Curves of zero self-intersection and Foliations

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Abstract

We study the holonomy group of a holomorphic foliation in a surface along a compact leaf. It is shown that linearization of a neighborhood of the curve implies strong restrictions in those groups.

We deal in this paper with suspensions along holomorphic curves of pseudogroups of local diffeomorphisms of $\mathbb{C}$ which fix $0 \in \mathbb{C}$.

Let C be a holomorphic, compact, smooth curve embedded in some holomorphic smooth surface S (we will be interested in small neighborhoods of C in S). Assume that S admits two regular holomorphic foliations $\mathcal{F}$ and $\mathcal{G}$ such that C is a leaf of $\mathcal{F}$ (so that the self intersection number $C \cdot C$ vanishes) and $\mathcal{G}$ is transverse to C; let H be the holonomy group of $\mathcal{F}$ along C (computed at some transverse section to C). Conversely, we may take the suspension of H along C, obtaining a new surface S' which is isomorphic to S; S' of course comes with two foliations $\mathcal{F}'$ and $\mathcal{G}'$ which are also isomorphic to $\mathcal{F}$ and $\mathcal{G}$.

The question that appears in a natural way in the first context is: if S is equivalent to the normal bundle of C (at least to a certain order), what can be said about H? The corresponding question in the context of suspensions is: which properties the pseudogroup has to satisfy so that the surface defined by its suspension along C is equivalent to the normal bundle of C (again, up to a certain order)?

Let us present a simple example that somehow illustrates the situation. Take in $C \times \mathbb{C}$ a foliation defined by

$$dt = \left(\sum_{k=1}^{\infty} \omega_k(z) t^k \right) dz$$

where $z \in \widetilde{C} \subset \mathbb{C}$, the universal covering space of C and any for each ω_k is a holomorphic 1-form of C. Let us assume that $\pi_1(C, p)$, the fundamental group of C with p as base point, is generated by the curves $a_1, \ldots, a_g$, $b_1, \ldots, b_g$ which form a sympletic basis of $H_1(C, \mathbb{Z})$. One sees easily that the linear holonomy group of the foliation has generators $\{t \mapsto \lambda_j t\}_{j=1}^{n}$ (associated to the a-curves $a_, \ldots, a_g$) and $\{t \mapsto \mu_j t\}_{j=1}^{n}$ (associated to the b-curves $b_1, \ldots, b_g$) satisfy:

$$\lambda_j = e^{2i\pi \int_{a_j} \omega_1} \quad \text{and} \quad \mu_j = e^{2i\pi \int_{b_j} \omega_1}.$$

Since the b-periods of a holomorphic 1-form are determined after the a-periods, we conclude that the b-generators of the linear holonomy group are also determined by the a-generators.

We may reformulate the example as: let us consider the suspension along C of a pseudogroup generated by

$$t \mapsto \lambda_j t + \ldots \quad \text{and} \quad t \mapsto \mu_j t + \ldots$$

to a foliation of a surface S; in order to get $C \times \mathbb{C}$ as the normal bundle of C in S (that is, S is $C \times \mathbb{C}$ up to order 1), we have a precise choise of $\mu_1, \ldots, \mu_g$ once we prescribe the values $\lambda_1, \ldots, \lambda_g$.

We proceed now to the statement of our Theorem. Consider again the surface S, a curve $C \subset S$ with normal bundle N_C and a foliation $\mathcal{G}$ transverse to C. We are interested in the foliations of S which have C as a leaf. Given such a foliation $\mathcal{F}$, we compute its holonomy group along C using a transverse section Σ to C (contained, for example, in a leaf of $\mathcal{G}$); let us parametrize it by a coordinate $w_0 \in \mathbb{C}$, and write the generators as $h_{a_j}(w_0) = \sum_{j=1}^{\infty} A_\ell^{(j)} w_0^\ell$, $h_{b_j}(w_0) = \sum_{j=1}^{\infty} B_\ell^{(j)} w_0^\ell$ for $j = 1, \ldots, g$. These maps depend on the foliation; we may write $h_{a_j}(\mathcal{F})$ and $h_{b_j}(\mathcal{F})$, as well as $A_l^j(\mathcal{F})$ and $B_l^j(\mathcal{F})$. We assume that $A_1^j(\mathcal{F}) = \lambda_j$ and $B_1^j(\mathcal{F}) = \mu_j$ are fixed (because they define N_C, as we have already seen).

Suppose that for a given $N \geq 2$, S has a system of coordinates of N-type. This concept was introduced in [3], pg. 587, and essentialy means that S is equivalent to N_C up to order N (see Section 1).

Theorem. *Assume S is equivalent to N_C up to order $N \geq 2$.*

(1) if $A_2 = \cdots = A_N = 0$ then $B_1 = \cdots = B_N = 0$.

(2) in general, the coefficients $B_2, \ldots, B_N$ are uniquely determined by the coefficients $A_2, \ldots, A_N$. In other words, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are holomorphic foliations which have C as a leaf and if $A_j(\mathcal{F}_1) = A_j(\mathcal{F}_2)$ then $B_j(\mathcal{F}_1) = B_j(\mathcal{F}_2)$, $2 \leq j \leq N$.

What seems to be behind this Theorem is the existence, for each $2 \leq \nu \leq N$, of a holomorphic differencial 1-form in the curve C whose periods are the coefficients of the terms of order ν of the generators of the holonomy group. We are not able to fully develop this idea, that is, write down the corresponding 1-forms, except in the proof of part (1) of the Theorem and in the cases $N = 2$ and $N = 3$, when N_C is the trivial line bundle. But anyway in the proof of part (2) it is used the principle of determination of b-periods once a-periods are known.

Using this Theorem we can obtain for each integer $N \geq 2$ examples of surfaces (obtained as suspensions of pseudogroups of diffeomorphisms) which have not coordinate systems of N-type.

We are grateful to M. Brunella for his conceptual proof concerning *periods* of holomorphic differential 1-forms taking values in line bundles over a curve.

1 Linearizing Coordinates

We consider the situation described in the Introduction: a compact, smooth, holomorphic curve C is embedded in some holomorphic surface S carrying a holomorphic fibration $\mathcal{G}$ transverse to C (we are in fact interested in small neighborhoods of C in S). Along C we use a coordinate $z \in \widetilde{C}$, where $\widetilde{C} \subset \mathbb{C}$ is the universal covering space of C; this coordinate can be extended to S making it constant along fibers of $\mathcal{G}$.

Given $N \geq 2$, a *system of coordinates for S of type N* is the following data:

(i) a covering $\{U'_i\}$ of a neighborhood of C by open sets,

(ii) for each U'_i there is a coordinate w'_i such that $U_i = C \cap U'_i$ is defined as $w'_i = 0$ and whenever $U_i \cap U_k \neq \emptyset$ one has

$$w'_k = \lambda_{ki} w'_i + h_{ik}(z)(w'_i)^{N+1} + \dots \tag{1}$$

This Definition appears in [3]; the coordinates (w'_i), together with the coordinate z, provide a linearization of S up to order N. Remark that the 1-cocycle $\{\lambda_{ki}\}$ defines the normal bundle N_C of C in S, which is a flat line bundle (see [1], pg. 134)

We will assume that the open sets U_i are small discs, hence distinguished neighborhoods for the covering map $\widetilde{C} \to C$.

For any different system of coordinates $\{w_i\}$ in $\{U'_i\}$

$$w'_i = w_i + P_i(z, w_i) \tag{2}$$

($P_i(z, w_i) = \sum\limits_{j=2}^{\infty} f_i^{(j)}(z) w_i^j$ is a holomorphic function for each i) we have of course a commutative diagram

$$\begin{array}{ccc} w_i & \rightarrow & w_k \\ \downarrow & & \downarrow \\ w'_i & \rightarrow & w'_k \end{array}$$

whenever $U_i \cap U_k \neq \emptyset$.

In the presence of a foliation $\mathcal{F}$ which has C as a leaf, we may choose $\{w_i\}$ in (2) in order to have U_i given by $w_i = 0$ and $\mathcal{F}|_{U'_i}$ defined by $dw_i = 0$. It follows that whenever $U_i \cap U_k \neq \emptyset$, the change of coordinates from w_i to w_k is given by

$$w_k = \lambda_{ki} w_i + P_{ki}(w_i) \tag{3}$$

where $P_{ki}(w_i) = \sum\limits_{j=2}^{\infty} f_{ki}^{(j)} w_i^j$ is a holomorphic function ($f_{ki}^{(j)} \in \mathbb{C}$).

Simultaneous existence of these two coordinate systems $\{w_i\}$ and $\{w'_i\}$ originates relations between the generators of the holonomy group of the foliation.

We may assume, without loss of generality, that $P_0(0, w_0) \equiv 0$ (we fix U_0 as the open set which contains $z = 0$).

We proceed now to make a modification in these coordinate systems aiming to prove our Theorem. Consider a simply connected open set $U \subset C$, with $U_0 \subset U$. The coordinate w_0 can be extended as W_0 to a neighborhood U' of U; we start with $W_0|_{U'} = w_0$ ands extend it constant way along each leaf of $\mathcal{F}|_{U'}$. We seek to extend also w_0' to U, but in fact only up to order N. Let $U_1 \subset U$, $U_0 \cap U_1 \neq \emptyset$. In the intersection $U_0' \cap U_1'$ (after (1), (2) and (3) above):

$$\begin{cases} w_0' = w_0 + \Sigma f_0^{(j)}(z) w_0^j, & z \in U_0 \\ w_1' = w_1 + \Sigma f_1^{(j)}(z) w_1^j, & z \in U_1 \\ w_1 = \lambda_{10} w_0 + \Sigma f_{10}^{(j)} w_0^j \\ w_1' = \lambda_{10} w_0' + h_{10}(z)(w_0')^{N+1} + \ldots . \end{cases}$$

This implies

$$w_1 + \Sigma f_1^{(j)}(z) w_1^j = \lambda_{10}(w_0 + \Sigma f_0^{(j)}(z) w_0^j) \quad \mod w_0^{N+1}$$

or

$$\sum_{j \geq 2} f_{10}^{(j)} + \sum_{j \geq 2} f_1^{(j)}(z)(\lambda_{10} w_0 + \sum_{\ell \geq 2} f_{10}^{(\ell)} w_0^\ell)^j = \lambda_{10} \sum_{j \geq 2} f_0^{(j)}(z) w_0^j \mod w_0^{N+1}.$$

We fix $2 \leq \nu \leq N$ and compare the coefficients of w_0^j in both sides, what leads to ($z \in U_0 \cap U_1$)

$$\lambda_{10} f_0^{(\nu)}(z) = \lambda_{10}^\nu f_1^{(\nu)}(z) + A_{\nu-1}^{(\nu)} f_1^{(\nu-1)}(z) + \cdots + A_2^{(\nu)} f_1^{(\nu)}(z) + f_{10}^{(\nu)}$$

where $A_{\nu-1}^{(\nu)}, \ldots, A_2^{(\nu)}$ are algebraic functions of $f_{10}^{(2)}, \ldots, f_{10}^{(\nu-1)}$(the precise expressions are not needed here).

Therefore $f_0^{(\nu)}(z)$ can be extended to U_1 as

$$\lambda_{10}^{\nu-1} f_1^{(\nu)}(z) + \lambda_{10}^{-1}[A_{\nu-1}^{(\nu)} f_1^{(\nu-1)}(z) + \cdots + A_2^{(\nu)} f_1^{(2)}(z)] + \lambda_{10}^{-1} f_{10}^{(\nu)}$$

so we may define $W_0' = W_0 + \sum\limits_{j=2}^{N} f_0^{(j)}(z) W_0^j$, $z \in U_0 \cap U_1$

Claim: The diagram

$$\begin{array}{ccc} W_0 & \to & w_1 \\ \downarrow & & \downarrow \\ W_0' & \to & w_1' \end{array}$$

commutes over U_1 up to order N.

This is quite obvious, because the diagram is commutative over $U_0 \cap U_1 \subset U_1$.

We repeat the reasoning for $U_2 \cap U_1 \neq \emptyset$, $U_2 \subset U$. From the commutativity of the above diagram and also of the diagram

$$\begin{array}{ccc} w_1 & \to & w_2 \\ \downarrow & & \downarrow \\ w_1' & \to & w_2' \end{array}$$

in $U_1 \cap U_2$ (up to order N), we see that

$$\begin{array}{ccc} W_0 & \to & w_2 \\ \downarrow & & \downarrow \\ W_0' & \to & w_2' \end{array}$$

is commutative over $U_1 \cap U_2$ (up to order N).

We go on until U is covered. We have proved then

Lemma 1. *Suppose $U_k \cap U \neq \emptyset$. Then*

$$\begin{array}{ccc} W_0 & \to & w_k \\ \downarrow & & \downarrow \\ W_0' & \to & w_k' \end{array}$$

is commutative (up to order N).

We remark that we have also proved:

Proposition 1. *The functions $f_0^{(2)}, \ldots, f_0^{(N)}$ have holomorphic extensions $F^{(2)}, \ldots, F^{(N)}$ to $\widetilde{C}$,* the universal covering space of C.

We can be more precise:

Proposition 2. *Let $Hol(\mathcal{F}, C)$ be computed in the section of coordinates $(0, W_0)$ as*

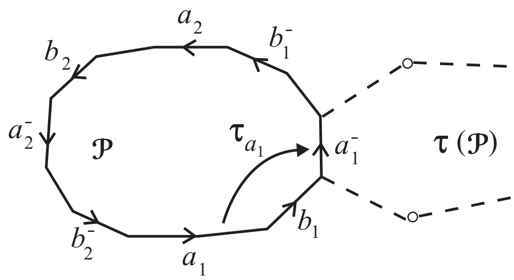
$$h_\tau(W_0) = \lambda_\tau W_0 + \sum_{j \geq 2} C_\tau^{(j)} W_0^j, \quad \tau \in \pi_1(C, p).$$

Therefore

(4) $F^{(\nu)}(\tau(z)) = \lambda_\tau^{\nu-1} F^{(\nu)}(z) + D_{\tau,\nu-1}^{(\nu)} F^{(\nu-1)}(z) + \cdots + D_{\tau,2}^{(\nu)} F^{(2)}(z) + \lambda_\tau^{-1} C_\tau^{(\nu)}$

for $z \in \widetilde{C}$; the coefficients $D^{(\nu)}_{\tau,\nu-1}, \dots, D^{(\nu)}_{\tau,2}$ are algebraic expressions in λ_τ, $C_\tau^{(2)}, \dots, C_\tau^{(\nu-1)}$.

Proof: Let us fix for example $\tau = \tau_{a_1}$ and apply Lemma 1 taking $\mathcal{P}$ or $\tau_{a_1}(\mathcal{P})$ as U; $\mathcal{P}$ is a fundamental polygon of sides $a_1, b_1, a_1^-, b_1^-, \dots, a_g, b_g, a_g^-, b_g^-$.



We link $\mathcal{P}$ and $\tau_{a_1}(\mathcal{P})$ by a small neighborhood U_k of some point of a_1^-. The coordinate W_0 of Lemma 1 can be used for neighborhoods $(\mathcal{P} \cup U_k)'$ of $\mathcal{P} \cup U_k$ and $\tau_{a_1}(\mathcal{P})'$ of $\tau_{a_1}(\mathcal{P})$; a leaf which is $\{W_0 = a\}$ in $\mathcal{P} \cup U_k$ arrives at $(\tau_{a_1}(\mathcal{P}))'$ as $W_0 = h_{a_1}(a)$. Let us use, for the sake of clarity, the notations $\overline{W}_0$ and $\overline{W}_0^1$ for coordinates of points in $\tau(\mathcal{P})'$.

The Proposition follows after applying the same computations we did in the proof of Lemma 1: we pass from U_k' to $\tau_{a_1}(\mathcal{P})'$ ($\overline{W}_0$ and $\overline{W}_0'$ play the role of w_1 and w_1') and the change of coordinates from w_k to $\overline{W}_0$ is given by h_{a_1}. □

2 Two Particular Cases

We assume that N_c is the trivial line bundle. The cases $\nu = 2$ and $\nu = 3$ are easy to deal with directly. Start with $\nu = 2$. Proposition 1 gives

$$F^{(2)}(\tau(z)) = F^{(2)}(z) + C_\tau^{(2)}.$$

Each $C_\tau^{(2)}$ is the τ-period of the holomorphic 1-form of C

$$\left[\frac{d}{dz} F^{(2)}(z)\right] dz.$$

Since a-periods determine b-periods, the numbers $C_{a_1}^{(2)}, \dots, C_{a_g}^{(2)}$ determine $C_{b_1}^{(2)}, \dots, C_{b_g}^{(2)}$.

Let us go to the next case $\nu = 3$. Then:

$$F^{(3)}(\tau(z)) = F^{(3)}(z) + D^{(3)}_{\tau,2} F^{(2)}(z) + C^{(3)}_\tau.$$

It can easily be seen that $D^{(3)}_{\tau,2} = 2C^{(3)}_\tau$, which implies:

$$\frac{d}{dz} F^{(3)}(\tau(z))\tau'(z) = \frac{d}{dz} F^{(3)}(z) + D^{(3)}_{\tau,2} \frac{d}{dz} F^{(2)}(z)$$

and

$$\frac{\frac{d}{dz} F^{(3)}(\tau(z))}{\frac{d}{dz} F^{(2)}(\tau(z))} = \frac{\frac{d}{dz} F^{(3)}(z)}{\frac{d}{dz} F^{(2)}(z)} + 2C^{(2)}_\tau = \frac{\frac{d}{dz} F^{(3)}(z)}{\frac{d}{dz} F^{(2)}(z)} + 2(F^{(2)}(\tau(z)) - F^{(2)}(z)).$$

Finally:

$$\frac{\frac{d}{dx} F^{(3)}(\tau(z))}{\frac{d}{dz} F^{(2)}(\tau(z))} = 2F^{(2)}(\tau(z)) = \frac{\frac{d}{dz} F^{(3)}(z)}{\frac{d}{dx} F^{(2)}(z)} - 2F^{(2)}(z).$$

We see that

$$\left[\frac{\frac{d}{dz} F^{(3)}(z)}{\frac{d}{dz} F^{(2)}(z)} - 2F^{(2)}(z)\right] \left[\frac{d}{dz} F^{(2)}(z)\right] dz$$

is a holomorphic 1-form of C', so the same is true for

$$\left[\frac{d}{dz} F^{(3)}(z) - 2F^{(2)}(z) \frac{d}{dz} F^{(2)}(z)\right] dz$$

or

$$\frac{d}{dz} [F^{(2)}(z) - (F^{(2)}(z))^2] dz.$$

The τ-period is $F^{(3)}(\tau(0)) - [F^{(2)}(\tau(0))]^2 = C^{(3)}_\tau - (C^{(2)}_\tau)^2$. Once more we conclude that $C^{(3)}_{a_1}, \ldots, C^{(3)}_{a_g}$ determine $C^{(3)}_{b_1}, \ldots, C^{(3)}_{b_g}$.

We are not able to exhibit appropriate holomorphic 1-forms for $\nu \geq 4$ allowing comparision between periods. We prove our Theorem by a less explicit method.

3 Proof of the Theorem

Let us take regular foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ in the surface S which have a smooth, holomorphic curve $C \subset S$ as a leaf. We assume that S is equivalent to N_C

up to order N, $N \geq 2$. We wish to prove that $\mathrm{Hol}(\mathcal{F}_1, C)$ and $\mathrm{Hol}(\mathcal{F}_2, C)$ are the same up to order N, under the hypothesis that the a-generators of both groups are the same op to order N (as before, the holonomy groups are computed for the transversal in $U_0$with coordinates $(0, w_0)$).

By Proposition 2 we have functions $F_1^{(2)}, \ldots, F_1^{(N)}$ and $F_2^{(2)}, \ldots, F_2^{(N)}$ in $\widetilde{C}$ associated to $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfying relations as in (4); we may assume $F_1^{(\nu)}(0) = F_2^{(\nu)}(0) \; \forall \; 2 \leq \nu \leq N$. We will prove that $F_1^{(\nu)} = F_2^{(\nu)} \; \forall \; 2 \leq \nu \leq N$ by induction. If $\nu = 2$, we observe that

$$\frac{d}{dz}[F_1^{(2)}(z) - F_2^{(2)}(z)]dz \in H^0(C, \Omega_C^1 \otimes N_C).$$

We have to show that this 1-form is zero. Next, if assumed that $F_1^{(\nu)} \equiv F_2^{(\nu)}$ for $2 \leq \nu \leq N-1$:

$$\frac{d}{dz}[F_1^{(N)}(z) - F_2^{(N)}(z)]dz \in H^0(C, \Omega_C^1 \otimes N_C^{\otimes(n-1)})$$

and this will be proven to be zero.

The hypothesis in the statement of the Theorem implies that the a-periods of these 1-forms vanish.

What we need then is

Lemma 2. *Let $\omega \in H^0(C, \Omega_C^1 \otimes L)$, where L is some flat line bundle over C. If ω has vanishing a-periods, then $\omega \equiv 0$.*

The case of a trivial line bundle is well-known (see [2],pg.142); we will adapt the proof. In that situation, periods appear as obstructions to exactness of holomorphic 1-forms.

In general, let L be defined by a cocycle $\{\lambda_{ij}\}$ associated to a covering by open sets $\{V_i\}$ of a compact curve C', which may have boundary components. A differential $\omega \in H^0(C', \Omega_{C'}^1 \otimes L)$ has local expressions $\{\omega_i\}$ related as $\omega_i = \lambda_{ij}\omega_j$; we try to find $\{f_i\} \in H^0(C', L)$ in order to have $\omega_i = df_i$. Fix $p \in V_0$ and $f_0 \in \mathcal{O}_{V_0}$ such that $\omega_0 = df_0$; we will take the analytic "continuation" of f_0 to C'. Given $q \in C'$ and a path joining p to q, we cover it by $V_0 \cup \cdots \cup V_\ell$ and select $f_1, \ldots, f_\ell$ satisfying $\omega_i = df_i$ and $f_{i+1} = \lambda_{i+1,i} f_i$. The problem arises when we try to do this for a different path from p to q. Or, equivalently, if we take a closed path γ passing through p and apply the same construction, we may arrive to a relation (became $V_\ell \cap V_0 \neq \emptyset$) of the kind

$$f_0 = \lambda_{0\ell} f_\ell + a$$

for some $a \in \mathbb{C}$ (this is the *γ-period of ω*; it is independent of the choice of γ in its homotopy class). Therefore, if all periods vanish, we may find $f \in H^0(C', L)$ such that $\omega = df$ (that is, $\omega_i = df_i$ and $f_i = \lambda_{ij} f_i$).

Proof of Lemma 2:

1) Since L is flat, we may assume $|\lambda_{ij}| = 1 \; \forall \; i, j$ (see [3], pg. 584). We take as C' the curve obtained after cutting C' along the a-curves $a_1, \ldots, a_g$; C' is a compact Riemann surface with boundary $a_1^+ \cup a_1^- \cup \cdots \cup a_g^+ \cup a_g^-$; its fundamental group $\pi_1(C', p)$ is generated by these curves (more precisely, we have to join the base point p to them). It follows that $\omega|_{C'} = df$, where $f \in H^0(C', L')$ and $L' = L|_{C'}$; we are using here that the a-periods of ω vanish.

2) The 2-form $\omega \wedge \bar{\omega}$ is well defined in C' (remember that $|\lambda_{ij}| = 1$); we wish to prove that

$$\iint_{C'} \omega \wedge \bar{\omega} = \iint_{C'} df \wedge df \bar{f} = 0.$$

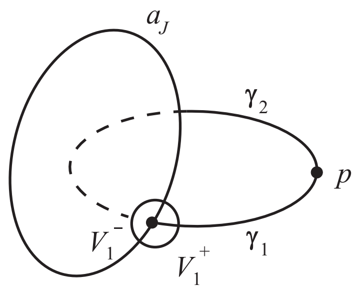
Since $df \wedge d\bar{f} = d(f \wedge d\bar{f})$ (again: $|\lambda_{ij}| = 1 \; \forall \; i, j$ implies that $\beta = f \wedge d\bar{f}$ is a well defined 1-form in C'). By Stokes' Theorem,

$$\iint_{C'} \omega \wedge \bar{\omega} = \int_{\delta C'} \beta.$$

Let us compare this 1-form β along the components of each pair (a_J^+, a_J^-). Cover a_J by open sets $V_1, \ldots, V_k$ with $V_1 \cap V_2 \neq \emptyset$, $V_{k-1} \cap V_k \neq \emptyset$, $V_k \cap V_1 \neq \emptyset$ (any other intersection between those set are supposed to be empty). In C the curve a_J has two sides; we decompose $V_1, \ldots, V_k$ according to each side. In this way a_J^+ is covered by $V_1^+ \cup \cdots \cup V_k^+$ and a_J^- by $V_1^- \cup \cdots \cup V_k^-$. We denote by f_i^+ and f_i^- the values of f in V_i^+ and V_i^-, respectively. They are obtained from the continuation of f_0; we start joining p to V_1^+ and V_1^- we get

$$f_1^+ = \lambda f_1^- + c$$

for some $|\lambda| = 1$ and some $c \in \mathbb{C}$



We proceed along a_J^+ and a_J^-. Let us use the transition cocycle of L along a_J (in fact along a_J^+ and a_J^-); then

$$\begin{aligned} f_2^+ &= \lambda_{21} f_1^+, \ldots, f_k^+ = \lambda_{k,k-1} f_{k-1}^+, \quad f_1^+ = \lambda_{1k} f_k^+ \\ f_2^- &= \lambda_{21} f_1^-, \ldots, f_k^- = \lambda_{k,k-1} f_{k-1}^-, \quad f_1^- = \lambda_{1k} f_k^- \\ f_i^+ &= \lambda f_i^- + \lambda^{(i)} c, \quad \text{where} \quad \lambda^{(i)} = \lambda_{i,i-1} \ldots \lambda_{21}. \end{aligned}$$

We see that the 2-form $f \wedge d\bar{f}$ along a_J is given by the collection $\{f_i^+ \wedge d\bar{f}_i^+\} = \{f_i^- \wedge d\bar{f}_i^- + c\lambda^{(i)} d\bar{f}_i^-\}$.

When we compute $\int_{a_J^+ \cup a_J^-} \beta$, because of the opposite orientations induced by C' in a_J^+ and a_J^-, $\int_{a_J^+} f_i^+ \wedge d\bar{f}_i^+$ and $\int_{a_J^-} f_i^- \wedge d\bar{f}_i^-$ cancel each other. We are left with $\{\lambda^{(i)} d\bar{f}_i^-\}$, which in fact is the differential of a well defined function along a_J^-:

$$\begin{aligned} \lambda^{(2)} \bar{f}_2^- &= \lambda_{21} \bar{f}_2^- = \lambda_{21} \bar{\lambda}_{21} \bar{f}_1^- = \bar{f}_1^- \quad \text{in } V_1^- \cap V_2^- \\ &\vdots \\ \lambda^{(i+1)} \bar{f}_{i+1}^- &= \lambda_{i_1,i} \ldots \lambda_2 \bar{f}_{i+1}^- = \lambda^{(i)} \lambda_{i+1,i} \bar{\lambda}_{i+1,i} \bar{f}_i^- = \lambda^{(i)} \bar{f}_i^- \text{ in } V_{i+1}^- \cap V_i^- \\ &\vdots \end{aligned}$$

It follows that $\int_{a_J^-} c\lambda^{(i)} d\bar{f}_i^- = 0$ and $\int_{a_J^+ \cup a_J^-} \beta = 0$ $\square$

4 Final Remarks

Let us start by observing that the Theorem of Riemann-Roch allows us to compute $\dim_{\mathbb{C}} H^0(C, \Omega_C^1 \otimes L)$, L a line bundle. When L is trivial, this dimension is g, the genus of C. Otherwise, the dimension is $g-1$.

Another observation is that the total space $[L]$ of a flat L carries a natural foliation $\mathcal{L}$ which has C as a leaf and whose holonomy group along C is linear. To see this, we notice that L is defined by coordinates $\{w_i\}$ related as $w_i = \lambda_{ij} w_j$ for $\lambda_{ij} \in \mathbb{C}$. The foliation $\mathcal{L}$ is given as $dw_i = 0$. We may wonder whether $[L]$ carries a different foliation having C as a leaf.

When $g = 1$ and L is no torsion (that is, $L^{\otimes n}$ is not trivial for all $n \in \mathbb{N}$) the answer is negative. In fact, $\frac{d}{dz} F^{(2)} \in H^0(C, \Omega^1_C)$ implies $\frac{d}{dz} F^{(2)} = 0$, so that $F^{(2)} = 0$. Assuming $F^{(i)} = 0$ for $2 \leq i \leq m$, we see from Proposition 2 that $\frac{d}{dz} F^{(m+1)} \in H^0(C, \Omega^1_C \otimes L^{\otimes m})$, so again $F^{(m+1)} = 0$.

Finally, let us explain how to use the Theorem to construct surfaces containing a curve C such that $C \cdot C = 0$ which are not linearizable. We consider the horizontal foliation in $C \times \mathbb{C}$; the holonomy group relatively to $C \times \{0\}$ is the Identity.. We consider then a nonlinear pseudogroup of diffeomorphisms which is the image of a representation of the fundamental group of C: we keep the a-generators and the b-generators as the Identity maps except for one of the b-generators which has a nonlinear 2-jet. It follows that the surface obtained by the suspension along C of this pseudogroup is not linearizable in order 2, so *a fortiori* not linearizable.

References

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