L^∞ SOLUTIONS FOR A MODEL OF NON-ISOTHERMAL POLYTROPIC GAS FLOW

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ABSTRACT. We establish the global existence of L^∞ solutions for a model of polytropic gas flow with varying temperature governed by a Fourier equation in the Lagrangian coordinates. The result is obtained by showing the convergence of a class of finite difference schemes, which includes the Lax–Friedrichs and Godunov schemes. Such convergence is achieved by proving the estimates required for the application of the compensated compactness theory.

1. INTRODUCTION

We consider the following system modeling a gas flow with a pressure-density-temperature equation of state of the form

$$p(\rho,\vartheta)=\kappa\vartheta\rho^\gamma,$$

where p denotes the pressure, ρ is the density, ϑ the temperature, $\gamma>1$ and $\kappa=\frac{1}{4\gamma}(\gamma-1)^2$. In the nomenclature of [7], this means that the thermal pressure $p_{th}(\rho,\vartheta)$ and the elastic pressure $p_e(\rho)$ satisfy $p_{th}(\rho,\vartheta)=p(\rho,\vartheta)$, $p_e(\rho)=0$. In particular, by Maxwell's relationship we get $e=Q(\vartheta)$, where e is the internal energy, that is, e is a function only of the temperature.

The model assumes that ϑ is governed by a Fourier equation in Lagrangian coordinates and, in Eulerian coordinates, reads

$$\rho_t+m_x=0, \tag{1.1}$$

$$m_t+\Big(\frac{m^2}{\rho}+p(\rho,\vartheta)\Big)_x=0, \tag{1.2}$$

$$T\#\left((\rho\vartheta)_t+(m\vartheta-\frac{1}{\rho}\vartheta_x)_x\right)=0, \tag{1.3}$$

where m is the momentum defined as $m=\rho u$, where u is the gas velocity. We denote by T the Lagrangian transformation determined modulo constants by $T(x,t)=(y(x,t),t)$, with y satisfying

$$\frac{\partial y}{\partial x}=\rho,\qquad \frac{\partial y}{\partial t}=-m, \tag{1.4}$$

and $T\#$ denotes the corresponding push-forward operator[1] from the distributions on $\mathbb{R}^2_+:=\mathbb{R}\times(0,\infty)$ in the (x,t) coordinates to the distributions on $\mathbb{R}^2_+$ in the (y,t)

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[1] The push-forward of a distribution of the form $\operatorname{div} F$, where $F\in L^\infty(\Omega;\mathbb{R}^n)$, with $\Omega\subseteq\mathbb{R}^n$, by a Lipschitz continuous mapping $f:\Omega\to\tilde\Omega\subseteq\mathbb{R}^N$, is the distribution $f\#(\operatorname{div} F)$ in $\tilde\Omega$ defined as $\langle f\#(\operatorname{div} F),\psi\rangle=-\langle F,\nabla(\psi\circ f)\rangle$ for all $\psi\in C_0^\infty(\tilde\Omega)$.

coordinates. We observe that, when vacuum occurs, T is not invertible and so we cannot get rid of the push-forward operation in (1.3).

The system (1.1)–(1.3) is a mathematical model intended to approximate the more physical model where equation (1.3) is replaced by the thermal energy equation (see, e.g., (1.36) in [7]), which reduces to (1.3) if the terms involving the velocity are neglected, and this motivates our mathematical model.

Initial data are given by

$$\rho(x,0) = \rho_0(x), \qquad m(x,0) = m_0(x), \tag{1.5}$$

$$\vartheta(x,0) = \vartheta_0(x) = \sigma(y_0(x)), \qquad y_0(x) = \int_0^x \rho_0(z)\,dz. \tag{1.6}$$

Assume that

$$\rho_0, m_0, \frac{m_0}{\rho_0} \in L^\infty(\mathbb{R}), \qquad \rho_0 \geq 0, \qquad 0 < \delta_0 \leq \sigma \in W^{3,2}_{\mathrm{loc}}(\mathbb{R}). \tag{1.7}$$

In particular, the initial data (and the solution) allows for the occurrence of vacuum. In addition, we also assume that σ is periodic with period, say, 2π, that is,

$$\sigma(y + 2\pi) = \sigma(y), \qquad y \in \mathbb{R}. \tag{1.8}$$

We remark that assumption (1.7), imposed on σ, implies that the solution of the heat equation with initial data σ,

$$\tilde{\sigma}(y,t) := \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}} e^{-(y-z)^2/4t} \sigma(z)\,dz, \tag{1.9}$$

satisfies

$$|\tilde{\sigma}(y,t) - \bar{\sigma}|, |\tilde{\sigma}_y(y,t)|, |\tilde{\sigma}_{yy}(y,t)| \leq C_0 e^{-t}, \quad \text{with} \quad \bar{\sigma} := \frac{1}{2\pi}\int_0^{2\pi} \sigma(z)\,dz, \tag{1.10}$$

for some absolute constant $C_0 > 0$. Indeed, (1.7) and (1.8) imply the absolute convergence of the Fourier series of σ, σ' and σ''. On the other hand, a straightforward calculation shows that

$$\begin{aligned} \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}} e^{-(y-z)^2/4t} e^{ikz}\,dz &= \frac{e^{(-y^2 + (y+2ikt)^2)/4t}}{(4\pi t)^{1/2}} \int_{\mathbb{R}} e^{\frac{-(z-(y+2ikt))^2}{4t}}\,dz \\ &= e^{iky - k^2 t}, \end{aligned}$$

for any $k \in \mathbb{R}$, which then gives the asserted asymptotic behavior, by plugging the Fourier series for σ, σ' and σ'' in (1.9) and the corresponding equations for $\tilde{\sigma}_y$ and $\tilde{\sigma}_{yy}$, obtained from (1.9) by replacing σ by σ' and σ'', respectively.

We have the following definition of weak solution.

Definition 1.1. We say that a function $(\rho, m, \vartheta) \in L^\infty(\mathbb{R}^2_+)$ is a weak solution to (1.1)–(1.6) if:

(i) $m/\rho \in L^\infty(\mathbb{R}^2_+)$;
(ii) $\vartheta \in W^{1,\infty}(\mathbb{R}^2_+)$, $\vartheta_x/\rho \in L^\infty(\mathbb{R}^2_+)$;
(iii) for all $\phi, \psi \in C_0^\infty(\mathbb{R}^2)$,

$$\begin{aligned} &\int_{\mathbb{R}^2_+} (\rho, m)(x,t)\phi_t + (m, \frac{m^2}{\rho} + p(\vartheta,\rho))(x,t)\phi_x\,dx\,dt \\ &\qquad + \int_{\mathbb{R}} (\rho_0, m_0)(x)\phi(x,0)\,dx = 0, \end{aligned} \tag{1.11}$$

$$\begin{aligned} &\int_{\mathbb{R}^2_+} \rho\vartheta \frac{\partial}{\partial t}\psi(y(x,t),t) + (m\vartheta - \frac{1}{\rho}\vartheta_x)\frac{\partial}{\partial x}\psi(y(x,t),t)\,dx\,dt \\ &\qquad + \int_{\mathbb{R}} \rho_0\vartheta_0\psi(y_0(x),x)\,dx = 0. \end{aligned} \tag{1.12}$$

We observe that, in Definition 1.1, we ask that $\rho, m, \vartheta, \vartheta_x/\rho \in L^\infty(\mathbb{R}^2_+)$ and so we have $(\rho\vartheta, m\vartheta - \frac{\vartheta_x}{\rho}) \in L^\infty(\mathbb{R}^2_+)$. Therefore, the push-forward operation $T\#\left(\rho\vartheta, m\vartheta - \frac{\vartheta_x}{\rho}\right)$ is well-defined by

(1.13)
$$\left\langle T\#\left((\rho\vartheta)_t + (m\vartheta - \frac{\vartheta_x}{\rho})_x\right), \psi\right\rangle := -\left\langle\left(\rho\vartheta, m\vartheta - \frac{\vartheta_x}{\rho}\right), (\frac{\partial}{\partial t}\psi\circ T, \frac{\partial}{\partial x}\psi\circ T)\right\rangle,$$

for all $\psi \in C_0^\infty(\mathbb{R}^2_+)$, where on the left-hand side $\langle\cdot,\cdot\rangle$ denotes the pairing $\langle W^{-1,\infty}, W^{1,1}\rangle$, while on the right-hand side it is the pairing $\langle (L^\infty)^2, (L^1)^2\rangle$. Item (iii) of Definition 1.1 is the natural extension of (1.13) in order to take into account the initial data.

Our main result reads as follows.

Theorem 1.1. *There exists a constant $r(\gamma) > 0$ such that if $\|(\rho_0, m_0)\|_\infty < r(\gamma)$, then there exists a global weak solution to the Cauchy problem (1.1)–(1.6) satisfying an entropy inequality of the form*

$$\eta_*(\rho, m, \vartheta)_t + q_*(\rho, m, \vartheta)_x \le Ce^{-t}, \tag{1.14}$$

in the sense of distributions, for some $C > 0$ depending on L^∞ bounds for ρ, m, ϑ, where

$$\eta_*(\rho, m, \vartheta) = \frac{1}{2}\rho u^2 + \frac{\kappa}{\gamma - 1}\vartheta\rho^\gamma, \qquad q_*(\rho, m, \vartheta) = u\eta_*(\rho, m, \vartheta) + pu. \tag{1.15}$$

Moreover, $r(\gamma) \to \infty$ as $\gamma \to 1+$. Further, if ρ_0, m_0 are periodic with period L such that $y_0(L) = 2\pi$, we have the following decay

$$\lim_{t\to\infty}\int_0^L |\,(\rho(x,t), m(x,t), \vartheta(x,t)) - (\bar\rho, \bar m, \bar\vartheta)|\, dx = 0, \tag{1.16}$$

where $\bar\rho, \bar m, \bar\vartheta$ are the mean values of ρ_0, m_0, ϑ_0, respectively.

The following sections of this paper are dedicated to the proof of Theorem 1.1. We will construct approximate solutions using an adaptation of Godunov's finite difference scheme which roughly runs as follows. We start with the approximate solution $(\rho^h, m^h, \vartheta^h)$ defined at $t = 0$ as a piecewise constant function with jumps located at the space grid points $x = (i + 1/2)\Delta x$, $i \in \mathbb{Z}$, setting ρ^h, m^h, ϑ^h on the space interval $((i - 1/2)\Delta x, (i + 1/2)\Delta x))$ constant equal to the mean values of ρ_0, m_0, ϑ_0 on that interval. Then, inductively, we assume that ρ^h, m^h, ϑ^h have been for $t = jh$, for some $j \in \mathbb{N}$, and are piecewise constant with jumps located at $x = (i + 1/2)\Delta x$, $i \in \mathbb{Z}$. Here h is the time-step, Δt. We assume that Δx and Δt satisfy a CFL-condition, which, in order to be justified, depends on a crucial L^∞ a priori bound for the approximate solution, which is a central point in our proof. We then define ρ^h, m^h, ϑ^h on the time-interval $[jh, (j + 1)h)$ by solving the Riemann problems for the 3×3 system (2.15), (2.16),(2.17), centered at the discontinuities on the points $((i + 1/2)\Delta x, jh)$, $i \in \mathbb{Z}$. Since vacuum may occur in the solutions of the Riemann problems, we need also to define ϑ^h on the vacuum zones, whenever this is the case for a certain Riemann problem solution. So, if vacuum occurs on a Riemann solution defined on the rectangle $[i\Delta x, (i + 1)\Delta x] \times [jh, (j + 1)h)$, for some $i \in \mathbb{Z}$, it takes place on a wedge

$$c_1(t - jh) \le (x - (i + \frac{1}{2})\Delta x)) \le c_2(t - jh), \qquad jh \le t < (j + 1)h,$$

for certain $-2\Delta t/\Delta x < c_1 < c_2 < 2\Delta t/\Delta x$. We then define ϑ^h on this wedge as $\left(\vartheta^h(i\Delta x, jh) + \vartheta^h((i + 1)\Delta x, jh)\right)/2$. Having defined $(\rho^h, m^h, \vartheta^h)$ on $[jh, (j + 1)h) \times \mathbb{R}$ as just indicated, we may define $y^h(x, t)$ on $\mathbb{R} \times [0, (j+1)h)$ by (3.1) and the auxiliary function $\sigma^h(x,t)$ by (3.2). We thus define ρ^h, m^h, ϑ^h on $\mathbb{R}\times\{t=(j+1)h\}$ by (3.3), (3.4) and (3.5).

As we mentioned, the crucial point in our proof is the obtainment of L^∞ a priori bounds for ρ^h, m^h, ϑ^h. These bounds allow us to apply the compensated compactness method following the ideas developed in [4], [5], [1], [10] and [9]. Using this framework we succeed to prove the convergence *a.e.* of ρ^h, m^h to certain ρ, m, which implies the uniform convergence of $y^h(x,t)$ to a certain Lipschitz continuous function $y(x,t)$ satisfying (1.4), as well as the uniform convergence of the auxiliary functions $\sigma^h(x,t)$, and hence that of $\vartheta^h(x,t)$, to $\tilde\sigma(y(x,t),t)$ where $\tilde\sigma(y,t)$ is the solution of the heat equation

$$\tilde\sigma_t = \tilde\sigma_{yy},$$

satisfying $\tilde\sigma(y,0)=\sigma(y)$. Therefore, we verify (1.11) and, using the Lagragian transformation $T(x,t)=(y(x,t),t)$, in a standard way, we easily verify (1.12). The entropy inequality (1.15)is also easily verified and based on it we can apply the proof of the main result in [2] to verify (1.16). The details of these procedures are given in the following sections.

Before passing to the proof of Theorem 1.1, we remark that in Lagrangian coordinates the model (1.1)–(1.3) reads

$$\begin{aligned} v_t - u_y &= 0,\\ u_t + p(v,\vartheta)_y &= 0,\\ \vartheta_t - \vartheta_{yy} &= 0, \end{aligned} \tag{1.17}$$

where $v=1/\rho$ is the specific volume. We observe that, despite the fact that system (1.17) has a form much simpler than (1.1),(1.2),(1.3), the possibility of occurrence of vacuum turns the direct analysis of the Cauchy problem for (1.17) a very difficult task and so, as in the isentropic case, a better strategy is to proceed with the analysis of the corresponding problem in Eulerian coordinates, that is, (1.1)–(1.6).

We now briefly describe the contents of the remaining sections. The main purpose of Section 2 is to describe the solution of the Riemann problem (2.15)–(2.18). In Section 3 we describe the construction of the approximate solutions to (1.1)–(1.6). In Section 4 we prove the L^∞ a priori bound for the approximate solutions, which is a central point in this paper. Section 5 is devoted to the proof of the convergence of the approximate solutions by means of the compensated compactness method. Finally, in Section 6, we conclude the proof of Theorem 1.1 by outlining the proof of the decay property (1.16).

2. Background results. The auxiliary Riemann problem.

Let us first recall results for the p-system for a polytropic gas in Eulerian coordinates. More precisely, we consider the system

$$\rho_t + m_x = 0, \tag{2.1}$$

$$m_t + \Big(\frac{m^2}{\rho} + p(\rho)\Big)_x = 0, \tag{2.2}$$

where the pressure is given by $p(\rho)=\kappa\vartheta\rho^\gamma$. For later use we observe that we can rewrite the conserved quantities in terms of the other variables, viz.,

$$\rho = \rho(p,\vartheta) = \Big(\frac{p}{\kappa\vartheta}\Big)^{1/\gamma}, \quad m = m(u,p,\vartheta) = \rho u = u\,\Big(\frac{p}{\kappa\vartheta}\Big)^{1/\gamma}. \tag{2.3}$$

Here we consider the isothermal case where the temperature ϑ is considered a constant. Recall that the functions

$$w = u + \frac{1}{\theta}\big(p_\rho\big)^{1/2} = u + \vartheta^{1/2}\rho^\theta = u + \Big(\frac{p}{\kappa}\Big)^{\theta/\gamma}\vartheta^{\frac{1}{2\gamma}}, \tag{2.4}$$

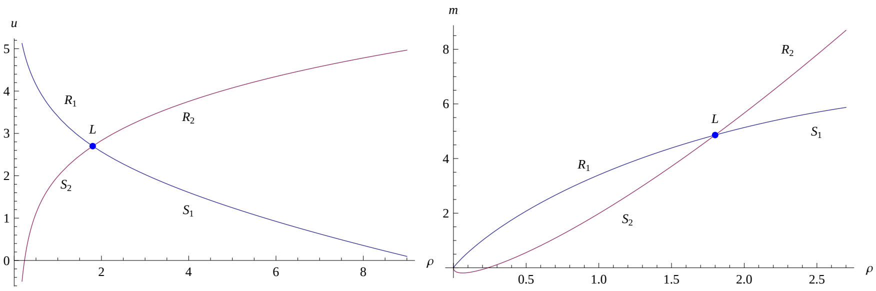

FIGURE 1. The wave curves for the p-system for a given left state.

$$z = u - \frac{1}{\theta}(p_\rho)^{1/2} = u - \vartheta^{1/2}\rho^\theta = u - \Big(\frac{p}{\kappa}\Big)^{\theta/\gamma}\vartheta^{\frac{1}{2\gamma}}, \tag{2.5}$$

with $\theta = \frac{1}{2}(\gamma-1)$, form a pair of Riemann invariants for system (2.1)–(2.2) in the isothermal case where ϑ is constant. A standard calculation (see, e.g., [8, 3]) yields that the rarefaction curves are given by

$$m = \frac{m_l}{\rho_l}\rho \pm \gamma^{1/2}\vartheta^{1/2}\rho(\rho^\theta - \rho_l^\theta),$$

while the Hugoniot locus reads

$$m = \frac{m_l}{\rho_l}\rho \pm \theta\vartheta^{1/2}\rho\Big(\frac{1}{\rho\rho_l}(\rho^\gamma - \rho_l^\gamma)(\rho-\rho_l)\Big)^{1/2},$$

from a given left state (ρ_l, m_l). When we involve the entropy condition we find that the wave curves equal

$$W_1(\rho_l, m_l): \quad m = \frac{m_l}{\rho_l}\rho - \begin{cases} \gamma^{1/2}\vartheta^{1/2}\rho(\rho^\theta - \rho_l^\theta) & \text{for } \rho \le \rho_l, \\ \theta\vartheta^{1/2}\rho\Big(\frac{1}{\rho\rho_l}(\rho^\gamma - \rho_l^\gamma)(\rho-\rho_l)\Big)^{1/2} & \text{for } \rho \ge \rho_l, \end{cases} \tag{2.6}$$

$$W_2(\rho_l, m_l): \quad m = \frac{m_l}{\rho_l}\rho + \begin{cases} \theta\vartheta^{1/2}\rho\Big(\frac{1}{\rho\rho_l}(\rho^\gamma - \rho_l^\gamma)(\rho-\rho_l)\Big)^{1/2} & \text{for } \rho \le \rho_l, \\ \gamma^{1/2}\vartheta^{1/2}\rho(\rho^\theta - \rho_l^\theta) & \text{for } \rho \ge \rho_l. \end{cases} \tag{2.7}$$

In the variables (ρ, u) we find

$$W_1(\rho_l, u_l): \quad u = u_l - \begin{cases} \gamma^{1/2}\vartheta^{1/2}(\rho^\theta - \rho_l^\theta) & \text{for } \rho \le \rho_l, \\ \theta\vartheta^{1/2}\Big(\frac{1}{\rho\rho_l}(\rho^\gamma - \rho_l^\gamma)(\rho-\rho_l)\Big)^{1/2} & \text{for } \rho \ge \rho_l, \end{cases} \tag{2.8}$$

$$W_2(\rho_l, u_l): \quad u = u_l + \begin{cases} \theta\vartheta^{1/2}\Big(\frac{1}{\rho\rho_l}(\rho^\gamma - \rho_l^\gamma)(\rho-\rho_l)\Big)^{1/2} & \text{for } \rho \le \rho_l, \\ \gamma^{1/2}\vartheta^{1/2}(\rho^\theta - \rho_l^\theta) & \text{for } \rho \ge \rho_l. \end{cases} \tag{2.9}$$

An important property of the p-system is that the Riemann invariants provide invariant regions. More specifically, (see, e.g., [3, Lemma 5]) if $(\rho_0(x), m_0(x)) \in \Omega = \{(\rho, m) \mid w \le w_0,\ z \ge z_0,\ w - z \ge 0\}$ for all $x \in \mathbb{R}$, then also the solution $(\rho(x,t), m(x,t))$ will remain in Ω, that is, $(\rho(x,t), m(x,t)) \in \Omega$ for $(x,t) \in \mathbb{R}\times[0,\infty)$.

An entropy-entropy flux pair (η, q) for the p-system satisfies for smooth solutions

$$\eta(\rho, m)_t + q(\rho, m)_x = 0.$$

Consistency with the system (2.1)–(2.2) requires

$$\nabla q(\rho, m) = \nabla\eta(\rho, m)\nabla F(\rho, m), \tag{2.10}$$

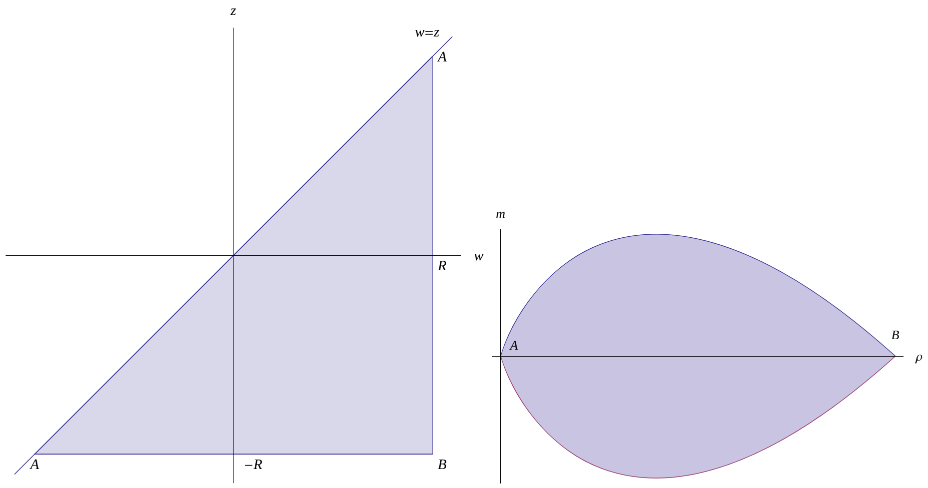

FIGURE 2. The triangle in the Riemann invariants (left) is mapped into the indicated region bounded by rarefaction waves in the (ρ, m)-plane.

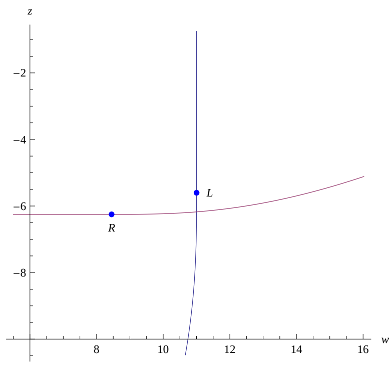

FIGURE 3. The solution of the Riemann problem using the Riemann invariants as coordinates. Through the right state backward Riemann invariants are drawn.

where $F = (m, \frac{m^2}{\rho} + p(\rho))$ is the flux function of the p-system. A particular choice of entropy-entropy flux pair (η_*, q_*) reads

$$\eta_* = \frac{m^2}{2\rho} + \frac{p}{\gamma - 1} = \frac{1}{2}\rho u^2 + \frac{\kappa}{\gamma - 1}\vartheta\rho^\gamma, \tag{2.11}$$

$$q_* = u\eta_* + pu = u\eta_* + u\kappa\vartheta\rho^\gamma. \tag{2.12}$$

More generally, the weak entropy-entropy flux pairs (η, q) constitute a class of entropy-entropy flux pairs of particular interest in isentropic gas dynamics, as first pointed out in [5], and they are characterized by the following conditions at the vacuum line:

$$\eta(\rho, u)|_{\rho=0} = 0, \qquad \eta_\rho(\rho, u)|_{\rho=0} = g(u),$$

for some continuous function g. Let us denote

$$\chi(\rho, u; \vartheta) = (\frac{p}{\rho} - u^2)_+^\lambda,$$

where $(a)_+ = \max\{0, a\}$ and $\lambda = \frac{3-\gamma}{2(\gamma-1)}$. As observed in [10], weak entropy-entropy flux pairs can be given by the integral formulas

$$\eta(\rho, u) = \int_{\mathbb{R}} g(\xi)\, \chi(\rho, \xi - u)\, d\xi, \tag{2.13}$$

$$q(\rho, u) = \int_{\mathbb{R}} g(\xi)\big(\theta\xi + (1-\theta)u\big)\chi(\rho, \xi - u)\, d\xi. \tag{2.14}$$

Remark 2.1. Observe that the entropy pair (η_*, q_*), defined in (1.15), is a weak convex entropy pair. Moreover, for any weak entropy pair (η, q) there exists a constant $C_\eta > 0$ such that $\eta + C_\eta \eta_*$ is convex.

Let us now turn to the full system

$$\rho_t + m_x = 0, \tag{2.15}$$

$$m_t + (\frac{m^2}{\rho} + p(\rho, \vartheta))_x = 0, \tag{2.16}$$

$$(\rho\vartheta)_t + (m\vartheta)_x = 0, \tag{2.17}$$

where the pressure p is given as above. The Riemann problem is the initial value problem for the system (2.15)–(2.17) with special initial data consisting of a single jump between two constant states, viz.

$$\begin{pmatrix} \rho \\ m \\ \vartheta \end{pmatrix}\Bigg|_{t=0}(x) = \begin{cases} \begin{pmatrix} \rho_l \\ m_l \\ \vartheta_l \end{pmatrix} & \text{for } x < 0, \\[2ex] \begin{pmatrix} \rho_r \\ m_r \\ \vartheta_r \end{pmatrix} & \text{for } x > 0. \end{cases} \tag{2.18}$$

The system (2.15)–(2.17) possesses three eigenfields associated with the eigenvalues

$$\lambda_1 = u - \sqrt{p_\rho}, \qquad \lambda_2 = u, \qquad \lambda_3 = u + \sqrt{p_\rho}.$$

The solution to a Riemann problem for system (2.15)–(2.17) may be described using the coordinates w, z, ϑ, that is, the Riemann invariants for the p-system and the temperature, in the following way. Consider first the case when the solution does not contain vacuum. The solution of the Riemann problem, starting from the left state $(\rho_l, m_l, \vartheta_l)$, consists of a slow wave in which the entropy ϑ remains constant (i.e., in the (w, z)-plane determined by $\vartheta = \vartheta_l$), followed by a contact discontinuity in which the velocity u and the pressure p remain unchanged, and finally a fast wave with constant temperature ϑ (i.e., in the (w, z)-plane determined by $\vartheta = \vartheta_r$) connected with the given right state $(\rho_r, m_r, \vartheta_r)$. Along the slow wave we can write the Riemann invariants as[2]

$$\begin{aligned} w &= u_1(\rho; \rho_l, u_l, \vartheta_l) + \vartheta_l^{1/2}\rho^\theta, \\ z &= u_1(\rho; \rho_l, u_l, \vartheta_l) - \vartheta_l^{1/2}\rho^\theta, \end{aligned} \tag{2.19}$$

where $u = u_1(\rho; \rho_l, u_l, \vartheta_l)$ is the slow wave given by (2.6). For the fast wave we consider the *backward* wave (i.e., consisting of the states that can be connected to a given right state from the left), and the Riemann invariants read

$$\begin{aligned} w &= \tilde{u}_2(\rho; \rho_r, u_r, \vartheta_r) + \vartheta_r^{1/2}\rho^\theta, \\ z &= \tilde{u}_2(\rho; \rho_r, u_r, \vartheta_r) - \vartheta_r^{1/2}\rho^\theta, \end{aligned} \tag{2.20}$$

[2]It turns out to be easier to describe the solution using the speed u rather than the momentum m as a variable.

where $u = \tilde{u}_2(\rho; \rho_r, u_r, \vartheta_r)$ is the fast *backward* wave corresponding to (2.7). The contact discontinuity, with pressure p^* and velocity u^*, jumps from a left density ρ_l^* to a right density ρ_r^* determined by

$$\begin{aligned} p^* &= \kappa\vartheta_l(\rho_l^*)^\gamma = \kappa\vartheta_r(\rho_r^*)^\gamma, \\ u^* &= u_1(\rho_l^*; \rho_l, u_l, \vartheta_l) = \tilde{u}_2(\rho_r^*; \rho_r, u_r, \vartheta_r), \end{aligned} \tag{2.21}$$

which yields

$$\frac{\rho_l^*}{\rho_r^*} = \left(\frac{\vartheta_r}{\vartheta_l}\right)^{1/\gamma}, \tag{2.22}$$

to be inserted in the second equation for the velocity, $u_1 = \tilde{u}_2$, to determine ρ_l^* and ρ_r^*. In terms of the Riemann invariants we find that w jumps from $u^* + (p^*/\kappa)^{\theta/\gamma}\vartheta_l^{\frac{1}{2\gamma}}$ to $u^* + (p^*/\kappa)^{\theta/\gamma}\vartheta_r^{\frac{1}{2\gamma}}$, and similarly z jumps from $u^* - (p^*/\kappa)^{\theta/\gamma}\vartheta_l^{\frac{1}{2\gamma}}$ to $u^* - (p^*/\kappa)^{\theta/\gamma}\vartheta_r^{\frac{1}{2\gamma}}$. An alternative way to describe the contact discontinuity is the following. Consider a point on the backward fast wave curve with Riemann invariants (w, z) given by (2.20), which we can write as $w = \tilde{u}_2 + (p/\kappa)^{\theta/\gamma}\,\vartheta_r^{\frac{1}{2\gamma}}$ and $z = \tilde{u}_2 - (p/\kappa)^{\theta/\gamma}\,\vartheta_r^{\frac{1}{2\gamma}}$. Construct now another curve $(\bar{w}, \bar{z})$, given as a Riemann invariant with the same velocity $\tilde{u}_2$ and pressure p as (w, z), but with the temperature ϑ_r replaced by ϑ_l, that is,

$$\bar{w} = \tilde{u}_2 + \left(\frac{p}{\kappa}\right)^{\frac{\theta}{\gamma}} \vartheta_l^{\frac{1}{2\gamma}}, \quad \bar{z} = \tilde{u}_2 - \left(\frac{p}{\kappa}\right)^{\frac{\theta}{\gamma}} \vartheta_l^{\frac{1}{2\gamma}}.$$

We find

$$\begin{aligned} w + z &= 2\tilde{u}_2 = \bar{w} + \bar{z}, \\ w - z &= 2\left(\frac{p}{\kappa}\right)^{\frac{\theta}{\gamma}} \vartheta_r^{\frac{1}{2\gamma}} = (\bar{w} - \bar{z})\left(\frac{\vartheta_r}{\vartheta_l}\right)^{\frac{1}{2\gamma}}, \end{aligned}$$

which yields

$$\begin{aligned} \bar{w} &= \frac{w}{2}\left(1 + \left(\frac{\vartheta_l}{\vartheta_r}\right)^{\frac{1}{2\gamma}}\right) + \frac{z}{2}\left(1 - \left(\frac{\vartheta_l}{\vartheta_r}\right)^{\frac{1}{2\gamma}}\right), \\ \bar{z} &= \frac{w}{2}\left(1 - \left(\frac{\vartheta_l}{\vartheta_r}\right)^{\frac{1}{2\gamma}}\right) + \frac{z}{2}\left(1 + \left(\frac{\vartheta_l}{\vartheta_r}\right)^{\frac{1}{2\gamma}}\right). \end{aligned}$$

The intersection between the slow wave curve in the Riemann invariants plane and the curve $(\bar{w}, \bar{z})$ determines the values of the variables to the left of the contact discontinuity, whose speed, in the physical space of the (x, t)-coordinates, is then $c_\vartheta := (\bar{w} + \bar{z})/2$. Through this intersection we draw the line where $w + z = 2c_\vartheta$, and the intersection between this line and the backward fast wave gives the values of the variables to the right of the contact discontinuity, cf. Figures 4 and 5.

The solution involves vacuum when the slow wave is a rarefaction wave that connects to a state on the vacuum line $w = z$; the velocity is then given by $u^* = u_l + \gamma^{1/2}\vartheta_l^{1/2}\rho_l^\theta$ and $w = z = u^*$. Similarly, the given right state connects via a rarefaction from a vacuum state with velocity $\tilde{u}^* = u_r - \gamma^{1/2}\vartheta_r^{1/2}\rho_r^\theta$ and $w = z = \tilde{u}^*$. This is possible only if $\tilde{u}^* > u^*$. On the physical space of the (x, t)-coordinates the vacuum region is the wedge $\mathcal{V} := \{(x, t) \,:\, u^*t \le x \le \tilde{u}^*t\}$; for definiteness, we then set $u := (u^* + \tilde{u}^*)/2$, and $\vartheta := (\vartheta_l + \vartheta_r)/2$ on $\mathcal{V}$. In this way, we define completely the Riemann solutions that will be used in the next section in the construction of the approximate solutions to (1.1)–(1.6).

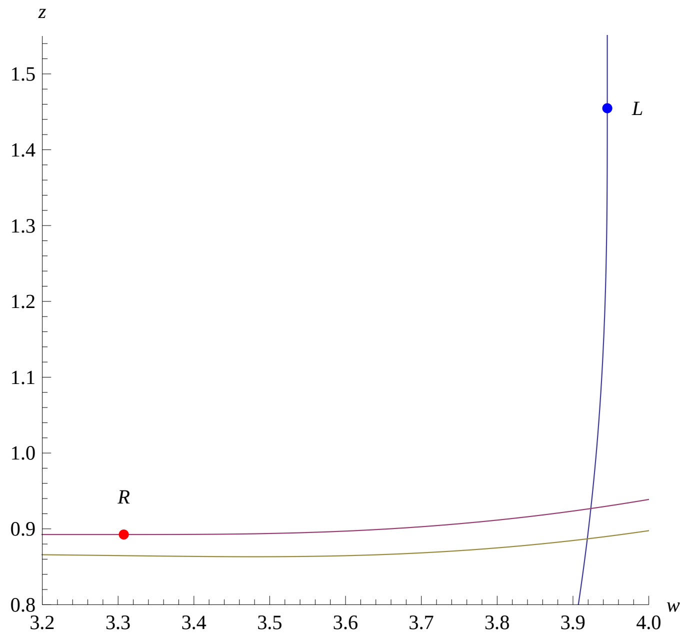

FIGURE 4. The slow Riemann invariant through the left state (blue curve), and the backward fast Riemann invariant through the right state (red curve). In addition the yellow curve $(\bar{w}, \bar{z})$, whose intersection with the slow Riemann invariant determines the contact discontinuity.

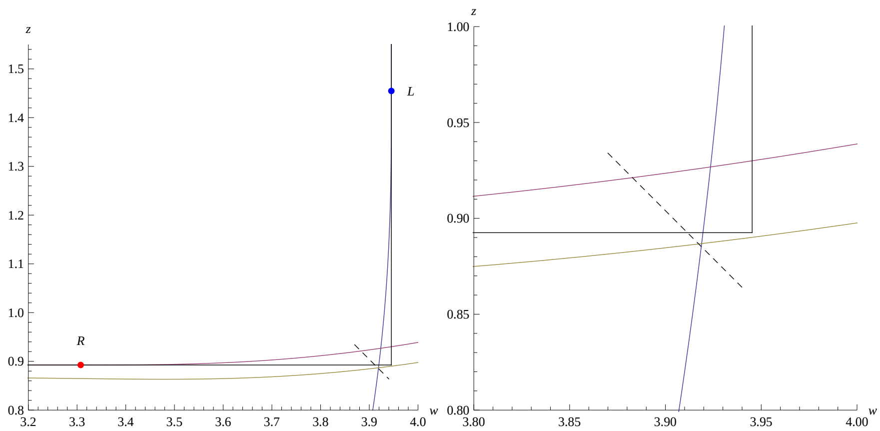

FIGURE 5. The same data is in Figure 4. Curves for the invariant region for the corresponding p-system are added (black). In addition, the dashed line is given by $w - z$ equals a constant determined by the intersection between the yellow and blue curves. The interaction of this straight line with the red curve gives the value on the right of the contact discontinuity. The right figure is a close-up near the intersection.

3. Construction of approximate solutions.

Here we provide the full proof of Theorem 1.1. We construct approximate solutions for (1.1)–(1.3) by using a Godunov-type finite difference scheme based on solving Riemann problems at each time step, updating the approximate ϑ using the Lagrange transformation, and averaging at the end of each time step.

Before we begin the proof, let us describe the fundamentals of the construction of the approximate solution. We discretize both in space and time. Let $h = \Delta t$, and $\Delta x = ch$ with $c > 0$ to be chosen by the CFL condition

$$c > \sup_{(x,t)\in\mathbb{R}\times[0,\infty)} \left| \frac{m^h(x,t)}{\rho^h(x,t)} \pm \sqrt{p_\rho(\rho^h(x,t),\vartheta^h(x,t))} \right|,$$

which is possible as long as we can obtain an L^∞ a priori bound for

$$\frac{m^h(x,t)}{\rho^h(x,t)} \pm \sqrt{p_\rho(\rho^h(x,t),\vartheta^h(x,t))}.$$

The initial data ρ_0, m_0, ϑ_0 is approximated by step functions with jumps at $x_{i-1/2} := (i-1/2)\Delta x$ for $i \in \mathbb{Z}$. The multiple Riemann problems are solved for $t \in [0,h)$. At $t = h$ a new step function is created with jumps at $x_{i-1/2}$ (details given below), and new Riemann problems are solved. More precisely, suppose the approximate solution $U^h = (\rho^h, m^h, \vartheta^h)$ has been defined for $t \le jh$ and that $U^h(x, jh)$ is constant for $x \in I_i$ where

$$I_i = (x_{i-1/2}, x_{i+1/2}), \quad i \in \mathbb{Z}.$$

For $t \in [jh,(j+1)h)$, setting $x_i = i\Delta x$, $i \in \mathbb{Z}$, we define $U^h(x,t)$ by glueing together the solutions of the Riemann problems for the system (2.15)–(2.17) defined at $[x_i, x_{i+1}] \times [jh,(j+1)h)$, determined by the discontinuities at the points $(x_{i+1/2}, jh)$, $i \in \mathbb{Z}$. Inductively this yields a function U^h defined on $\mathbb{R}\times[0,\infty)$, as long as we are able to obtain the necessary a priori bound mentioned above.

We describe the construction of the approximate solution as follows. Assume that we have constructed the approximate solution U^h for $x \in \mathbb{R}$ and $t < jh$, and have defined it at time $t = jh$ as a piecewise constant function with jumps at $x_{i+1/2}$ for $i \in \mathbb{Z}$. For $(x,t) \in [x_i, x_{i+1}] \times [jh,(j+1)h)$, $i \in \mathbb{Z}$, let $U^h(x,t)$ be the solution of the Riemann problem (2.15)–(2.17) as described in the previous section. Set

$$y^h(x,t) = \int_0^x \rho^h(z,t)\,dz - \int_0^t m^h(0,s)\,ds, \quad x \in \mathbb{R},\ t \in [jh,(j+1)h), \tag{3.1}$$

and

$$\sigma^h(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(y^h(x,t)-z)^2}{4t}} \sigma(z)\,dz = \tilde\sigma(y^h(x,t),t), \quad (x,t) \in \mathbb{R}\times[jh,(j+1)h). \tag{3.2}$$

We then define[3]

$$\rho^h(x,(j+1)h) = \frac{1}{\Delta x}\int_{I_i} \rho^h(\tilde x,(j+1)h - 0)\,d\tilde x, \tag{3.3}$$

$$m^h(x,(j+1)h) = \frac{1}{\Delta x}\int_{I_i} m^h(\tilde x,(j+1)h - 0)\,d\tilde x, \tag{3.4}$$

$$\vartheta^h(x,(j+1)h) = \frac{1}{\Delta x}\int_{I_i} \sigma^h(\tilde x,(j+1)h - 0)\,d\tilde x, \tag{3.5}$$

for $x \in I_i$.

[3] We use the standard notation $f(x \pm 0) = \lim_{\epsilon\downarrow 0} f(x \pm \epsilon)$.

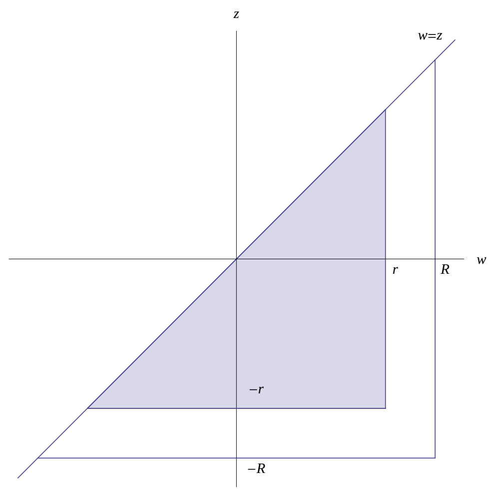

FIGURE 6. Assuming that the initial data are in the shaded region, we show the existence of an R such that the solution remains in the larger triangle. The vacuum line is $w = z$.

4. L^∞ A PRIORI ESTIMATE.

We now investigate the problem of obtaining an a priori L^∞ bound for the approximate solution U^h. Let us denote

$$w^h(x,t) = w(U^h(x,t)), \qquad z^h(x,t) = z(U^h(x,t)).$$

Let $r > 0$ be such that

$$w^h(x,0) \le r, \qquad z^h(x,0) \ge -r, \qquad x \in \mathbb{R}.$$

We assume for the moment that w^h, z^h satisfies an a priori bound of the form

$$w^h(x,t) \le R, \qquad z^h(x,t) \ge -R, \qquad (x,t) \in \mathbb{R} \times [0,\infty), \tag{4.1}$$

for some constants $R > r$, and we will find a condition relating r and R under which (4.1) can be justified.

We first observe that if (4.1) holds, then, for any $(x_1,t_1),(x_2,t_2) \in \mathbb{R} \times [0,\infty)$,

$$|y^h(x_1,t_1) - y^h(x_2,t_2)| \le C(R)(|x_1 - x_2| + |t_1 - t_2| + h), \tag{4.2}$$

for some constant $C(R) > 0$ depending only on R. In what follows, $C(R)$ will always represent a positive constant depending on R that may differ from one occurrence to the next one.

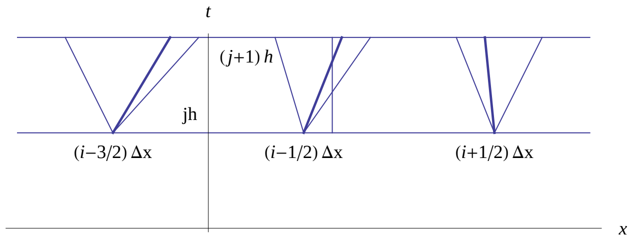

FIGURE 7. Schematic figure of the solution of the Riemann problem. Contact discontinuities are indicated by thick lines. We see that the vertical line x equals a constant first intersects a slow or a fast wave before it crosses the contact discontinuity.

We also observe that

$$
\begin{aligned}
&|\sigma^h(x_1,t_1)-\sigma^h(x_2,t_2)| \\
&\qquad = |\tilde\sigma(y^h(x_1,t_1),t_1)-\tilde\sigma(y^h(x_2,t_2),t_2)| \\
&\qquad \le |\tilde\sigma(y^h(x_1,t_1),t_1)-\tilde\sigma(y^h(x_2,t_2),t_1)| \\
&\qquad\qquad + |\tilde\sigma(y^h(x_2,t_2),t_1)-\tilde\sigma(y^h(x_2,t_2),t_2)| \\
&\qquad \le |y^h(x_1,t_1)-y^h(x_2,t_2)| \int_0^1 |\tilde\sigma_y(\tau y_2^h+(1-\tau)y_1^h,t_1)|\,d\tau \qquad (4.3)\\
&\qquad\qquad + |t_1-t_2| \int_0^1 |\tilde\sigma_t(y_2^h,\theta t_2+(1-\theta)t_1)|\,d\theta \\
&\qquad \le C(R)\Big((|x_1-x_2|+|t_1-t_2|+h)e^{-t_1}+|t_1-t_2|e^{-\min(t_1,t_2)}\Big) \\
&\qquad \le C(R)(|x_1-x_2|+|t_1-t_2|+h)e^{-\min(t_1,t_2)},
\end{aligned}
$$

where we have used (1.10) and denoted $y_i^h=y^h(x_i,t_i)$, $i=1,2$.

Assume inductively that

$$
w^h(x,t)\le r_j, \qquad z^h(x,t)\ge -r_j, \qquad (x,t)\in\mathbb{R}\times[0,jh],
$$

for some constant r_j. For $t\in[jh,(j+1)h)$ the approximate solution is defined by solving the Riemann problems given by the discontinuities at the points $(x_{i+1/2},jh)$, $i\in\mathbb{Z}$. Since the p-system enjoys an invariant region given in terms of w and z, the only possible increase in w beyond r_j, and, similarly, the only possible decrease in z beyond $-r_j$, may occur across the contact discontinuity. Here both the velocity and the pressure remain unchanged, and the sole change is in the entropy. Observe first that since the slow Riemann invariant is increasing in w, there can be no increase in the value of w. Fix x and let $t\in[jh,(j+1)h)$. We see from Figure 7 that the vertical line x equals a constant crosses slow or fast waves before it crosses the contact discontinuity. Let $jh<\tilde t<\bar t<(j+1)h$ denote two times such that $\tilde t$ is after the fast or slow wave, but prior to the contact discontinuity, while $\bar t$ is after the contact discontinuity. Then we find

$$
\begin{aligned}
z^h(x,\bar t) &= z^h(x,\tilde t)+(z^h(x,\bar t)-z^h(x,\tilde t)) \\
&\ge z^h(x,\tilde t)-\left|z^h(x,\bar t)-z^h(x,\tilde t)\right| \\
&\ge -r_j-\left|z^h(x,\bar t)-z^h(x,\tilde t)\right|,
\end{aligned}
$$

since the solution of the p-system remains within the invariant region. Furthermore,

$$
\begin{aligned}
&\left|z^h(x,\bar t)-z^h(x,\tilde t)\right|\\
&\le \left|\left(\frac{p^h}{\kappa}\right)^{\frac{\theta}{\gamma}}\left(\vartheta^h\right)^{\frac{1}{2\gamma}}(x,\bar t)-\left(\frac{p^h}{\kappa}\right)^{\frac{\theta}{\gamma}}\left(\vartheta^h\right)^{\frac{1}{2\gamma}}(x,\tilde t)\right|\\
&\le \left(\frac{p^h}{\kappa}\right)^{\frac{\theta}{\gamma}}\left(\vartheta^h\right)^{\frac{1}{2\gamma}}(x,\bar t)\,\left(\vartheta^h\right)^{-\frac{1}{2\gamma}}(x,\bar t)\\
&\qquad\times\left|\left(\vartheta^h\right)^{\frac{1}{2\gamma}}(x,\tilde t)-\left(\vartheta^h\right)^{\frac{1}{2\gamma}}(x,\bar t)\right|\\
&\le \frac12(w^h-z^h)(x,\bar t)\,\left(\vartheta^h\right)^{\frac{-1}{2\gamma}}(x,\bar t)\\
&\qquad\times\left|\left(\vartheta^h\right)^{\frac{1}{2\gamma}}(x,\tilde t)-\left(\vartheta^h\right)^{\frac{1}{2\gamma}}(x,\bar t)\right|\\
&\le r_jC(R)(\gamma-1)\left|[\![\vartheta^h(\bar t)]\!]\right|\\
&= r_jC(R)(\gamma-1)\left|[\![\vartheta^h(jh)]\!]\right|,
\end{aligned}
\tag{4.4}
$$

where we have used the mean value theorem and estimated the resulting factor multiplying the jump in ϑ^h times $\left(\vartheta^h\right)^{-\frac{1}{2\gamma}}(x,\bar t)$ by a constant $C(R)$. Next we estimate the jump in the temperature. Let x_1 and x_2 be two points on the left and right side of a jump, respectively, thus $x_1<x_{i-1/2}<x_2$, with $x_2-x_1<\Delta x$. We obtain

$$
\begin{aligned}
\left|[\![\vartheta^h(jh)]\!]\right| &= \left|\vartheta^h(x_2,jh)-\vartheta^h(x_1,jh)\right|\\
&\le \frac{1}{\Delta x}\int_{I_i}\left|\sigma^h(\tilde x+\Delta x,jh)-\sigma^h(\tilde x,jh)\right|d\tilde x\\
&\le C(R)h\,e^{-jh},
\end{aligned}
\tag{4.5}
$$

by (4.3), where, for the last inequality, we have used (1.10).

This yields

$$
z^h(x,\bar t)\ge -r_j(1+C(R)(\gamma-1)h\,e^{-jh}),
\tag{4.6}
$$

and we conclude that

$$
z^h(x,t)\ge -r_j(1+C(R)(\gamma-1)h\,e^{-jh}),\quad t\in[jh,(j+1)h).
\tag{4.7}
$$

A similar calculation leads to

$$
w^h(x,t)\le r_j(1+C(R)(\gamma-1)h\,e^{-jh}),\quad t\in[jh,(j+1)h).
\tag{4.8}
$$

At $t=(j+1)h$ we average the approximate solution as described in (3.3)–(3.5). Here we argue as follows. We first observe that the averaging of the values of $(\rho^h(x,(j+1)h-0),m^h(x,(j+1)h-0))$ in the intervals $I_i^{j+1}:=I_i\times\{t=(j+1)h-0\}$, $i\in\mathbb{Z}$, in order to obtain the values of $(\rho^h(x,(j+1)h),m^h(x,(j+1)h))$ in these intervals, does not affect the bounds (4.7) and (4.8). More precisely, at each such interval, $\vartheta^h(x,(j+1)h-0)$ assumes at most 3 values, due to the possibility that two contact discontinuities, departing from $(x_{i-1/2},jh)$ and $(x_{i+1/2},jh)$, respectively, end inside I_i^{j+1}. This means that the values of (ρ^h,m^h) in each interval I_i^{j+1} belong to the union of at most 3 regions of the form

$$
R_\alpha:=\{(\rho,m)\,:\,-C\rho+{\vartheta_\alpha}^{1/2}\rho^{\theta+1}\le m\le C\rho-{\vartheta_\alpha}^{1/2}\rho^{\theta+1}\},\qquad \alpha=1,2,3,
$$

for some constant $C>0$ common to all regions R_α, $\alpha=1,2,3$. But, one easily check that $\vartheta_1<\vartheta_2$ implies $R_1\supset R_2$, that is, the regions R_α, $\alpha=1,2,3$, are contained in that one corresponding to $\vartheta_*=\min\{\vartheta_1,\vartheta_2,\vartheta_3\}$. In particular, if we define

$$
\vartheta_*^h(x,(j+1)h):=\min\{\vartheta^h(\xi,(j+1)h-0)\,:\,\xi\in I_i\},\qquad \text{in } I_i^{j+1},\ i\in\mathbb{Z},
$$

then, from the convexity of the regions R_α, we have

$$z(\rho^h, u^h, \vartheta_*^h)(x, (j+1)h) \geq -r_j(1 + C(R)(\gamma - 1)h\, e^{-jh}), \tag{4.9}$$

and also

$$w(\rho^h, u^h, \vartheta_*^h)(x, (j+1)h) \leq r_j(1 + C(R)(\gamma - 1)h\, e^{-jh}), \tag{4.10}$$

where

$$u^h(x, (j+1)h) := \begin{cases} \frac{m^h(x,(j+1)h)}{\rho^h(x,(j+1)h)}, & \text{if } \rho^h(x, (j+1)h) > 0 \\ u^h(x, (j+1)h - 0), & \text{otherwise} \end{cases}$$

and we agree that the value of $u^h(x, (j+1)h - 0)$ at a vacuum interval is the mean value between its values at the extremes of the interval, which determines precisely the values of $u^h(x, (j+1)h - 0)$ for all $x \in \mathbb{R}$. Observe also that the case in which I_i^{j+1} is contained in a vacuum interval is trivial since $\rho^h = m^h = 0$ in such an interval, and so the values of ρ^h and m^h do not change through averaging on I_i^{j+1}.

Now, we need to check how the bounds (4.9) and (4.10) change when we replace $\vartheta_*^h(x, (j+1)h)$ by the values of $\vartheta^h(x, (j+1)h)$ given by (3.5). For this, we first estimate the change in ϑ^h from $\vartheta^h(x, (j+1)h - 0)$, to $\vartheta^h(x, (j+1)h)$, given by (3.5). As already mentioned, $\vartheta^h(x, (j+1)h - 0)$ can be one of three values; either the value $\vartheta^h(x, jh)$, or the values of ϑ in the neighboring intervals, that is, $\vartheta^h(x \pm \Delta x, jh)$. In any of the three cases, the entropy is given by a formula similar to (3.5), but with $(j+1)h$ replaced by jh. We consider the most representative case where the value is in a neighboring interval. Thus

$$\begin{aligned} &\left|\vartheta^h(x, (j+1)h) - \vartheta^h(x - \Delta x, jh)\right| \\ &\quad \leq \frac{1}{\Delta x} \int_{I_i} \left|\sigma^h(\tilde{x}, (j+1)h) - \sigma^h(\tilde{x} - \Delta x, jh)\right| \, d\tilde{x} \\ &\quad \leq C(R)h\, e^{-jh}, \end{aligned} \tag{4.11}$$

again by (4.3). Since,

$$\begin{aligned} z^h(x, (j+1)h) &= z(\rho^h, u^h, \vartheta^h)(x, (j+1)h), \\ w^h(x, (j+1)h) &= z(\rho^h, u^h, \vartheta^h)(x, (j+1)h), \end{aligned}$$

we conclude as above that

$$\begin{aligned} z^h(x, t) &\geq z(\rho^h, u^h, \vartheta_*^h)(x, (j+1)h) \\ &\qquad - |z(\rho^h, u^h, \vartheta_*^h)(x, (j+1)h) - z(\rho^h, u^h, \vartheta^h)(x, (j+1)h)| \\ &\geq -r_j(1 + C(R)(\gamma - 1)h\, e^{-jh})^2 =: -r_{j+1}, \\ w^h(x, t) &\leq w(\rho^h, u^h, \vartheta_*^h)(x, (j+1)h) \\ &\qquad + |w(\rho^h, u^h, \vartheta_*^h)(x, (j+1)h) - w(\rho^h, u^h, \vartheta^h)(x, (j+1)h)| \\ &\leq r_j(1 + C(R)(\gamma - 1)h\, e^{-jh})^2 = r_{j+1}. \end{aligned} \tag{4.12}$$

It remains to estimate the r_j. From the inductive formula (4.12) for the r_j, we find

$$
\begin{aligned}
r_j &= r \prod_{k=1}^{j} (1 + C(R)(\gamma-1)h\, e^{-kh})^2 \\
&\le r \exp\big(2C(R)(\gamma-1)\sum_{k=1}^{j} e^{-kh}h)\big) \\
&\le r \exp\big(2C(R)(\gamma-1)\int_0^\infty e^{-s}ds\big) \\
&\le re^{2C(R)(\gamma-1)}.
\end{aligned} \tag{4.13}
$$

Therefore, we see from (4.13) that the condition relating r and R under which the a priori bound (4.1) holds is

$$
Re^{-2(\gamma-1)C(R)} \ge r. \tag{4.14}
$$

We may easily check that $C(R)$ may be defined as a continuous increasing function of $R \in [0,\infty)$ such that $C(0)=0$ and $C(R)\to\infty$ as $R\to\infty$. Hence, the left-hand side of (4.14) attains a maximum value for some $R_* \in (0,\infty)$ and by (4.14) the initial bound r can take the largest possible value given by the left-hand side of (4.14) for $R = R_*$. In particular, (4.14) may be viewed as a restriction on the initial bound r which amounts to a restriction on $\|\rho_0\|_\infty$ and $\|m_0\|_\infty$, assuming given ϑ_0. We also verify that the initial bound can be taken as large as we wish provided that $\gamma - 1$ is sufficiently small.

5. Convergence of the approximate solutions.

Now we proceed to prove the compactness of the sequence of approximate solutions U^h. The proof is based on the general analysis carried out by DiPerna in [4] and we are going to apply the compactness result in [5] and its extensions in [1], [10] and [9], which together cover the whole range $\gamma > 1$.

Now, let $V^h = (\rho^h, m^h)$ and $F^h = (m^h, \rho^h(u^h)^2 + p(\rho^h, \vartheta^h))$. For any $\phi \in C_0^\infty(\mathbb{R}^2)$ we have

$$
\begin{aligned}
\iint_{\mathbb{R}\times[0,\infty)} V^h\phi_t + F^h\phi_x\,dx\,dt &= \sum_{j=0}^{\infty}\int_{jh}^{(j+1)h}\int_{\mathbb{R}} V^h\phi_t + F^h\phi_x\,dx\,dt \\
&= \sum_{j=0}^{\infty}\int_{\mathbb{R}} [\![V^h(jh)]\!]\phi(x,jh)\,dx \\
&= \sum_{j=1}^{\infty}\int_{\mathbb{R}} [\![V^h(jh)]\!]\phi(x,jh)\,dx - \int_{\mathbb{R}} V^h(x,0)\phi(x,0)\,dx,
\end{aligned} \tag{5.1}
$$

where

$$
[\![V^h(jh)]\!] = V^h(x,jh-0) - V^h(x,jh+0).
$$

Further, if (η, q) is an arbitrary entropy pair for (1.1)–(1.2), with ϑ constant, we have

$$
\begin{aligned}
&\iint_{\mathbb{R}\times[0,\infty)} \eta^h\phi_t + q^h\phi_x\,dx\,dt = \sum_{j=0}^{\infty}\int_{jh}^{(j+1)h}\int_{\mathbb{R}} \eta^h\phi_t + q^h\phi_x\,dx\,dt \\
&= -\int_{\mathbb{R}} \eta^h(x,0)\phi(x,0)\,dx + \sum_{j=1}^{\infty}\int_{\mathbb{R}} [\![\eta^h(jh)]\!]\phi(x,jh)\,dx + \int_0^\infty \mathcal{S}(\phi)\,dt + \int_0^\infty \mathcal{C}(\phi)\,dt,
\end{aligned} \tag{5.2}
$$

where, for reasons of brevity, we write $\eta^h = \eta(V^h, \vartheta^h)$ and $q^h = q(V^h, \vartheta^h)$. Here

$$[\![\eta^h(jh)]\!] = \eta^h(x, jh - 0) - \eta^h(x, jh + 0),$$

and $\mathcal{S}(\phi)$ is defined as

$$\mathcal{S}(\phi) = \sum_{\text{shocks}} \big(s[\![\eta^h]\!] - [\![q^h]\!]\big)\phi(x(t), t),$$
$$[\![\eta^h]\!] = \eta^h(x(t) - 0, t) - \eta^h(x(t) + 0, t),$$

where the sum is over all shock discontinuities $(x(t), t)$ at time t, $s = x'(t)$ denoting the shock speed, while $\mathcal{C}(\phi)$ is defined as

$$\mathcal{C}(\phi) = \sum_{\substack{\text{contact}\\ \text{discontinuities}}} \big(u^h[\![\eta^h]\!] - [\![q^h]\!]\big)\phi(x(t), t),$$

with sum running over all contact discontinuities $(x(t), t)$ at time t, where u^h is the velocity. The latter is defined over a vacuum interval as the arithmetic mean between the velocity at the end of the 1-rarefaction wave bounding the vacuum interval on the left-hand side and the velocity at the beginning of the 2-rarefaction wave bounding the vacuum interval on the right-hand side.

We recall that if (η, q) is a convex entropy pair for the isentropic system (1.1)–(1.2) where ϑ is constant, then

$$s[\![\eta^h]\!] - [\![q^h]\!] \geq 0, \tag{5.3}$$

across each shock wave. Since ϑ^h is constant across waves of the first and third family, inequality (5.3) also holds here. Therefore, for any weak entropy pair (η, q), we find that the functional

$$\int_0^\infty \mathcal{S}(\phi)\, dt$$

is a (signed) measure with locally finite total variation, as a consequence of Remark 2.1.

Concerning the functional

$$\int_0^\infty \mathcal{C}(\phi)\, dt,$$

if (η, q) is a smooth entropy pair, we have, in view of previous calculations,

$$|u^h[\![\eta^h(jh)]\!] - [\![q^h(jh)]\!]| \leq C_\eta e^{-jh} h,$$

and so

$$\left| \int_0^\infty \mathcal{C}(\phi)\, dt \right| \leq C_\eta \operatorname{diam}(K) \|\phi\|_\infty,$$

where K is any compact containing the support of ϕ, which gives that this functional is also a measure with locally finite total variation.

Observe that the weak entropies may be also written as

$$\eta(\rho, u) = \rho \int_{-1}^{1} g\left(\frac{m}{\rho} + z\vartheta^{1/2}\rho^{(\gamma-1)/2}\right)(1 - z^2)_+^\lambda\, dz,$$

while a similar formula holds for q. In particular, η, q are Lipschitz up to vacuum if g is smooth.

We also observe that for the special entropy pair (η_*, q_*) we have $\int_0^\infty \mathcal{C}(\phi)\,dt = 0$. Also, for this entropy pair, for nonnegative $\phi \in C_0^\infty(\mathbb{R}^2)$ we have

$$
\begin{aligned}
&\sum_{j=1}^{\infty} \int_{\mathbb{R}} [\![\eta_*^h(jh)]\!]\phi(x,jh)\,dx \\
&\quad = \sum_{j=1}^{\infty}\sum_{i\in\mathbb{Z}} \int_{I_i} \Big(\eta_*(V^h(x,jh-0)), \vartheta^h(x,jh+0)) \\
&\qquad\qquad\qquad - \eta_*(V^h(x,jh+0), \vartheta^h(x,jh+0))\Big)\phi(x,jh)\,dx \\
&\quad - \sum_{j=1}^{\infty}\sum_{i\in\mathbb{Z}} \int_{I_i} \Big(\eta_*(V^h(x,jh-0)), \vartheta^h(x,jh+0)) \\
&\qquad\qquad\qquad - \eta_*(V^h(x,jh-0), \vartheta^h(x,jh-0))\Big)\phi(x,jh)\,dx.
\end{aligned}
\tag{5.4}
$$

The first sum in the right-hand side of equation (5.4) is nonnegative for nonnegative ϕ, since $V^h(x,jh+0)$ is the average of $V^h(x,jh-0)$, in each interval I_i, and η_* is convex. Therefore, we get

$$
\begin{aligned}
&\sum_{j=1}^{\infty} \int_{\mathbb{R}} [\![\eta_*^h(jh)]\!]\phi(x,jh)\,dx \\
&\quad \geq -\sum_{j=1}^{\infty}\sum_{i\in\mathbb{Z}} \int_{I_i} \eta_{*\vartheta}^h(\cdots)(\vartheta^h(x,jh+0) - \vartheta^h(x,jh-0))\phi(x,jh)\,dx \\
&\quad \geq -\sum_{j=1}^{\infty} Ce^{-jh}h \int_{\mathbb{R}} \phi(x,jh)\,dx,
\end{aligned}
\tag{5.5}
$$

where $\eta_{*\vartheta}^h(\cdots) = \int_0^1 \eta_{*\vartheta}^h(V^h(x,jh-0), A(\theta))\,d\theta$ is the coefficient of the linear remaining term in the trivial Taylor expansion of zero order in the variable ϑ and $A(\theta) = (1-\theta)\vartheta^h(x,jh-0) + \theta\vartheta^h(x,jh+0)$. In particular, both the left-hand side as well as the second term of the right-hand side of (5.4) are measures of locally finite total variation. As a consequence, we may apply equality (5.4) with ϕ replaced by the characteristic function of any suitably chosen rectangle $|x| \leq L = M\Delta x$, $0 \leq t \leq T = Nh$, to find that

$$
\sum_{jh\leq N}\ \sum_{|i\Delta x|\leq M} \int_{I_i} D_V^2\eta_*^h(\cdots)(V^h(x,jh-0)) - V^h(x,jh+0))^2\,dx \leq \text{const.},
\tag{5.6}
$$

for any $M, N > 0$, the constant depending on M, N, where $D_V^2\eta_*^h(\cdots) = \int_0^1(1-\theta)D_V^2\eta_*(B(\theta), \vartheta^h(x,jh+0))\,d\theta$ is the coefficient of the quadratic remaining term in the Taylor expansion of first order and $B(\theta) = (1-\theta)V^h(x,jh+0) + \theta V^h(x,jh-0)$.

Since for all weak entropy η we have $|D_V^2\eta| \leq C_\eta D_V^2\eta_*$, for some $C_\eta > 0$, it follows from (5.6) that

$$
\left| \sum_{jh\leq N}\ \sum_{|i\Delta x|\leq M} \int_{I_i} |D_V^2\eta|(V^h(x,jh-0)) - V^h(x,jh+0))^2\,dx \right| \leq \text{const.},
\tag{5.7}
$$

for any $M, N > 0$, the constant depending on M, N.

We can then use DiPerna's method in [4] to prove the $W_{\mathrm{loc}}^{-1,2}$ compactness of the distributions $\eta_t^h + q_x^h$ by decomposing the functional

$$
L(\phi) = \sum_{j=1}^{\infty} \int_{\mathbb{R}} [\![\eta^h(jh)]\!]\phi(x,jh)\,dx
$$

as

$$
\begin{aligned}
L(\phi) &= \sum_{j=1}^{\infty}\sum_{i\in\mathbb{Z}}\int_{I_i} [\![\eta^h(jh)]\!]\phi(x,jh)\,dx \\
&= \sum_{j=1}^{\infty}\sum_{i\in\mathbb{Z}}\Big(\phi(x_i,jh)\int_{I_i}[\![\eta^h(jh)]\!]\,dx \\
&\qquad\qquad + \int_{I_i}[\![\eta^h(jh)]\!]\big(\phi(x,jh)-\phi(x_i,jh)\big)dx\Big) \\
&= L_1(\phi)+L_2(\phi).
\end{aligned} \tag{5.8}
$$

We consider the two terms separately. We have

$$
L_1(\phi) = \sum_{j=1}^{\infty}\sum_{i\in\mathbb{Z}}\phi(x_i,jh)\int_{I_i}[\![\eta^h(jh)]\!]_V + [\![\eta^h(jh)]\!]_\vartheta\,dx =: L_{11}(\phi)+L_{12}(\phi),
$$

where, if $[\![\eta(V,\vartheta)]\!] = \eta(V_-,\vartheta_-)-\eta(V_+,\vartheta_+)$, we denote

$$
[\![\eta(V,\vartheta)]\!]_V = \eta(V_-,\vartheta_-)-\eta(V_+,\vartheta_-), \qquad [\![\eta(V,\vartheta)]\!]_\vartheta = \eta(V_+,\vartheta_-)-\eta(V_+,\vartheta_+).
$$

Since $|[\![\eta^h(jh)]\!]_\vartheta| \le Ce^{-jh}h$, we clearly have

$$
|L_{12}(\phi)| \le C\|\phi\|_\infty.
$$

Concerning $L_{11}(\phi)$, we have, cf. (5.7),

$$
\begin{aligned}
|L_{11}(\phi)| &\le \left|\sum_{j=1}^{\infty}\sum_{i\in\mathbb{Z}}\phi(i,jh)\int_{I_i}[\![\eta^h(jh)]\!]_V\,dx\right| \\
&= \left|\sum_{j=1}^{\infty}\sum_{i\in\mathbb{Z}}\phi(i,jh)\int_{I_i}D_V^2\eta^h(\cdots)(V^h(x,jh-0)-V^h(x,jh+0))^2\,dx\right| \\
&\le C\,\|\phi\|_\infty.
\end{aligned} \tag{5.9}
$$

Hence, we have

$$
|L_1(\phi)| \le C_1\|\phi\|_\infty.
$$

Next, exactly as in [4], we find, assuming that the test function ϕ satisfies $\operatorname{supp}\phi \subseteq [-N,N]\times[-J,J]$ and keeping $\theta>0$ sufficiently small,

(5.10)

$$\begin{aligned}
|L_2(\phi)| &\le \sum_{|j|\le J}\sum_{|i|\le N}\int_{I_i}\left|[\![\eta^h(jh)]\!]\right| |\phi(x,jh)-\phi(i,jh)|\,dx\\
&\le \|\phi\|_{C^\alpha}\sum_{\substack{|j|\le J\\|i|\le N}}\int_{I_i}\left|[\![\eta^h(jh)]\!]\right|\Delta x^\alpha\,dx\\
&\le \|\phi\|_{C^\alpha}\sum_{\substack{|j|\le J\\|i|\le N}}\int_{I_i}\Big(\frac{\Delta x^{2\alpha}}{\Delta x^\theta}+\Delta x^\theta\left|[\![\eta^h(jh)]\!]\right|^2\Big)dx\\
&\le \|\phi\|_{C^\alpha}\sum_{|j|\le J}\sum_{|i|\le N}\Big(\frac{\Delta x^{2\alpha+1}}{\Delta x^\theta}+\Delta x^\theta\int_{I_i}\left|[\![\eta^h(jh)]\!]\right|^2\,dx\Big)\\
&\le \|\phi\|_{C^\alpha}\Big(\frac{\Delta x^{2\alpha+1}}{\Delta x^\theta}(2J+1)(2N+1)+\Delta x^\theta\sum_{|j|\le J}\int_{\mathbb{R}}\left|[\![\eta^h(jh)]\!]\right|^2\,dx\Big)\\
&\le \|\phi\|_{C^\alpha}\Big(\frac{\Delta x^{2\alpha+1}}{\Delta x^\theta}\mathcal{O}\big(\frac{1}{\Delta x\Delta t}\big)+\Delta x^\theta\sum_{|j|\le J}\int_{\mathbb{R}}\left|[\![\eta^h(jh)]\!]\right|^2\,dx\Big)\\
&\le C_2\,\|\phi\|_{C^\alpha}\Big(\frac{\Delta x^{2\alpha+1}}{\Delta x^{\theta+2}}+\Delta x^\theta\Big)\\
&\le C_2\,\|\phi\|_{C^\alpha}\,\Delta x^{\alpha-1/2}
\end{aligned}$$

where C^α denotes the Hölder space with seminorm

$$\|\phi\|_{C^\alpha}=\sup_{x,y\in\mathbb{R}}|\phi(x)-\phi(y)|\,/\,|x-y|^\alpha\,,\qquad \alpha>1/2,$$

and where C_2 depends on the support of ϕ. Thus

$$|L_1(\phi)|\le C_1\|\phi\|_\infty,\qquad\text{and}\qquad |L_2(\phi)|\le C_2(\Delta x)^\beta\|\phi\|_{C^\alpha}$$

for appropriate $\alpha,\beta\in(0,1)$, for some positive constants C_1,C_2 depending on $\operatorname{supp}\phi$, but independent of ϕ, and through the Sobolev imbedding theorem

$$L_2(\phi)\le C_2(\Delta x)^\beta\|\phi\|_{W^{1,q}},$$

for an appropriate $q\in(1,2)$ and constant depending on the support of ϕ.

In this way we obtain by the usual interpolation argument that for any weak entropy pair (η,q) for (1.1)–(1.2) we have

$$\eta(V^h,\vartheta^h)_t+q(V^h,\vartheta^h)_x\in\{\text{ compact of } W^{-1,2}_{\mathrm{loc}}(\mathbb{R}\times[0,\infty))\ \}. \tag{5.11}$$

By (4.2), (4.3), (3.5), (3.2), it easily follows the uniform convergence of $y^h(x,t)$ and $\vartheta^h(x,t)$, by passing to subsequences if necessary, to Lipschitz continuous functions $y(x,t)$ and $\vartheta(x,t)$ with $\vartheta(x,t)=\tilde\sigma(y(x,t),t)$, where

$$\tilde\sigma(y,t)=\frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}}e^{-\frac{(y-z)^2}{4t}}\sigma(z)\,dy,$$

is the solution of the heat equation

$$\tilde\sigma_t=\tilde\sigma_{yy},$$

with initial data

$$\tilde\sigma(y,0)=\sigma(y).$$

By (5.11) and the uniform convergence of $\vartheta^h(x,t)$ to $\vartheta(x,t)$, we can then use the compactness results in [5, 1, 10, 9] to deduce that we may extract a subsequence of (ρ^h,m^h,ϑ^h) converging in $L^1_{\mathrm{loc}}(\mathbb{R}\times[0,\infty))$ to a weak solution $(\rho(x,t),m(x,t),\vartheta(x,t))$

to (1.1)–(1.6). Indeed, (5.1) implies (1.11) by a calculation similar to the estimate for $L_2(\phi)$ above. Moreover, we have

$$
\begin{aligned}
0 &= \int_0^\infty \int_{\mathbb{R}} \tilde\sigma(y,t)\psi_t(y,t) - \tilde\sigma_y(y,t)\psi_y(y,t)\,dy\,dt + \int_{\mathbb{R}} \sigma(y)\psi(y,0)\,dy \\
&= \int_0^\infty \int_{\mathbb{R}} \left(\tilde\sigma(y(x,t),t)\psi_t(y(x,t),t) - \tilde\sigma_y(y(x,t),t)\psi_y(y(x,t),t)\right)\rho(x,t)\,dx\,dt \\
&\quad + \int_{\mathbb{R}} \sigma(y_0(x))\psi(y_0(x),0)\rho_0(x)\,dx \\
&= \int_0^\infty \int_{\rho(x,t)>0} \big(\rho(x,t)\tilde\sigma(y(x,t),t)\psi_t(y(x,t),t) - \rho(x,t)m(x,t)\tilde\sigma(y(x,t),t)\psi_y(y(x,t),t) \\
&\qquad\qquad + \rho(x,t)m(x,t)\tilde\sigma(y(x,t),t)\psi_y(y(x,t),t) \\
&\qquad\qquad - \frac{1}{\rho(x,t)}\rho(x,t)\tilde\sigma_y(y(x,t),t)\rho(x,t)\psi_y(y(x,t),t)\big)\,dx\,dt \\
&\quad + \int_{\mathbb{R}} \sigma(y_0(x))\psi(y_0(x),0)\rho_0(x)\,dx \\
&= \int_{\mathbb{R}\times(0,\infty)} \rho\vartheta\frac{\partial}{\partial t}\psi(y(x,t),t) + (m\vartheta - \frac{1}{\rho}\vartheta_x)\frac{\partial}{\partial x}\psi(y(x,t),t)\,dx\,dt \\
&\quad + \int_{\mathbb{R}} \rho_0\vartheta_0\psi(y_0(x),x)\,dx,
\end{aligned}
$$

where we have used the coarea formula (see, e.g., [6]) and (1.4), thus proving (1.12).

Also, (5.5) implies the entropy inequality (1.15),

6. Conclusion of the proof of Theorem 1.1.

To conclude the proof of Theorem 1.1 it remains to verify (1.16), which we do as follows. First, from the above discussion, we deduce that for any weak entropy pair we have

$$|\langle \eta(\rho,m,\vartheta)_t + q(\rho,m,\vartheta)_x, \phi\rangle| \le C_1\|\phi\|_\infty,$$

with C_1 depending only on $\operatorname{supp}\phi$ and bounds for (ρ,m,ϑ). Hence, if $U^T = (\rho^T, m^T, \vartheta^T)$ is the self-scaling sequence $U^T(x,t) = U(Tx,Tt)$, we see that for any entropy pair

$$|\langle \eta(\rho^T,m^T,\vartheta^T)_t + q(\rho^T,m^T,\vartheta^T)_x, \phi\rangle| \le C_1\|\phi\|_\infty,$$

while from (1.15) we have, for $0 \le t \le T$,

$$
\begin{aligned}
\int_{[0,L]} \eta_*(\rho,m,\vartheta)(x,t)\,dx &\ge \int_{[0,L]} \eta_*(\rho,m,\vartheta)(x,T)\,dx - C\int_t^T \int_{[0,L]} e^{-s}\,dx\,ds \\
&\ge \int_{[0,L]} \eta_*(\rho,m,\vartheta)(x,T)\,dx - CLe^{-t}.
\end{aligned}
$$

Hence, we can apply the decay result in [2] to deduce (1.16), which then concludes the proof.

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