound for t given in* (7)*).*

Remark 2. *Since $1/\tau - 1$ converges to 1 as τ goes to $(1/2)^-$, and $1/\tau - 1$ tends to $+\infty$ as τ goes to 0^+, continuity of $1/\tau - 1$ ensures that any expansion is of type (QL) for an appropriate $\tau \in (0, 1/2)$.*

Proposition 1. *If $T : H \to \mathcal{P}(H)$ is maximal τ-(QL) then its graph is closed (in the strong topology).*

Proof. Elementary. ☐

Class (QL) contains a graph symmetric class, which is the class of those operators with semimonotone additive inverse. Let us denote this class as (SQL) (this acronym stands for "Symmetric Quadratic Lipschitz"). First we recall the definition of semimonotonicity, introduced in [2].

Definition 5. *A set-valued operator $T : H \to \mathcal{P}(H)$ is said to be ρ-semimonotone, $\rho \in (0, 1)$, when*

$$\langle x - y, u - v \rangle \geq -\frac{\rho}{2} \left(\|x - y\|^2 + \|u - v\|^2 \right) \quad \forall (x, u), (y, v) \in G(T). \tag{8}$$

Moreover, T is said to be maximal ρ-semimonotone if it is ρ-semimonotone and additionally $G(T) = G(T')$ for all ρ-semimonotone operator $T' : H \to \mathcal{P}(H)$ such that $G(T) \subset G(T')$.

Definition 6. *A set-valued operator $T : H \to \mathcal{P}(H)$ belong to class (SQL) with parameter $\rho \in (0,1)$ if and only if*

$$\langle x-y, u-v\rangle \leq \frac{\rho}{2}\left(\|x-y\|^2 + \|u-v\|^2\right) \quad \forall (x,u),(y,v) \in G(T). \tag{9}$$

Note that T is ρ-(SQL) if and only if $-T$ is ρ-semimonotone. Hence we say that T is maximal ρ-(SQL) if and only if $-T$ is maximal ρ-semimonotone.

Proposition 2. *If $T : H \to \mathcal{P}(H)$ is ρ-(SQL), with $\rho \in (0,1)$, then T is τ-(QL) with $\tau = \rho/2$.*

Proof. Assume that T is of (SQL) type for some $\rho \in (0,1)$. Then, for all $(x,u),(y,v) \in G(T)$,

$$\langle x-y, u-v\rangle \leq \frac{\rho}{2}\left(\|x-y\|^2 + \|u-v\|^2\right) \leq \left(1-\frac{\rho}{2}\right)\|x-y\|^2 + \frac{\rho}{2}\|u-v\|^2,$$

where the second inequality follows from the fact that $0 < \rho/2 \leq 1-\rho/2$, because $0<\rho<1$. $\square$

At this point we clarify the choice of $\tau < 1/2$ in Definition 4. This restriction in the value of τ is essential because otherwise any set-valued operator fits the class. In fact, any set-valued operator is 1-semimonotone and, in view of Proposition 2, of type 1/2-(QL).

It is also true that any contraction has semimonotone additive inverse and consequently is of (SQL) type for certain $\rho \in (0,1)$, but the same is not true for nonexpansive mappings in general. For example, the function F defined as $F(x) = x$ for all $x \in H$ is obviously nonexpansive and consequently of type (QL), but it is not of (SQL) type. In fact, for all $\rho \in (0,1)$ and all $x \neq y$, it holds that

$$\|x-y\|^2 = \langle F(x)-F(y), x-y\rangle \not\leq \rho\|x-y\|^2 = \frac{\rho}{2}\left(\|x-y\|^2 + \|F(x)-F(y)\|^2\right).$$

We give next some equivalent definitions of classes (SQL) and (QL).

Proposition 3.
1. *$T : H \to \mathcal{P}(H)$ is ρ-(SQL) if and only if*

$$\|(x-y)+(u-v)\| \leq \sqrt{1+\rho}\,\|(x-y,u-v)\| \quad \forall (x,u),(y,v) \in G(T)$$

if and only if

$$\|(x-y)-(u-v)\| \geq \sqrt{1-\rho}\,\|(x-y,u-v)\| \quad \forall (x,u),(y,v) \in G(T).$$

2. *$T : H \to \mathcal{P}(H)$ is τ-(QL) if and only if, for all $(x,u),(y,v) \in G(T)$,*

$$\|(x-y)+(u-v)\|^2 \leq (1+2\tau)\|(x-y,u-v)\|^2 + 2(1-2\tau)\|x-y\|^2$$

if and only if, for all $(x,u),(y,v) \in G(T)$,

$$\begin{aligned}\|(x-y)-(u-v)\|^2 &\geq (1-2\tau)\left(\|u-v\|^2 - \|x-y\|^2\right)\\ &= (1-2\tau)\|(x-y,u-v)\|^2 - 2(1-2\tau)\|x-y\|^2.\end{aligned}$$

Proof. Elementary. $\square$

Semimonotone operators have nice regularity properties; see [2]. We quote next some of them, which will be used in the sequel.

Proposition 4. *Let I be the identity operator in H. Take $\rho \in (0,1)$ and define $\beta, \gamma, \eta \in \mathbb{R}_{++}$ as*

$$\beta = \beta(\rho) = \frac{1 - \sqrt{1-\rho^2}}{\rho}, \tag{10}$$

$$\gamma = \gamma(\rho) = \frac{\rho}{2\sqrt{1-\rho^2}}, \tag{11}$$

$$\eta = \eta(\rho) = \frac{1 + \sqrt{1-\rho^2}}{\rho}. \tag{12}$$

i) An operator $T : H \to \mathcal{P}(H)$ is ρ-semimonotone if and only if the operator $(T+\beta I)^{-1}$ is γ-hypomonotone.

ii) Moreover, T is maximal ρ-semimonotone if and only if $(T+\beta I)^{-1}$ is maximal γ-hypomonotone.

iii) If $T : H \to \mathcal{P}(H)$ is ρ-semimonotone then the operator $(T+\lambda I)^{-1}$ is Lipschitz continuous for all $\lambda \in (\beta(\rho), \eta(\rho))$, with Lipschitz constant $L(\lambda)$ given by (13).

$$L(\lambda) = \frac{|1-\rho\lambda| + \sqrt{1-\rho^2}}{2\lambda - \rho(1+\lambda^2)}, \tag{13}$$

iv) If in addition T is maximal ρ-semimonotone, then $T + \lambda I$ is onto for all $\lambda \in (\beta(\rho), \eta(\rho))$.

Proof. See Theorem 2, Corollary 2 and Theorem 3 of [2]. $\square$

One of the consequences of these regularity properties is that $F(T)$ is nonempty and single valued for any maximal element of the class (SQL), as we prove next.

Corollary 1. *If $T : H \to \mathcal{P}(H)$ is maximal ρ-(SQL) for some $\rho \in (0,1)$ then T has a unique fixed point.*

Proof. Assume that $T : H \to \mathcal{P}(H)$ is maximal ρ-(SQL) with $\rho \in (0,1)$. Then $-T$ is maximal ρ-semimonotone and, in view of Proposition 4(iv), $-T + \lambda I$ is onto for all $\lambda \in (\beta(\rho), \eta(\rho))$, where β and η are given by equations (10) and (12) respectively. Observe now that $\lambda = 1$ belongs to $(\beta(\rho), \eta(\rho))$ for all $\rho \in (0,1)$. Hence, $-T + I$ is onto, and therefore there exist $x \in D(T) \subseteq H$ such that $0 \in (I-T)(x)$, or equivalently there exists $x \in H$ with $x \in T(x)$. Uniqueness follows from Proposition 4(iii), since Lipschitz continuity of $[I-T]^{-1}$ implies that the nonempty set $F(T) = [I-T]^{-1}(0)$ is a singleton. $\square$

Of course, operators in class (QL), even when they are maximal, my lack in general fixed points, because this class contains all nonexpansive mappings. For instance, for any $p \in H$, $p \neq 0$, the operator F defined as $F(x) = x+p$, for all $x \in H$, is nonexpansive and continuous with full domain, and hence maximal τ-(QL) for all $\tau \in (0, 1/2)$ (see Lemma 1), but F has no fixed points.

Another interesting consequence of the regularity properties associated to class (SQL), through its connection with semimonotonicity, is that $I - T$, with T of type (SQL), has hypomonotone inverse.

Corollary 2. *If $T : H \to \mathcal{P}(H)$ is maximal ρ-(SQL) for some $\rho \in (0, 1)$, then $(I-T)^{-1}$ is maximal ζ-hypomonotone, where*

$$\zeta = \frac{1 - \rho + \sqrt{1 - \rho^2}}{2(1 - \rho)} = \frac{1}{2}\left(1 + \sqrt{\frac{1 - \rho}{1 + \rho}}\right). \tag{14}$$

Proof. Apply Proposition 4(iii) to the ρ-semimonotone operator $-T$ in order to get Lipschitz continuity of $(-T + \lambda I)^{-1}$ with Lipschitz constant ζ given by (14), using (13) with $\lambda = 1 \in (\beta(\rho), \eta(\rho))$. Moreover, maximal ρ-semimonotonicity of $-T$ implies that $I - T$ is onto, in view of Proposition 4(iv). Then, $(I - T)^{-1}$ has full domain, is continuous and ζ-hypomonotone. Hence, $(I - T)^{-1} + \zeta I$ has full domain, is continuous and monotone; thus, maximal monotone. It follows that $(I - T)^{-1}$ is maximal ζ-hypomonotone. □

Though class (QL) is strictly bigger than class (SQL), each element of class (QL) can be identified with an element of class (SQL) and reciprocally, as we show next.

Proposition 5. *$T : H \to \mathcal{P}(H)$ satisfies property (QL) with $\tau \in (0, 1/2)$ if and only if $\beta(\rho(\tau))T$ is $\rho(\tau)$-(SQL), where $\beta(\rho)$ is given by equation* (10) *and $\rho(\tau)$ is defined as*

$$\rho(\tau) = \sqrt{1 - (1 - 2\tau)^2} = 2\sqrt{\tau(1 - \tau)}. \tag{15}$$

Proof. The functions $\rho(\cdot) : (0, 1/2) \to (0, 1)$ and $\beta(\cdot) : (0, 1) \to (0, 1)$, defined in (15) and (10) respectively, are bijections, so that $(\beta \circ \rho)(\cdot) : (0, 1/2) \to (0, 1)$ is a bijection too. Fix $\tau \in (0, 1/2)$ and assume that $T : H \to \mathcal{P}(H)$ is such that $\beta(\rho(\tau))T$ is $\rho(\tau)$-(SQL), i.e.,

$$\langle x - y, \hat{u} - \hat{v}\rangle \leq \frac{\rho(\tau)}{2}\left(\|x - y\|^2 + \|\hat{u} - \hat{v}\|^2\right) \tag{16}$$

for all $(x, \hat{u}), (y, \hat{v}) \in G\left(\beta(\rho(\tau))T\right)$. Observe now that $\hat{u} \in \beta(\rho(\tau))T(x)$ if and only if $\hat{u} = \beta(\rho(\tau))u$ with $u \in T(x)$ and, similarly, $\hat{v} \in \beta(\rho(\tau))T(y)$ if and only if $\hat{v} = \beta(\rho(\tau))v$ with $v \in T(y)$, so that (16) is equivalent to

$$\langle x - y, \beta(\rho(\tau))(u - v)\rangle \leq \frac{\rho(\tau)}{2}\left(\|x - y\|^2 + \|\beta(\rho(\tau))(u - v)\|^2\right)$$

for all $(x,u),(y,v) \in G(T)$, which is itself equivalent to

$$\langle x-y, u-v\rangle \leq \frac{\rho(\tau)}{2\beta(\rho(\tau))}\|x-y\|^2 + \frac{\rho(\tau)\beta(\rho(\tau))}{2}\|u-v\|^2$$

for all $(x,u),(y,v) \in G(T)$. Finally, in order to prove that T is of class τ-(QL), note that

$$\frac{\rho(\tau)}{2\beta(\rho(\tau))} = 1-\tau$$

and

$$\frac{\rho(\tau)\beta(\rho(\tau))}{2} = \tau.$$

□

Corollary 3. *$T : H \to \mathcal{P}(H)$ is maximal τ-(QL) with $\tau \in (0,1/2)$ if and only if $\beta(\rho(\tau))T$ is maximal $\rho(\tau)$-(SQL), where $\beta(\rho)$ is given by equation* (10) *and $\rho(\tau)$ is given by* (15)*.*

Proof. It follows from Proposition 5. □

2.1 Smooth operators of classes (QL) and (SQL)

It is worthwhile to look at classes (QL) and (SQL) in the case of smooth operators. For the sake of simplicity we assume that $H = \mathbb{R}^n$. We start with the affine case, for which we have the following complete characterization.

Proposition 6. *Define $T : \mathbb{R}^n \to \mathbb{R}^n$ as $T(x) = Ax+b$ with $A \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^n$. For $\tau \in (0,1/2), \rho \in (0,1)$, define $L_{T,\tau} = (1-\tau)I + \tau A^tA - (1/2)(A^t+A)$, $M_{T,\rho} = (\rho/2)[I + A^tA] - (1/2)(A^t+A)$. Then*

i) T is τ-(QL) if and only if $L_{T,\tau}$ is positive semidefinite.

ii) T is ρ-(SQL) if and only if $M_{T,\rho}$ is positive semidefinite.

Proof. For statement (i), note that

$$(x-y)^tL_{T,\tau}(x-y) = (1-\tau)\|x-y\|^2 + \tau\|T(x)-T(y)\|^2 - \langle x-y, T(x)-T(y)\rangle \tag{17}$$

for all $x,y \in \mathbb{R}^n$. If $L_{T,\tau}$ is positive semidefinite, then the left hand side of (17) is nonnegative, and hence (5) holds. If T is τ-(QL) then the right hand side is nonnegative, implying positive semidefiniteness of $L_{T,\tau}$. A similar calculation establishes statement (ii). □

In the symmetric case, we get the characterization in terms of the location of the eigenvalues of A.

Corollary 4. *Define $T : \mathbb{R}^n \to \mathbb{R}^n$ as $T(x) = Ax + b$ with $A \in \mathbb{R}^{n\times n}$ symmetric, $b \in \mathbb{R}^n$.*

i) T is τ-(QL) if and only if all eigenvalues of A lie outside the interval $(1, 1/\tau - 1)$.

ii) T is ρ-(SQL) if and only if all eigenvalues of A lie outside the interval (β, η), with $\beta = \beta(\rho), \eta = \eta(\rho)$ as in (10), (12) respectively.

Proof. The results follow from Proposition 6, noting that in the symmetric case we have $L_{T,\tau} = (1-\tau)I + \tau A^2 - A$, $M_{T,\rho} = (\rho/2)(I + A^2) - A$. Let $\omega_1, \dots, \omega_n$ be the eigenvalues of A. Define

$$\bar{\omega}_i = (1-\tau) + \tau\omega_i^2 - \omega_i \quad (1 \le i \le n), \tag{18}$$

$$\hat{\omega}_i = (\rho/2)(1 + \omega_i^2) - \omega_i \quad (1 \le i \le n). \tag{19}$$

It follows easily that $\bar{\omega}_1, \dots, \bar{\omega}_n$ are the eigenvalues of $L_{T,\tau}$ and $\hat{\omega}_1, \dots, \hat{\omega}_n$ are the eigenvalues of $M_{T,\rho}$. It is an immediate consequence of (18), (19) that the eigenvalues of $L_{T,\tau}, M_{T,\rho}$ are nonnegative if and only if the eigenvalues of A lie outside the respective open intervals. □

It follows from Corollary 4 that every affine operator with symmetric linear part belongs to class (QL) for some $\tau \in (0, 1/2)$: it suffices to take $\tau > 1/(\bar{\omega} + 1)$, where $\bar{\omega}$ is the smallest eigenvalue of A larger than 1 (if all eigenvalues of A are less than or equal to 1, then T is in class (QL) for all $\tau \in (0, 1/2)$). On the other hand, since $1 \in (\beta(\rho), \eta(\rho))$ for all $\rho \in (0, 1)$, not all affine operators with symmetric linear part belong to class (SQL): if 1 is an eigenvalue of A then T does not belong to class (SQL) for any $\rho \in (0, 1)$; if 1 is not an eigenvalue then T belongs to class (SQL) for ρ close enough to 0 so that the interval $(\beta)\rho), \eta(\rho))$ does not contain any eigenvalue of A.

We consider now the case of nonlinear smooth operators. For $T : \mathbb{R}^n \to \mathbb{R}^n$ of class $\mathcal{C}^1$, we will denote as $J_T(x) \in \mathbb{R}^{n\times n}$ the Jacobian matrix of T at x, and as $\widetilde{J}_T(x)$ its symmetrization, i.e.,

$$\widetilde{J}_T(x) = \frac{1}{2}\left[J_T(x)^t + J_T(x)\right].$$

The reasonable conjecture is that Proposition 6 holds also in this case, after replacing A by $J_T(x)$ in the definitions of $L_{T,\tau}, M_{T,\rho}$. We have only a partial proof of this extension, in the sense that we will prove the "if" statement only for the symmetric case, and with an additional technical hypothesis. We conjecture that the "if" statement holds also without these limitations. There is no trouble with the "only if" statement, as we show next.

For $\tau \in (0, 1/2), \rho \in (0, 1)$, define the matrix functions $L_{T,\tau}, M_{T,\rho} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ as

$$L_{T,\tau}(x) = (1-\tau)I + \tau J_T(x)^t J_T(x) - \widetilde{J}_T(x), \tag{20}$$

$$M_{T,\rho}(x) = \frac{\rho}{2}\left[I + J_T(x)^t J_T(x)\right] - \widetilde{J}_T(x). \tag{21}$$

Proposition 7. *Assume that $T : \mathbb{R}^n \to \mathbb{R}^n$ is twice differentiable.*

i) If T is τ-(QL), then $L_{T,\tau}(x)$ is positive semidefinite for all $x \in \mathbb{R}^n$.

ii) If T is ρ-(SQL), then $M_{T,\rho}(x)$ is positive semidefinite for all $x \in \mathbb{R}^n$.

Proof. i) Note that in the smooth case (5) can be rewritten as

$$0 \leq (1-\tau)\|x-y\|^2 + \tau\|T(y)-T(x)\|^2 - \langle y-x, T(y)-T(x)\rangle \quad \forall x, y \in \mathbb{R}^n. \tag{22}$$

Take $x, z \in \mathbb{R}^n$ and consider (22) with $y = x + tz$, $t \in \mathbb{R}$, so that (22) becomes

$$0 \leq (1-\tau)t^2\|z\|^2 + \tau\|T(x+tz)-T(x)\|^2 - t\langle T(x+tz)-T(x), z\rangle. \tag{23}$$

Expanding $T(x+tz)$ around $t=0$, we get from (23),

$$0 \leq (1-\tau)t^2\|z\|^2 + \tau t^2\|J_T(x)z\|^2 - t^2 z^t J_T(x)z + o(t^2) =$$

$$t^2\left\{z^t\left[(1-\tau)t^2 I + \tau J_T(x)^t J_T(x) - \widetilde{J}_T(x)\right]z\right\} + o(t^2) = t^2\left[z^t L_{T,\tau}(x)z\right] + o(t^2), \tag{24}$$

using (20) in the last equality. Dividing the extreme expressions of (24) by t^2 and taking the limit as $t \to 0$, we get $0 \leq z^t L_{T,\tau}(x)z$ for all $x, z \in \mathbb{R}^n$, i.e. positive definiteness of $L_{T,\tau}(x)$ for all $x \in \mathbb{R}^n$.

ii) Similar to (i), starting from (9) instead of (5).

□

Corollary 5. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be twice differentiable with $J_T(x)$ symmetric for all $x \in \mathbb{R}^n$.*

i) If T is τ-(QL), then for all $x \in \mathbb{R}^n$ all eigenvalues of $J_T(x)$ lie outside the open interval $(1, 1/\tau - 1)$.

ii) If T is ρ-(SQL), then for all $x \in \mathbb{R}^n$ all eigenvalues of $J_T(x)$ lie outside the open interval (β, η), with $\beta = \beta(\rho), \eta = \eta(\rho)$ as in (10), (12) respectively.

Proof. i) If $J_T(x)$ is symmetric then $\widetilde{J}_T(x) = J_T(x)$ and $J_T^t(x)J_T(x) = [J_T(x)]^2$, so that $L_{T,\tau}(x) = (1-\tau)I + \tau[J_T(x)]^2 - J_T(x)$. Note that if ω is an eigenvalue of $J_T(x)$ then $(1-\tau) + \tau\omega^2 - \omega$ is an eigenvalue of $L_{T,\tau}(x)$. Take any eigenvalue ω of $J_T(x)$. If T is τ-(QL), then by Proposition 7(i) $0 \leq (1-\tau) + \tau\omega^2 - \omega$ and hence $\omega \notin (1, 1/\tau - 1)$.

ii) Similar to (i), using Proposition 7(ii). □

In connection with the symmetry assumption in Corollary 5, we recall the following well known fact: $J_T(x)$ is symmetric for all $x \in \mathbb{R}^n$ if and only if $T(x) = \nabla h(x)$ for some smooth $h : \mathbb{R}^n \to \mathbb{R}$.

For the converse result, we will invoke Propositions 4 and 5, which will force us to exclude, due to technical reasons, the values 1 and β as eigenvalues of $J_T(x)$, in the case of τ-(QL) and ρ-(SQL) operators respectively.

Proposition 8. *Assume that $T : \mathbb{R}^n \to \mathbb{R}^n$ is of class $\mathcal{C}^1$, with $J_T(x)$ symmetric for all $x \in \mathbb{R}^n$.*

i) If for all $x \in \mathbb{R}^n$ all eigenvalues of $J_T(x)$ lie outside the interval $[\beta, \eta)$, with $\beta = \beta(\rho), \eta = \eta(\rho)$ as in (10), (12)*, then T is ρ-(SQL).*

ii) If all eigenvalues of $J_T(x)$ lie outside the interval $[1, 1/\tau - 1)$, for all $x \in \mathbb{R}^n$, then T is τ-(QL).

Proof. We prove statement (i). In view of Definitions 5 and 6, we must prove that $-T$ is ρ-semimonotone. In view of Proposition 4(i), it suffices to check that the operator $(\beta I - T)^{-1}$ is γ-hypomononote, with $\gamma = \gamma(\rho)$ as in (11), which, as observed right after Definition 3, is equivalent to monotonicity of the operator S defined as

$$S(x) = [(\beta I - T)^{-1} + \gamma I](x). \tag{25}$$

We mention that the equivalence given in Proposition 4(i) holds even when the domain of $(\beta I - T)^{-1}$ is not the full space $\mathbb{R}^n$. But at this point we need $S(x)$ to be differentiable for all $x \in \mathbb{R}^n$, for which we invoke the Inverse Function Theorem, requiring nonsingularity of the Jacobian matrix of the operator $\beta I - T$, namely $\beta I - J_T$, on the whole $\mathbb{R}^n$. Since β is not an eigenvalue of $J_T(x)$ for all $x \in \mathbb{R}^n$, by assumption, such Jacobian matrix is indeed nonsingular, and we conclude that S is point-to-point and indeed of class $\mathcal{C}^1$ on the whole $\mathbb{R}^n$. Define $V(x) = S(x) - \gamma x$. Note that (25) can be rewritten as

$$T(V(x)) = \beta S(x) - (1 + \beta\gamma)x. \tag{26}$$

Let $J_S(x)$ denote the Jacobian matrix of S at x. Differentiating (26) we get

$$J_T(V(x))\,[J_S(x) - \gamma I] = \beta J_S(x) - (1 + \beta\gamma)I = \beta\,[J_S(x) - \gamma I] - I. \tag{27}$$

We claim that $J_S(x) - \gamma I$ is nonsingular for all $x \in \mathbb{R}^n$. Otherwise, there exists $u \neq 0$ such that $[J_S(x) - \gamma I]u = 0$, and multiplying the extreme expressions of (27) by u we get $0 = 0 - u$, a contradiction. The claim is established, and multiplying the extreme expressions of (27) by $[J_S(x) - \gamma I]^{-1}$ we get

$$J_T(V(x)) = \beta I - [J_S(x) - \gamma I]^{-1} = \beta I + [\gamma I - J_S(x)]^{-1}. \tag{28}$$

Note that symmetry of $J_T(x)$ implies symmetry of $J_S(x)$, and hence all eigenvalues of $J_S(x)$ are real numbers. Let ω be an eigenvalue of $J_S(x)$. It follows from (28) that $\beta + 1/(\gamma - \omega)$ is an eigenvalue of $J_T(V(x))$. By assumption, the eigenvalues of $J_T(V(x))$ lie outside the interval $[\beta, \eta)$, i.e., either $\beta + 1/(\gamma - \omega) < \beta$, in which case $\gamma - \omega < 0$, and hence $\omega > \gamma > 0$, or $\beta + 1/(\gamma - \omega) \geq \eta$, i.e. $1/(\gamma - \omega) \geq \eta - \beta > 0$, so that $\omega \geq \gamma - 1/(\eta - \beta) = 0$, taking into account (10), (11) and (12). We have proved that all eigenvalues of $J_S(x)$ are nonnegative, and therefore S is monotone, which as explained above, implies that T is ρ-(SQL).

We prove now statement (ii). By assumption, the eigenvalues of $J_T(x)$ lie outside the interval $[1, 1/\tau - 1)$ for all $x \in \mathbb{R}^n$. Consider the operator T' defined as $T'(x) = \beta(\rho(\tau))T(x) = \sqrt{\tau/(1-\tau)}T(x)$. It is immediate that the eigenvalues of the Jacobian matrix of T' lie outside the interval

$$\left[\sqrt{\frac{\tau}{1-\tau}}, \sqrt{\frac{1-\tau}{\tau}}\right) = [\beta(\rho(\tau)), \eta(\rho(\tau))).$$

It follows from (i) that $T' = \beta(\rho(\tau))T$ is $\rho(\tau)$-(SQL). Proposition 5 entails now that T is τ-(QL). $\square$

3 Algorithms for operators of type (QL)

In this section we present predictor-corrector methods for finding fixed points of set-valued operators that satisfy property (QL) for some $\tau \in (0, 1/2)$. The predictor step will consist of an approximate solution of the inclusion in (4). The error criteria use the constants $\beta = \beta(\rho(\tau))$, provided by Proposition 5, and $\gamma = \gamma(\rho(\tau))$, where $\gamma(\rho)$ is given by (11) and $\rho(\tau)$ is given by (15), i.e.

$$\beta = \sqrt{\frac{\tau}{1-\tau}}$$

and

$$\gamma = \frac{\sqrt{\tau(1-\tau)}}{1-2\tau}.$$

The algorithms requires also an exogenous sequence $\{\lambda_k\} \subset \mathbb{R}_{++}$ bounded away from zero, and constants $\sigma \in [0, 1)$, $\bar{\lambda} = \dfrac{1}{\beta} \inf\{\lambda_k\}$ and

$$\nu = \frac{\sqrt{\sigma + (1-\sigma)\left(2\gamma/\bar{\lambda}\right)^2} - 2\gamma/\bar{\lambda}}{1 + 2\gamma/\bar{\lambda}}. \tag{29}$$

Algorithms:

1. Choose $x^0 \in H$.

2. Given x^k, find $\tilde{x}^k$ and v^k satisfying

$$\begin{aligned} &v^k \in T(\tilde{x}^k) \\ &e^k = (\lambda_k + 1)\tilde{x}^k - \lambda_k v^k - x^k, \end{aligned} \tag{30}$$

and such that

$$\left\|e^k\right\| \leq \sigma\beta\left(\frac{\bar{\lambda}}{2} - \gamma\right)\left\|\tilde{x}^k - v^k\right\|, \tag{31}$$

or

$$\left\|e^k\right\| \leq \nu\left\|\tilde{x}^k - x^k\right\|, \tag{32}$$

3. Let

$$x^{k+1} = \tilde{x}^k - e^k \tag{33}$$

From now on, Algorithm 1 refers to the algorithm with error criterium given by (31), and Algorithm 2 to the one with error criterium given by (32).

We will show that, through a particular change of variables, Algorithms 1 and 2 reduce to two proximal point methods for hypomonotone operators developed in [4]. Thus, we proceed to describe these methods and its convergence properties.

Consider a set-valued operator $T : H \to \mathcal{P}(H)$ whose inverse is maximal ρ-hypomonotone. Take a sequence $\{\xi_k\} \subset \mathbb{R}_+$, bounded away from 0.

Given $x^k \in H$, find $(y^k, v^k) \in H \times H$ such that

$$v^k \in T(y^k), \tag{34}$$

$$\xi_k v^k + y^k - x^k = e^k, \tag{35}$$

where the error term e^k satisfies either

$$\left\| e^k \right\| \leq \sigma \left(\frac{\hat{\xi}}{2} - \rho \right) \left\| v^k \right\|, \tag{36}$$

or

$$\left\| e^k \right\| \leq \nu \left\| y^k - x^k \right\|, \tag{37}$$

with

$$\nu = \frac{\sqrt{\sigma + (1-\sigma)\left(2\rho/\hat{\xi}\right)^2} - 2\rho/\hat{\xi}}{1 + 2\rho/\hat{\xi}}, \tag{38}$$

where $\sigma \in [0, 1)$, $\hat{\xi} = \inf\{\xi_k\}$, and ρ is the hypomonotonicity constant of T^{-1}. Then, under any of our two error criteria, the next iterate x^{k+1} is given by

$$x^{k+1} = x^k - \xi_k v^k. \tag{39}$$

We will denote as Algorithm A the method given by (34)–(36) and (39), and as Algorithm B to the one given by (34), (35) and (37)–(39).

It has been proved in Corollary 2 of [4], that if $2\rho < \hat{\xi}$ and T has zeroes, then the sequence $\{x_k\}$ generated by either Algorithm A or Algorithm B is weakly convergent to a zero of T, starting from any $x^0 \in H$.

Also, it has been proved in Theorem 1 of [4] that under the same assumptions, the sequences $\{w_k\}$ and $\{x_k - y_k\}$ generated by these algorithms are strongly convergent to 0.

We present now the convergence results for Algorithms 1 and 2.

Theorem 1. *Consider Algorithms 1 and 2 applied to a maximal τ-(QL) operator T, with $\tau \in (0,1/2)$. If the regularization parameters $\{\lambda_k\}$ are chosen so that $\inf\{\lambda_k\} > 2\beta\gamma = 2\tau/(1-2\tau)$ for all k (i.e. so that $\bar{\lambda} > 2\gamma$), then the sequences $\{x^k\}$ generated by either Algorithm 1 or Algorithm 2 are weakly convergent to a fixed point of T, starting from any $x^0 \in H$, under the only assumption of existence of fixed points of T.*

Proof. Assume that T is maximal τ-(QL) for some $\tau \in (0,1/2)$ and has fixed points. Fix $\rho(\tau)$ as in (15), $\beta(\rho)$ as in (10) and $\gamma(\rho)$ as in (11). Define $\beta = \beta(\rho(\tau)) = \sqrt{\tau/(1-\tau)}$ and $\gamma = \gamma(\rho(\tau)) = \sqrt{\tau(1-\tau)}/(1-2\tau)$. Then, in view of Proposition 5, $-\beta(\rho(\tau))T$ is maximal $\rho(\tau)$-semimonotone and $T_\beta = -\beta(\rho(\tau))T + \beta(\rho(\tau))I$ has maximal $\gamma(\rho(\tau))$-hypomonotone inverse (see Proposition 4(ii)) and zeros. Moreover, it follows easily from the definitions of Algorithms 1, 2, A and B that $\{v^k\}$, $\{\tilde{x}^k\}$, $\{x^k\}$, $\{e^k\}$ are sequences generated by Algorithm 1 (respectively, Algorithm 2) if and only if $w^k = \beta(y^k - v^k) \in T_\beta(\tilde{x}^k)$, $y^k = \tilde{x}^k$, $\{x^k\}$ and $\{e^k\}$ are sequences generated by Algorithm A (respectively, Algorithm B) of [4] with regularizing parameters $\xi_k = \lambda_k/\beta$. Since $\inf\{\xi_k\} = \bar{\lambda}/\beta > 2\gamma$, where $\gamma = \gamma(\rho(\tau))$ is the hypomonotonicity constant of T_β^{-1}, we can apply the above mentioned result established in Corollary 2 of [4], in order to ensure that $\{x^k\}$ converges weakly to a point $\bar{x} \in T_\beta^{-1}(0)$. Thus, $0 \in T_\beta(\bar{x}) = -\beta T(\bar{x}) + \beta\bar{x}$, concluding that $\bar{x} \in T(\bar{x})$. □

We comment now on some features of Algorithms 1 and 2. Note that for $e^k = 0$, in which case we have the "exact" versions of both algorithms, they reduce to solving the implicit inclusion:

$$x^{k+1} + \lambda_k^{-1}(x^{k+1} - x^k) \in T(x^{k+1}), \tag{40}$$

which is equivalent to

$$x^{k+1} \in \left[I + \lambda_k(I - T)\right]^{-1}(x^k),$$

that is to say, to the basic scheme (4) with $\beta_k = \lambda_k$.

We also mention that class (QL) becomes indeed quite large as τ approaches $1/2$ (as commented earlier, for $\tau = 1/2$ class (QL) contains all set-valued operators in H). Algorithms 1 and 2 do indeed converge for operators in class (QL) having fixed points with any $\tau \in (0,1/2)$, but when τ approaches the value $1/2$ there is a price to be paid: according to Theorem 1, in order to obtain convergence the regularization parameters λ_k must satisfy

$$\lambda_k > \frac{2\tau}{1-2\tau}. \tag{41}$$

As τ approaches $1/2$, the right hand side of (41) tends to $+\infty$ and hence the same holds for the regularization parameters λ_k. In view of (40), for very large values of λ_k the regularization effect of the basic scheme vanishes, since in the limit with $\lambda_k \to +\infty$, (40) reduces to the original (and possibly ill-conditioned) problem $0 \in (I-T)(x^{k+1})$. In other words, as τ approaches $1/2$ Algorithms 1 and 2 gradually lose their regularization effects (and at the same time the operator T is farther away from being a contraction, i.e. it becomes less and less "regularizable").

3.1 Strongly convergent algorithms for (SQL) type operators.

This subsection is devoted to a slight variation of Algorithms 1 and 2, better suited to the smaller class of (SQL) operators. Assume that $T : H \to \mathcal{P}(H)$ is maximal ρ-(SQL) with $\rho \in (0,1)$. Of course, T is $\rho/2$-(QL) maximal and the results of the previous section still hold. But in this case we invoke Corollaries 1 and 2 for ensuring that T has a unique fixed point and $I-T$ has Lipschitz inverse with constant ζ given by (14). Hence, $(I-T)^{-1}$ is maximal ζ-hypomonotone and the result of Theorem 1 still holds. Nevertheless, we want to modify both algorithms in order to allow the choices of $\beta = 1$ and $\gamma = \zeta$ as parameters of the algorithms. Thus, we replace the error criterium given in (31) by

$$\left\| e^k \right\| \le \sigma \left(\frac{\bar{\lambda}}{2} - \zeta \right) \left\| \tilde{x}^k - v^k \right\|, \tag{42}$$

and, in (32), we change the value of ν given in (29), using instead

$$\nu = \frac{\sqrt{\sigma + (1-\sigma)\left(2\zeta/\bar{\lambda}\right)^2} - 2\zeta/\bar{\lambda}}{1 + 2\zeta/\bar{\lambda}}, \tag{43}$$

where $\bar{\lambda} = \inf\{\lambda_k\}$.

Theorem 2. *If T is maximal ρ-(SQL) then the modified Algorithms 1 and 2 (i.e., with* (31) *replaced by* (42)*, and ν given by* (43) *instead of* (29)*), generate sequences $\{x^k\}$ which are strongly convergent to the unique fixed point of T, starting from any $x^0 \in H$, assuming that $2\zeta < \bar{\lambda}$.*

Proof. The argument is similar to that in the proof of Theorem 1, but now applied to the maximal ζ-hypomonotone operator $(I-T)^{-1}$, which has a unique zero, so that $w^k = (y^k - v^k) \in (I - T)(\tilde{x}^k)$, $y^k = \tilde{x}^k$, x^k, e^k are sequences generated by the above described Algorithm A (respectively, Algorithm B) of [4] with regularizing parameters $\xi_k = \lambda_k$ and $2\zeta < \bar{\lambda} = \inf\{\lambda_k\}$. Then, we can apply once again Corollary 2 of [4] for ensuring that $\{x^k\}$ converges weakly to $\bar{x} \in (I-T)^{-1}(0)$. Invoking the above mentioned result in Theorem 1 of [4], we get that w^k and $y^k - x^k$ are strongly convergent to 0. Thus, Lipschitz continuity of $(I-T)^{-1}$ implies that

$$\left\| \tilde{x}^k - \bar{x} \right\| = \left\| (I-T)^{-1}(w^k) - (I-T)^{-1}(0) \right\| \le \zeta \left\| w^k - 0 \right\|.$$

It follows that $y^k = \tilde{x}^k$ converges strongly to $\bar{x}$ and so does $\{x^k\}$. □

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