Degree theory of immersed hypersurfaces

15 September 2011

Harold Rosenberg

IMPA,
Estrada Dona Castorina 110,
Rio de Janeiro,
BRASIL 22460-320

Graham Smith

IMPA,
Estrada Dona Castorina 110,
Rio de Janeiro,
BRASIL 22460-320

Abstract: We develop a degree theory for compact immersed hypersurfaces of prescribed K-curvature immersed in a compact, orientable Riemannian manifold, where K is any elliptic curvature function. We apply this theory to count the (algebraic) number of immersed hyperspheres in various cases: where K is mean curvature; extrinsic curvature and special Lagrangian curvature, and we show that in all these cases, this number is equal to $-\chi(M)$, where $\chi(M)$ is the Euler characteristic of M.

This paper is a considerably revised version of our earlier paper (arXiv:1010.1879v1)

Key Words: Degree Theory, Immersions, Convexity, Prescribed Curvature, Non-Linear Elliptic PDEs.

AMS Subject Classification: 58D10 (58B05, 58C40, 58J05)

1 - Introduction.

Let $M := M^{n+1}$ be a compact, orientable, $(n+1)$-dimensional, Riemannian manifold, and let $\Sigma := \Sigma^n$ be a compact, orientable, n-dimensional manifold. We develop a degree theory for certain immersions of prescribed curvature of Σ in M. More precisely, let $\mathcal{I}$ be an open set of immersions $i : \Sigma \to M$. We say that the immersion i is simple if for any $p \neq q \in \Sigma$ and for all sufficiently small neighbourhoods U and V of p and q respectively, we have $i(U) \neq i(V)$. Observe that this is a weaker notion than injectivity which allows for self-intersections but not multiple covers. We henceforth assume that $\mathcal{I}$ consists only of simple immersions. We identify two elements i and $\tilde{i}$ in $\mathcal{I}$ whenever they differ by a diffeomorphism of Σ and we furnish $\mathcal{I}$ with the topology of smooth convergence modulo reparametrisation.

Let K be a curvature function, such as **mean curvature**, or **extrinsic curvature** (c.f. Section 2.1 for a precise definition), and for $i \in \mathcal{I}$, define $K(i) : \Sigma \to \Sigma$, the K-curvature of i by:

$$K(i)(p) = K(A_i(p)),$$

where A_i is the shape operator of i.

We henceforth assume:

Ellipticity: for all $[i] \in \mathcal{I}$, the Jacobi operator over $[i]$ of the K-curvature is an elliptic, second order partial differential operator. In other words, it is a generalised Laplacian.

Now let $\mathcal{O} \subseteq C^\infty(M)$ be an open connected set of functions. We define the **solution space** $\mathcal{Z}$ to be the set of all pairs $([i], f) \in \mathcal{I} \times \mathcal{O}$ where the K-curvature of $[i]$ is prescribed by f, that is:

$$\mathcal{Z} = \{([i], f) \mid K(i) = f \circ i\}.$$

Let $\pi : \mathcal{Z} \to \mathcal{O}$ by the projection onto the second factor:

$$\pi([i], f) = f,$$

and we suppose:

Properness: the projection $\pi : \mathcal{Z} \to \mathcal{O}$ is a proper mapping.

When K is elliptic and π is proper, we show that π has an integer valued degree. Indeed, we show that for f in an open dense subset $\mathcal{O}'$ of $\mathcal{O}$, $\pi^{-1}(f)$ is finite, and each $([i], f) \in \pi^{-1}(f)$ has a well defined signature taking values in $\{-1, 1\}$ and we thus define:

$$\mathrm{Deg}(\pi; f) = \sum_{[i] \in \pi^{-1}(f)} \mathrm{sig}([i], f).$$

We prove that $\mathrm{Deg}(\pi; f)$ is independant of $f \in \mathcal{O}'$ and therefore defines a degree for π.

The hypothesis that every element of $\mathcal{I}$ is simple is required to exclude the possibility of orbifold points arising on the space of immersions. We will show in our forthcoming paper [21] how multiply covered immersions may be allowed by permitting the degree to take rational values. In the context of our current applications, this allows us to drop the 1/4-pinched condition on the ambient spaces in Theorems 1.2 and 1.3 below.

Our degree is inspired by the beautiful degree theory developed by Brian White for mean curvature and parametric elliptic integrals in [33], [34], [35] and [36]. The main idea is to view π as a "Fredholm map of index zero between Banach manifolds". Taking $\mathcal{O}'$ to be the set of regular values of π and by applying a "Sard-Smale" type theorem, we see that this set is both open and dense. For $f \in \mathcal{O}'$ and for $[i] \in \pi^{-1}(f)$, $D\pi_{([i],f)}$ is an isomorphism, its index is defined to be the sum of the algebraic (that is, nilpotent) multiplicities of its real, negative eigenvalues, and its signature is defined to be (-1) raised to the power of its index, i.e. the parity of the index. We then prove that $\sum_{[i]\in\pi^{-1}(f)} \mathrm{sig}([i], f)$ is well defined and is independant of $f \in \mathcal{O}'$, thus yielding the degree.

Making the above discussion precise requires in-depth analsysis forming the content of Sections 2 and 3.1 through to 3.4. There are two major difficulties. The first lies in defining a "Banach manifold" structure for which π is Fredholm of index zero and then proving a Sard/Smale theorem for π, and the second then lies in proving that the degree does not depend on the regular value chosen. For the reader's convenience, in Appendix A we provide a discussion of the functional analytic framework used, showing, in particular how a Sard/Smale theorem works in this context.

This paper was initially motivated by the question of the existence of embeddings of constant mean curvature of $\Sigma = S^n$ into (S^{n+1}, g). Indeed, we conjecture that for any $c \geqslant 0$ and any metric g on S^3 of positive sectional curvature, there is an embedding of S^2 into S^3 of constant mean curvature c. This result is known for $c = 0$ (c.f. [25] and [5]) and also for large values of c, since solutions to the isoperimetric problem for small volumes are embedded spheres of large constant curvature (c.f. [37]). However, even when $n = 1$ and g is a positive curvature metric on S^2, we do not know if there exist embeddings of S^1 having any prescribed, positive geodesic curvature. Nonetheless, Anne Robeday (c.f. [19]), and independantly Matthias Schneider (c.f. [22]) proved that there always exists an Alexandrov embedding (i.e. an immersion that extends to an immersion of the disk) of S^1 into (S^2, g) of any prescribed, constant geodesic curvature, assuming that g has positive curvature. Anne Robeday's approach used the degree theory of Brian White to prove this result whilst M. Schneider developed a different degree theory for his proof. Schneider's theory applies to immersions of the circle into any Riemannian surface and has yielded many interesting results (c.f. [23]). Additionally, we mention that the first author and M. Schneider have proven that given a metric of positive curvature on S^2, there is an $\epsilon > 0$, such that for any $c \in]0, \epsilon[$, there are at least two embeddings of S^1 into (S^2, g) of geodesic curvature equal to c (c.f. [20]).

Applications: In Sections 4 and 5, we give applications of our degree theory. We say that a property holds for **generic** $f \in \mathcal{O}$ if and only if it holds for all f in an open, dense subset of $\mathcal{O}$, and we prove four theorems which count, generically and under appropriate hypotheses, the algebraic number of immersions of prescribed curvature of S^n into a compact, orientable, Riemannian manifold (M^{n+1}, g). In each case, we will see that this algebraic number (which is the degree of π) is equal to $-\chi(M)$, where $\chi(M)$ is the Euler characteristic of M. Indeed, this formula may be anticipated by the following discussion. In the case where the scalar curvature function R of M is a Morse function. Ye proved in [37] that a punctured neighbourhood of each critical point of R is foliated by a family of constant mean curvature spheres $\Sigma(H)$ where H varies over an interval $[B,\infty[$, where B is large and depends on (M,g). Ye's result readily extends to general notions of curvature, and in [28], the second author calculated the signature of such a so-called Ye-sphere, showing it to be equal to $(-1)^n$ times the signature of the corresponding critical point of the scalar curvature of M. Thus, if $H \geqslant B$ is a regular value of π and if the only immersions in $\pi^{-1}(H)$ are these Ye-spheres, then $\mathrm{Deg}(\pi;H)=(-1)^n\chi(M)=-\chi(M)$. In our applications, we will make the above argument precise, and as one may expect, the main difficulty lies in showing that π is a proper map.

The first theorem concerns the case where $K=H$ is mean curvature. An immersion $i:S^n\to M^{n+1}$ is said to be pointwise 1/2-pinched if for each $p\in\Sigma=S^n$:

$$\lambda_1(p) > \frac{1}{2n}(\lambda_1+...+\lambda_n)(p)=\frac{1}{2}H(p),$$

where $0<\lambda_1\leqslant ...\leqslant\lambda_n$ are its principal curvatures. Observe that, in particular, a pointwise 1/2-pinched immersion is locally strictly convex. Let $\mathcal{I}:=\mathcal{C}_{1/2}$ be the space of Alexandrov embeddings of S^n into M that are pointwise 1/2-pinched. Let $H_0=4\mathrm{Max}(\|R\|^{1/2},\|\nabla R\|^{1/3})$, where R is the curvature tensor of M and ∇R is its covariant derivative, and define the space $\mathcal{O}$ by.

$$\mathcal{O}:=\left\{f\in C^\infty(M)\ \middle|\ \begin{matrix} f & >H_0, \text{ and} \\ \|\mathrm{Hess}(f)\| & <\frac{3n}{3n-2}H_0^2. \end{matrix}\right\}.$$

Let $\pi:\mathcal{Z}\to\mathcal{O}$ be the projection of the solution space onto $\mathcal{O}$. By proving that π is a proper map, we obtain:

Theorem 1.1

For generic $f\in\mathcal{O}$, the algebraic number of Alexandrov embedded, hyperspheres in M of prescribed mean curvature equal to f (i.e. the degree of π) is equal to $-\chi(M)$.

Our next theorem concerns the case where $K=K_e$ is extrinsic curvature (also referred to as Gauss-Krönecker curvature). We say that the manifold M is pointwise 1/2-pinched if $\sigma_{\mathrm{Max}}(p)<2\sigma_{\mathrm{Min}}(p)$ for all $p\in M$, where $\sigma_{\mathrm{Max}}(p)$ is the maximum of the sectional curvatures of M at p and $\sigma_{\mathrm{Min}}(p)$ is the minimum. Let $\mathcal{I}$ be the space of strictly convex embeddings of S^n into M, and define the space $\mathcal{O}$ by:

$$\mathcal{O}=\{f\in C^\infty(M)\mid f>0\}$$

Let $\mathcal{Z}$ be the corresponding solution space. By proving that $\pi:\mathcal{Z}\to\mathcal{O}$ is a proper map, we obtain:

Theorem 1.2

Let M be $1/4$-pinched and pointwise $1/2$-pinched. Then for generic $f\in\mathcal{O}$, the algebraic number of embeddings $[i]\in\mathcal{I}$ having prescribed extrinsic curvature f (i.e. the degree of π) is equal to $-\chi(M)$.

Our third application concerns the case where $K = R_\theta$ is special Lagrangian curvature (c.f. [29] for a detailled definition). Letting $\mathcal{I}$ and $\mathcal{O}$ be exactly as in Theorem 1.2, we prove:

Theorem 1.3

Let M^{n+1} be $1/4$-pinched and $n \geqslant 3$. Then for generic $f \in \mathcal{O}$, the algebraic number of embeddings $[i] \in \mathcal{I}$ of prescribed special Lagrangian curvature f (i.e. the degree of π) is equal to $-\chi(M)$.

The final theorem we prove (and which for us is the deepest result), concerns the case where $K = K_e$ is the extrinsic curvature of a surface in a 3-dimensional manifold. Let (M^3, g) be a compact, orientable Riemannian 3-manifold and define $K_0 > 0$ by:

$$K_0^2 = \frac{1}{2}\left(\left|\sigma_{\mathrm{Min}}^-\right| + \sqrt{\left|\sigma_{\mathrm{Min}}^-\right|^2 + \|T\|_O^2}\right),$$

where T is the trace free Ricci curvature of M, $\|T\|_O$ is its operator norm, viewed as an endomorphism of TM, and σ_{Min}^- is defined to be 0, or the infimum of the sectional curvatures of M, whichever is lower. We let $\mathcal{I}$ be the space of locally strictly convex immersions of S^2 into M and we define $\mathcal{O}$ by:

$$\mathcal{O} = \left\{ f \in C^\infty(M) \;\middle|\; \begin{array}{ll} f & > K_0, \text{ and} \\ \|Df\| & \text{is controlled.} \end{array} \right\},$$

where a formal description of "controlled" is given in Section 5. Letting $\mathcal{Z}$ be the solution space, and proving that the projection $\pi : \mathcal{Z} \to \mathcal{O}$ is a proper map, we obtain:

Theorem 1.4

For generic $f \in \mathcal{O}$, the algebraic number of locally strictly convex immersions of prescribed extrinsic curvature equal to f (i.e. the degree of π) is equal to zero.

Remark: An important tool used in the proof of the properness of the projection π in this case is the result [14] of François Labourie, where by viewing surfaces of positive extrinsic curvature as pseudo-holomorphic curves in a contact manifold he is able to apply a variant of Gromov's compactness theory to obtain a general compactness result for these surfaces.

This paper is structured as follows:

(i) in Section 2, we define the degree;

(ii) in Section 3, we prove that the degree is independant of the regular value chosen;

(iii) in Section 4, we apply the degree to immersions of prescribed mean, extrinsic and special Lagrangian curvatures;

(iv) in Section 5, we apply the degree to immersed surfaces of prescribed extrinsic curvature, which requires a deeper analysis than the cases studied in Section 4; and

(v) in Appendix A, we review the functional analytic framework within which we work, proving, in particular, a Sard-Smale type theorem within this context.

We would like to thank Bernard Helffer for detailled correspondence concerning spectral theory. The second author would like to thank IMPA, Rio de Janeiro, Brazil and CRM, Barcelona, Spain, for providing the conditions required to prepare this work. In addition, the second author's work in the CRM was supported by a Marie Curie postdoctoral fellowship.

2 - Degree Theory.

2.1 Defining the Degree.

Let $M := M^{n+1}$ be a compact, oriented, $(n+1)$-dimensional, Riemannnian manifold. Let $\Sigma := \Sigma^n$ be a compact, oriented, n-dimensional manifold. Let $C^\infty_{\mathrm{imm}}(\Sigma, M)$ be the space of smooth immersions from Σ into M. We introduce the following more general definition of simple immersions:

(i) if Σ is simply connected, then the immersion $i : \Sigma \to M$ is said to be **simple** if and only if there exists no non-trivial diffeomorphism $\alpha : \Sigma \to \Sigma$ such that $i \circ \alpha = i$. In other words, it is not a multiple cover; and

(ii) if Σ is not simply connected, we let $\tilde{\Sigma}$ be the universal cover of Σ and $p : \tilde{\Sigma} \to \Sigma$ the canonical projection. Let $\pi_1(\Sigma)$ be the fundamental group of Σ which we consider as a subgroup of the diffeomorphism group of $\tilde{\Sigma}$. We say that an immersion $i : \Sigma \to M$ is **simple** if and only if its lift $\tilde{\imath} := i \circ p$ to $\tilde{\Sigma}$ is invariant only under the action of elements of $\pi_1(\Sigma)$. In other words, if $\alpha : \tilde{\Sigma} \to \tilde{\Sigma}$ is a diffeomorphism such that $\tilde{\imath} \circ \alpha = \tilde{\imath}$, then $\alpha \in \pi_1(\Sigma)$.

Thus (ii) constitutes the natural extension of (i) to the non-simply connected case. Although this notion of simplicity is different (in fact, weaker) than that given in the introduction, we shall see presently (c.f. Corollary 2.3 below) that the two notions coincide for immersions of prescribed elliptic curvature. Non-simple immersions correspond to orbifold points in the space of immersions, and since this adds unnecessary complexity for our current purposes, we defer their study to a later paper. Let $\mathrm{Simp} := \mathrm{Simp}(\Sigma, M) \subseteq C^\infty_{\mathrm{imm}}(\Sigma, M)$ be an open subset of the space of smooth immersions from Σ into M consisting only of simple immersions. Throughout the sequel, we use the terminology of Appendix A, with which the reader should familiarise himself before continuing.

Let $\mathrm{Diff}^\infty(\Sigma)$ be the group of smooth, orientation preserving diffeomorphisms of Σ. This group acts on Simp by composition. We assume that Simp is invariant under the action of $\mathrm{Diff}^\infty(\Sigma)$, and we define $\mathcal{I} := \mathcal{I}(\Sigma, M)$ to be the quotient of Simp by this group action:

$$\mathcal{I} = \mathrm{Simp}/\mathrm{Diff}^\infty(\Sigma).$$

$\mathcal{I}$ is thus an open set of unparametrised, simple immersions. We furnish $\mathcal{I}$ with the quotient topology which coincides with the topology of smooth convergence modulo reparametrisation. $\mathrm{Diff}^\infty(\Sigma)$ also acts on $C^\infty(\Sigma)$ by composition. We thus define the space $\mathrm{Smooth} := \mathrm{Smooth}(\Sigma, M)$ as follows:

$$\mathrm{Smooth} = \mathrm{Simp} \times C^\infty(\Sigma)/\mathrm{Diff}^\infty(\Sigma).$$

We also furnish Smooth with the quotient topology. Let p be the canonical projection which makes Smooth into a topological vector bundle over $\mathcal{I}$. By identifying, for any given immersion, smooth functions over Σ with smooth, infinitesimal, normal deformations of that immersion, we obtain the canonical identification of Smooth with the tangent bundle $T\mathcal{I}$. Given $i \in \mathrm{Simp}$, we define the **canonical identification** of $C^\infty(\Sigma)$ with the fibre of Smooth over $[i]$ by identifying the function $\phi \in C^\infty(\Sigma)$ with the vector $[i, \phi] \in \mathrm{Smooth}$. A continuous functional $\mathcal{F} : \mathcal{I} \to \mathrm{Smooth}$ is said to be a **section** of this bundle if and only if $p \circ \mathcal{F} = \mathrm{Id}$.

We recall the definition of a curvature function (c.f. [4] and [31]). Let $\mathrm{Symm}(\mathbb{R}^n)$ be the space of symmmetric, n-dimensional matrices over $\mathbb{R}^n$ and observe that the orthogonal group $\mathrm{O}(n)$ acts on $\mathrm{Symm}(\mathbb{R}^n)$ by conjugation. Let $\Gamma \subseteq \mathrm{Symm}(\mathbb{R}^n)$ be an open, convex cone based on 0 which is invariant under this action. A smooth function $K : \Gamma \to]0, \infty[$ is said to be a **curvature function** whenever it is **invariant** under the action of $\mathrm{O}(n)$ and **elliptic** in the sense that its gradient at any point of Γ is a positive, definite, symmetric matrix. By invariance, for any matrix A, $K(A)$ only depends on the eigenvalues of A, and we therefore consider K also as a smooth function from an open subset of $\mathbb{R}^n$ into $]0, 1[$.

Let K be a curvature function defined on the open cone Γ. For $[i] \in \mathcal{I}$, we define the K-curvature of $[i]$ by:

$$K(i)(p) = K(A_i(p)),$$

where A_i is the shape operator of i. We henceforth assume that $\mathcal{I}$ only consists of immersions whose shape operators are elements of Γ and the ellipticity of K now implies the following important property:

Ellipticity: For every $[i] \in \mathcal{I}$, the Jacobi operator of K at $[i]$ is an elliptic, second order, partial differential operator.

As examples, when $\Gamma = \mathrm{Symm}(\mathbb{R}^n)$ and $K(A) = H(A) := \mathrm{Tr}(A)/n$, we recover the **mean curvature**, and when Γ is the cone of positive definite, symmetric matrices, and $K(A) = \mathrm{Det}(A)^{1/n}$, we recover the **extrinsic curvature** (sometimes referred to as the Gauss-Krönecker curvature).

We define the functional $\mathcal{F}_{\mathrm{equiv}}(K) : \mathrm{Simp} \to C^\infty(\Sigma)$ such that for all immersions $i \in \mathrm{Simp}$ and for all points $p \in \Sigma$:

$$\mathcal{F}_{\mathrm{equiv}}(K)(i)(p) = K_i(p),$$

where $K_i(p)$ is the K-curvature of the immersion i at the point p. By Lemma A.1, $\mathcal{F}_{\mathrm{equiv}}(K)$ is smooth of order 2. It is equivariant under the action of $\mathrm{Diff}(\Sigma)$ and quotients down to a smooth section $\mathcal{F}(K)$ of Smooth over $\mathcal{I}$. Moreover, since the Jacobi Operator of K is elliptic, $\mathcal{F}(K)$ is also elliptic as a section of Smooth. In the sequel, where no ambiguity arises, we abuse notation and denote $\mathcal{F}(K)([i])$ merely by $K(i)$.

The zeroes of $K(i)$ are those immersions of constant K-curvature equal to 0. More generally, let $\mathcal{O} \subseteq C^\infty(M)$ be an open subset of functions. We define the evaluation functional $\mathcal{E}_{\mathrm{equiv}} : \mathrm{Simp} \times \mathcal{O} \to C^\infty(\Sigma)$ by:

$$\mathcal{E}_{\mathrm{equiv}}(i, f) = f \circ i.$$

By Lemma A.1, $\mathcal{E}_{\text{equiv}}$ is smooth with respect to the first component and weakly smooth with respect to the second. Since it is equivariant under the action of $\text{Diff}^\infty(\Sigma)$, it quotients down to a mapping $\mathcal{E} : \mathcal{I} \times \mathcal{O} \to \text{Smooth}$. $\mathcal{E}$ is a family of smooth sections of Smooth over $\mathcal{I}$ which is weakly smooth with respect to $\mathcal{O}$. In the sequel, where no ambiguity arises, we abuse notation and denote $\mathcal{E}([i], f)$ merely by $f \circ i$. We define $\hat{\mathcal{F}}(K) : \mathcal{I} \times \mathcal{O} \to \text{Smooth}$ by:

$$\hat{\mathcal{F}}(K)([i], f) = \mathcal{F}(K)([i]) - \mathcal{E}([i], f) = K(i) - f \circ i,$$

and where no ambiguity arises, we abuse notation and denote $\hat{\mathcal{F}}(K)([i], f)$ merely by $\hat{K}(i, f)$.

The Solution Space: For any $f \in \mathcal{O}$, let $\mathcal{Z}_f \subseteq \mathcal{I}$ be the set of zeroes of the section $\hat{K}(\cdot, f)$. $\mathcal{Z}_f$ consists of those immersions whose K-curvature is prescribed by f. In other words $\mathcal{Z}_f$ consists of those immersions $[i] \in \mathcal{I}$ such that:

$$K(i) = f \circ i.$$

We are interested in the number of elements of $\mathcal{Z}_f$ counted with appropriate signature which, in a similar manner to [33], we interpret as the degree of a mapping between two spaces as follows: we define the **solution space** $\mathcal{Z} \subseteq \mathcal{I} \times \mathcal{O}$ by:

$$\mathcal{Z} = \hat{K}^{-1}(\{0\}).$$

Let $\pi : \mathcal{Z} \to \mathcal{O}$ be the canonical projection. We suppose henceforth:

Properness: The projection π is a proper mapping from the solution space $\mathcal{Z}$ into the space $\mathcal{O}$ of data.

We will see presently that π has a well defined integer valued degree (when multiple covers are permitted, the degree may also take rational values, as will be shown in a forthcoming paper). For generic $f \in \mathcal{O}$ we will see that $\mathcal{Z}_f$ is finite and that this degree is defined to be equal to the number of elements of $\mathcal{Z}_f$ counted with appropriate signature.

Remark: an alternative interpretation is to view this degree as the number of zeroes of certain vector fields over the space $\mathcal{I}$. Indeed, identifying Smooth with $T\mathcal{I}$, we see that $\mathcal{O}$ parametrises a family of vector fields over $\mathcal{I}$ by associating to every $f \in \mathcal{O}$ the vector field $[i] \mapsto K(i) - f \circ i$. For all $f \in \mathcal{O}$, the zero set of its corresponding vector field is $\mathcal{Z}_f$, and for generic $f \in \mathcal{O}$, as we shall see, these zeroes are non-degenerate and isolated and their number, counted with signature, is then precisely the degree. It is this perspective that Schneider adopts in [22] and [23], reflecting the earlier work [6] of Elworthy and Tromba (see also [32]).

We now describe the geometry of elements of $\mathcal{Z}$, which will be of importance in the sequel. For $i \in \text{Simp}$, let $J^\infty(i)$ be the C^∞-jet of i.

Proposition 2.1

For all $([i], f) \in \mathcal{Z}$, $J^\infty(i)$ is injective.

Proof: Suppose the contrary. Choose $p \neq q \in \Sigma$ such that $J^\infty(i)(p) = J^\infty(i)(q)$. Since K is a second order, elliptic operator, and since i satisfies $K_i = f \circ i$, by Aronszajn's Unique Continuation Theorem (c.f. [1]), there exists a diffeomorphism α sending a neighbourhood of p to a neighbourhood of q such that $\alpha(p) = q$ and $i \circ \alpha = i$. Let $\tilde{\Sigma}$ be the universal cover of Σ and let $\tilde{\imath} : \tilde{\Sigma} \to M$ be the lift of i. Applying Aronszajn's Unique Continuation Theorem again, we extend α to a diffeomorphism $\tilde{\alpha}$ of $\tilde{\Sigma}$ such that $\tilde{\imath} \circ \tilde{\alpha} = \tilde{\imath}$ over the whole of $\tilde{\Sigma}$. However, since i is simple, $\alpha \in \pi_1(\Sigma)$, and so, in particular, returning to the quotient, $q = \alpha(p) = p$. This is absurd, and the result follows. $\square$

We say that a point $p \in \Sigma$ is an **injective point** of i if and only if $i(q) \neq i(p)$ for all $q \neq p$.

Proposition 2.2

For all $([i], f) \in \mathcal{Z}$, the set injective points of i is open and dense.

Proof: We denote the set of injective points of i by Ω. For all $p \in \Sigma$ and for all $r > 0$, let $B_r(p)$ be the intrinsic ball of radius r about p in Σ. By compactness, there exists $\epsilon > 0$ such that, for all $p \in \Sigma$, the restriction of i to $B_{2\epsilon}(p)$ is injective. Choose $p \in \Omega$ and denote $B := B_\epsilon(p)$. By definition:

$$i(p) \notin i(B^c).$$

Since B^c is compact, there exists a neighbourhood U of p in B such that:

$$i(U) \cap i(B^c) = \emptyset.$$

Since the restriction of i to B is injective, $U \subseteq \Omega$, and this proves that Ω is open.

Suppose that Ω is not dense. Let U be an open subset of Ω^c and choose $p \in U$. Since i is everywhere locally injective, the set of points distinct from p but having the same image as p is discrete and therefore finite. Define B as before and let $q_1, ..., q_n \in B^c$ be these points. We define $V \subseteq \Sigma$ by:

$$V = \bigcup_{1 \leqslant k \leqslant n} B_\epsilon(q_k).$$

Then by definition:

$$i(p) \notin i((B \cup V)^c).$$

Since $(B \cup V)^c$ is compact, there exists a neighbourhood, W of p in U such that:

$$i(W) \cap i((B \cup V)^c) = \emptyset.$$

We now claim that, for each k, $i(B) \cap i(B_{2\epsilon}(q_k))$ does not contain any open subset of $i(B)$. Indeed, suppose the contrary. Then i would have the same C^∞-jet at two distinct points. This is absurd by Proposition 2.1, and the assertion follows. Thus, for each k, there exists a dense subset $\tilde{B}_k \subseteq B$ such that:

$$i(\tilde{B}_k) \cap i(B_{2\epsilon}(q_k)) = \emptyset.$$

Thus, for each k, there exists an open dense subset $B_k \subseteq B$ such that:

$$i(B_k) \cap i(B_\epsilon(q_k)) \subseteq i(B_k) \cap i(\overline{B_\epsilon(q_k)}) = \emptyset.$$

We define $B_0 \subseteq B$ by:

$$B_0 = \bigcap_{1 \leqslant k \leqslant n} B_k.$$

$B_0 \cap W$ is a non-trivial subset of U consisting of injective points of i. However, by definition, $B_0 \cap W \subseteq U \subseteq \Omega^c$. This is absurd, and this completes the proof. $\square$

In particular, for immersions of prescribed curvature, we recover the notion of simplicity given in the introduction:

Corollary 2.3

If $([i], f) \in \mathcal{Z}$ then for all $p \neq q \in \Sigma$, and for all sufficiently small neighbourhoods U and V of p and q respectively:

$$i(U) \neq i(V).$$

The Degree of the Projection: We now proceed to the construction of the degree. Choose $f \in \mathcal{O}$. We need to associate a signature to each element $[i] \in \mathcal{Z}_f$ in a canonical manner. We do this as follows: for $[i] \in \mathcal{Z}_f$, we define $J(K, f)_i : C^\infty(\Sigma) \to C^\infty(\Sigma)$, the Jacobi Operator of the pair (K, f) at i, by:

$$J(K, f)_i \cdot \varphi = JK \cdot \varphi - \langle \nabla f, \mathsf{N}_i \rangle \varphi,$$

where JK is the Jacobi operator of K at i and N_i is the unit, normal vector field over i compatible with the orientation. Observe that, when we identify the fibre of Smooth $= T\mathcal{I}$ over $[i]$ with $C^\infty(\Sigma)$ in the canonical manner, $J(K, f)_i$ identifies with the operator $\mathcal{L}_1 \hat{K}_{([i], f)}$, which is the partial linearisation at $([i], f)$ of $\hat{K}$ with respect to the first component.

By the ellipticity hypothesis on K, $J(K, f)_i$ is a linear, elliptic, partial differential operator of order 2. Since it acts on $C^\infty(\Sigma)$, and since Σ is compact, it is Fredholm of index 0 and has compact resolvent (c.f. [10]) and therefore has discrete spectrum (c.f. [12]). Bearing in mind that pseudo-differential operators generalise differential operators (c.f. [8]), we recall the following result from this more general setting which is central to the sequel (c.f. [12]):

Lemma 2.4, Algebraic Multiplicity

Let Σ be compact, and let $L : C^\infty(\Sigma) \to C^\infty(\Sigma)$ be an elliptic pseudo-differential operator. For all $\lambda \in \mathrm{Spec}(L)$, there exists a decomposition of $C^\infty(\Sigma)$:

$$C^\infty(\Sigma) = E \oplus R,$$

where:

(i) E is finite dimensional;

(ii) L preserves both E and R;

(iii) the restriction of $L - \lambda\mathrm{Id}$ to E is nilpotent; and

(iv) the restriction of $L - \lambda\mathrm{Id}$ to R is injective.

We call this decomposition the **nilpotent decomposition** of $C^\infty(\Sigma)$ with respect to the eigenvalue λ of the operator L. We define the **algebraic multiplicity** of the eigenvalue λ to be the dimension of E. We distinguish this from the **geometric multiplicity** of λ, which is defined to be equal to the dimension of the kernel of $L - \lambda\mathrm{Id}$ (that is, the dimension of the eigenspace of the eigenvalue λ). In general, the algebraic multiplicity is bounded below by the geometric multiplicity.

Remark: It is important to observe that the nilpotent decomposition varies continuously with L. In particular, as L varies, even though a given eigenvalue λ of L may perturb to a family of distinct eigenvalues, the sums of their algebraic multiplicities will always be equal to the algebraic multiplicity of λ (c.f. Lemma 3.7, Proposition 3.10 and, more generally, Proposition 3.21). Moreover, since L is real, complex eigenvalues only exist in conjugate pairs, and so, even though real eigenvalues may perturb to complex eigenvalues, they do so two at a time. The number of strictly negative real eigenvalues counted with multiplicity is therefore constant modulo 2 unless some eigenvalue pases through 0, and we thus see how the signature, which we will define presently (c.f. Definition 2.8) varies in a controlled manner.

The spectrum of $J(K, f)$ is further controlled by the following result which, for later use, we state in a slightly more general context than is required here:

Lemma 2.5

Let $L : f \mapsto -a^{ij}f_{;ij} + b^i f_{;i} + cf$ be a generalised Laplacian over Σ. For $h \in C^\infty(\Sigma\times\Sigma)$, define $L_h : C^\infty(\Sigma) \to C^\infty(\Sigma)$ by:

$$(L_h f)(p) = (Lf)(p) + \int_\Sigma h(p,q)f(q)\mathrm{dVol}_\Sigma.$$

There exists $B > 0$, which only depends on the metric on Σ as well as:

(i) the C^1 norm of a;

(ii) the C^0 norms of a^{-1}, b and c; and

(iii) the L^2 norm of h;

such that the real eigenvalues of L lie in $]-B, +\infty[$.

Proof: Suppose first that $h = 0$. At each point $p \in \Sigma$, consider a^{ij} as a scalar product over $T_p^*\Sigma$. This induces a scalar product over $T_p\Sigma$, and thus yields a metric, a_{ij}, over Σ. Let ${\Gamma^k}_{ij}$ be the relative Christophel symbol of the Levi-Civita covariant derivative of a_{ij} with respect to that of the standard metric. Thus, if "," denotes covariant differentiation with respect to the Levi-Civita covariant derivative of a_{ij}, then, for all f:

$$f_{,ij} = f_{;ij} - {\Gamma^k}_{ij}f_{;k}.$$

Observe that Γ depends on the first derivative of a. We now denote:

$$\tilde{b}^i = b^i - {\Gamma^i}_{pq}a^{pq},$$

and so, for all f:

$$\begin{aligned} Lf &= -a^{ij}f_{,ij} + \tilde{b}^i f_{,i} + cf \\ &= -\Delta^a f + \tilde{b}^i f_{,i} + cf, \end{aligned}$$

where Δ^a is the Laplacian of the metric a_{ij}.

Now let λ be a real eigenvalue of L and let $f \in C^\infty(\Sigma, \mathbb{R})$ be a corresponding real eigenvector. Suppose that:

$$\|f\|_{L^2}^2 = \int_\Sigma f^2 \mathrm{dVol}^a = 1.$$

Then, bearing in mind Stokes' Theorem and the Cauchy-Schwarz Inequality:

$$\begin{aligned} \lambda &= \int_\Sigma fLf\mathrm{dVol}^a \\ &= \int_\Sigma -f\Delta^a f + f\tilde{b}^i f_{,i} + cf^2 \mathrm{dVol}^a \\ &= \int_\Sigma \|\nabla f\|_a^2 + f\tilde{b}^i f_{,i} + cf^2 \mathrm{dVol}^a \\ &\geqslant \|\nabla f\|_{L^2}^2 - \|\tilde{b}\|_{L^\infty}\|\nabla f\|_{L^2} - \|c\|_{L^\infty} \\ &= (\|\nabla f\|_{L^2} - (1/2)\|\tilde{b}\|_{L^\infty})^2 - \|c\|_{L^\infty} - \tfrac{1}{4}\|\tilde{b}\|_{L^\infty}^2 \\ &\geqslant -\|c\|_{L^\infty} - \tfrac{1}{4}\|\tilde{b}\|_{L^\infty}^2. \end{aligned}$$

The result follows for the case where $h = 0$. For general h, choose $\phi \in C^\infty(\Sigma)$ such that:

$$\phi\mathrm{dVol}^a = \mathrm{dVol}_\Sigma,$$

and define $\tilde{h}$ by:

$$\tilde{h}(p,q) = h(p,q)\phi(q).$$

For $\|f\|_{L^2} = 1$:

$$\begin{aligned} \left|\int f(p)\int h(p,q)f(q)\mathrm{dVol}_\Sigma \mathrm{dVol}^a\right| &= \left|\int f(p)\int \tilde{h}(p,q)f(q)\mathrm{dVol}^a \mathrm{dVol}^a\right| \\ &\leqslant \|\tilde{h}\|_{L^2}\|f(p)f(q)\|_{L^2} \\ &= \|\tilde{h}\|_{L^2}\|f\|_{L^2}^2 \\ &= \|\tilde{h}\|_{L^2}. \end{aligned}$$

Thus:

$$\lambda \geqslant -\|c\|_{L^\infty} - \frac{1}{4}\|\tilde{b}\|_{L^\infty}^2 - \|\tilde{h}\|_{L^2},$$

and the general case follows. □

By Lemma 2.5, since the spectrum of $J(K,f)_i$ is discrete, its set of strictly negative real eigenvalues is finite.

Definition 2.6, Index and Signature

Suppose that $([i], f) \in \mathcal{Z}$:

(i) define $\mathrm{index}([i], f)$, the index of $([i], f)$, to be the number of strictly negative, real eigenvalues of $J(K,f)_i$ counted with algebraic multiplicity; and

(ii) define $\mathrm{sig}([i], f)$, the signature of $([i], f)$, by:

$$\mathrm{sig}([i], f) = (-1)^{\mathrm{index}([i],f)}.$$

The following result permits us to use differential topological techniques to study the degree:

Proposition 2.7

For all $([i],f)\in\mathcal{Z}$ the linearisation $\mathcal{L}\hat{K}_{([i],f)}$ of $\hat{K}$ at $([i],f)$ is surjective.

Proof: We identify the fibre of Smooth over $[i]$ with $C^\infty(\Sigma)$ in the canonical manner. Let $L:=\mathcal{L}_1\hat{K}_{([i],f)}$ be the partial linearisation of $\hat{K}$ with respect to the first component at $([i],f)$. L is a second order, linear, elliptic, partial differential operator. Let L^* be its L^2 dual. Since $\mathrm{Ker}(L^*)$ is finite dimensional, there exists a finite family $(p_k)_{1\leqslant k\leqslant n}\in\Sigma$ of points in Σ such that the mapping $A:\mathrm{Ker}(L^*)\rightarrow\mathbb{R}^n$ given by:

$$A(\alpha)_k=\alpha(p_k),$$

is an isomorphism. By Proposition 2.2, we may assume that for all k, p_k is an injective point of i and moreover that there exists a neighbourhood U_k of p_k in Σ such that every point of U_k is also an injective point of i. For each k, choose $\beta_k\in C_0^\infty(U_k)$, and define the mapping $A_\beta:\mathrm{Ker}(L^*)\rightarrow\mathbb{R}^n$ by:

$$A_\beta(\alpha)_k=\int_\Sigma\beta_k\alpha\mathrm{dVol}.$$

For each k, let δ_k be the Dirac delta function supported on p_k. As $(\beta_1,...,\beta_n)$ converges to $(\delta_1,...,\delta_n)$ in the weak sense, A_β converges to A. Thus, choosing $(\beta_1,...,\beta_n)$ sufficiently close to $(\delta_1,...,\delta_n)$ in the weak sense, A_β is an isomorphism.

For each k, let $\pi:M\rightarrow i(\Sigma)$ be the nearest point projection. Choose $1\leqslant k\leqslant n$. Since the restriction of i to U_k is an embedding, π_k is smooth near $i(U_k)$, and we define $a_k\in C^\infty(M)$ such that near $i(\Sigma)$:

$$a_k(x)=(\beta_k\circ\pi)(x).$$

In particular:

$$a_k\circ i=\beta_k.$$

For each k, we define the strong tangent vector V_k to $\mathcal{I}\times\mathcal{O}$ at $([i],f)$ by:

$$V_k=\partial_s([i],f+sa_k)|_{s=0}.$$

Trivially:

$$\mathcal{L}\hat{K}_{([i],f)}\cdot V_k=\beta_k.$$

Thus, if we define $E\subseteq C^\infty(\Sigma)$ to be the finite dimensional subspace spanned by $\beta_1,...,\beta_n$, then:

$$\mathrm{Im}(L)+E\subseteq\mathrm{Im}(\mathcal{L}\hat{K}_{([i],f)}).$$

Observe that the mapping A_β is conjugate to the orthogonal projection from E onto $\mathrm{Ker}(L^*)$, and thus, since A_β is an isomorphism, so is this projection. It follows that $\mathrm{Dim}(E)=\mathrm{Dim}(\mathrm{Ker}(L^*))=\mathrm{Dim}(\mathrm{Coker}(L))$ and:

$$E\cap\mathrm{Im}(L)=E\cap\mathrm{Ker}(L^*)^\perp=\{0\}.$$

Consequently:

$$C^\infty(\Sigma) = \mathrm{Im}(L) \oplus E \subseteq \mathrm{Im}(\mathcal{L}\hat{K}_{([i],f)}) \subseteq C^\infty(\Sigma).$$

Surjectivity follows, and this completes the proof. $\square$

Suppose that f is a regular value of $\pi : \mathcal{Z} \to \mathcal{O}$. By Proposition 2.7, $\mathcal{L}\hat{K}$ is surjective at every point of $\mathcal{Z}$. Thus, by the Implicit Function Theorem (Theorem A.10), $\mathcal{Z}_f := \pi^{-1}(\{0\})$ is a (possibly empty) compact, 0-dimensional submanifold of $\mathcal{I}$. In other words, it is a finite subset. Moreover, by Lemma A.11, $J(K, f)_i$ is non-degenerate at every point $[i] \in \mathcal{Z}_f$. We thus define:

Definition 2.8, The Degree of the Projection

If $\pi : \mathcal{Z} \to \mathcal{O}$ is proper and if $f \in \mathcal{O}$ is a regular value of π, then we define, $\mathrm{Deg}(\pi; f)$, the degree of π at f by:

$$\mathrm{Deg}(\pi; f) = \sum_{[i] \in \mathcal{Z}_f} \mathrm{sig}([i]),$$

and $\mathrm{Deg}(\pi; f)$ is defined to be equal to 0 when $\mathcal{Z}_f$ is empty.

We will show that regular values of π are generic. This would normally be acheived using the Sard-Smale Theorem for smooth functionals between Banach Manifolds (c.f. [24]). Since the spaces we study are not however Banach manifolds (c.f. Appendix A), we require Theorem A.12, which provides a version of the Sard-Smale Theorem better adapted to our context. We obtain:

Proposition 2.9

The set of regular values of π is open and dense in $\mathcal{O}$.

Proof: Since π is proper, for all $f \in \mathcal{O}$, $\pi^{-1}(\{f\})$ is compact. Thus if f is a regular value of π, then so is every function in a neighbourhood of f. The set of regular values of π is therefore open. By Proposition 2.7, $\mathcal{L}\hat{K}$ is surjective at every point of $\mathcal{Z}$. Let $X = \{0\}$ be the 0-dimensional manifold consisting of a single point. For $f \in \mathcal{O}$, define $\mathcal{G}_f : X \to \mathcal{O}$ by $\mathcal{G}_f(0) = f$. Observe that $\mathcal{G}_f$ is transverse to π if and only if f is a regular value of π. By Theorem A.12, there exists $f' \in \mathcal{O}$ as close to f as we wish such that $\mathcal{G}_{f'}$ is transverse to π and so f' is a regular value of π. The set of regular values of π is therefore dense, and this completes the proof. $\square$

This allows us to define the degree for generic f:

Theorem 2.10

For generic $f \in \mathcal{O}$:

(i) $\mathcal{Z}_f = \pi^{-1}(\{f\})$ is finite; and

(ii) for all $([i], f) \in \mathcal{Z}_f$, the Jacobi operator $J(K, f)_i$ is non-degenerate.

In particular, the degree $\mathrm{Deg}(\pi; f)$ is well defined at f.

Proof: By Proposition 2.9, the set of regular values of π is generic in $\mathcal{O}$. Let $f \in \mathcal{O}$ be a regular value. The result now follows by the discussion preceeding Definition 2.8. $\square$

Varying the metric: Before proceeding to prove the independance of this degree on the regular value of π chosen, which will constitute the content of the next four sections, we briefly outline how the same approach may be generalised to allow the metric of the ambient manifold to vary.

Let $\mathcal{G}$ be an open subset of the space of Riemannian metrics over M. We define $\mathrm{Simp}_{\mathcal{G}} := \mathrm{Simp}_{\mathcal{G}}(\Sigma, M) \subseteq C^\infty_{\mathrm{imm}}(\Sigma, M) \times \mathcal{G}$ to be an open subset consisting of pairs (i, g) where i is a simple immersion. For all $g \in \mathcal{G}$, we define $\mathrm{Simp}_g := \mathrm{Simp}_g(\Sigma, M)$ to be its fibre over g:

$$\mathrm{Simp}_g = \mathrm{Simp}_{\mathcal{G}} \cap (C^\infty_{\mathrm{imm}}(\Sigma, M) \times \{g\}).$$

We assume that every fibre of $\mathrm{Simp}_{\mathcal{G}}$ is invariant under the action of $\mathrm{Diff}^\infty(\Sigma)$, and we define $\mathcal{I}_{\mathcal{G}} := \mathcal{I}_{\mathcal{G}}(\Sigma, M)$ to be the quotient of $\mathrm{Simp}_{\mathcal{G}}$ under this group action (where the action on the second component is trivial). For all $g \in \mathcal{G}$, we likewise define $\mathcal{I}_g := \mathcal{I}_g(\Sigma, M)$ to be its fibre over g:

$$\mathcal{I}_{\mathcal{G}} = \mathrm{Simp}_{\mathcal{G}}/\mathrm{Diff}^\infty(\Sigma), \qquad \mathcal{I}_g = \mathrm{Simp}_g/\mathrm{Diff}^\infty(\Sigma).$$

Remark: A typical example is the set $\mathrm{Simp}_{\mathcal{G}}$ of all pairs (i, g) such that i is locally strictly convex with respect to g. On the one hand, the fibre Simp_g is always $\mathrm{Diff}^\infty(\Sigma)$ invariant, but on the other, since convexity depends on the metric, we see that the fibre Simp_g and $\mathcal{I}_g$ depend on the metric g.

Given an elliptic curvature function K we define the functional $\mathcal{F}_{\mathcal{G},\mathrm{equiv}} : \mathrm{Simp}_{\mathcal{G}} \to C^\infty(\Sigma)$ by:

$$\mathcal{F}_{\mathcal{G},\mathrm{equiv}}(i, g)(p) = K_g(i)(p),$$

where $K_g(i)(p)$ is the K-curvature of the immersion i with respect to the metric g at the point p. By Lemma A.1, $\mathcal{F}_{\mathcal{G},\mathrm{equiv}}$ is smooth of order 2 with respect to the first component and weakly smooth with respect to the second. It is equivariant under the action of $\mathrm{Diff}(\Sigma)$ and quotients down to a family $\mathcal{F}_{\mathcal{G}}$ of smooth sections of Smooth over $\mathcal{I}_{\mathcal{G}}$ which is weakly smooth in the $\mathcal{G}$ direction.

Let $\mathcal{O}_{\mathcal{G}} \subseteq C^\infty(M) \times \mathcal{G}$ be an open subset and for all $g \in \mathcal{G}$, we define $\mathcal{O}_g$ to be its fibre over g.

Remark: As we shall see, the conditions required on $\mathcal{O}$ in order to prove properness of the projection $\pi : \mathcal{Z} \to \mathcal{O}$ typically depend on the metric of the ambient space, and it is for this reason that the fibre $\mathcal{O}_g$ is also allowed to depend on the metric g.

We define $\mathcal{U}_{\mathcal{G}}$ by:

$$\mathcal{U}_{\mathcal{G}} = \{([i], f, g) \mid ([i], g) \in \mathcal{I}_{\mathcal{G}} \ \&\ (f, g) \in \mathcal{O}_{\mathcal{G}}\}.$$

For all $g \in \mathcal{G}$, we define $\mathcal{U}_g$ to be the fibre of $\mathcal{U}_{\mathcal{G}}$ over g. Trivially, for all $g \in \mathcal{G}$:

$$\mathcal{U}_g = \mathcal{I}_g \times \mathcal{O}_g.$$

We define the evaluation functional $\mathcal{E}$ in the same way as before, and we define $\hat{\mathcal{F}}_{\mathcal{G}} : \mathcal{U}_{\mathcal{G}} \to$ Smooth by:

$$\hat{\mathcal{F}}_{\mathcal{G}}([i], f, g) = \mathcal{F}_{\mathcal{G}}([i], g) - \mathcal{E}([i], f) = K_g(i) - f \circ i.$$

We abuse notation and denote $\hat{\mathcal{F}}_{\mathcal{G}}([i], f, g)$ merely by $\hat{K}(i, f, g)$. We define the **solution space** $\mathcal{Z}_{\mathcal{G}} \subseteq \mathcal{U}_{\mathcal{G}}$ by:

$$\mathcal{Z}_{\mathcal{G}} = \hat{K}^{-1}(\{0\}).$$

Let $\pi_{\mathcal{G}} : \mathcal{Z}_{\mathcal{G}} \to \mathcal{O}_{\mathcal{G}}$ be the projection onto the second and third factors. We now suppose:

Properness: The projection $\pi_{\mathcal{G}}$ is a proper mapping from the solution space $\mathcal{Z}_{\mathcal{G}}$ onto the space $\mathcal{O}_{\mathcal{G}}$ of data.

Remark: Suppose that for $g \in \mathcal{G}$, we denote by $\mathcal{Z}_g$ the fibre of $\mathcal{Z}_{\mathcal{G}}$ over g and by $\pi_g : \mathcal{Z}_g \to \mathcal{O}_g$ the projection onto the second factor. Then the properness of $\pi_{\mathcal{G}}$ implies in particular that π_g is also proper for all g, and we thus recover a g-dependant version of our original framework.

We leave the reader to verify that in all our applications, the techniques used to show that the projection $\pi_g : \mathcal{Z}_g \to \mathcal{O}_g$ is a proper mapping readily extend to show that the projection $\pi_{\mathcal{G}} : \mathcal{Z}_{\mathcal{G}} \to \mathcal{O}_{\mathcal{G}}$ is also a proper mapping for an appropriately chosen set $\mathcal{O}_{\mathcal{G}}$ of data. It then follows, as before, that for generic $(f, g) \in \mathcal{O}_{\mathcal{G}}$, (f, g) is a regular value of $\pi_{\mathcal{G}}$, the degree of the projection $\pi_{\mathcal{G}}$ is well defined and that this degree is independant of the regular value (f, g) of $\pi_{\mathcal{G}}$ chosen. Moreover, the degree of $\pi_{\mathcal{G}}$ thus defined is readily shown to be equal to the degree of π_g for all g, and we thus see how our degree theory extends to allow for varying metrics.

3 - The Degree is Constant.

3.1 Integral Operators.

Let $f_0, f_1 \in \mathcal{O}$ be regular values of π. Let $p : [0, 1] \to \mathcal{O}$ be a smooth, injective functional such that $p(0) = f_0$ and $p(1) = f_1$. We denote:

$$\mathcal{P} := p([0, 1]).$$

We define $\mathcal{Z}_p \subseteq \mathcal{I} \times \mathcal{P} \subseteq \mathcal{I} \times \mathcal{O}$ by:

$$\mathcal{Z}_p = \pi^{-1}(\mathcal{P}) = \left\{ (i, f) \in \mathcal{I} \times \mathcal{P} \mid \hat{K}([i], f) = 0 \right\}.$$

By Proposition 2.7, $\mathcal{L}\hat{K}$ is surjective at every point of $\mathcal{Z}$. Thus, since $\pi : \mathcal{Z} \to \mathcal{O}$ is proper, by the Sard-Smale Theorem (Theorem A.12), we may assume that p is transverse to π. It then follows from the Implicit Function Theorem (Theorem A.10) that $\mathcal{Z}_p$ is a compact, smooth, 1-dimensional, embedded submanifold of $\mathcal{I} \times \mathcal{P} \subseteq \mathcal{I} \times \mathcal{O}$ whose boundary is contained in $\mathcal{I} \times \partial\mathcal{P}$.

This suffices to prove that the degree is constant modulo 2. In order to prove that the degree itself is constant, we will show in the following sections that $\mathcal{Z}_p$ also carries a canonical orientation such that the degree of the projection onto $\mathcal{P}$ (which itself carries a canonical orientation form from its natural identification with $[0,1]$) coincides at the respective end points with the degree of π at f_0 and f_1.

In this section, we modify the problem by restricting our data set in one direction and enlargening it in another. This allows us in the sequel to impose strong geometric properties on the solution space. Using local liftings, we define an integration functional near any point of $\mathcal{Z}$ as follows: choose $z := ([i], f) \in \mathcal{Z}$, $\epsilon > 0$ and define Σ_ϵ by:

$$\Sigma_\epsilon = \Sigma \times]-\epsilon, \epsilon[.$$

Let N_i be the unit, normal vector field over i compatible with the orientation and define $I_z : \Sigma_\epsilon \to M$ by:

$$I_z(p,t) = \mathrm{Exp}(t\mathsf{N}_i),$$

where Exp is the exponential map of M. By reducing ϵ if necessary, we may assume that I_z is an immersion. Moreover, since $\mathcal{Z}_p$ is compact, after reducing ϵ even further, we may assume that I_w is an immersion for all $w \in \mathcal{Z}_p$, and even for all w in a neighbourhood of $\mathcal{Z}_p$.

We furnish Σ_ϵ with the pull-back metric $I_z^* g$ and we define the integration functional $\hat{\mathcal{S}}_{\mathrm{equiv},z} : \mathrm{Simp}(\Sigma, \Sigma_\epsilon) \times C^\infty(\Sigma_\epsilon \times \Sigma_\epsilon) \to C^\infty(\Sigma)$ by:

$$\hat{\mathcal{S}}_{\mathrm{equiv},z}(j,h)(p) = \int_\Sigma h(j(p), j(q))\mathrm{dVol}(j)(q),$$

where $\mathrm{dVol}(j)$ is the volume form over Σ induced by the immersion j. By Lemma A.1, $\hat{\mathcal{S}}_{\mathrm{equiv},z}$ is smooth with respect to $\mathrm{Simp}(\Sigma, \Sigma_\epsilon)$ and weakly smooth with respect to $C^\infty(\Sigma_\epsilon \times \Sigma_\epsilon)$. Moreover, it is equivariant under the action of $\mathrm{Diff}^\infty(\Sigma)$ and thus quotients down to a mapping $\hat{\mathcal{S}}_z : \mathcal{I}(\Sigma, \Sigma_\epsilon) \times C^\infty(\Sigma_\epsilon \times \Sigma_\epsilon) \to \mathrm{Smooth}(\Sigma, \Sigma_\epsilon)$ which defines a family of smooth sections of $\mathrm{Smooth}(\Sigma, \Sigma_\epsilon)$ over $\mathcal{I}(\Sigma, \Sigma_\epsilon)$ which is weakly smooth with respect to $C^\infty(\Sigma_\epsilon \times \Sigma_\epsilon)$.

By Proposition 2.2, the set of injective points of i is non-empty. Thus, by Proposition A.6, there exists a neighbourhood $\mathcal{U}_z$ of $[i]$ in $\mathcal{I}$ such that every element $[j] \in \mathcal{U}$ lifts uniquely to an embedding $[\hat{\jmath}] \in \mathcal{I}(\Sigma, \Sigma_\epsilon)$ such that:

$$j = I_z \circ \hat{\jmath}.$$

We denote the lifting map by $\mathcal{L}_z$.

Pulling $\hat{\mathcal{S}}_z$ back through $\mathcal{L}_z$ yields a family of smooth sections of Smooth over $\mathcal{U}_z$ which is weakly smooth with respect to $C^\infty(\Sigma_\epsilon \times \Sigma_\epsilon)$. Let $\chi_z : \mathcal{I} \to \mathbb{R}$ be a smooth bump functional supported in $\mathcal{U}_z$ and equal to 1 near $[i]$. We define the mapping $\mathcal{S}_z : \mathcal{I} \times C^\infty(\Sigma_\epsilon \times \Sigma_\epsilon) \to$ Smooth by:

$$\mathcal{S}_z([i], h) = \chi_z([i])(\mathcal{L}_z^* \hat{\mathcal{S}}_z)([i], h).$$

$\mathcal{S}_z$ is a family of smooth sections of $\mathrm{Smooth}(\Sigma, M)$ over $\mathcal{I}$ which is weakly smooth with respect to $C^\infty(\Sigma_\epsilon \times \Sigma_\epsilon)$.

We take particular care near $\partial\mathcal{Z}_p$ in order to ensure that the perturbations of $\mathcal{Z}_p$ constructed in the sequel share the same boundary as $\mathcal{Z}_p$: let $\mathcal{L}_1\hat{K}$ be the partial linearisation of $\hat{K}$ with respect to the first component. Recalling that $p(0)$ and $p(1)$ are regular values, since invertibility of elliptic operators is an open property (c.f. Proposition A.3), there exists a closed subset, $\mathcal{Z}_p' \subseteq \mathcal{Z}_p$ such that:

(i) $\mathcal{Z}_p' \cap \partial\mathcal{Z}_p = \emptyset$; and

(ii) $\mathcal{L}_1\hat{K}_{([i],f)}$ is non-degenerate for all $([i],f) \in \mathcal{Z}_p \setminus \mathcal{Z}_p'$.

Since $\mathcal{Z}_p$ is compact, so is $\mathcal{Z}_p'$ and so there exist finitely many points $z_1, ..., z_n \in \mathcal{Z}_p$ such that:

$$\mathcal{Z}_p' \subseteq \left(\bigcup_{1\leqslant k\leqslant n} \mathrm{Int}(\chi_{z_k}^{-1}(\{1\}))\right) \times \mathcal{P}^o.$$

For all $1 \leqslant k \leqslant n$, we denote $I_k := I_{z_k}$, $\mathcal{S}_k := \mathcal{S}_{z_k}$ and $\chi_k := \chi_{z_k}$. Let Ω be a neighbourhood of $\mathcal{Z}_p$ in $\mathcal{I} \times \mathcal{P}$, and let $\Delta\mathcal{O}$ be a neighbourhood of 0 in $C^\infty(\Sigma_\epsilon \times \Sigma_\epsilon)^n$. Let $\eta \in C_0^\infty(\mathcal{P}^o)$ be a smooth function equal to 1 near $\pi(\mathcal{Z}_p')$ and define $\Delta\hat{\mathcal{F}}(K) : \Omega \times \Delta\mathcal{O} \to \mathrm{Smooth}$ by:

$$\Delta\hat{\mathcal{F}}(K)([i], f, h) = \hat{\mathcal{F}}(K)([i], f) - \eta(t) \sum_{1\leqslant k\leqslant n} \mathcal{S}_k([i], h_k).$$

$\Delta\hat{\mathcal{F}}(K)$ is a family of smooth sections of Smooth over Ω which is weakly smooth with respect to $\Delta\mathcal{O}$. In the sequel, where no ambiguity arises, we abuse notation and denote the element $\Delta\hat{\mathcal{F}}(K)([i], f, h)$ merely by $\Delta\hat{K}(i, f, h)$. We define the **solution space** $\Delta\mathcal{Z} \subseteq \Omega \times \Delta\mathcal{O}$ by:

$$\Delta\mathcal{Z} = \Delta\hat{K}^{-1}(\{0\}).$$

Let $\Delta\pi : \Delta\mathcal{Z} \to \Delta\mathcal{O}$ be the canonical projection. For all $h \in \Delta\mathcal{O}$, we define $\Delta\mathcal{Z}_h \subseteq \Omega$ by $\Delta\mathcal{Z}_h = (\Delta\pi)^{-1}(\{h\})$.

In summary, we obtain a new framework that resembles the original framework in all important respects, having merely replaced $\mathcal{I}$ with Ω, $\mathcal{O}$ with $\Delta\mathcal{O}$, $\hat{K}$ with $\Delta\hat{K}$, $\mathcal{Z}$ with $\Delta\mathcal{Z}$ and π with $\Delta\pi$. We now show how the new framework has the same basic properties as the original:

Proposition 3.1

Reducing $\Delta\mathcal{O}$ and Ω if necessary, $\Delta\pi : \Delta\mathcal{Z} \to \Delta\mathcal{O}$ is also proper.

Proof: Choose $z := ([i], f) \in \mathcal{Z}_p$. Let $(i, U, V, \mathcal{E})$ be a graph chart of $\mathcal{I}$ about $[i]$. Smooth pulls back through $\mathcal{E}$ to the trivial bundle $U \times C^\infty(\Sigma)$. We identify sections of $U \times C^\infty(\Sigma)$ over U with functions from U into $C^\infty(\Sigma)$ in the canonical manner, and we denote by $\mathcal{F} : U \times \mathcal{P} \times \Delta\mathcal{O} \to C^\infty(\Sigma)$ the pull back of $\Delta\hat{K}$ through $\mathcal{E}$. $\mathcal{F}$ is smooth with respect to $U \times \mathcal{P}$ and weakly smooth with respect to $\Delta\mathcal{O}$. In fact, chosing appropriate Hölder completions of $\Delta\mathcal{O}$ and $C^\infty(\Sigma)$, we may assume that $\mathcal{F}$ is C^1 (c.f. Lemma A.1).

If we denote by $D_1\mathcal{F}$ the partial derivative of $\mathcal{F}$ with respect to U, then $D_1\mathcal{F}$ is a second order, elliptic, linear, partial differential operator. In particular, it is Fredholm. Thus, by

Proposition A.4, there exist neighbourhoods N_z and M_z of $(0, f)$ in $U \times \mathcal{P}$ and of 0 in $\Delta\mathcal{O}$ respectively such that if $\pi : U \times \mathcal{P} \times \Delta\mathcal{O} \to \Delta\mathcal{O}$ is the projection onto the third factor, then the restriction of π to $(\overline{N}_z \times \overline{M}_z) \cap \mathcal{F}^{-1}(\{0\})$ is proper. Identifying now N_z with its image under $\mathcal{E}$, since $\mathcal{F}^{-1}(\{0\}) = \mathcal{E}^{-1}(\Delta\mathcal{Z})$, we deduce that the restriction of $\Delta\pi$ to $(\overline{N}_z \times \overline{M}_z) \cap \Delta\mathcal{Z}$ is proper.

Since $\mathcal{Z}_p$ is compact, there exist finitely many points $z_1, ..., z_n \in \mathcal{Z}_p$ such that:

$$\mathcal{Z}_p \subseteq N := \bigcup_{i=1}^{n} N_{z_i}.$$

We denote:

$$M = \bigcap_{i=1}^{n} M_{z_i}.$$

The restriction of $\Delta\pi$ to $(\overline{N} \times \overline{M}) \cap \Delta\mathcal{Z}$ is proper. Replacing $\Delta\mathcal{O}$ with M, we may therefore assume that the restriction of $\Delta\pi$ to $(\overline{N} \times \Delta\mathcal{O}) \cap \Delta\mathcal{Z}$ is proper. Reducing Ω if necessary, we may suppose that it is contained in N, and so the restriction of $\Delta\pi$ to $\Delta\mathcal{Z}$ is relatively proper in $\overline{\Omega} \times \Delta\mathcal{O}$. However, by definition, $\mathcal{Z}_p$ lies in the interior of Ω and so, in particular:

$$\mathcal{Z}_p \cap \partial\Omega = \emptyset.$$

Thus, by properness, reducing $\Delta\mathcal{O}$ further if necessary:

$$\Delta\pi^{-1}(\Delta\mathcal{O}) \cap \partial\Omega = \emptyset,$$

and so the restriction of $\Delta\pi$ to $\Delta\mathcal{Z}$ is proper. This completes the proof. $\square$

The following result will also be useful in the sequel:

Proposition 3.2

Let $\mathcal{L}\Delta\hat{K}$ be the linearisation of $\Delta\hat{K}$. After reducing $\Delta\mathcal{O}$ and Ω if necessary, $\mathcal{L}\Delta\hat{K}$ is surjective at every point of $\Delta\mathcal{Z}$.

Proof: Since surjectivity of elliptic functionals is an open property (c.f. Proposition A.3) and since $\Delta\pi$ is proper, it suffices to prove that $\mathcal{L}\Delta\hat{K}$ is surjective at every point of $\mathcal{Z}_p = \Delta\mathcal{Z}_0$.

Choose $z := ([i], f) \in \mathcal{Z}_p$. Choose k such that $\chi_k = 1$ near $[i]$. Let $[j]$ be the k'th lifting of $[i]$ into $\mathrm{Simp}(\Sigma, \Sigma_\epsilon)$. That is:

$$i = I_{z_k} \circ j.$$

Choose $\varphi \in C^\infty(\Sigma)$. Let $\pi : \Sigma_\epsilon \to \Sigma$ be the nearest point projection onto the image of j. Since j is an embeddeding, π is smooth near $j(\Sigma)$ and we define $a \in C^\infty(\Sigma_\epsilon)$ by:

$$a(x) = (\varphi \circ \pi)(x).$$

Trivially:

$$(a \circ j) = \varphi.$$

Define $g \in C^\infty(\Sigma_\epsilon \times \Sigma_\epsilon)$ by:

$$g(x,y) = -a(x).$$

Identifying the fibres of Smooth and Smooth$(\Sigma, \Sigma_\epsilon)$ over $[i]$ and $[j]$ respectively with $C^\infty(\Sigma)$ in the canonical manner, we obtain:

$$\begin{aligned}\mathcal{S}_k([i], g) &= -\chi_k([i])\hat{\mathcal{S}}_k([j], g)\\ &= -(a \circ i)\\ &= -\varphi.\end{aligned}$$

We thus define $h \in C^\infty(\Sigma_\epsilon \times \Sigma_\epsilon)^n$ by:

$$h_i = \begin{cases} g \text{ if } i = k; \text{ and}\\ 0 \text{ otherwise.}\end{cases}$$

Bearing in mind that the bump function η is equal to 1 near f, we obtain, for all $s \in \mathbb{R}$:

$$\begin{aligned}\Delta\hat{K}([i], f, sh) &= \hat{K}([i], f) - \mathcal{S}_k([i], sh_k)\\ &= \hat{K}([i], f) + s\varphi.\end{aligned}$$

We define the strong tangent vector X to $\Omega \times \Delta\mathcal{O}$ at z by:

$$X := \partial_s([i], f, sh)|_{s=0}.$$

Then:

$$\mathcal{L}\Delta\hat{K}_{(z,0)} \cdot X = \partial_s \Delta\hat{K}([i], f, sh)|_{s=0} = [i, \varphi],$$

and it follows that $\mathcal{L}\Delta\hat{K}$ is surjective.

Suppose now that $z \in \mathcal{Z}_p \setminus \mathcal{Z}_p'$. Let $\mathcal{L}_1\hat{K}$ and $\mathcal{L}_2\hat{K}$ be the partial linearisations of $\hat{K}$ with respect to the first and second components respectively and let $\mathcal{L}_3\Delta\hat{K}$ be the partial linearisation of $\Delta\hat{K}$ with respect to the third component. For all (α, β, γ) tangent to $\mathcal{I} \times \mathcal{P} \times \Delta\mathcal{O}$ at $([i], f, 0)$:

$$\mathcal{L}\Delta\hat{K}_{([i],f,0)} \cdot (\alpha, \beta, \gamma) = \mathcal{L}_1\hat{K}_{([i],f)} \cdot \alpha + \mathcal{L}_2\hat{K}_{([i],f)} \cdot \beta + \mathcal{L}_3\Delta\hat{K}_{([i],f,0)} \cdot \gamma.$$

By definition of $\mathcal{Z}_p'$, $\mathcal{L}_1\hat{K}_{([i],f)}$ is surjective, and thus so is $\mathcal{L}\Delta\hat{K}_{(z,0)}$. This completes the proof. $\square$

The Sard-Smale Theorem may be used to show that regular values of $\Delta\pi$ are generic in $\Delta\mathcal{O}$. However, in the current setting, this is not necessary:

Proposition 3.3

Reducing $\Delta\mathcal{O}$ and Ω if necessary, every $h \in \Delta\mathcal{O}$ is a regular value of $\Delta\pi$.

Proof: Since surjectivity of elliptic functionals is an open property (c.f. Proposition A.3), and since $\Delta\pi$ is proper, it suffices to show that 0 is a regular value of $\Delta\pi$.

Choose $z := ([i], f) \in \mathcal{Z}_p = \Delta\mathcal{Z}_0$. Choose $h \in \Delta\mathcal{O}$ and define $\varphi \in C^\infty(\Sigma)$ such that:

$$[i, \varphi] = \eta(f) \sum_{1 \leqslant k \leqslant n} \mathcal{S}_k([i], h_k).$$

Let $\mathcal{L}\hat{K}$ be the linearisation of $\hat{K}$. By Proposition 2.7, $\mathcal{L}\hat{K}$ is surjective at $([i], f)$. There therefore exists a strong tangent vector (α, β) to $\mathcal{I} \times C^\infty(M)$ at $([i], f)$ such that:

$$\mathcal{L}\hat{K}_{([i],f)} \cdot (\alpha, \beta) = [i, \varphi].$$

Since p is transverse to π, there exists a strong tangent vector (γ, δ) to $\mathcal{Z}$ at $([i], f)$ and a tangent vector V to $\mathcal{P}$ at f such that:

$$\beta = \delta + V.$$

Since (γ, δ) is a strong tangent vector to $\mathcal{Z}$:

$$\mathcal{L}\hat{K}_{([i],f)} \cdot (\gamma, \delta) = 0.$$

Thus:

$$\begin{aligned} & \mathcal{L}\hat{K}_{([i],f)} \cdot (\alpha - \gamma, \beta - \delta) && = [i, \varphi] \\ \Rightarrow\ & \mathcal{L}\hat{K}_{([i],f)} \cdot (\alpha - \gamma, V) && = [i, \varphi] \\ \Rightarrow\ & \mathcal{L}\Delta\hat{K}_{([i],f,0)} \cdot (\alpha - \gamma, V, 0) && = [i, \varphi]. \end{aligned}$$

And so:

$$\mathcal{L}\Delta\hat{K}_{([i],f,0)} \cdot (\alpha - \gamma, V, h) = 0.$$

It follows that $(\alpha - \gamma, V, h)$ is a strong tangent vector to $\Delta\mathcal{Z}$ at z and so $h \in \mathrm{Im}(D\Delta\pi)$, where $D\Delta\pi$ is the derivative of $\Delta\pi$. 0 is thus a regular value of $\Delta\pi$ and this completes the proof. $\square$

3.2 The Degree is Constant.

Using the notation of the preceeding section, we now construct a canonical orientation over $\mathcal{Z}_p$. This will require genericity results in the form of Propositions 3.5 and 3.9 below. The proofs of these results are moderately long and technical, and since, in particular, they distract from the main flow of the argument, we defer them to Sections 3.3 and 3.4 respectively.

Choose $h \in \Delta\mathcal{O}$. For $z := ([i], f) \in \Delta\mathcal{Z}_h$, we identify the fibre of Smooth $= T\mathcal{I}$ over $[i]$ with $C^\infty(\Sigma)$ in the canonical manner, and we define $\Delta J(K, h)_{(i,f)} : C^\infty(\Sigma) \to C^\infty(\Sigma)$, the Jacobi operator of (K, h) at (i, f), to be equal to $\mathcal{L}_1\Delta\hat{K}_{([i],f,h)}$, the partial linearisation at $([i], f, h)$ of $\Delta\hat{K}$ with respect to the first component.

We now recall that a (smooth) **generalised Laplacian** is an operator $L : C^\infty(\Sigma) \to C^\infty(\Sigma)$ which may be expressed in local coordinates in the form:

$$L\varphi = -a^{ij}\partial_i\partial_j\varphi + b^i\partial_i\varphi + c\varphi,$$

where the summation convention is assumed, a, b and c are smooth functions, and there exists $K > 0$ such that, for all $p \in \Sigma$ and for every vector X tangent to Σ at p:

$$\frac{1}{K}\|X\|^2 \leqslant a^{ij}X_iX_j \leqslant K\|X\|^2.$$

We define a **second order, elliptic, integro-differential operator** to be an operator $L : C^\infty(\Sigma) \to C^\infty(\Sigma)$ of the form:

$$(L\varphi)(p) = (L_0\varphi)(p) + \int_\Sigma h(p,q)\varphi(q)\mathrm{dVol}_q,$$

where L_0 is a generalised Laplacian, and h is a smooth function. Let X be a finite dimensional manifold. A family $(L_x)_{x\in X}$ of integro-differential operators is said to be **smooth** if and only if:

$$(L_x\varphi)(p) = (L_{0,x}\varphi)(p) + \int_\Sigma h(p,q,x)\varphi(q)\mathrm{dVol}_q,$$

where $h \in C^\infty(\Sigma\times\Sigma\times X)$ is a smooth function, and the coefficients of $L_{0,x}$ vary smoothly with x.

Returning to $\Delta J(K,h)_{(i,f)}$, although its explicit formula is too cumbersome for us to want to bother the reader with it, we may now observe that it is a second order, linear, elliptic, integro-differential operator with smooth coefficients, as follows from the fact that $\Delta\hat{K}$ is obtained from $\hat{K}$ by adding a finite sum of products of integral operators and smooth bump functions, all of whose linearisations are integral operators (c.f. Section A.3 and Lemma A.5). In particular, as in Section 2.1, $\Delta J(K,h)_{(i,f)}$ has compact resolvent and therefore discrete spectrum. Moreover, by Lemma 2.5, it has only finitely many strictly negative real eigenvalues, counted with multiplicity, and thus, as in Definition 2.8, we denote by $\mathrm{sig}(z,h)$ the signature of the operator $\Delta J(K,h)_{(i,f)}$.

By Propositions 3.1 and 3.3 and Theorem A.7, after reducing $\Delta\mathcal{O}$ if necessary, for all $h \in \Delta\mathcal{O}$, $\Delta\mathcal{Z}_h$ is a strongly smooth, one-dimensional, embedded submanifold of Ω whose boundary is contained in $\mathcal{I}\times\partial\mathcal{P}$. Moreover, by Proposition 3.1, $\Delta\mathcal{Z}_h$ converges to $\Delta\mathcal{Z}_0 = \mathcal{Z}_p$ as h tends to 0. Observe in addition that the bump function η is supported away from $\partial\mathcal{P}$, and so recalling the definition of $\Delta\hat{K}$, we find that, for all $h \in \Delta\mathcal{O}$:

$$\Delta\mathcal{Z}_h \cap (\mathcal{I}\times\partial\mathcal{P}) = \mathcal{Z}_p \cap (\mathcal{I}\times\partial\mathcal{P}).$$

Moreover, when $h = 0$, for all $z := ([i], f) \in \Delta\mathcal{Z}_0 = \mathcal{Z}_p$:

$$\Delta J(K,0)_{(i,f)} = \mathcal{L}_1\Delta\hat{K}_{([i],f,0)} = \mathcal{L}_1\hat{K}_{([i],f)} = J(K,f)_i.$$

We thus immediately obtain:

Proposition 3.4

For all $z := ([i], f) \in \partial\mathcal{Z}_p = \partial\Delta\mathcal{Z}_0$:

$$\mathrm{sig}(\Delta J(K,0)_{(i,t)}) = \mathrm{sig}(J(K,f)_i).$$

In particular:

$$\mathrm{sig}(z,0) = \mathrm{sig}(z).$$

Recalling that p is bijective, we define $t : \mathcal{I} \times \mathcal{P} \to [0,1]$ by:

$$t([i], f) = p^{-1}(f).$$

Thus p is essentially the projection onto the second factor. We denote also by t its restriction to Ω.

Proposition 3.5

After reducing $\Delta\mathcal{O}$ if necessary, for generic $h \in \Delta\mathcal{O}$, all critical points of $t : \Delta\mathcal{Z}_h \to [0,1]$ are non-degenerate.

Proof: This follows immediately from Proposition 3.17, whose statement and proof we defer to Section 3.3. □

Thus, without loss of generality, we assume that all critical points of $t : \Delta\mathcal{Z}_h \to [0,1]$ are non-degenerate. In particular, they are isolated, and since $\Delta\mathcal{Z}_h$ is compact, there are only finitely many. The tangent space to $\Delta\mathcal{Z}_h$ is related to the kernel of $\Delta J(K,h)_{(i,f)}$ by the following result:

Proposition 3.6

$\Delta J(K,h)_{(i,f)}$ is degenerate if and only if $dt = 0$. Moreover:

$$\mathrm{Dim}(\mathrm{Ker}(\Delta J(K,h)_{(i,f)})) \leqslant 1.$$

Proof: Let $\mathcal{L}_1\Delta\hat{K}$ be the partial linearisation of $\Delta\hat{K}$ with respect to the first component. By Proposition A.11, for $([i], f) \in \Delta\mathcal{Z}_h$:

$$\mathrm{Ker}(\mathcal{L}_1\Delta\hat{K}_{([i],f,h)}) = T_{([i],f)}\Delta\mathcal{Z}_h \cap T_{([i],f)}(\mathcal{I} \times \mathcal{P}).$$

Thus:

$$\mathrm{Dim}(\mathrm{Ker}(\mathcal{L}_1\Delta\hat{K}_{([i],f,h)})) \leqslant \mathrm{Dim}(T_{([i],f)}\Delta\mathcal{Z}_h) = 1.$$

Moreover, this space is non-trivial if and only if the projection from $T_{([i],f)}\Delta\mathcal{Z}_h$ onto $T_f\mathcal{P}$ is trivial, which holds if and only if $dt = 0$. Since $\Delta J(K,h)_{(i,f)} = \mathcal{L}_1\Delta\hat{K}_{([i],f,h)}$, this completes the proof. □

We observe that $\Delta\mathcal{Z}_h$ inherits a metric from its canonical embedding into $\mathcal{I} \times \mathcal{P}$, and we define the form μ whenever $dt \neq 0$ by:

$$\mu = \mathrm{sig}([i], t, h)\frac{dt}{\|dt\|}.$$

We will show that μ extends to a smooth form over the whole of $\Delta\mathcal{Z}_h$, thus defining an orientation form. We first show in what manner the spectrum of an operator varies continuously with that operator. Let L be a second order, linear, elliptic, integro-differential operator acting on $C^\infty(\Sigma)$. Let $\mathrm{Spec}(L) \subseteq \mathbb{C}$ be the spectrum of L. We recall from Section 2.1 that, since Σ is compact, L has compact resolvent, and its spectrum is discrete. We define the function $\mathrm{Mult}(L;\cdot) : \mathbb{C} \to \mathbb{N}_0$ such that $\mathrm{Mult}(L;\zeta)$ is the algebraic multiplicity of ζ whenever ζ is an eigenvalue of L and $\mathrm{Mult}(L;\zeta) = 0$ otherwise. $\mathrm{Mult}(L;\zeta)$ varies continuously with L in the following manner: let $\gamma : S^1 \to \mathbb{C}$ be a simple, closed curve that does not intersect the spectrum of L. Let U be the interior of γ. Let X be a smooth finite dimensional manifold, let $(L_x)_{x\in X}$ be a smooth family of second order, linear, uniformly elliptic, integro-differential operators such that $L_{x_0} = L$ for some $x_0 \in X$. From classical spectral theory we obtain (c.f. [12]):

Lemma 3.7

For x sufficiently close to x_0:

$$\sum_{\zeta\in U}\mathrm{Mult}(L_x;\zeta) = \sum_{\zeta\in U}\mathrm{Mult}(L;\zeta).$$

Using this we prove that μ is locally constant away from critical points of t:

Proposition 3.8

Suppose that $dt(z) \neq 0$, then for all $w \in \Delta\mathcal{Z}_h$ sufficiently close to z:

$$\mathrm{sig}(w, h) = \mathrm{sig}(z, h).$$

Proof: Let $\mathcal{L}_1\Delta\hat{K}$ be the partial linearisation of $\Delta\hat{K}$ with respect to the first component. Since $dt \neq 0$, by Proposition 3.6, $\mathcal{L}_1\Delta\hat{K}_{(z,h)} = \Delta J(K,h)_z$ is non-singular. In particular, 0 is not an eigenvalue. Let $B > 0$ be as in Lemma 2.5. Let $\Omega \subseteq \mathbb{C}$ be a relatively compact neighbourhood of $]-B, 0]$ such that:

(i) $\partial\Omega$ is smooth;

(ii) Ω is symmetric about $\mathbb{R}$;

(iii) no point of $\mathrm{Spec}(\mathcal{L}_1\Delta\hat{K}_{(z,h)})$ lies on $\partial\Omega$; and

(iv) the only points in $\mathrm{Spec}(\mathcal{L}_1\Delta\hat{K}_{(z,h)}) \cap \Omega$ are real.

Suppose, moreover, that there exists $\delta > 0$ such that:

(i) $(\Omega \setminus \overline{B_\delta(0)}) \cap \mathbb{R}$ is an open subset of the negative real axis; and

(ii) no point of $\mathrm{Spec}(\mathcal{L}_1 \Delta \hat{K}_{(z,h)})$ lies in $\overline{B_\delta(0)}$.

By Lemma 2.5, all negative, real eigenvalues of $\mathcal{L}_1 \Delta \hat{K}_{(w,h)}$ lie in Ω. By Lemma 3.7, for all $w \in \Delta \mathcal{Z}_h$ close to z:

$$\sum_{\zeta \in B_\delta(0)} \mathrm{Mult}(\mathcal{L}_1 \Delta \hat{K}_{(w,h)}; \zeta) = \sum_{\zeta \in B_\delta(0)} \mathrm{Mult}(\mathcal{L}_1 \Delta \hat{K}_{(z,h)}; \zeta) = 0.$$

Thus, for w sufficiently close to z, all real eigenvalues of $\mathcal{L}_1 \Delta \hat{K}_{(w,h)}$ lying in Ω also lie in the complement of $B_\delta(0)$ and are therefore strictly negative. Thus:

$$\mathrm{index}(w,h) = \sum_{\zeta \in \Omega \cap \mathbb{R}} \mathrm{Mult}(\mathcal{L}_1 \Delta \hat{K}_{(w,h)}; \zeta).$$

However, since $\mathcal{L}_1 \Delta \hat{K}_{(w,h)}$ is real, all its complex eigenvalues exist in conjugate pairs with equal multiplicity, and so, for all w close to z:

$$\mathrm{index}(w,h) = \sum_{\zeta \in \Omega} \mathrm{Mult}(\mathcal{L}_1 \Delta \hat{K}_{(w,h)}; \zeta) \bmod 2.$$

However, by Lemma 3.7 again:

$$\sum_{\zeta \in \Omega} \mathrm{Mult}(\mathcal{L}_1 \Delta \hat{K}_{(w,h)}; \zeta) = \sum_{\zeta \in \Omega} \mathrm{Mult}(\mathcal{L}_1 \Delta \hat{K}_{(z,h)}; \zeta) = \mathrm{index}(z,h).$$

Thus:

$$\begin{array}{lll} & \mathrm{index}(\Delta J(K,h)_w) & = \mathrm{index}(\Delta J(K,h)_z) \bmod 2 \\ \Leftrightarrow & \mathrm{index}(w,h) & = \mathrm{index}(z,h) \bmod 2 \\ \Leftrightarrow & \mathrm{sig}(w,h) & = \mathrm{sig}(z,h). \end{array}$$

The result now follows. □

We now consider points where dt vanishes. At these points, by Proposition 3.6, $\Delta J(K,h)$ is degenerate with 1-dimensional kernel. However, since $\Delta J(K,h)$ is not self-adjoint, the algebraic multiplicity of the eigenvalue 0 may be greater than 1. We first deal with this possibility:

Proposition 3.9

For generic $h \in \Delta \mathcal{O}$, at any critical point $z \in \Delta \mathcal{Z}_h$ of $t : \Delta \mathcal{Z}_h \to [0,1]$ the algebraic multiplicity of the eigenvalue 0 of $\Delta J(K,h)_z$ is equal to 1.

Proof: This follows immediately from Proposition 3.24, whose statement and proof we defer to Section 3.4. □

Thus, without loss of generality, we may suppose that at any critical point $z \in \Delta\mathcal{Z}_h$ of t the algebraic multiplicity of the eigenvalue 0 of $\Delta J(K,h)_z$ is equal to 1. We determine how this eigenvalue varies near the critical point. We will see in Proposition 3.11 below that this eigenvalue passes through 0 with non-zero velocity as z passes through this critical point. As before, let X be a smooth manifold, and let $(L_x)_{x\in X}$ be a smooth family of second order, linear, uniformly elliptic, integro-differential operators such that $L_{x_0} = L$ for some $x_0 \in X$. We recall:

Proposition 3.10

Suppose that λ is an eigenvalue of L with algebraic multiplicity 1. Let $f \in C^\infty(\Sigma)$ be such that:

$$\|f\| = 1, \qquad Lf = \lambda.$$

There exists a neighbourhood Ω of x_0 and two smooth families $(f_x)_{x\in\Omega}$ and $(\lambda_x)_{x\in\Omega}$ such that $f_{x_0} = f$, $\lambda_{x_0} = \lambda$, and, for all x:

$$\|f_x\| = 1, \qquad L_x f_x = \lambda_x f_x.$$

Moreover, for all $V \in T_{x_0}X$:

$$L(D_V f)_{x_0} + (D_V L)_{x_0} f = \lambda (D_V f)_{x_0} + (D_V\lambda)_{x_0} f.$$

We now prove that μ is locally constant near critical points of t:

Proposition 3.11

Choose $z \in \Delta\mathcal{Z}_h$. Suppose that $dt(z) = 0$. Let $\sigma : \Delta\mathcal{Z}_h \to \mathbb{R}$ be a path length parametrisation of $\Delta\mathcal{Z}_h$ near z. For all z^-, z^+ sufficiently close to z such that:

$$\sigma(z^-) < \sigma(z) < \sigma(z^+),$$

we have:

$$\mathrm{sig}(z^-,h) = (-1)\mathrm{sig}(z^+,h).$$

Proof: We identify $\mathcal{P}$ with $[0,1]$ by identifying every $f \in \mathcal{P}$ with the unique $t \in [0,1]$ such that $p(t) = f$. Let $(i_\sigma, t_\sigma)_{\sigma\in]-\epsilon,\epsilon[}$ be a smooth family such that $i_0 = 0$ and $\sigma \mapsto z_\sigma := ([i_\sigma], t_\sigma)$ is a path length parametrisation of $\Delta\mathcal{Z}_h$ near z. In the sequel, we identify $\sigma \in]-\epsilon,\epsilon[$ with the point $([i_\sigma], t_\sigma) \in \Delta\mathcal{Z}_h$ which it parametrises. For all σ, we identify $C^\infty(\Sigma)$ with $T_{[i_\sigma]}\mathcal{I} = \mathrm{Smooth}_{[i_\sigma]}$ by identifying the function ϕ with the vector $[i_\sigma, \phi]$. For all σ, we define $\varphi_\sigma \in C^\infty(\Sigma) = T_{[i_\sigma]}\mathcal{I}$ by:

$$\varphi_\sigma = \partial_\sigma [i_\sigma].$$

By definition, for all σ, $(\varphi_\sigma, \partial_\sigma t_\sigma) \in T_\sigma\mathcal{I} \times [0,1]$ is a strong tangent vector to $\Delta\mathcal{Z}_h$ at σ.

Let $\mathcal{L}_1\Delta\hat{K}$ be the partial linearisation of $\Delta\hat{K}$ with respect to the first component. By Proposition 3.6, $\mathcal{L}_1\Delta\hat{K}_{(z,h)}$ is singular, and so 0 is an eigenvalue. Moreover, its kernel is 1-dimensional. Let $\mathcal{L}_2\Delta\hat{K}$ be the partial linearisation of $\Delta\hat{K}$ with respect to t, and, for all σ, let $\psi_\sigma \in C^\infty(\Sigma)$ be such that:

$$\psi_\sigma = \mathcal{L}_2\Delta\hat{K}_\sigma \cdot \partial_t.$$

Then, since $\Delta\hat{K}$ vanishes over $\Delta\mathcal{Z}_h$, by the chain rule:

$$\mathcal{L}_1\Delta\hat{K}_\sigma\varphi_\sigma + (\partial_\sigma t)\psi_\sigma = 0.$$

We claim that ψ_0 does not lie in the image of $\mathcal{L}_1\Delta\hat{K}_{(z,h)}$. Indeed, suppose the contrary. There exists $\phi \in C^\infty(\Sigma)$ such that:

$$\begin{aligned} & \mathcal{L}_1\Delta\hat{K}_{(z,h)} \cdot \phi + \psi_0 &= 0 \\ \Leftrightarrow\quad & \mathcal{L}_{1,2}\Delta\hat{K}_{(z,h)} \cdot (\phi, \partial_t) &= 0 \end{aligned}$$

(ϕ, ∂_t) is thus a strong tangent vector to $\Delta\mathcal{Z}_h$ at z. However, since $(\partial_\sigma t)_0 = 0$, $(\varphi_0, 0)$ is also a strong tangent vector to $\Delta\mathcal{Z}_h$ at z, and $\Delta\mathcal{Z}_h$ is therefore 2-dimensional, which is absurd, and the assertion follows.

Differentiating with respect to σ, and using again the fact that $\partial_\sigma t = 0$ at $\sigma = 0$, we obtain:

$$(\partial_\sigma\mathcal{L}_1\Delta\hat{K})_0\varphi_0 + \mathcal{L}_1\Delta\hat{K}_0(\partial_\sigma\varphi)_0 + (\partial_\sigma^2 t)_0\psi_0 = 0.$$

By Proposition 3.9, the eigenvalue 0 of $\mathcal{L}_1\Delta\hat{K}_{(z,h)}$ has algebraic multiplicity equal to 1. By Proposition 3.10, there exist smooth families $(\phi_\sigma)_{\sigma\in]-\epsilon,\epsilon[} \in C^\infty(\Sigma)$ and $(\lambda_\sigma)_{\sigma\in]-\epsilon,\epsilon[} \in \mathbb{R}$ such that:

(i) $\phi_0 = \varphi_0$; and

(ii) for all σ sufficiently close to 0, $\mathcal{L}_1\Delta\hat{K}_\sigma\phi_\sigma = \lambda_\sigma\phi_\sigma$.

Let $\delta > 0$ be such that 0 is the only eigenvalue of $\mathcal{L}_1\Delta\hat{K}_{(z,h)}$ in the closure of $B_\delta(0)$. By Lemma 3.7, for σ sufficiently close to 0, λ_σ is the only eigenvalue of $\mathcal{L}_1\Delta\hat{K}_{(\sigma,h)}$ in $B_\delta(0)$. In particular, since complex eigenvalues arise in conjugate pairs, λ_σ is real for all σ. Moreover, differentiating, we obtain:

$$(\partial_\sigma\mathcal{L}_1\Delta\hat{K})_0\phi_0 + \mathcal{L}_1\Delta\hat{K}_0(\partial_\sigma\phi)_0 = (\partial_\sigma\lambda)_0\phi_0 + \lambda_0(\partial_\sigma\phi)_0.$$

Since $\lambda_0 = 0$, and $\phi_0 = \varphi_0$, this yields:

$$(\partial_\sigma^2 t)_0\psi_0 + (\partial_\sigma\lambda)_0\phi_0 \in \mathrm{Im}(\mathcal{L}_1\Delta\hat{K}_{(z,h)}).$$

By hypothesis, $z = \sigma(0)$ is a non-degenerate critical point of t, and so $(\partial_\sigma^2 t)_0 \neq 0$. Consequently, since $\psi_0 \notin \mathrm{Im}(\mathcal{L}_1\Delta\hat{K}_{(z,h)})$ it follows that $(\partial_\sigma\lambda)_0 \neq 0$. We conclude that, for σ near 0, the eigenvalue λ_σ is real and changes sign as σ passes through 0. The remaining eigenvalues are treated as in the proof of Proposition 3.8, and the result follows. □

Proposition 3.12

For generic $h \in \Delta\mathcal{O}$, μ extends to a smooth non-vanishing 1-form over $\Delta\mathcal{Z}_h$.

Proof: This follows from Propositions 3.8 and 3.11. □

Proposition 3.13

There exists a canonical smooth non-vanishing 1-form μ over $\mathcal{Z}_p$ such that, for $z = ([i], t) \in \partial\mathcal{Z}_p$:

$$\mu = \mathrm{sig}(z, h)\frac{dt}{\|dt\|} = \mathrm{sig}(z)\frac{dt}{\|dt\|}.$$

Proof: By Proposition 3.4, for $z \in \partial\mathcal{Z}_p$, $\mathrm{sig}(z, h) = \mathrm{sig}(z)$. Existence follows from Proposition 3.12 by taking limits as h tends to 0. Canonicity follows since the sign of the form is determined by its signs at the end-points and its norm is determined by the embedding of $\mathcal{Z}_p$ into $\mathcal{I} \times \mathcal{P}$. □

This allows us to prove that the degree is constant:

Theorem 3.14

For any two generic $f_0, f_1 \in \mathcal{O}$ in the same path connected component:

$$\mathrm{Deg}(\pi; f_0) = \mathrm{Deg}(\pi; f_1).$$

Proof: We define $t : \mathcal{Z}_p \to [0, 1]$ by:

$$t([i], f) = p^{-1}(f).$$

We furnish $\mathcal{Z}_p$ with the canonical orientation form as given by Proposition 3.13. For every regular value $s \in [0, 1]$ of t, we define $\mathrm{Deg}(t; s)$ in the canonical manner for smooth maps between oriented finite dimensional manifolds. By definition of the orientation:

$$\mathrm{Deg}(\pi; f_0) = \mathrm{Deg}(t; 0), \qquad \mathrm{Deg}(\pi; f_1) = \mathrm{Deg}(t; 1).$$

However, by classical differential topology, the degree of a smooth, proper map between two oriented manifolds is constant. Thus:

$$\mathrm{Deg}(\pi; f_0) = \mathrm{Deg}(t; 0) = \mathrm{Deg}(t; 1) = \mathrm{Deg}(\pi; f_1).$$

This completes the proof. □

3.3 Genericity of Non-Degenerate Critical Points.

We prove the first genericity result of Section 3.2, being Proposition 3.5. We continue to use the notation of the preceeding sections. We define the equivalence relation $\sim$ over $\mathbb{R}^2$ by:

$$(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow x_1 = \pm x_2.$$

We furnish $\mathbb{R}^2/\sim$ with the quotient topology and differential structure. Thus, given an open interval I and a mapping $\gamma : I \to \mathbb{R}^2/\sim$, we say that γ is continuous (resp. smooth) if and only if it lifts everywhere locally to a continuous (resp. smooth) mapping $\hat{\gamma} : I \to \mathbb{R}^2$.

We identify $\mathcal{P}$ with $[0,1]$ by identifying $t \in [0,1]$ with $p(t) \in \mathcal{P}$. We thus view Ω henceforth as an open subset of $\mathcal{I} \times [0,1]$. We define the functional $\alpha : \Delta\mathcal{Z} \to \mathbb{R}^2/\sim$ as follows: choose $p := ([i], t, h) \in \Delta\mathcal{Z}$. By definition $\Delta\mathcal{Z}_h$ is a strongly smooth, compact, one dimensional submanifold of $\mathcal{I} \times [0,1]$. Using the canonical L^2 metric over $\mathcal{I}$ and the canonical metric over $[0,1]$, we define a smooth path length parametrisation $\sigma : \Delta\mathcal{Z}_h \to \mathbb{R}$. σ is well defined up to a choice of base point and orientation on every connected component. We define $\alpha(p) \in \mathbb{R}/\sim$ by:

$$\alpha(p) = [\partial_\sigma t, \partial_\sigma^2 t],$$

where $t : \mathcal{I} \times [0,1] \to [0,1]$ is projection onto the second factor. $\alpha(p)$ is independant of the orientations and base points chosen, and thus defines a well defined functional from $\Delta\mathcal{Z}$ into $\mathbb{R}^2/\sim$ which is trivially weakly smooth.

We observe that if $([i], t) \in \Delta\mathcal{Z}_h$, then $([i], t, h) \in \Delta\mathcal{Z}$, and $([i], t)$ is a degenerate critical point of t along $\Delta\mathcal{Z}_h$ if and only if $\alpha(p) = 0$. The following two propositions show that $D\alpha$ is surjective at every point of $\alpha^{-1}(\{0\})$ in $\Delta\mathcal{Z}_0 = \mathcal{Z}_p$:

Proposition 3.15

For all $p := ([i], t) \in \Delta\mathcal{Z}_0$ such that $\alpha(p) = 0$, there exists a strong tangent vector X to $\Delta\mathcal{Z}$ at $(p, 0)$ such that:

$$X \cdot (\partial_\sigma t) \neq 0.$$

Proof: Let $p = ([i], t) \in \Delta\mathcal{Z}_0 = \mathcal{Z}_p$ be such that $\alpha(p) = 0$. Recalling the notation of Section 3.1, we observe that $\alpha \neq 0$ over $\mathcal{Z}_p \setminus \mathcal{Z}_p'$, and thus $p \in \mathcal{Z}_p'$. We identify $C^\infty(\Sigma)$ with $T\mathcal{I}_{[i]} = \mathrm{Smooth}_{[i]}$ by identifying the function $\phi \in C^\infty(\Sigma)$ with the vector $[i, \phi]$. Let $\mathcal{L}_1\Delta\hat{K}$ and $\mathcal{L}_2\Delta\hat{K}$ be the partial linearisations of $\Delta\hat{K}$ with respect to $\mathcal{I}$ and $[0,1]$ respectively, and denote $\mathcal{L}_{1,2}\Delta\hat{K} = \mathcal{L}_1\Delta\hat{K} + \mathcal{L}_3\Delta\hat{K}$. By Proposition 3.6, since p is a critical point of t along $\Delta\mathcal{Z}_0$:

$$\mathrm{Ker}(\mathcal{L}_1\Delta\hat{K}_p) \neq 0.$$

Choose $\varphi \subseteq C^\infty(\Sigma)$ such that $\|\varphi\|_2 = 1$ and:

$$\mathcal{L}_1\Delta\hat{K}_p \cdot \varphi = 0.$$

In particular, $\mathcal{L}_{1,2}\Delta\hat{K}_p \cdot (\varphi, 0) = 0$, and so, by Proposition A.8, $(\varphi, 0)$ is a strong tangent vector to $\Delta\mathcal{Z}_0$ at p. Since $(\varphi, 0)$ has length 1, by definition of σ:

$$D\sigma \cdot (\varphi, 0) = 1.$$

Choose $\psi \in C^\infty(\Sigma)$ such that, for all $s \in \mathbb{R}$:

$$\mathcal{L}_2\Delta\hat{K}_p \cdot s = s\psi,$$

Recalling again the construction of Section 3.1, we choose k such that $\chi_k = 1$ near $[i]$, and we let $[j]$ be the k'th lifting of $[i]$ into $\mathrm{Simp}(\Sigma, \Sigma_\epsilon)$, that is:

$$i = I_k \circ j.$$

Let d_j be the signed distance in Σ_ϵ to $j(\Sigma)$ and let $\pi_j : \Sigma_\epsilon \to j(\Sigma)$ be the nearest point projection. Since j is an embedding, both d_j and π_j are smooth near $j(\Sigma)$. We define $a, b \in C^\infty(\Sigma_\epsilon)$ such that, near $j(\Sigma)$:

$$a(x) = d_j(x)(\varphi \circ \pi_j)(x), \qquad b(x) = (\psi \circ \pi_j)(x).$$

The functions a and b have the following properties:

(i) $a \circ j = 0$;

(ii) $(\nabla a) \circ j = \varphi \mathsf{N}_j$, where N_j is the unit normal vector field over j compatible with the orientation; and

(iii) $b \circ j = \psi$.

Define $g \in C^\infty(\Sigma_\epsilon \times \Sigma_\epsilon)$ by:

$$g(x, y) = a(y)b(x).$$

Define $h := (h_1, ..., h_n) \in C^\infty(\Sigma_\epsilon \times \Sigma_\epsilon)^n$ by:

$$h_l = \begin{cases} g \text{ if } l = k; \text{ and} \\ 0 \text{ otherwise.} \end{cases}$$

For s near 0, denote $p_s = ([i], t, sh)$. Since $h_k = g$ vanishes along j, $p_s \in \Delta\mathcal{Z}$ for all s. We define the strong tangent vector X to $\Delta\mathcal{Z}$ at p by:

$$X := \partial_s p_s|_{s=0} = \partial_s([i], t, sg)|_{s=0}.$$

We show that X has the desired properties. Consider the functional $\hat{A} : \mathcal{I}(\Sigma, \Sigma_\epsilon) \times [0, 1] \to \mathrm{Smooth}(\Sigma, \Sigma_\epsilon)$ given by:

$$\begin{aligned} A([\tilde{\jmath}], t) &= \textstyle\int_\Sigma g(\tilde{\jmath}(\cdot), \tilde{\jmath}(q))\mathrm{dVol}_q \\ &= \textstyle\int_\Sigma (a \circ \tilde{\jmath})(q)\mathrm{dVol}_q(b \circ \tilde{\jmath}). \end{aligned}$$

Let $\mathcal{L}\hat{A}$ be the linearisation of $\hat{A}$. For all strong tangent vectors θ to $\mathcal{I}(\Sigma, \Sigma_\epsilon)$ at $[j]$ (c.f. Appendix A):

$$\mathcal{L}\hat{A}_{[j]} \cdot \theta = \int_\Sigma \varphi\theta \mathrm{dVol}\psi.$$

Let $A : \mathcal{I} \to \mathrm{Smooth}$ be the pull back of $\hat{A}$ through the lifting map. Since $\chi_k = 1$ near $[i]$ and $\eta = 1$ near f, for all sufficiently small s, we readily obtain:

$$\mathcal{L}_{1,2}\Delta\hat{K}_{p_s} = \mathcal{L}_{1,2}\Delta\hat{K}_p + s\mathcal{L}A_{[i]}.$$

Thus, for every strong tangent vector (θ, y) to $\mathcal{I} \times [0, 1]$ at p_s:

$$\begin{aligned} \mathcal{L}_{1,2}\Delta\hat{K}_{p_s} \cdot (\theta, y) &= \mathcal{L}_{1,2}\Delta\hat{K}_p \cdot (\theta, y) + s \textstyle\int_\Sigma \varphi\theta\mathrm{dVol}_p\psi \\ &= \mathcal{L}_1\Delta\hat{K}_p \cdot \theta + \mathcal{L}_2\Delta\hat{K}_p \cdot y + s\langle\varphi, \theta\rangle\psi \\ &= \mathcal{L}_1\Delta\hat{K}_p \cdot \theta + (y + s\langle\varphi, \theta\rangle)\psi. \end{aligned}$$

Thus, since $\mathcal{L}_1\Delta\hat{K}_p \cdot \varphi = 0$, for all s:

$$\mathcal{L}_{1,2}\Delta\hat{K}_{p_s} \cdot (\varphi, -s\|\varphi\|_2^2) = 0.$$

The vector $(\varphi, -s\|\varphi\|_2^2)$ is therefore a strong tangent vector to $\Delta\mathcal{Z}_{sg}$ at p_s. We extend σ to a weakly smooth functional $\sigma : \Delta\mathcal{Z} \to \mathbb{R}$ such that, for all s, p_s is the base point of the path length parametrisation. In other words:

$$\sigma(p_s) = 0.$$

Bearing in mind that $\|\varphi\|_2 = 1$, since σ is the path length parametrisation:

$$\begin{array}{lll} & D\sigma \cdot (\varphi, -s) & = (1+s^2)^{1/2} \\ \Rightarrow & \partial_\sigma & = (1+s^2)^{-1/2}(\varphi, -s) \\ \Rightarrow & \partial_\sigma t & = -s(1+s^2)^{-1/2} \\ \Rightarrow & \partial_s\partial_\sigma t|_{s=0} & = -1. \end{array}$$

Thus:

$$X \cdot (\partial_\sigma t) = \partial_s\partial_\sigma t|_{s=0} \neq 0.$$

This completes the proof. □

Proposition 3.16

For all $p := ([i], t) \in \Delta\mathcal{Z}$ such that $\alpha(p) = 0$, there exists a strong tangent vector X to $\Delta\mathcal{Z}$ at p such that:

$$X \cdot (\partial_\sigma^2 t) \neq 0, \qquad X \cdot (\partial_\sigma t) = 0.$$

Proof: Let $p := ([i], t) \in \Delta\mathcal{Z}_0 = \mathcal{Z}_p$ be such that $\alpha(p) = 0$. Recalling the notation of Section 3.1, we observe that $\alpha \neq 0$ over $\mathcal{Z}_p \setminus \mathcal{Z}_p'$, and thus $p \in \mathcal{Z}_p'$. We identify $C^\infty(\Sigma)$ with $T\mathcal{I}_{[i]} = \mathrm{Smooth}_{[i]}$ by identifying the function $\phi \in C^\infty(\Sigma)$ with the vector $[i, \phi]$. Let $\mathcal{L}_1\Delta\hat{K}$ and $\mathcal{L}_2\Delta\hat{K}$ be the partial linearisations of $\Delta\hat{K}$ with respect to $\mathcal{I}$ and $[0,1]$ respectively, and denote $\mathcal{L}_{1,2}\Delta\hat{K} = \mathcal{L}_1\Delta\hat{K} + \mathcal{L}_2\Delta\hat{K}$. By Proposition 3.6, since p is a critical point of t along $\Delta\mathcal{Z}(\mathcal{E}(f,h))$:

$$\mathrm{Ker}(\mathcal{L}\Delta\hat{K}_p) \neq 0.$$

Choose $\varphi \in C^\infty(\Sigma)$ such that $\|\varphi\| = 1$ and:

$$\mathcal{L}_1\Delta\hat{K}_p \cdot \varphi = 0.$$

In particular, $\mathcal{L}_{1,2}\Delta\hat{K}_p \cdot (\varphi, 0)$ and so, by Proposition A.8, $(\varphi, 0)$ is a strong tangent vector to $\Delta\mathcal{Z}_0$ at p. Since $(\varphi, 0)$ has length 1, by definition of σ:

$$D\sigma \cdot (\varphi, 0) = 1.$$

Since $\mathcal{L}_1\Delta\hat{K}_p$ is Fredholm of index 0, $\mathrm{Im}(\mathcal{L}_1\Delta\hat{K}_p)$ has codimension 1. Recalling the construction of subsection 3.1, we choose $\psi \in \mathrm{Im}(\mathcal{L}_1\Delta\hat{K}_p)^{\perp}$. Choose k such that $\chi_k = 1$ near $[i]$ and we let $[j]$ be the k'th lifting of $[i]$ into $\mathrm{Simp}(\Sigma, \Sigma_\epsilon)$, that is:

$$i = I_k \circ j.$$

Let d_j be the signed distance in Σ_ϵ to $j(\Sigma)$ and let $\pi_j : \Sigma_\epsilon \to j(\Sigma)$ be the nearest point projection. Since j is an embedding, both d_j and π_j are smooth near $j(\Sigma)$. We define $a, b \in C^\infty(\Sigma_\epsilon)$ such that, near $j(\Sigma)$:

$$a(x) = \frac{1}{2}d_j^2(x), \qquad b(x) = (\psi \circ \pi_j)(x).$$

The functions a and b have the following properties:

(i) $a \circ j = 0$;

(ii) $(\nabla a) \circ j = 0$;

(iii) $(\nabla_{\mathsf{N}_j}\nabla a) = \mathsf{N}_j$, where N_j is the unit normal vector field over j compatible with the orientation; and

(iv) $(b \circ j) = \psi$.

Define $g \in C^\infty(\Sigma_\epsilon \times \Sigma_\epsilon)$ by:

$$g(x, y) = a(y)b(x).$$

Define $h := (h_1, ..., h_n) \in C^\infty(\Sigma_\epsilon \times \Sigma_\epsilon)^n$ by:

$$h_l = \begin{cases} g \text{ if } l = k; \text{ and} \\ 0 \text{ otherwise.} \end{cases}$$

For s near 0, denote $p_s = ([i], t, sh)$. Since $h_k = g$ vanishes along j, $p_s \in \Delta\mathcal{Z}$ for all s. We define the strong tangent vector X to $\Delta\mathcal{Z}$ at p by:

$$X := \partial_s p_s|_{s=0} = \partial_s([i], t, sh)|_{s=0}.$$

We show that X has the desired properties. Consider first the functional $\hat{A} : \mathcal{I}(\Sigma, \Sigma_\epsilon) \to \mathrm{Smooth}(\Sigma, \Sigma_\epsilon)$ given by:

$$\begin{aligned} \hat{A}(\tilde{\jmath}) &= \textstyle\int_\Sigma g(\tilde{\jmath}(\cdot), \tilde{\jmath}(q))\mathrm{dVol}_q \\ &= \textstyle\int_\Sigma (a \circ \tilde{\jmath})(q)\mathrm{dVol}_q(b \circ \tilde{\jmath}). \end{aligned}$$

Let $\mathcal{L}\hat{A}$ be the linearisation of $\hat{A}$. Since both a and ∇a vanish along j, for all strong tangent vectors θ to $\mathcal{I}(\Sigma, \Sigma_\epsilon)$ at $[j]$ (c.f. Appendix A):

$$\mathcal{L}\hat{A}_{[j]} \cdot \theta = 0.$$

Calculating the second order linearisation, we obtain:

$$\mathcal{L}^2\hat{A}_{[j]}(\varphi,\theta) = \int_\Sigma \varphi\theta \mathrm{dVol}\psi.$$

Let $A : \mathcal{I} \to \mathrm{Smooth}$ be the pull back of $\hat{A}$ through the lifting map. Since $\chi_k = 1$ near $[i]$ and $\eta = 1$ near f, for all sufficiently small s, we readily obtain:

$$\mathcal{L}_{1,2}\Delta\hat{K}_{p_s} = \mathcal{L}_{1,2}\Delta\hat{K}_p + s\mathcal{L}A_{[i]}.$$

Thus, for all s:

$$\begin{aligned}\mathcal{L}_{1,2}\Delta\hat{K}_{p_s}\cdot(\theta,y) \quad &= \mathcal{L}_{1,2}\Delta\hat{K}_p\cdot(\theta,y) + s\mathcal{L}A_{[i]}\cdot\theta\\ &= \mathcal{L}_{1,2}\Delta\hat{K}_p\cdot(\theta,y).\end{aligned}$$

In particular:

$$\mathcal{L}_{1,2}\Delta\hat{K}_{p_s}\cdot(\varphi,0) = 0.$$

The vector $(\varphi,0)$ is therefore a strong tangent vector to $\Delta\mathcal{Z}_{sh}$ at p_s for all s. We extend σ to a weakly smooth functional $\sigma : \Delta\mathcal{Z} \to \mathbb{R}$ defined in a neighbourhood of p such that, for all s, p_s is the base point of the path length parametrisation. In other words:

$$\sigma(p_s) = 0.$$

Since $\|\varphi\| = 1$, by definition of σ, for all s:

$$D\sigma_{p_s}\cdot(\varphi,0) = 1.$$

In other words, for all s, $\partial_\sigma(p_s) = (\varphi,0)$. We now restrict attention to a strongly smooth embedded 2-dimensional submanifold, $S \subseteq \Delta\mathcal{Z}$ passing through p such that, for all s sufficiently close to 0:

$$\Delta\mathcal{Z}_{sh} \subseteq S.$$

(s,σ) defines a local coordinate system of S near p. Calculating the second order partial linearisation of $\Delta\hat{K}$, we obtain:

$$\begin{aligned}(\partial_\sigma\mathcal{L}_{1,2}\Delta\hat{K})_{p_s}\cdot(\theta,y) \quad &= (\mathcal{L}^2_{1,2}\Delta\hat{K})_{p_s}((\varphi,0),(\theta,y))\\ &= (\mathcal{L}^2_{1,2}\Delta\hat{K})_p((\varphi,0),(\theta,y)) + s(\mathcal{L}^2A)(\varphi,\theta)\\ &= (\mathcal{L}^2_{1,2}\Delta\hat{K})_p((\varphi,0),(\theta,y)) + s\int_\Sigma\varphi\theta\mathrm{dVol}\psi.\\ \Rightarrow \quad (\partial_s\partial_\sigma\mathcal{L}_{1,2}\Delta\hat{K})_p\cdot(\theta,y) \quad &= \int_\Sigma\varphi\theta\mathrm{dVol}\psi.\end{aligned}$$

For all $(s,\sigma)\in S$, let $(\varphi_{s,\sigma},(\partial_\sigma t)_{s,\sigma})$ be the strong tangent vector to $\Delta\mathcal{Z}_{sh}$ in $\mathcal{I}\times[0,1]$ at (s,σ) satisfying:

$$D\sigma\cdot(\varphi_{s,\sigma},(\partial_\sigma t)_{s,\sigma}) = 1.$$

Since $S \subseteq \Delta\mathcal{Z}$, for all (s,σ):

$$\begin{aligned}&\Delta\hat{K}([i_{s,\sigma}],t_{s,\sigma},sh) \quad &&= 0\\ \Rightarrow \quad &(\mathcal{L}_{1,2}\Delta\hat{K})_{s,\sigma}\cdot(\varphi_{s,\sigma},(\partial_\sigma t)_{s,\sigma}) \quad &&= 0.\end{aligned}$$

Differentiating twice more yields:

$$\begin{aligned}(\partial_s\partial_\sigma\mathcal{L}_{1,3}\Delta\hat{K})_{s,\sigma}\cdot(\varphi_{s,\sigma},(\partial_\sigma t)_{s,\sigma}) + (\partial_\sigma\mathcal{L}_{1,3}\Delta\hat{K})_{s,\sigma}\cdot(\partial_s\varphi_{s,\sigma},(\partial_s\partial_\sigma t)_{s,\sigma})\\ +(\partial_s\mathcal{L}_{1,3}\Delta\hat{K})_{s,\sigma}\cdot(\partial_\sigma\varphi_{s,\sigma},(\partial_\sigma^2 t)_{s,\sigma}) + (\mathcal{L}_{1,3}\Delta\hat{K})_{s,\sigma}\cdot(\partial_s\partial_\sigma\varphi_{s,\sigma},(\partial_s\partial_\sigma^2 t)_{s,\sigma}) = 0.\end{aligned}$$

By construction, $(\varphi_{s,0},(\partial_\sigma t)_{s,0}) = (\varphi, 0)$ for all s, and so:

$$(\partial_s\varphi)_{0,0},(\partial_s\partial_\sigma t)_{0,0} = 0$$

Moreover, since a and ∇a both vanish along j (c.f. Appendix A):

$$(\partial_s\mathcal{L}_{1,3}\Delta\hat{K})_{0,0} = \mathcal{L}A_{0,0} = 0.$$

Thus, using the formula for $(\partial_s\partial_\sigma\mathcal{L}_{1,3}\Delta\hat{K})_p$ determined above:

$$(\mathcal{L}_1\Delta\hat{K}_p)(\partial_s\partial_\sigma\varphi)_{0,0} + (\partial_s\partial_\sigma^2 t)_{0,0}\omega + \int_N \varphi^2\mathrm{dVol}\psi = 0,$$

for some function $\omega \in C^\infty(\Sigma)$. However, by definition, $\psi \notin \mathrm{Im}(\mathcal{L}_1\Delta\hat{K}_p)$, and therefore:

$$\begin{aligned} & (\partial_s\partial_\sigma^2 t)_{0,0}\omega & \neq 0\\ \Rightarrow\ & (\partial_s\partial_\sigma^2 t)_{0,0} & \neq 0.\end{aligned}$$

Moreover, recalling that $(\varphi, 0)$ is tangent to $\Delta\mathcal{Z}_{sh}$ at p_s for all s, we obtain:

$$\begin{aligned} & (\partial_\sigma t)_{0,s} & = 0 \text{ for all } s\\ \Rightarrow\ & (\partial_s\partial_\sigma t)_{0,0} & = 0.\end{aligned}$$

Thus:

$$\begin{aligned} X\cdot(\partial_\sigma^2 t) &= (\partial_s\partial_\sigma^2 t)_{0,0} & \neq 0\\ X\cdot(\partial_\sigma t) &= (\partial_s\partial_\sigma t)_{0,0} & = 0.\end{aligned}$$

This completes the proof. □

We thus obtain:

Proposition 3.17

After reducing $\Delta\mathcal{O}$ if necessary, the set of all points $h \in \Delta\mathcal{O}$ such that $t : \Delta\mathcal{Z}_h \to [0,1]$ only has non-degenerate critical points is open and dense.

Proof: Let $Z := \{0\}$ be the 0-dimensional manifold consisting of a single point, and for $h \in \Delta\mathcal{O}$, define $\mathcal{G}_h : Z \to \Delta\mathcal{O}$ by:

$$\mathcal{G}_h(0) = h.$$

We claim that if $\mathcal{G}_h$ is transverse to the restriction of $\Delta\pi$ to $\alpha^{-1}(\{0\})$, then the function $t : \Delta\mathcal{Z}_h \to [0,1]$ only has non-degenerate critical points. Indeed, by Proposition A.10:

$$\mathcal{Z}(\mathcal{F},\alpha) = \emptyset,$$

since, otherwise, it would be a smooth, embedded manifold of dimension equal to -1, which is absurd. However, $p \in \Delta\mathcal{Z}_h$ is a degenerate critical point of t if and only if $\alpha(p)$ equals 0, and the assertion follows.

We thus claim that the set of all points $h \in \Delta\mathcal{O}$ such that $\mathcal{G}_h$ is transverse to the restriction of $\Delta\pi$ to $\alpha^{-1}(\{0\})$ is open and dense. However, by Propositions 3.15 and 3.16, $D\alpha$ is surjective at every point of $\alpha^{-1}(\{0\})$ in $\Delta\mathcal{Z}_0$. Thus, by properness of $\Delta\pi : \Delta\mathcal{Z} \to \Delta\mathcal{O}$, since surjectivity is an open property, reducing $\Delta\mathcal{O}$ if necessary $D\alpha$ is surjective at every point of $\alpha^{-1}(\{0\})$ in $\Delta\mathcal{Z}$. Thus, since $\Delta\pi : \Delta\mathcal{Z} \to \Delta\mathcal{O}$ is a proper mapping, and since $\mathcal{L}\Delta\hat{K}$ is surjective at every point of $\Delta\mathcal{Z}$, it follows from the Sard-Smale Theorem (Theorem A.12) that $\mathcal{G}_h$ is transverse to the restriction of $\Delta\pi$ to $\alpha^{-1}(\{0\})$ for generic h. Openness follows by the properness of $\Delta\pi$, and this completes the proof. $\square$

3.4 Genericity of Points of Trivial Nilpotency.

We continue to use the notation of the preceeding sections. Let $\mathcal{L}_1\Delta\hat{K}$ be the partial linearisation of $\Delta\hat{K}$ with respect to the first component. We define the functional $\tilde{N} : \Delta\mathcal{Z} \to \mathbb{N}_0$ by:

$$\tilde{N}(p) = \mathrm{Mult}(\mathcal{L}_1\Delta\hat{K}_p, 0).$$

By definition, $N(p)$ equals zero when $\mathcal{L}_1\Delta\hat{K}_p$ is invertible.

Proposition 3.18

The functional $\tilde{N} : \Delta\mathcal{Z} \to \mathbb{N}_0$ is upper semi-continuous.

Proof: Choose $p \in \Delta\mathcal{Z}$. Let $\delta > 0$ be such that the only possible eigenvalue of $\mathcal{L}_1\Delta\hat{K}_p$ in the closed ball of radius δ about 0 is 0 itself (which may have multiplicity 0). By Lemma 3.7, for all $q \in \Delta\mathcal{Z}$ sufficiently close to p:

$$\tilde{N}(q) \leqslant \sum_{z \in B_\delta(0)} \mathrm{Mult}(\mathcal{L}_1\Delta\hat{K}_q, z) = \sum_{z \in B_\delta(0)} \mathrm{Mult}(\mathcal{L}_1\Delta\hat{K}_p, z) = \tilde{N}(p).$$

This completes the proof. $\square$

We define the functional $N : \Delta\mathcal{O} \to \mathbb{N}_0$ by:

$$N(h) = \operatorname*{Sup}_{(i,f)\in\Delta\mathcal{Z}_h} \tilde{N}(i, f, h).$$

Proposition 3.19

The functional $N : \Delta\mathcal{O} \to \mathbb{N}_0$ is everywhere finite and upper semi-continuous.

Proof: Since $\Delta\pi : \Delta\mathcal{Z} \to \Delta\mathcal{O}$ is proper, for all $h \in \Delta\mathcal{O}$, $\Delta\mathcal{Z}_h$ is compact. Since $\tilde{N}$ is upper-semicontinuous, N is finite. Upper semi-continuity of N follows similarly. $\square$

We aim to show that $N \leqslant 1$ over an open, dense subset of $\Delta\mathcal{O}$. We first require more refined information concerning the nilpotent decomposition of an operator. Let L be a second order, elliptic, integro-differential operator defined over Σ. Let λ be an eigenvalue of L. Let $C^\infty(\Sigma) = E \oplus R$ be the nilpotent decomposition of $C^\infty(\Sigma)$ with respect to this eigenvalue. We readily obtain:

Proposition 3.20

Let L^* be the L^2 adjoint of L. Let $C^\infty(\Sigma) = E^* \oplus R^*$ be the nilpotent decomposition of $C^\infty(\Sigma)$ with respect to the eigenvalue $\overline{\lambda}$ of L^*. Then, with respect to the L^2 metric:

$$E^* = R^\perp.$$

In particular, E^* may be identified with the dual space to E.

Let X be a finite dimensional manifold. Let $(V_x)_{x\in X}$ be a family of finite dimensional subspaces of $C^\infty(\Sigma)$. We say that this family is **strongly smooth** if and only if there exists everywhere locally a strongly smooth family $F := (f_1, ..., f_n) : X \to C^\infty(\Sigma)^n$ of bases. Likewise, let $(W_x)_{x\in X}$ be a family of finite codimension subspaces of $C^\infty(\Sigma)$. We say that this family is **strongly smooth** if and only if the family of L^2 dual spaces $(W_x^*)_{x\in X} \subseteq C^\infty(\Sigma)$ is strongly smooth. When these subspaces depend on elements of a function space, we define weak smoothness of this dependence as in Appendix A. The following partial generalisation of Proposition 3.10 shows how degenerate eigenvalues perturb (c.f [12]):

Proposition 3.21

Let $\mathrm{IDO}_2(\Sigma)$ denote the space of second order, elliptic, integro-differential operators over $C^\infty(\Sigma)$ with smooth coefficients. There exists a neighbourhood Ω of L in $\mathrm{IDO}_2(\Sigma)$ and two weakly smooth families of subspaces $(E(L'))_{L'\in\Omega}$ and $(R(L'))_{L'\in\Omega}$ such that, for all $L' \in \Omega$:

(i) $C^\infty(\Sigma) = E(L') \oplus R(L')$;

(ii) L' preserves $E(L')$ and $R(L')$; and

(iii)the restriction of $(L' - \lambda\mathrm{Id})$ to $R(L')$ is invertible.

Moreover, in particular, $E(L')$ has the same finite dimension for all $L' \in \Omega$.

This allows us to control $\tilde{N}$ using weakly smooth functionals:

Proposition 3.22

Choose $n > 1$. Let $p \in \mathcal{Z}'$ be such that $N(p) = n$. There exists a neighbourhood, U of p in $\Delta\mathcal{Z}$ and a weakly smooth functional $T : \Delta\mathcal{Z} \to \mathbb{R}$ such that, for all $q \in U$:

(i)$\tilde{N}(q) \leqslant \tilde{N}(p)$; and

(ii)$\tilde{N}(q) = \tilde{N}(p)$ only if $T(q) = 0$.

Proof: Choose $p = ([i], t, h) \in \Delta\mathcal{Z}$. Let $(i, U_i, V_i, \mathcal{E}_i)$ be a graph chart of $\mathcal{I}$ about $[i]$. We identify $T\mathcal{I}$ with Smooth, and, for all $q = \mathcal{E}_i(\varphi) \in V_i$, we identify $C^\infty(\Sigma)$ with $T\mathcal{I}_q$ by identifying the function $\psi \in C^\infty(\Sigma)$ with the vector $[\hat{\mathcal{E}}_i(\varphi), \psi]$. Denote $L = \mathcal{L}_1\Delta\hat{K}_p$. Let $C^\infty(\Sigma) = E \oplus R$ be the nilpotent decomposition of $C^\infty(\Sigma)$ with respect to the eigenvalue

0 of L. By Proposition 3.21, there exists a neighbourhood V of L in $\mathrm{IDO}_2(\Sigma)$ such that, for $L' \in V$, $C^\infty(\Sigma)$ decomposes as $C^\infty(\Sigma) = E' \oplus R'$ where:

(i) E' and R' are preserved by L'; and

(ii) the restriction of L' to R' is invertible.

Moreover, E' depends in a weakly smooth manner on L'. In particular, for $L' \in V$, the multiplicity of the eigenvalue 0 of L' is at most $\mathrm{Dim}(E') = n$. We define the weakly smooth mapping $T_0 : V \to \mathbb{R}$ by:

$$T_0(L') = \mathrm{Tr}(L'|_{E'}).$$

For $L' \in V$, if $T_0(L') \neq 0$, then the multiplicity of the eigenvalue 0 of L' is at most $n-1$. Let $W \subseteq \mathcal{I} \times [0,1] \times \Delta\mathcal{O}$ be a neighbourhood of p such that for all $q \in W$:

$$\mathcal{L}_1\hat{K}_q \in V.$$

Let $\chi : \mathcal{I} \times [0,1] \times \Delta\mathcal{O} \to \mathbb{R}$ be a smooth functional supported in W and equal to 1 near p (c.f. Appendix A.6). Define the functional $T : \mathcal{I} \times [0,1] \times \Delta\mathcal{O} \to \mathbb{R}$ by:

$$T(p) = \chi(p)T_0(p).$$

Let $U \subseteq W$ be such that $\chi(q) \neq 0$ for all $q \in U$. U and T are the required open set and weakly smooth mapping, and this completes the proof. $\square$

Proposition 3.23

For all $p = ([i], t) \in \Delta\mathcal{Z}_0$ such that $\tilde{N}(p) \geqslant 2$, if T is defined as in Proposition 3.22, then there exists a strong tangent vector X to $\Delta\mathcal{Z}$ at p such that:

$$X \cdot T \neq 0, \qquad X \cdot (\partial_\sigma t) = 0.$$

Proof: Let $p := ([i], t) \in \Delta\mathcal{Z}_0 = \mathcal{Z}_p$ be such that $n := \tilde{N}(p) \geqslant 2$. Recalling the notation of Section 3.1, we observe that $\tilde{N}(p) = 0$ over $\mathcal{Z}_p \setminus \mathcal{Z}'_p$ and thus $p \in \mathcal{Z}'_p$. We identify $C^\infty(\Sigma)$ with $T\mathcal{I}_{[i]} = \mathrm{Smooth}_{[i]}$ by identifying the function $\phi \in C^\infty(\Sigma)$ with the vector $[i, \phi]$. Let $\mathcal{L}_1\Delta\hat{K}$ and $\mathcal{L}_2\Delta\hat{K}$ be the partial linearisations of $\Delta\hat{K}$ with respect to $\mathcal{I}$ and $[0,1]$ respectively, and denote $\mathcal{L}_{1,2}\Delta\hat{K} = \mathcal{L}_1\Delta\hat{K} + \mathcal{L}_2\Delta\hat{K}$.

Denote $L = \mathcal{L}_1\hat{K}_p$. Let L^* be the L^2 dual of L. Let $C^\infty(\Sigma) = E \oplus R$ and $C^\infty(\Sigma) = E^* \oplus R^*$ be the nilpotent decompositions of $C^\infty(\Sigma)$ with respect to the eigenvalue 0 of the operators L and L^* respectively. By definition:

$$\begin{aligned} E &= \left\{\varphi \in L^2(\Sigma) \mid L^m\varphi = 0 \text{ for some } m \geqslant 1\right\}, \\ E^* &= \left\{\varphi \in L^2(\Sigma) \mid (L^*)^m\varphi = 0 \text{ for some } m \geqslant 1\right\}. \end{aligned}$$

By Proposition 3.20:

$$E^* = R^\perp,$$

and we identify E^* with the dual space to E. Let $(\phi_k)_{1\leqslant k\leqslant n}$ be a basis of E and let $(\psi_l)_{1\leqslant l\leqslant n}$ be the dual basis of E^* with respect to the L^2 pairing. Thus, for all k, l:

$$\int_N \phi_k\psi_l \mathrm{dVol} = \delta_{kl}.$$

Recalling the construction of Section 3.1, we choose k such that $\chi_k = 1$ near $[i]$ and let j be the k'th lifting of $[i]$ into $\mathrm{Simp}(\Sigma, \Sigma_\epsilon)$, that is:

$$i = I_k \circ j.$$

Let d_j be the signed distance in Σ_ϵ to $j(\Sigma)$ and let $\pi_j : \Sigma_\epsilon \to j(\Sigma)$ be the nearest point projection. Since j is an embedding, both d_j and π_j are smooth near $j(\Sigma)$. We define $\alpha_k, \beta_l \in C^\infty(\Sigma_\epsilon)$ such that, near $j(\Sigma)$:

$$\alpha_k(x) = d_j(x)(\psi_k \circ \pi_j)(x), \qquad \beta_l(x) = (\phi_l \circ \pi_j)(x)$$

For all k and l, the functions α_k and β_l have the following properties:

(i) $(\alpha_k \circ j) = 0$;

(ii) $(\nabla \alpha_k) \circ j = \psi_k \mathsf{N}_j$, where N_j is the unit normal vector field over j compatible with the orientation; and

(i) $(\beta_l \circ j) = \phi_l$.

For all r, s, define $g_{rs} \in C^\infty(\Sigma_\epsilon \times \Sigma_\epsilon)$ by:

$$g_{rs}(p, q) = \beta_l(p)\alpha_k(q),$$

and define $h_{rs} := (h_{rs,1}, ..., h_{rs,n}) \in C^\infty(\Sigma_\epsilon \times \Sigma_\epsilon)^n$ by:

$$h_{rs,m} = \begin{cases} g_{rs} \text{ if } m = k;\ \text{and} \\ 0 \text{ otherwise.}\end{cases}$$

For $a \in \mathbb{R}^{n\times n}$, denote:

$$h_a = \sum_{r,s=1}^n a^{rs} h_{rs}, \qquad g_a = \sum_{r,s=1}^n a^{rs} g_{rs}.$$

For s near 0, denote $p_{a,s} := ([i], t, sg_a)$. Since $h_{a,k} = g_a$ vanishes along j, $p_{a,s} \in \Delta\mathcal{Z}$ for all s. We define the strong tangent vector X_a to $\Delta\mathcal{Z}$ at p by:

$$X_a := \partial_s p_{a,s}|_{s=0} = \partial_s([i], t, sh_a)|_{s=0}.$$

We show that X_a the desired properties for some a. As in the proof of Propositions 3.15 and 3.16, bearing in mind that α_k vanishes over i_0 for all k, for all (θ, y) tangent to $\mathcal{I} \times \mathcal{P}$ at $([i], t)$:

$$\mathcal{L}_{1,2}\Delta\hat{K}_{p_{a,s}} \cdot (\theta, y) = \mathcal{L}_{1,2}\Delta\hat{K}_p \cdot (\theta, y) + s \sum_{k,l=1}^n a^{kl} \int_N \psi_k \theta \mathrm{dVol} \phi_l \circ i.$$

Since the restriction of L to E is nilpotent and its kernel is 1-dimensional, we may suppose that:

$$L\phi_{k+1} = \phi_k \text{ for all } k < n, \qquad L\phi_1 = 0,$$

and we may thus assume that $\phi_1 = \varphi$. Choose a such that $a_{kl} = \delta_{kn}\delta_{ln}$. Let T be defined as in Proposition 3.22. Observe that, for all s, the operator $\mathcal{L}_{1,2}\Delta\hat{K}_{p_{a,s}}$ preserves both E and R. Thus, for all s:

$$\begin{array}{rlll} & T(p_{a,s}) & = T_0(\mathcal{L}_1\Delta\hat{K}_{p_{a,s}}) & = s \\ \Rightarrow & (\partial_s T)(p) & = 1 & \neq 0. \end{array}$$

Moreover, since $n \geqslant 2$:

$$\langle\psi_n, \varphi\rangle = \langle\psi_n, \phi_1\rangle = 0.$$

Thus, for all s:

$$\mathcal{L}_{1,2}\Delta\hat{K}_{p_{a,s}} \cdot (\varphi, 0) = \mathcal{L}_{1,2}\Delta\hat{K}_p \cdot (\varphi, 0) = 0.$$

By Lemma A.8, $(\varphi, 0)$ is thus a strong tangent vector to $\Delta\mathcal{Z}_{sh_a}$ at $p_{a,s}$ for all s. We thus extend σ to a weakly smooth functional over $\Delta\mathcal{Z}$ such that, for all s, $p_{a,s}$ is the base point of the path length parametrisation. In other words:

$$\sigma(p_{a,s}) = 0.$$

Since $\|\varphi\| = 1$, by definition of σ, for all s:

$$D\sigma_{p_{a,s}}(\varphi, 0) = 1.$$

In other words $\partial_\sigma = (\varphi, 0)$, and so:

$$\begin{array}{rll} & (\partial_\sigma t)(p_{a,s}) & = 0 \text{ for all } s \\ \Rightarrow & (\partial_s\partial_\sigma t)(p) & = 0. \end{array}$$

Thus:

$$\begin{array}{lll} X_a \cdot T & = \partial_s T & \neq 0 \\ X_a \cdot (\partial_\sigma t) & = \partial_s\partial_\sigma t & = 0. \end{array}$$

This completes the proof. □

Define the equivalence relation $\sim$ over $\mathbb{R}^2$ by:

$$(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow x_1 = \pm x_2.$$

Given T as constructed in Proposition 3.22, we define the weakly smooth functional $\beta : \Delta\mathcal{Z} \to \mathbb{R}/\sim$ as follows: choose $z = (i, t, h) \in \Delta\mathcal{Z}$. Let $\sigma : \Delta\mathcal{Z}_h \to \mathbb{R}$ be a path length parametrisation. Define $\beta(z) \in \mathbb{R}^2/\sim$ by:

$$\beta(z) = (\partial_\sigma t, T),$$

where $t : \mathcal{I} \times [0,1] \times \Delta\mathcal{O} \to \mathbb{R}$ is projection onto the second factor. β is independant of the orientations and base points chosen. It is thus a well defined functional from $\Delta\mathcal{Z}$ into $\mathbb{R}^2/\sim$ which is trivially weakly smooth. Let U be as in Proposition 3.22. If $q \in \Delta\mathcal{Z} \cap U$, then $\tilde{N}(q) \leqslant \tilde{N}(p)$, with equality only if $\beta(q) = 0$. Moreover, by reducing U if necessary, by Propositions 3.15 and 3.23, we may assume that $D\beta$ is surjective at every point of $\Delta\mathcal{Z} \cap U \cap \beta^{-1}(\{0\})$.

Proposition 3.24

After reducing $\Delta\mathcal{O}$ if necessary, the set of all points $h \in \Delta\mathcal{O}$ such that $N(f, h) \leqslant 1$ is open and dense.

Proof: By Proposition 3.19, N is upper semi-continuous, and this set is therefore open. Choose $h \in \Delta\mathcal{O}$ such that $N(h) \geqslant 2$. Since $\Delta\pi$ is proper, $\Delta\mathcal{Z}_h$ is compact. Thus, by the preceeding discussion, there exists a finite family $(U_l)_{1\leqslant l\leqslant m} \subseteq \mathcal{I}\times[0,1]\times\Delta\mathcal{O}$ of open sets, and, for each l, a weakly smooth functional $\beta_l : \Delta\mathcal{Z} \to \mathbb{R}$ such that:

(i) $\Delta\mathcal{Z}_h \subseteq U_1 \cup ... \cup U_m$; and

for all l:

(ii) $D\beta_l$ is surjective at every point of $U_l \cap \beta_l^{-1}(\{0\})$; and

(iii) for all $q \in U_l$, $\tilde{N}(q) \leqslant N(f,h)$, with equality only if $\beta_l(q) = 0$.

Since $\Delta\pi$ is proper, there exists a neighbourhood, V of h in $\Delta\mathcal{O}$ such that for all $h' \in V$:

$$\Delta\mathcal{Z}_{h'} \subseteq U_1 \cup ... \cup U_m.$$

Let $Y = \{0\}$ be the 0-dimensional manifold consisting of one element. For $h' \in V$, define $\mathcal{G}_{h'} : Y \to \Delta\mathcal{O}$ by $\mathcal{G}_{h'}(0) = h'$. Suppose that $\mathcal{G}(h')$ is transverse to the restriction of $\Delta\pi$ to $\beta_l^{-1} \cap U_l$ for all l. We claim that $N(h') \leqslant N(h) - 1$. Indeed, by Proposition A.10, for each l:

$$U_l \cap \beta_l^{-1}(\{0\}) \cap \Delta\mathcal{Z}_{h'} = \emptyset,$$

since otherwise it would be a smooth embedded submanifold of dimension -1, which is absurd. However, for $q \in \Delta\mathcal{Z}_{h'}$, by definition $\tilde{N}(q) \leqslant N(h)$ with equality only if $\beta_l(q) = 0$ for all k, and the assertion follows.

We claim that the set of all points $(h') \in V$ such that $\mathcal{G}_{h'}$ is transverse to the restriction of $\Delta\pi$ to $\beta_l^{-1}(\{0\}) \cap U_l$ for all l is dense. Observe first that, for all l, the restriction of $\Delta\pi$ to the closure of $U_l \cap \Delta\mathcal{Z}$ is proper. We thus say that it is relatively proper, and the Sard-Smale Theorem (Theorem A.12) readily adapts to this setting. Thus, for all l, there exists an open dense subset $V_l \subseteq V$ such that, for all $h' \in V_l$, $\mathcal{G}_{h'}$ is transverse to the restriction of $\Delta\pi$ to $\beta_l^{-1}(\{0\}) \cap U_l$. The intersection $V_1 \cap ... \cap V_n$ is thus the required dense subset of V, and the assertion follows.

There therefore exists $h' \in \Delta\mathcal{O}$, as close to h as we wish, such that $N(h') \leqslant N(h) - 1$. By induction, there exists $h' \in \mathcal{O}'$ as close to h as we wish such that $N(h') \leqslant 1$. It follows that $N^{-1}(\{0,1\})$ is dense in $\Delta\mathcal{O}$, and this completes the proof. □

4 - Existence Results.

We now apply the degree theory developed in the preceeding section to prove the existence results given in the introduction.

Let $M := M^{n+1}$ be a compact, orientable, $(n+1)$-dimensional Riemannian manifold. We suppose that $n \geqslant 2$ (the case $n = 1$ having been studied by Schneider in [22] and [23]). Let $\Sigma := \Sigma^n$ be the standard n-dimensional sphere, and let $B := B^{n+1}$ be the standard $(n+1)$-dimensional closed ball. In particular:

$$\Sigma = \partial B.$$

We say that a smooth immersion $i : \Sigma \to M$ is an **Alexandrov embedding** if and only if it extends to an immersion of B into M. Let $\mathrm{Conv} := \mathrm{Conv}(\Sigma, M) \subseteq C^\infty_{\mathrm{imm}}(\Sigma, M)$ be the subset consisting of locally strictly convex Alexandrov embeddings. Conv is trivially open and $\mathrm{Diff}^\infty(\Sigma)$-invariant and we define $\mathcal{C} \subseteq \mathcal{I}$ to be the quotient of Conv by this group action:

$$\mathcal{C} = \mathrm{Conv}/\mathrm{Diff}^\infty(\Sigma).$$

Let K be a curvature function, let $\mathcal{O} \subseteq C^\infty(M,]0, \infty[)$ be an open set of smooth, positive functions over M and define the solution space $\mathcal{Z} := \mathcal{Z}(K; \mathcal{O}) \subseteq \mathcal{C} \times \mathcal{O}$ by:

$$\mathcal{Z} = \{([i], f) \mid K(i) = f \circ i\}.$$

Let $\pi : \mathcal{Z} \to \mathcal{O}$ be the projection onto the second factor. We show that under appropriate hypotheses on M, K and $\mathcal{O}$, and after restricting $\mathcal{C}$ in different ways, the projection $\pi : \mathcal{Z} \to \mathcal{O}$ becomes a proper mapping, and so the properties of ellipticity and properness outlined in Section 2.1 are satisfied and our degree theory may be applied. The results of [28] allow us to determine the degree in these cases. Indeed, we identify $\mathbb{R}$ with the constant functions in $C^\infty(M)$. Bearing in mind that within the present context all immersions in $\mathcal{C}$ are locally strictly convex, we obtain:

Proposition 4.1

Suppose that K is mean curvature, extrinsic curvature or special Lagrangian curvature. Let g be the metric on M. Suppose that for all metrics g' sufficiently close to g there exists $C > 0$ and $T > 0$ such that:

(i) $]T, +\infty[\subseteq \mathcal{O}$; and

(ii) if $t \in]T, +\infty[$ and $([i], t) \in \mathcal{Z}$, then $\mathrm{Diam}(\Sigma; i^*g) \leqslant Ct^{-1}$.

If π is proper, then:

$$\mathrm{Deg}(\pi) = -\chi(M),$$

where $\chi(M)$ is the Euler Characteristic of M.

Remark: It follows from the definitions that if $([i], t) \in \mathcal{Z}$, then, in particular, i is locally strictly convex. This condition turns out to play an important role in the calculation of the degree. Indeed, we show that under these hypotheses each locally strictly convex, immersed sphere of sufficiently large constant curvature is, in fact, a leaf of a foliation of a neighbourhood of a critical point of the scalar curvature function of M, analogous to the foliation constructed by Ye in [37]. We say that such spheres are of Ye type, and by identifying them with their corresponding critical point we obtain a formula for the degree. However, when the local strict convexity hypothesis is dropped, immersed spheres of large constant curvature are no longer necessarily of Ye type. Indeed, even in the case where the manifold is 3-dimensional and K is mean curvature, Pacard and Malchiodi (c.f. [17]) construct under general conditions immersed hyperspheres of arbitrarily large, constant curvature which are dumbbell shaped, and not of Ye type. In this case, the asymptotic behaviour is described in the recent work [16] of Laurain. It would be interesting to know how this affects the calculation of the degree.

Proof: We recall the framework of [28]. Let R be the scalar curvature function of M. Bearing in mind that the degree theory readily extends to a context where the metric on M is allowed to vary, we perturb this metric slightly and thus suppose that R is a Morse Function. Let $\mathrm{Crit}(R) \subseteq M$ be the set of critical points of R. Theorem 1.1 of [37] readily extends to show that there exists $\epsilon > 0$ such that for every critical point p of R in M, there exists a nieghbourhood U_p of p in M and a foliation $(\Sigma_{p,s})_{s\in]0,\epsilon]}$ of $U_p \setminus \{p\}$ such that for all p and for all s, the K-curvature of $\Sigma_{p,s}$ is constant and equal to s^{-1} (c.f. [28] and [18] for more details). For all p and s let $i_{p,s} : \Sigma \to M$ be an immersion parametrising $\Sigma_{p,s}$. In particular, for all $t \geqslant 1/\epsilon$:

$$\left\{([i_{p,t^{-1}}], t) \mid p \in \mathrm{Crit}(R)\right\} \subseteq \mathcal{Z}_t.$$

By Property (ii) and Theorem III of [28], for sufficiently large t:

$$\mathcal{Z}_t \subseteq \left\{([i_{p,t^{-1}}], t) \mid p \in \mathrm{Crit}(R)\right\},$$

and these two sets therefore coincide. Finally, by Theorem II of [28], for all p and for all sufficiently large t, the Jacobi operator of (K, t) is non-degenerate at $i_{p,t^{-1}}$ and:

$$\mathrm{sig}([i_{p,t^{-1}}], t) = (-1)^n \mathrm{sig}(R; p),$$

where $\mathrm{sig}(R; p)$ is the signature of the critical point p of R. Choosing t sufficiently large, and using classical Morse Theory, we thus obtain:

$$\mathrm{Deg}(\pi) = (-1)^n \sum_{p\in\mathrm{Crit}(R)} \mathrm{sig}(R; p) = (-1)^n \chi(M),$$

where $\chi(M)$ is the Euler Characteristic of M. Since $\chi(M) = 0$ when $n = \mathrm{Dim}(M) - 1$ is even the result follows. $\square$

4.1 Prescribed Mean Curvature.

Let $K := H$ be mean curvature. Thus:

$$K(\lambda_1, ..., \lambda_n) = H(\lambda_1, ..., \lambda_n) = \frac{1}{n}(\lambda_1 + ... + \lambda_n).$$

We say that a locally strictly convex immersion $i : \Sigma \to M$ is **pointwise 1/2-pinched** if and only if for every $p \in S$ if $0 < \lambda_1 \leqslant ... \leqslant \lambda_n$ are the principal curvatures of i at p, then:

$$\lambda_1 > \frac{1}{2n}(\lambda_1 + ... + \lambda_n) = \frac{1}{2}H(i)(p).$$

We denote by $\mathrm{Conv}_{1/2} \subseteq \mathrm{Conv} \subseteq C^\infty_{\mathrm{imm}}(\Sigma, M)$ the set of locally strictly convex, pointwise 1/2-pinched Alexandrov embeddings from Σ into M. This set is trivially open and invariant under the action of $\mathrm{Diff}^\infty(\Sigma)$ and we define $\mathcal{C}_{1/2} \subseteq \mathcal{C}$ to be its quotient under this group action:

$$\mathcal{C}_{1/2} = \mathrm{Conv}_{1/2}/\mathrm{Diff}^\infty(\Sigma).$$

Let R and ∇R be the Riemann curvature tensor of M and its covariant derivative respectively and let $\|R\|$ and $\|\nabla R\|$ be their respective operator norms. In other words:

$$\begin{aligned} \|R\| &= \text{Sup}\left\{R(e_1,e_2,e_3,e_4) \mid e_k \in TM \ \&\ \|e_k\| = 1\ \forall k\right\};\ \text{and} \\ \|\nabla R\| &= \text{Sup}\left\{(\nabla R)(e_1,e_2,e_3,e_4;e_5) \mid e_k \in TM \ \&\ \|e_k\| = 1\ \forall k\right\}. \end{aligned}$$

We define $H_0 := H_0(M) \geqslant 0$ by:

$$H_0 = 4\text{Max}(\|R\|^{1/2}, \|\nabla R\|^{1/3}).$$

We identify $\mathbb{R}$ with the constant functions in $C^\infty(M)$ and thus consider the interval $]H_0, +\infty[$ as a subset of $C^\infty(M)$. We define the neighbourhood $\mathcal{O}$ of $]H_0, +\infty[$ in $C^\infty(M)$ by:

$$\mathcal{O} = \left\{ f \ \middle| \ \begin{matrix} f & > H_0,\ \text{and} \\ \|\text{Hess}(f)\| & < 3nH_0^3/16(3n-2) \end{matrix} \right\}.$$

Define the solution space $\mathcal{Z}_{1/2} := \mathcal{Z}_{1/2}(K;\mathcal{O}) \subseteq \mathcal{C}_{1/2} \times \mathcal{O}$ by:

$$\mathcal{Z}_{1/2} = \left\{([i], f) \mid H(i) = f \circ i\right\}.$$

Let $\pi : \mathcal{Z}_{1/2} \to \mathcal{O}$ be the projection onto the second factor. We obtain:

Theorem 1.1

For generic $f \in \mathcal{O}$, the algebraic number of locally strictly convex, pointwise $1/2$-pinched, Alexandrov embedded hyperspheres in M of prescribed mean curvature equal to f is equal to $-\chi(M)$, where $\chi(M)$ is the Euler Characteristic of M.

The remainder of this subsection is devoted to proving Theorem 1.1. Bearing in mind Theorems 2.10 and 3.14, we acheive this by proving the properness of π. The main ingredient is the following highly technical lemma which is of independant interest. We provisionally return to the general framework where K is any elliptic curvature function. We recall that K is given by a smooth function acting on the space of positive definite, symmetric matrices which is invariant under the action of $O(n)$. Let DK be the derivative of K. Let A be a positive definite, symmetric matrix and let B be the gradient of K at A. In other words, for any other symmetric matrix M:

$$DK_A(M) = \langle B, M\rangle = \text{Tr}(BM).$$

It follows from the ellipticity of K that B is positive definite and from the $O(n)$-invariance of K that A and B commute (c.f. [31] for details). Let $\lambda_1, ..., \lambda_n$ and $\mu_1, ..., \mu_n$ be the eigenvalues of A and B respectively with respect to some orthonormal basis of eigenvectors. We define the K**-Laplacian** $\Delta^K : C^\infty(\Sigma) \to C^\infty(\Sigma)$ by:

$$\Delta^K \varphi = DK_A(\text{Hess}(\varphi)) = \sum_{k=1}^n \mu_k f_{;kk}.$$

In addition, we require a weak notion of differential inequalities: for continuous functions $\varphi, \psi \in C^0(\Sigma)$, we say that $\Delta^K\varphi \geqslant \psi$ in the **weak sense** if and only if for all $p \in \Sigma$ there exists a neighbourhood U of p in M and a smooth function $\xi \in C^\infty(U)$ such that $\varphi \geqslant \xi$, $\varphi(p) = \xi(p)$ and:

$$(\Delta^K\xi)(p) \geqslant \psi(p).$$

We now state the result:

Lemma 4.2

Choose $f \in C^\infty(M)$ and suppose that the K-curvature of Σ is prescribed at every point by f. Then, throughout Σ, in the weak sense:

$$\begin{aligned}\Delta^K\lambda_n \quad &\geqslant f_{;nn} - \lambda_n f_{;\nu} + \textstyle\sum_{i=1}^n \mu_i(\nabla R_{in\nu n;i} + \nabla R_{in\nu i;n})\\ &\quad - \textstyle\sum_{i=1}^n \mu_i(\lambda_n R_{i\nu\nu i} - \lambda_i R_{n\nu\nu n})\\ &\quad +2\textstyle\sum_{i=1}^n \mu_i(\lambda_n - \lambda_i)R_{inni} + \sum_{i=1}^n \mu_i\lambda_n\lambda_i(\lambda_n - \lambda_i),\end{aligned}$$

where ν denotes the direction normal to the immersion, R is the Riemann curvature tensor of M and ∇R is its covariant derivative.

In the case where $K = H$ is mean curvature, the same inequality holds in the reverse sense for the K-Laplacian of λ_1.

Remark: Bearing in mind the Maximum Principle, a priori bounds follow by determining under which conditions the term on the right hand side in the above expression is positive. The behaviour of this term depends on the structure of K, but 4 important features stand out:

(i) the first two terms, involving the derivatives of f, do not qualitatively affect the expression when f is C^2-close to a constant function;

(ii) the third and fourth terms do not qualitatively affect the expression when $\sum_{i=1}^n \mu_i$ is bounded, as in the case of mean curvature, but they do when it is unbounded;

(iii) when the ambient manifold is pointwise 1/2-pinched, the fifth term provides a strong positive contribution which may cancel the preceeding terms;

(iv) in all cases studied below, the sixth term, which is the only non-linear term in λ_n, provides a strong positive contribution which may also cancel the preceeding terms.

Proof: This follows by taking the second derivative of the shape operator of i and applying the appropriate commutation relations to the derivatives. See the proof of Proposition 6.6 of [27] for details. □

We now return to the specific case of this section, where $K = H$ is mean curvature. We first show that no element of $\mathcal{Z}_{1/2}$ is a multiple cover:

Proposition 4.3

If $([i], f) \in \mathcal{Z}_{1/2}$, then $[i]$ is not a multiple cover.

Remark: we use the mean curvature flow as studied by Huisken in [11]. The alert reader will notice that although the hypotheses of Huisken's result are stated in terms of the norm of ∇R, it is not explicit which norm he actually uses. However, closer examination of [11], in particular of lines 11 to 14 on p472 in the proof of Theorem 4.2, shows that Huisken's result is valid for the operator norm of ∇R, as defined above.

Proof: We denote $K = \|R\|$ and $L = \|\nabla R\|$. Observe that K is an upper bound for the absolute value of the sectional curvatures of M. Let A be the shape operator of i and let g be the metric induced over Σ by i. Since $([i], f) \in \mathcal{Z}_{1/2}$:

$$H := H(i) = f \circ i > H_0 = 4\mathrm{Max}(K^{1/2}, L^{1/3}).$$

Thus, since i is pointwise 1/2-pinched, we obtain:

$$\begin{aligned} HA_{ij} &\geqslant \tfrac{1}{2}H^2 g_{ij} \\ &\geqslant 4Kg_{ij} + \tfrac{1}{H}16Lg_{ij}. \end{aligned}$$

Denoting $\hat{H} = nH$, we obtain:

$$\hat{H}A_{ij} \geqslant nKg_{ij} + \frac{n^2}{\hat{H}}Lg_{ij}.$$

Observe that this is the condition given by Huisken in [11] for the existence of a unique, smooth mean curvature flow $i : \Sigma \times [0, T[\to M$ such that:

(i) $i_0 = i$; and

(ii) $(i_t)_{t \in [0,T[}$ is asymptotic to a family of round spheres about a point in M as $t \to T$.

We now deduce that i_0 is simple. Indeed, assume the contrary. There exists a non-trivial diffeomorphism $\alpha : \Sigma \to \Sigma$ such that $i \circ \alpha = i$. By uniqueness, for all $t \in [0, T[$, $i_t \circ \alpha = i_t$. However, by (ii), for t sufficiently close to T, i_t is embedded, and the only diffeomorphism α of Σ such that $i_t \circ \alpha = i_t$ is the identity. This is absurd, and the assertion follows. This completes the proof. $\square$

We now show that the hypotheses of Proposition 4.1 are satisfied:

Proposition 4.4

For any metric g on M, there exists $C > 0$ and $T > 0$ such that:

(i) $]T, +\infty[\subseteq \mathcal{O}$; and

(ii) if $t > T$ and $([i], t) \in \mathcal{Z}$, then $\mathrm{Diam}(\Sigma; i^*g) \leqslant Ct^{-1}$.

Proof: (i) is trivial for sufficiently large T. Choose $t > 0$ and choose $[i] \in \mathcal{C}_{1/2}$ such that $([i], f_t) \in \mathcal{Z}_{1/2}$. For sufficently large t, by the pointwise 1/2-pinched condition, the sectional curvature of i is bounded below by $t^2/4$. Its intrinsic diameter is therefore bounded above by $2/t$. (ii) follows and this completes the proof. $\square$

It remains to prove properness:

Proposition 4.5

The projection $\pi : \mathcal{Z} \to \mathcal{O}$ is a proper mapping.

Proof: Let $(f_m)_{m\in\mathbb{N}}, f_0 \in \mathcal{O}$ be such that $(f_m)_{m\in\mathbb{N}}$ converges to f_0 in the C^∞ sense, and let $([i_m])_{m\in\mathbb{N}} \in \mathcal{C}_{1/2}$ be such that, for all m, the mean curvature of i_m is prescribed at every point by the function f_m. We first show that there exists $i_0 : \Sigma \to M$ towards which $(i_m)_{n\in\mathbb{N}}$ subconverges in the C^∞ sense modulo reparametrisation. Indeed, choose $\epsilon > 0$ such that:

$$f_0^2/4 > \|R\| + 2\epsilon.$$

Choose $m \in \mathbb{N}$ sufficiently large such that:

$$f_m^2/4 > \|R\| + \epsilon.$$

Choose $p \in \Sigma$ and let $0 < \lambda_1 \leqslant ... \leqslant \lambda_n$ be the principal curvatures of i_m at p. By definition:

$$\lambda_1 \geqslant f_m/2.$$

Thus, for all i, j:

$$\lambda_i\lambda_j \geqslant \lambda_1^2 \geqslant f_m^2/4 > \|R\| + \epsilon.$$

It follows that the sectional curvature of i_m is bounded below by ϵ, and so by classical comparison theory its intrinsic diameter is bounded above by $\epsilon^{-1/2}$. By strict convexity, the norm of the shape operator is bounded above by the mean curvature and uniform bounds for this norm follow immediately. We thus conclude by the Arzela-Ascoli Theorem for immersed hypersurfaces (c.f. [26]) and elliptic regularity that there exists a smooth, locally convex immersion $i_0 : \Sigma \to M$ towards which $(i_m)_{m\in\mathbb{N}}$ subconverges in the C^∞ sense modulo reparametrisation. This proves the assertion.

It remains to show that $i := i_0 \in \mathrm{Conv}_{1/2}$. Suppose the contrary. By rescaling the metric of the ambiant manifold by a constant factor $\lambda = (H_0/4)^2$, we may suppose that:

$$\mathrm{Max}(\|R\|^{1/2}, \|\nabla R\|^{1/3}) = 1.$$

We henceforth work with respect to the rescaled metric. Observe that this has the effect of rescaling the mean curvature of i by $1/\lambda = 4/H_0$. In particular, the mean curvature of i is now prescribed by $\hat{f} := 4f/H_0$. Trivially:

$$\hat{f} > 4, \qquad \|\mathrm{Hess}(\hat{f})\| < 12n/(3n-2).$$

Let A be the shape operator of i, let $0 < \lambda_1 \leqslant ... \leqslant \lambda_n$ be the principal curvatures of i. Let $p \in S$ be a point where $\lambda_1/\hat{f}$ is minimised. Since i is a limit point of $\mathrm{Conv}_{1/2}$, λ_1 is equal to $\hat{f}/2$ at p. Assume first that λ_1 is smooth near p, and consider the Laplacian of λ_1/f at p:

$$\Delta(\lambda_1/\hat{f}) = \frac{1}{\hat{f}^2}(\hat{f}\Delta\lambda_1 - \lambda_1\Delta\hat{f}) - \frac{2}{\hat{f}}\sum_{k=1}^{n}(\lambda_1/\hat{f})_{;k}\hat{f}_{;k}.$$

Since $\lambda_1/\hat{f}$ is minimised at p, $\nabla(\lambda_1/\hat{f}) = 0$, and so:

$$\Delta(\lambda_1/\hat{f}) = \frac{1}{\hat{f}^2}(\hat{f}\Delta\lambda_1 - \lambda_1\Delta\hat{f}).$$

The Hessian of the restriction of $\hat{f}$ to $i(\Sigma)$ is given by:

$$\mathrm{Hess}^{\Sigma}(f)_{ij} = \mathrm{Hess}(f)_{ij} - f_{;\nu}A_{ij},$$

where the index ν denotes the unit direction normal to the immersion i at p. The Laplacian at p of the restriction of $\hat{f}$ to $i(\Sigma)$ is thus given by:

$$\Delta\hat{f} = \sum_{k=1}^{n}(\hat{f}_{;kk} - \lambda_k\hat{f}_{;\nu}) = \sum_{k=1}^{n}\hat{f}_{;kk} - nH\hat{f}_{;\nu} = \sum_{k=1}^{n}\hat{f}_{;kk} - n\hat{f}\hat{f}_{;\nu},$$

Since $DK := B = (1/n)\mathrm{Id}$, $\mu_1 = ... = \mu_n = 1/n$ and $\Delta^K = (1/n)\Delta$. Thus, combining the above relations with Lemma 4.2, we obtain:

$$\begin{array}{rl}\hat{f}^2\Delta(\lambda_1/\hat{f}) & \leqslant n\hat{f}\hat{f}_{;11} - \lambda_1\sum_{k=1}^n\hat{f}_{;kk} \\ & +\hat{f}\sum_{k=1}^n(\nabla R_{k1\nu1;i} + \nabla R_{k1\nu k;1}) \\ & -\hat{f}\sum_{k=1}^n(\lambda_1 R_{k\nu\nu k} - \lambda_k R_{1\nu\nu1}) \\ & +2\hat{f}\sum_{k=1}^n R_{k11k}(\lambda_1 - \lambda_k) + \hat{f}\sum_{k=1}^n\lambda_1\lambda_k(\lambda_1 - \lambda_k),\end{array}$$

Since $\lambda_1 = \hat{f}/2$, and bearing in mind the bound on $\mathrm{Hess}(\hat{f})$, we obtain:

$$\begin{array}{rl}n\hat{f}\hat{f}_{;11} - \lambda_1\sum_{k=1}^n\hat{f}_{;kk} & = \frac{(2n-1)\hat{f}}{2}\hat{f}_{;11} - \frac{\hat{f}}{2}\sum_{k=2}^n\hat{f}_{;kk} \\ & \leqslant \frac{(3n-2)\hat{f}}{2}\|\mathrm{Hess}(\hat{f})\| \\ & < 6n\hat{f}.\end{array}$$

Since $\lambda_k \geqslant 0$ for all k and since $\|R\| \leqslant 1$:

$$\begin{array}{rl}\hat{f}\sum_{k=1}^n(\lambda_k R_{1\nu\nu1} - \lambda_1 R_{k\nu\nu k}) & \leqslant \hat{f}\sum_{k=1}^n(\lambda_1 + \lambda_i) \\ & = \hat{f}(n\lambda_1 + nH) \\ & = (3n/2)\hat{f}^2.\end{array}$$

Since $\lambda_i - \lambda_1 \geqslant 0$ for all i and since $\|R\| \leqslant 1$:

$$\begin{array}{rl}2\hat{f}\sum_{k=1}^n R_{k11k}(\lambda_1 - \lambda_k) & \leqslant 2\hat{f}\sum_{k=1}^n(\lambda_k - \lambda_1) \\ & = 2n\hat{f}(H - \lambda_1) \\ & = n\hat{f}^2.\end{array}$$

Using Lagrange Multipliers and convexity, since $\lambda_1 = \hat{f}/2$, we readily show that:

$$\sum_{k=2}^n\lambda_k(\lambda_k - \lambda_1) \geqslant \frac{(2n-1)(3n-2)}{4(n-1)}\hat{f}^2 \geqslant \frac{n\hat{f}^2}{2}.$$

Combining these relations we obtain:

$$(\hat{f}/n)\Delta\lambda_1 < 6 + \frac{5}{2}\hat{f} - \frac{1}{4}\hat{f}^3.$$

Since $\hat{f} > 4$, $\Delta\lambda_1 < 0$ at this point, which is absurd by the Maximum Principal, and it follows that $i_0 \in \mathrm{Conv}_{1/2}$. The case where λ_1 is not smooth follows similarly, since Lemma 4.2 is valid in the weak sense even when the function φ is only continuous. This completes the proof. □

We now prove Theorem 1.1:

Proof of Theorem 1.1: By Proposition 4.3, $\mathcal{Z}_{1/2} \subseteq \mathcal{C}_{\mathrm{simp},1/2} \times \mathcal{O}$, where $\mathcal{C}_{\mathrm{simp},1/2} = \mathcal{C}_{1/2} \cap \mathcal{I}$ consists of those immersions in $\mathcal{C}_{1/2}$ which are simple. By Proposition 4.5, the projection $\pi : \mathcal{Z}_{1/2} \to \mathcal{O}$ is proper and so, by Theorems 2.10 and 3.14, $\mathrm{Deg}(\pi)$ is well defined. By Proposition 4.4, the hypotheses of Proposition 4.1 are satisfied, and it follows that $\mathrm{Deg}(\pi) = -\chi(M)$, where $\chi(M)$ is the Euler characteristic of M. This completes the proof. □

4.2 Extrinsic Curvature of Hyperspheres.

Let K be (the n'th root of) the extrinsic curvature. Thus:

$$K(\lambda_1, ..., \lambda_n) = (\lambda_1 \cdot ... \cdot \lambda_n)^{1/n}.$$

We now restrict the geometry of the ambient space. We say that M is 1/4**-pinched** if and only if:

$$\sigma_{\mathrm{Max}}(M) < 4\sigma_{\mathrm{Min}}(M),$$

where $\sigma_{\mathrm{Max}}(M)$ and $\sigma_{\mathrm{Min}}(M)$ are the maximum and minimum values respectively of the scalar curvatures of planes tangent to M. We say that M is pointwise 1/2**-pinched** if and only if:

$$\sigma_{\mathrm{Max}}(M;p) < 2\sigma_{\mathrm{Min}}(M;p),$$

for all $p \in M$ where $\sigma_{\mathrm{Max}}(M;p)$ and $\sigma_{\mathrm{Min}}(M;p)$ are the maximum and minimum values respectively of the scalar curvatures of planes tangent to M at p. We now suppose that M is both 1/4-pinched and pointwise 1/2-pinched. The 1/4-pinched condition is not strictly speaking necessary (c.f. our subsequent paper [21]). It is imposed merely in order to exclude multiply covered immersed immersions. Under these hypotheses, M is diffeomorphic to the standard sphere. Indeed, this follows immediately from the 1/4-pinched condition, but also follows from the pointwise 1/2-pinched condition even when the 1/4-pinched condition is dropped (we refer the reader to [2] for a discussion of these facts).

We denote by $\mathrm{Conv}_{\mathrm{emb}} \subseteq \mathrm{Conv} \subseteq C^\infty_{\mathrm{imm}}(\Sigma, M)$ the set of strictly convex embeddings from Σ into M which bound an open set. This set is trivially open and $\mathrm{Diff}^\infty(\Sigma)$-invariant, and we define $\mathcal{C}_{\mathrm{emb}} \subseteq \mathcal{C}$ to be its quotient under this group action:

$$\mathcal{C}_{\mathrm{emb}} = \mathrm{Conv}_{\mathrm{emb}}/\mathrm{Diff}^\infty(\Sigma).$$

Let $\mathcal{O} \subseteq C^\infty(M,]0,\infty[)$ be the set of all smooth, strictly positive functions over M and define the solution space $\mathcal{Z}_{\mathrm{emb}} := \mathcal{Z}_{\mathrm{emb}}(K;O) \subseteq \mathcal{C}_{\mathrm{emb}} \times \mathcal{O}$ by:

$$\mathcal{Z}_{\mathrm{emb}} = \{([i], f) \mid K([i]) = f \circ i\}.$$

Let $\pi : \mathcal{Z}_{\mathrm{emb}} \to \mathcal{O}$ be projection onto the second factor. We obtain:

Theorem 1.2

Suppose that M is both $1/4$-pinched and pointwise $1/2$-pinched. Then, for generic $f \in \mathcal{O}$, the algebraic number of locally strictly convex embedded hyperspheres of prescribed extrinsic curvature equal to f is equal to $-\chi(M)$ where $\chi(M)$ is the Euler Characteristic of M.

The remainder of this subsection is devoted to proving Theorem 1.2. We first bound the shape operator of immersions in $\mathcal{Z}_{\mathrm{emb}}$:

Proposition 4.6

Choose $([i], f) \in \mathcal{Z}_{\mathrm{emb}}$. There exists $\Lambda := \Lambda(s,t)$, which only depends on M such that if:

$$\|D^2 f\|/f < \inf_{p\in M}(2\sigma_{\mathrm{Min}}(p) - \sigma_{\mathrm{Max}}(p))/2,$$

then, if A_i is the shape operator of i, then, throughout Σ:

$$\|A_i\| \leqslant f\Lambda(f^{-1}, \|D^2 f\| f^{-2}).$$

Moreover $\Lambda(s,t)$ tends to 1 as (s,t) tends to 0.

Remark: Observe that the hypotheses of this proposition are trivially satisfied when f is positive and constant.

Proof: Denote $\delta = (2\sigma_{\mathrm{Min}} - \sigma_{\mathrm{Max}})$. Choose $\epsilon > 0$ such that $\sigma_{\mathrm{Max}} + \epsilon < 2\sigma_{\mathrm{Min}}$. We define $\Lambda(s,t)$ by:

$$\Lambda(s,t) = \mathrm{Max}(4\|\nabla R\| s/\delta, 1 + nt + n\|R\| s)$$

Let $p \in \Sigma$ be the point maximising $\|A\|/f$. Let $L = \|A\|/f \geqslant 1$ be the value that this function takes at p. We claim that if $\|D^2 f\|/f < \delta/2$, then $L \leqslant \Lambda(f^{-1}, \|D^2 f\| f^{-2})$. Indeed, assume the contrary. Let $0 < \lambda_1 \leqslant ... \leqslant \lambda_n = Lf$ be the eigenvalues of A_i at p. Assume first that λ_n is smooth near p, and consider the K-Laplacian of λ_n/f at p:

$$\Delta^K(\lambda_n/f) = \frac{1}{f^2}(f\Delta^K\lambda_n - \lambda_n\Delta^K f) - \frac{2}{f}\sum_{k=1}^n \mu_k(\lambda_n/f)_{;k} f_{;k}.$$

Since λ_n/f is maximised at p, $\nabla(\lambda_n/f) = 0$, and so:

$$\Delta^K(\lambda_n/f) = \frac{1}{f^2}(f\Delta^K\lambda_n - \lambda_n\Delta^K f).$$

The Hessian of the restriction of f to Σ is given by:

$$\mathrm{Hess}^\Sigma(f) = \mathrm{Hess}(f) - f_{;\nu}A_i,$$

where the subscript ν denotes the direction normal to the immersion R. In this case:

$$B := DK = \frac{1}{n}KA^{-1}.$$

Thus, for all $1 \leqslant k \leqslant n$, $\mu_k = f/(n\lambda_i)$. The K-Laplacian of the restriction of f to Σ is thus given by:

$$\Delta^K f = \sum_{k=1}^{n} \frac{1}{n\lambda_k} f f_{;kk} - f f_{;\nu}.$$

Thus, by Lemma 4.2:

$$\begin{array}{rl} f^2\Delta^K(\lambda_n/f) & \geqslant f f_{;nn} - \lambda_n \sum_{k=1}^n \frac{1}{n\lambda_k} f f_{;kk} \\ & \quad +(f^2/n)\sum_{k=1}^n \lambda_k^{-1}(\nabla R_{kn\nu n;k} + \nabla R_{kn\nu k;n}) \\ & \quad -(f^2/n)\sum_{k=1}^n \lambda_k^{-1}(\lambda_n R_{k\nu\nu k} - \lambda_k R_{n\nu\nu n}) \\ & \quad +2(f^2/n)\sum_{k=1}^n \lambda_k^{-1} R_{knnk}(\lambda_n - \lambda_k) + (f^2/n)\sum_{k=1}^n \lambda_n(\lambda_n - \lambda_k). \end{array}$$

Then:

$$\begin{array}{rl} f^2\Delta^K(\lambda_n/f) & \geqslant -f\|D^2 f\| - L\|D^2 f\|(f^2/n)\sum_{k=1}^n \frac{1}{\lambda_k} - 2(f^2/n)\|\nabla R\|\sum_{k=1}^n \lambda_k^{-1} \\ & \quad -2f^2\|R\| + \delta L(f^3/n)\sum_{k=1}^n \lambda_k^{-1} + L(f^3/n)\sum_{k=1}^n(\lambda_n - \lambda_k). \end{array}$$

Thus, since $f > 2\|D^2 f\|/\delta$ and $L > 4\|\nabla R\|/\delta f$, we obtain:

$$f^2\Delta^K(\lambda_n/f) \geqslant -f\|D^2 f\| - 2f^2\|R\| + L(f^3/n)\sum_{k=1}^n(\lambda_n - \lambda_k).$$

Since $\lambda_1 < f$ and bearing in mind that $L \geqslant 1$, this yields, in particular

$$\begin{array}{rl} f^2\Delta^K(\lambda_n/f) & \geqslant -(f\|D^2 f\| + 2f^2\|R\|) + L(L-1)(f^3/n) \\ & \geqslant -(f\|D^2 f\| + 2f^2\|R\|) + (L-1)(f^3/n). \end{array}$$

Thus, if $L > (1 + n\|D^2 f\|/f^2 + 2n\|R\|f^{-1})$, then:

$$f^2\Delta^K(\lambda_n/f) > 0.$$

This is absurd by the maximum principal, and the result follows in the case where λ_n is smooth near p. The case where λ_n is not smooth near p follows similarly, since Lemma 4.2 remains valid in the weak sense even when the function φ is only continuous. This completes the proof. □

We refine Proposition 4.6 for small f to obtain:

Proposition 4.7

Choose $([i], f) \in \mathcal{Z}_{\text{emb}}$. There exists $\Lambda := \Lambda(r, s, t)$, which only depends on M such that if A_i is the shape operator of i, then, throughout Σ:

$$\|A_i\| \leqslant \Lambda(\|f\|, \|Df\|, \|\text{Hess}(f)\|).$$

Proof: Let $p \in \Sigma$ be the point maximising $\|A\|$. Let $0 < \lambda_1 \leqslant ... \leqslant \lambda_n = \|A\|$ be the eigenvalues of A at p. Suppose first that λ_n is smooth near p. Thus, by Lemma 4.2, there exists $C > 0$ which only depends on M and the derivatives of f up to order 2 such that:

$$\Delta^K \lambda_n \geqslant -C(1 + \lambda_n + \sum_{k=1}^n \mu_k) - \sum_{k=1}^n \mu_k(\lambda_n R_{k\nu\nu k} - \lambda_k R_{n\nu\nu n}) + 2\sum_{k=1}^n \mu_k(\lambda_n - \lambda_k) R_{knnk}.$$

As before, for all k:

$$\mu_k = \frac{f}{n\lambda_k}.$$

Thus, increasing C if necessary, and denoting $\delta = (2\sigma_{\mathrm{Min}}(p) - \sigma_{\mathrm{Max}}(p))$, we obtain:

$$\Delta^K \lambda_n \geqslant -C(1 + \lambda_n + \sum_{k=1}^n \lambda_k^{-1}) + \frac{f\delta\lambda_n}{n}\sum_{k=1}^n \lambda_k^{-1}.$$

Since:

$$(\lambda_1 \cdot ... \cdot \lambda_n)^{1/n} = f,$$

the sum $\sum_{k=1}^n \lambda_k^{-1}$ tends to infinity as λ_n tends to infinity. There therefore exists $B > 0$ which only depends on M and the derivatives of f up to order 2 such that if $\lambda_n \geqslant B$, then $\Delta^K \lambda_n > 0$. In particular, by the maximum principal $\lambda_n < B$. The result follows in the case where λ_n is smooth near p. The general case follows in the same manner since Lemma 4.2 remains valid in the weak sense even when the function φ is only continuous. $\square$

We now show that the hypotheses of Proposition 4.1 are satisfied:

Proposition 4.8

Let g be the metric on M. For all g' sufficiently close to g, there exists $C > 0$ and $T > 0$ such that:

(i) $]-T, +\infty[\subseteq \mathcal{O}$; and

(ii) if $t > T$ and $([i], t) \in \mathcal{Z}$, then $\mathrm{Diam}(\Sigma; i^*g) \leqslant Ct^{-1}$.

Proof: (i) is trivial. Choose $t > 0$ and choose $[i] \in \mathcal{C}_{\mathrm{emb}}$ such that $([i], t) \in \mathcal{Z}_{\mathrm{emb}}$. Let A_i be the shape operator of i, and let $0 < \lambda_1 \leqslant ... \leqslant \lambda_n$ be its eigenvalues. Choose $\Lambda > 1$. By Proposition 4.6, there exists $T > 0$ such that for $t > T$:

$$\lambda_n = \|A_i\| < \Lambda f.$$

Since $K(i) = \mathrm{Det}(A_i)^{1/n} = f$, this yields:

$$\lambda_1 > f\Lambda^{1-n}.$$

And so, for all $1 \leqslant i, j \leqslant n$:

$$\lambda_i\lambda_j > f^2\Lambda^{2(1-n)}.$$

Increasing T if necessary, we deduce that the sectional curvature of i^*g is bounded below by $t^2/4\Lambda^{2(1-n)}$ and its intrinsic diameter is therefore bounded above by $2\Lambda^{1-n}/t$. (ii) now follows with $C = 2\Lambda^{1-n}$, and this completes the proof. $\square$

Properness also follows readily:

Proposition 4.9

The projection $\pi : \mathcal{Z}_{\mathrm{emb}} \to \mathcal{O}$ is a proper mapping.

Proof: Let $(f_m)_{m\in\mathbb{N}}, f_0 \in \mathcal{O}$ be such that $(f_m)_{m\in\mathbb{N}}$ converges to f_0 in the C^∞ sense, and let $([i_m])_{m\in\mathbb{N}} \in \mathcal{C}_{\text{emb}}$ be such that, for all m, i_m has prescribed extrinsic curvature equal to f_m. Proposition 4.7 yields a uniform upper bound for the norms of the shape operators of the $(i_m)_{m\in\mathbb{N}}$. Since, for all m, the determinant of the shape operator of i_m is equal to $(f_m)^n$, we also obtain uniform lower bounds for the principal curvatures of the $(i_m)_{n\in\mathbb{N}}$. Thus, since the ambient space has positive sectional curvature, we obtain lower bounds for the sectional curvatures of the metrics generated over Σ by the $(i_m)_{m\in\mathbb{N}}$ and this in turn yields uniform upper bounds for the intrinsic diameters of the $(i_m)_{m\in\mathbb{N}}$. It now follows by the Arzela-Ascoli Theorem for immersed hypersurfaces (c.f. [26]) and elliptic regularity that there exists a locally strictly convex immersion $i_0 \in \text{Conv}_{\text{emb}}$ towards which $(i_m)_{m\in\mathbb{N}}$ subconverges. Finally, since M is 1/4-pinched it follows by [7] that i_0 is embedded, and this completes the proof. $\square$

We now prove Theorem 1.2:

Proof of Theorem 1.2: Since all immersions in $\mathcal{Z}_{\text{emb}}$ are embedded, they are trivially simple. By Proposition 4.9, the projection $\pi : \mathcal{Z}_{1/2} \to \mathcal{O}$ is proper and so, by Theorems 2.10 and 3.14, $\text{Deg}(\pi)$ is well defined. By Proposition 4.8, the hypotheses of Proposition 4.1 are satisfied, and it follows that $\text{Deg}(\pi) = -\chi(M)$, where $\chi(M)$ is the Euler characteristic of M. This completes the proof. $\square$

4.3 Special Lagrangian Curvature.

Let K be special Lagrangian curvature (c.f. [29]). Thus:

$$K(\lambda_1, ..., \lambda_n) = R_\theta(\lambda_1, ..., \lambda_n).$$

We recall that special Lagrangian curvature depends on an angle parameter $\theta \in [0, n\pi/2[$. Moreover, when $\theta \in [(n-1)\pi/2, n\pi/2[$, it is convex and possesses strong regularity properties described in detail in [29]. Of particular interest is the case when $\theta = (n-1)\pi/2$, since it is here that the special Lagrangian curvature has the simplest expression. For example, when $n = 3$ and $\theta = \pi$:

$$R_\pi = (K/H)^{1/2} = (\lambda_1\lambda_2\lambda_3/(\lambda_1 + \lambda_2 + \lambda_3))^{1/2}.$$

We now suppose only that M has strictly positive sectional curvature and is 1/4-pinched. As before, the 1/4-pinched condition is not strictly speaking necessary (c.f. our subsequent paper [21]) and is merely imposed in order to exclude multiply covered immersions.

As before, let $\text{Conv}_{\text{emb}} \subseteq \text{Conv}$ be the set of strictly convex embeddings of Σ into M which bound an open set, let $\mathcal{C}_{\text{emb}} \subseteq \mathcal{C}$ be its quotient under the action of $\text{Diff}^\infty(M)$ and let $\mathcal{O} \subseteq C^\infty(M,]0, \infty[)$ be the set of all smooth, strictly positive functions over M and define the solution space $\mathcal{Z}_{\text{emb}} := \mathcal{Z}_{\text{emb}}(K; O) \subseteq \mathcal{C}_{\text{emb}} \times \mathcal{O}$ by:

$$\mathcal{Z}_{\text{emb}} = \{([i], f) \mid K([i]) = f \circ i\}.$$

Let $\pi : \mathcal{Z}_{\text{emb}} \to \mathcal{O}$ be projection onto the second factor. We obtain:

Theorem 1.3

Suppose that $n \geqslant 3$ and M is $1/4$-pinched. Then, for generic $f \in \mathcal{O}$, the algebraic number of locally strictly convex, embedded hypersurfaces of prescribed special Lagrangian curvature equal to f is equal to $-\chi(M)$, where $\chi(M)$ is the Euler Characteristic of M.

The remainder of this subsection is devoted to proving Theorem 1.3.

Proposition 4.10

Let g be the metric on M. Suppose that $\theta > (n-1)\pi/2$. Then, for all g' sufficiently close to g, there exists $C > 0$ and $T > 0$ such that:

(i) $]T, +\infty[\subseteq \mathcal{O}$; and

(ii) if $t > T$ and $([i], t) \in \mathcal{Z}$, then $\mathrm{Diam}(\Sigma; i^*g) \leqslant Ct^{-1}$.

Proof: (i) is trivial. Since $\theta > (n-1)\pi/2$, by Lemma 2.2 of [30], there exists $\epsilon > 0$ such that, for all $0 < \lambda_1 \leqslant ... \leqslant \lambda_n$:

$$\lambda_1 \geqslant \epsilon R_\theta(\lambda_1, ..., \lambda_n).$$

Thus, for sufficiently large t, the sectional curvature of i^*g is everywhere bounded below by $t^2/4$, and so its intrinsic diameter is bounded above by $2/t$. (ii) follows and this completes the proof. $\square$

We now prove properness:

Lemma 4.11

If $n \geqslant 3$ and $\theta \geqslant (n-1)\pi/2$, then the projection $\pi : \mathcal{Z} \to \mathcal{O}$ is a proper mapping.

Proof: Let $(f_m)_{m\in\mathbb{N}}, f_0 \in \Omega$ be such that $(f_m)_{m\in\mathbb{N}}$ converges to f_0. For all m, let $[i_m] \in \mathcal{C}_{\mathrm{emb}}$ have prescribed R_θ-curvature equal to f_m. Since the sectional curvature of M is bounded below by ϵ, say, and since i_m is locally strictly convex, the sectional curvature of i_m^*g is also bounded below by ϵ, and its intrinsic diameter is therefore bounded above by $\epsilon^{-1/2}$. Let $UM \subseteq TM$ be the bundle of unit vectors over M. For all m, let $\mathsf{N}_m : \Sigma \to UM$ be the outward pointing unit normal vector field over i_m, and let A_m be its shape operator. For all $m \in \mathbb{N}$, let $p_m \in \Sigma$ be the point maximising the norm of A_m. Consider the sequence $(\Sigma, \mathsf{N}_m, p_m)_{m\in\mathbb{N}}$ of complete, pointed immersed submanifolds of UM. By Theorem 1.4 of [29] there exists a complete, pointed immersed submanifold, (Σ_0, j_0, p_0) of UM towards which $(\Sigma, \mathsf{N}_m, p_m)_{m\in\mathbb{N}}$ subconverges in the C^∞-Cheeger/Gromov sense.

Suppose first that $\theta > (n-1)\pi/2$. Let $\pi : UM \to M$ be the canonical projection. By Theorem 1.3 of [29], $\pi \circ j_0$ is an immersion. In particular, $(A_m(p_m))_{m\in\mathbb{N}}$ converges towards the shape operator of $(\pi \circ j_0)(p_0)$ and is therefore bounded. Now suppose that $\theta = (n-1)\pi/2$. If $\pi \circ j_0$ is an immersion, then we conclude as before that $(A_m(p_m))_{m\in\mathbb{N}}$ is bounded. Otherwise, by Theorem 1.3 of [29], there exists a complete geodesic $\Gamma \subseteq M$ such that (Σ_0, j_0, p_0) is a covering of the sphere bundle of unit, normal vectors over Γ. Since $n \geqslant 3$, the fibres are spheres of dimension at least 2 and are therefore simply connected.

Thus, since the diameter of $(\Sigma, \mathsf{N}_m, p_m)_{m\in\mathbb{N}}$ is uniformly bounded above, Γ is closed and this covering is finite. In particular Σ_0 is compact and diffemorphic to $S^1\times S^{n-1}$. However, since the limit is compact, it follows from the definition of Cheeger/Gromov convergence that Σ_0 is also diffeomorphic to $\Sigma = S^n$. This is absurd, and we thus exclude the possibility that $(\pi\circ j_0)$ is not an immersion.

We conclude that the norms of the shape operators of the $(i_m)_{m\in\mathbb{N}}$ are uniformly bounded above, and since their intrinsic diameters are also uniformly bounded above, the Arzela-Ascoli Theorem for immersed submanifolds (c.f. [26]), implies the existence of an immersion $i_0 : S \to M$ towards which $(i_m)_{m\in\mathbb{N}}$ subconverges after reparametrisation. Trivially i_0 is locally strictly convex and of prescribed R_θ-curvature equal to f_0. Finally, since M is 1/4-pinched, it follows from [7] that i_0 is embedded and bounds an open set, and this completes the proof. □

We now prove Theorem 1.3:

Proof of Theorem 1.3: Since all immersions in $\mathcal{Z}_{\mathrm{emb}}$ are embedded, they are trivially simple. By Proposition 4.11, the projection $\pi : \mathcal{Z}_{\mathrm{emb}} \to \mathcal{O}$ is proper and so, by Theorems 2.10 and 3.14, $\mathrm{Deg}(\pi)$ is well defined. By Proposition 4.10, the hypotheses of Proposition 4.1 are satisfied, and it follows that $\mathrm{Deg}(\pi) = -\chi(M)$, where $\chi(M)$ is the Euler characteristic of M. This completes the proof. □

5 - Extrinsic Curvature.

5.1 The Framework.

We now consider locally strictly convex, 2-dimensional spheres immersed inside compact, 3-dimensional manifolds. Let $M := M^3$ be a compact, orientable, 3-dimensional Riemannian manifold. Let $\Sigma := S^2$ be the 2-dimensional sphere. Let K denote (the square root of) the extrinsic curvature. Thus:

$$K(\lambda_1, \lambda_2) = K_e(\lambda_1, \lambda_2) := (\lambda_1\lambda_2)^{1/2}.$$

Let T be the trace-free Ricci curvature tensor of M. Define $\sigma^-_{\mathrm{Min}}(M)$ by:

$$\sigma^-_{\mathrm{Min}} = \mathop{\mathrm{Inf}}_{P\subseteq TM} \mathrm{Min}(\sigma(P), 0),$$

where P ranges over all tangent planes in TM and $\sigma(P)$ is the sectional curvature of P. We define K_0 by:

$$K_0^2 = \frac{1}{2}\left(\left|\sigma^-_{\mathrm{Min}}\right| + \sqrt{\left|\sigma^-_{\mathrm{Min}}\right|^2 + \|T\|_O^2}\right).$$

where $\|T\|_O$ is the operator norm of T when viewed as an endomorphism of TM.

Remark: Observe that $K_0^2 \geqslant \left|\sigma^-_{\mathrm{Min}}\right|$. Moreover, when M has non-negative sectional curvature, this simplifies to $K_0^2 = \|T\|_O/2$. In addition $K_0 = 0$ if and only if M is a space form of non-negative curvature.

Let Conv $:= \mathrm{Conv}(\Sigma, M) \subseteq C^\infty_{\mathrm{imm}}(\Sigma, M)$ be the subset consisting of locally strictly convex immersions. this set is trivially open and $\mathrm{Diff}^\infty(\Sigma)$-invariant, and we define $\mathcal{C} \subseteq \mathcal{I}$ to be its quotient under this group action:

$$\mathcal{C} = \mathrm{Conv}/\mathrm{Diff}^\infty(\Sigma).$$

We define $\mathcal{O} \subseteq C^\infty(M,]0, \infty[)$ by:

$$\mathcal{O} = \left\{ f \;\middle|\; \begin{array}{l} f > K_0;\ \text{and} \\ \|df\|^2 < 4\sqrt{2f^4 - 2\left|\sigma^-_{\mathrm{Min}}\right| f^2 + \|T\|^2_O/2} - \|T\|_O. \end{array} \right\}.$$

Remark: Identifying $\mathbb{R}$ with the constant functions in $C^\infty(M)$, we see that $]K_0, +\infty[\subseteq \mathcal{O}$. Observe moreover that the quantity on the right hand side of the inequality for $\|df\|^2$ grows quadratically with the size of f as f tends to $+\infty$. In addition, it follows from the definition of K_0 that this quantity is always positive for $f > K_0$.

We define the solution space $\mathcal{Z} \subseteq \mathcal{C} \times \mathcal{O}$ by:

$$\mathcal{Z} = \left\{ ([i], f) \mid K([i]) = f \circ i \right\}.$$

Let $\pi : \mathcal{Z} \to \mathcal{O}$ be the projection onto the second factor. We obtain:

Theorem 1.4

For generic $f \in \mathcal{O}$, the algebraic number of locally strictly convex, immersed spheres in M of prescribed extrinsic curvature equal to f is equal to 0.

Remark: This theorem requires a much deeper compactness result than that underlying the previous three theorems.

The remainder of this section is devoted to proving Theorem 1.4.

5.2 Basic Relations in Riemannian Manifolds.

We consider briefly a more general framework. Let $\Sigma := \Sigma^n$ be an n-dimensional Riemannian manifold. Let g, ∇, R, Ric be the metric, the Levi-Civita covariant derivative and Riemannian and Ricci curvatures of Σ respectively. Let p be a point in Σ, and let $(e_1, ..., e_n)$ be an orthonormal basis for $T_p\Sigma$. Here and in the sequel, we adopt the convention:

$$R_{e_ie_j}e_k = R_{ijk}{}^l e_l, \qquad \mathrm{Ric}_{ik} = -\frac{1}{(n-1)}R_{ijk}{}^j, \qquad \mathrm{Scal} = \frac{1}{n}\mathrm{Ric}_i{}^i.$$

Remark: With this convention, the Ricci and scalar curvatures of the unit sphere in $(n+1)$-dimensional Euclidean space are equal to δ_{ij} and 1 respectively.

We use here a semi-colon to denote covariant differentiation with respect to ∇. Thus, for example, if $T = T_{ij}$ is a $(0,2)$-tensor, then:

$$T_{ij;k} = (\nabla T)(e_i, e_j; e_k) = (\nabla_{e_k}T)(e_i, e_j).$$

Let ∇' be another covariant derivative over Σ. We define the **relative Christophel tensor** Ω of ∇' with respect to ∇ by:

$$\Omega^k{}_{ij} e_k = \nabla'_{e_i} e_j - \nabla_{e_i} e_j.$$

Let R' be the Riemannian curvature tensor of ∇'.

Proposition 5.1

R' is given by:

$$R'_{ijk}{}^l = R_{ijk}{}^l + \Omega^l{}_{jk;i} - \Omega^l{}_{ik;j} + \Omega^l{}_{im}\Omega^m{}_{jk} - \Omega^l{}_{jm}\Omega^m{}_{ik}.$$

Proof: This is a direct calculation. □

Let A be a positive definite, symmetric matrix over Σ. Let g^A be the metric over Σ defined by A. Thus, for all $X, Y \in T\Sigma$:

$$\langle X, Y\rangle^A = g^A(X, Y) = \langle X, AY\rangle.$$

Let ∇^A be the Levi-Civita covariant derivative of g^A.

Proposition 5.2

The relative Christophel tensor of ∇^A with respect to ∇ is given by:

$$\Omega^k{}_{ij} = \frac{1}{2} B^{kp}(A_{pi;j} + A_{pj;i} - A_{ij;p}),$$

where B^{ij} is the inverse matrix of A_{ij}.

Proof: This follows from the Koszul formula. □

5.3 Basic Relations in Hypersurfaces.

Let $M := M^{n+1}$ be an $(n+1)$-dimensional Riemannian manifold. Let $\overline{g}$ and $\overline{R}$ be respectively the metric and the Riemannian curvature tensor of M. Let $\Sigma := \Sigma^n$ be a compact, n-dimensional manifold, and let $i : \Sigma \to M$ be locally strictly convex, immersion. Here and in the sequel, we use a semi-colon to denote covariant differentiation with respect to the Levi-Civita covariant derivative of $i^*\overline{g}$. Let A be the shape operator of i. Since i is locally strictly convex, A is positive definite, and thus defines a metric over S.

Proposition 5.3

$A_{ij;k}$ is symmetric in the first two terms, and:

$$A_{ij;k} = A_{kj;i} + \overline{R}_{ki\nu j},$$

where ν is the unit outward pointing normal vector of Σ.

Proof: The first assertion follows from the symmetry of A, which is preserved by covariant differentiation. Let N be the unit outward pointing, unit, normal vector field over i. Choose $p \in \Sigma$ and let X, Y and Z be vector fields over Σ which are parallel at p. At p:

$$\begin{aligned}(\nabla_Z A)(X,Y) &= \langle(\nabla_Z A)X, Y\rangle \\ &= \langle\nabla_Z\nabla_X\mathsf{N} - \nabla_{\nabla_Z X}\mathsf{N}, Y\rangle \\ &= \langle\overline{R}_{ZX}\mathsf{N} + \nabla_X\nabla_Z\mathsf{N} - \nabla_{\nabla_X Z}\mathsf{N}, Y\rangle \\ &= \langle\overline{R}_{ZX}\mathsf{N}, Y\rangle + (\nabla_X A)(Z,Y).\end{aligned}$$

This proves the second assertion, which completes the proof. $\square$

Let ∇ be the Levi-Civita covariant derivative of $g := i^*\overline{g}$ over Σ, and let ∇^A be the Levi-Civita covariant derivative of the metric $g(A\cdot,\cdot)$.

Proposition 5.4

If $\Omega^k{}_{ij}$ is the relative Christophel tensor of ∇ with respect to ∇^A, then:

$$\Omega^k{}_{ij} = \frac{1}{2}B^{kp}\overline{R}_{pi\nu j} - \frac{1}{2}B^{kp}A_{pi;j},$$

where ν is the unit, outward pointing normal vector of i.

Proof: Let $\hat{\Omega}$ be the relative Christophel tensor of ∇^A with respect to ∇. By Proposition 5.2:

$$\hat{\Omega}^k_{ij} = \frac{1}{2}B^{kp}(A_{pi;j} + A_{pj;i} - A_{ij;p}).$$

Thus, by Proposition 5.3:

$$\begin{aligned}\hat{\Omega}^k_{ij} &= \tfrac{1}{2}B^{kp}(A_{pi;j} + \overline{R}_{ij\nu p} - \overline{R}_{pj\nu i}) \\ &= \tfrac{1}{2}B^{kp}(A_{pi;j} + \overline{R}_{ij\nu p} + \overline{R}_{jp\nu i}).\end{aligned}$$

Thus, by the first Bianchi identity:

$$\hat{\Omega}^k_{ij} = \frac{1}{2}B^{kp}(A_{pi;j} - \overline{R}_{pi\nu j}).$$

Finally:

$$\hat{\Omega} = \nabla^A - \nabla = -(\nabla - \nabla^A) = -\Omega.$$

This completes the proof. $\square$

5.4 Differential Formula for Curvature.

Let $h : M \to \mathbb{R}$ be a smooth function. We now suppose that the extrinsic curvature of i is prescribed by $e^{h/2}$. In other words:

$$\mathrm{Det}(A) = K_e(i)^2 = e^{h\circ i}.$$

We recall that since i is locally strictly convex, A is positive definite and thus defines a metric over Σ. We aim to determine a formula for the scalar curvature of this metric in terms of the Ricci curvature of the metric induced by the immersion in M. Observe that A is invertible. We denote its inverse by B.

Proposition 5.5

For all m:

$$B^{ij}A_{ij;m} = h_{;m}.$$

Proof: This follows by differentiating the relation $\mathrm{Log}(\mathrm{Det}(A)) = h$. $\square$

Let R and R^A denote the respective curvature tensors of $g := i^*\overline{g}$ and the metric $g(A\cdot,\cdot)$. Let Ric and Ric^A and Scal and Scal^A denote their respective Ricci and Scalar curvatures respectively.

Theorem 5.6

There exists a 1-form, α over Σ such that, for all $\lambda > 0$:

$$\mathrm{Scal}^A \geqslant \frac{1}{n}B^{ik}\mathrm{Ric}_{ik} + \nabla^A\cdot\alpha - \frac{1+\lambda}{4n(n-1)}\|dh\|_A^2 - \frac{1+n\lambda^{-1}}{4n(n-1)}\|\overline{R}\cdot\cdot\nu\cdot\|_A^2,$$

where $\nabla^A\cdot$ is the divergence operator of ∇^A.

Proof: Let the subscripts ";" and "," denote covariant differentiation with respect to ∇ and ∇^A respectively. Let Ω be the relative Christophel tensor of ∇ with respect to ∇^A. By Proposition 5.1:

$$R_{ijk}{}^l = R^A_{ijk}{}^l + \Omega^l{}_{jk,i} - \Omega^l{}_{ik,j} + \Omega^l{}_{im}\Omega^m{}_{jk} - \Omega^l{}_{jm}\Omega^m{}_{ik}.$$

We raise and lower indices using $B := A^{-1}$. Thus, for all i,j:

$$A^{ij}A_{jk} = B^{im}A_{mn}B^{nj}A_{jk} = \delta^i{}_k.$$

Denoting $\kappa = n(n-1)$, contracting then yields:

$$\begin{aligned}\tfrac{1}{n}B^{ik}\mathrm{Ric}_{ik} &= -\tfrac{1}{\kappa}B^{mn}R_{mpn}{}^p\\ &= \mathrm{Scal}^A - \tfrac{1}{\kappa}B^{mn}(\Omega^j{}_{jm,n} - \Omega^j{}_{mn,j} + \Omega^j{}_{mp}\Omega^p{}_{jn} - \Omega^j{}_{jp}\Omega^p{}_{mn}).\end{aligned}$$

We first claim that the second and third terms on the right hand side combine to yield an exact form. Indeed, by Propositions 5.4 and 5.5 and bearing in mind that B is symmetric and that $\overline{R}$ is antisymmetric in the first two components:

$$\Omega^j{}_{jm} = \frac{1}{2}B^{jp}(\overline{R}^M_{pj\nu m} - A_{pj;m}) = -\frac{1}{2}h_{;m} = -\frac{1}{2}h_{,m}.$$

Likewise, bearing in mind in addition Proposition 5.3:

$$\begin{aligned}B^{mn}\Omega^j{}_{mn} &= \tfrac{1}{2}B^{jp}B^{mn}(\overline{R}_{pm\nu n} - A_{pm;n})\\ &= -\tfrac{1}{2}B^{jp}B^{mn}(A_{nm;p} + \overline{R}_{np\nu m} + \overline{R}_{mp\nu n})\\ &= -\tfrac{1}{2}B^{jp}h_{;p} - B^{jp}B^{mn}\overline{R}_{mp\nu n}\\ &= -\tfrac{1}{2}B^{jp}h_{,p} - B^{jp}B^{mn}\overline{R}_{mp\nu n}.\end{aligned}$$

We denote:

$$\alpha_i = \frac{1}{\kappa}B^{mn}\overline{R}_{mi\nu n}.$$

Then:

$$\begin{array}{rll} & B^{mi}\Omega^j{}_{jm} - B^{mn}\Omega^i{}_{mn} & = \kappa\alpha^i \\ \Rightarrow & B^{mn}\Omega^j{}_{jm,n} - B^{mn}\Omega^j{}_{mn,j} & = \kappa\nabla^A\cdot\alpha, \end{array}$$

which when multiplied by dVol^A yields an exact form as asserted. We now consider the last two terms on the right hand side. First, using the above relation again:

$$\begin{array}{rl} B^{mn}\Omega^j{}_{jp}\Omega^p{}_{mn} & = \frac{1}{4}B^{pq}h_{,p}(h_{,q} + 2\kappa\alpha_q) \\ & = \frac{1}{4}\|dh\|_A^2 + \frac{\kappa}{2}B^{pq}h_{,p}\alpha_q \end{array}$$

Finally, recall that, for all i, j and k:

$$\Omega^k{}_{ij} = \Omega^k{}_{ji}, \qquad A_{ij;k} = A_{ji;k}.$$

Thus, bearing in mind Proposition 5.4:

$$\begin{array}{rl} B^{mn}\Omega^j{}_{mp}\Omega^p{}_{jn} & = B^{mn}\Omega^i{}_{jm}\Omega^j{}_{in} \\ & = \frac{1}{4}B^{ip}B^{jq}B^{kr}(A_{ij;k} - \overline{R}_{ij\nu k})(A_{qp;r} - \overline{R}_{qp\nu r}) \\ & = \frac{1}{4}B^{ip}B^{jq}B^{kr}(A_{ij;k} - \overline{R}_{ij\nu k})(A_{pq;r} + \overline{R}_{pq\nu r}) \\ & = \frac{1}{4}\|\nabla A\|_A^2 - \frac{1}{4}\|\overline{R}_{\cdot\cdot\nu\cdot}\|_A^2. \end{array}$$

Combining these terms therefore yields:

$$\begin{array}{rl} \frac{1}{n}B^{ik}\mathrm{Ric}_{ik} & = \mathrm{Scal}^A - \nabla^A\cdot\alpha - \frac{1}{4\kappa}\|\nabla A\|_A^2 \\ & \quad + \frac{1}{4\kappa}\|\overline{R}_{\cdot\cdot\nu\cdot}\|_A^2 + \frac{1}{2}B^{mn}h_{,m}\alpha_n + \frac{1}{4\kappa}\|dh\|_A^2 \end{array}$$

However, for all $\mu > 0$:

$$\left|\frac{1}{2}B^{mn}\phi_{,m}\alpha_n\right| \leqslant \frac{\mu^{-1}}{4}\|\alpha\|_A^2 + \frac{\mu}{4}\|dh\|_A^2.$$

Recall that, for any matrix, M:

$$\mathrm{Tr}(M)^2 \leqslant n\mathrm{Tr}(MM^t).$$

Thus, bearing in mind that B is positive definite:

$$\begin{array}{rll} & \mathrm{Tr}(BM)^2 & = \mathrm{Tr}(B^{1/2}MB^{1/2})^2 \\ & & \leqslant n\mathrm{Tr}(BMBM^t) \\ \Rightarrow & \alpha_i^2 & = \frac{1}{n^2(n-1)^2}(\sum_{i=1}^n \frac{1}{\lambda_j}\overline{R}_{ji\nu j})^2 \\ & & \leqslant \sum_{p,q=1}^n \frac{1}{\lambda_p\lambda_q n(n-1)^2}(\overline{R}_{pi\nu q})^2. \end{array}$$

Thus:

$$\|\alpha\|_A^2 \leqslant \frac{1}{n(n-1)^2}\|\overline{R}_{\cdot\cdot\nu\cdot}\|_A^2.$$

In conclusion, taking $\mu = \lambda/n(n-1)$, we obtain:

$$\frac{1}{n}B^{ik}\mathrm{Ric}_{ik} \leqslant \mathrm{Scal}^A - \nabla^A\cdot\alpha + \frac{1+\lambda}{4n(n-1)}\|dh\|_A^2 + \frac{1+n\lambda^{-1}}{4n(n-1)}\|\overline{R}_{\cdot\cdot\nu\cdot}\|_A^2.$$

This completes the proof. $\square$

5.5 L^1 Mean Curvature Bounds.

We now return to the case where $n = 2$. Thus $M := M^3$ is a 3-dimensional manifold and $\Sigma := \Sigma^2$ is a compact surface.

Lemma 5.7

If M is 3-dimensional and $i : \Sigma \to M$ is an immersed surface, then, for all $\lambda > 0$:

$$\mathrm{Scal}^A \geqslant \mathrm{Scal} H_i e^{-h} + \nabla^A \cdot \alpha - \frac{1+\lambda}{8}\|dh\|_A^2 - \frac{1+2\lambda^{-1}}{4} e^{-h}\|\overline{T}(J_0\cdot, \mathsf{N})\|_A^2,$$

where:

(i) H_i is the mean curvature of the immersion i;

(ii) $\overline{T}$ is the trace free Ricci tensor of M;

(iii) N is the outward pointing unit normal over i; and

(iv) J_0 is the unique complex structure over Σ compatible with the orientation of Σ and the metric $g := i^*\overline{g}$.

Proof: When Σ is 2-dimensional:

$$\mathrm{Ric}_{ij} = \mathrm{Scal}\delta_{ij}.$$

Moreover, if λ_1, λ_2 are the principal curvatures of i, then:

$$\mathrm{Tr}(B) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} = \frac{\lambda_1 + \lambda_2}{\lambda_1\lambda_2} = H_i e^{-h},$$

where H_Σ is the mean curvature of Σ. Finally:

$$\begin{aligned} \|\overline{R}_{\cdot\cdot\nu\cdot}\|_A^2 &= \tfrac{2}{\lambda_1^2\lambda_2}(\overline{R}_{21\nu2})^2 + \tfrac{2}{\lambda_1\lambda_2^2}(\overline{R}_{12\nu2})^2 \\ &= \tfrac{2}{\lambda_1\lambda_2}\tfrac{1}{\lambda_2}(\mathrm{R\overline{ic}}_{1\nu})^2 + \tfrac{2}{\lambda_1\lambda_2}\tfrac{1}{\lambda_1}(\mathrm{R\overline{ic}}_{2\nu})^2 \\ &= 2e^{-h}\|\mathrm{R\overline{ic}}(J_0\cdot, \mathsf{N})\|_A^2. \end{aligned}$$

However, for all X tangent to Σ:

$$\mathrm{R\overline{ic}}(J_0X, \mathsf{N}) = \overline{T}(J_0X, \mathsf{N}).$$

Combining these relations yields the desired result. $\square$

In the special case where $\Sigma = S^2$ is the sphere, we use the Gauss-Bonnet Theorem to obtain:

Proposition 5.8

Suppose that Σ is the sphere. Let $K \subseteq \mathcal{O}$ be a compact subset. There exists $B > 0$ such that if $f \in K$ and $[i] \in \mathcal{C}$ are such that $([i], f) \in \mathcal{Z}$, then:

$$\int_\Sigma f H_i \mathrm{dVol} < B,$$

where H_i is the mean curvature of the immersion i.

Proof: Choose $f \in K$. We denote $h = 2\mathrm{Log}(f)$. Thus:

$$K^2 = \mathrm{Det}(A) = f^2 \circ i = e^{h \circ i}.$$

Using the notation of Proposition 5.7, we obtain, for all $\lambda > 0$:

$$\mathrm{Scal}^A \geqslant \mathrm{Scal} H_i e^{-h} + \nabla^A \cdot \alpha - \frac{1+\lambda}{8}\|dh\|_A^2 - \frac{1+2\lambda^{-1}}{4}e^{-h}\|\overline{T}(J_0\cdot, \mathsf{N})\|_A^2,$$

Let dVol^A be the volume form of A. Observe that $\mathrm{dVol}^A = f\mathrm{dVol}$. Since Σ is the sphere, by the Gauss-Bonnet Formula, integrating the above relation with respect to dVol^A yields:

$$\int_\Sigma \mathrm{Scal} H_i \frac{1}{f}\mathrm{dVol} - \frac{1+\lambda}{2}\int_\Sigma \|df\|_A^2 \frac{1}{f}\mathrm{dVol} - \frac{1+2\lambda^{-1}}{4}\int_\Sigma \|T(J_0\cdot, \mathsf{N})\|_A^2 \frac{1}{f}\mathrm{dVol} \leqslant 4\pi.$$

Let $\sigma(T\Sigma)$ be the sectional curvature of the tangent plane to $i(\Sigma)$. We obtain:

$$\int_\Sigma \mathrm{Scal} H_i \frac{1}{f}\mathrm{dVol} = \int_\Sigma (f^2 + \sigma(T\Sigma)) H_i \frac{1}{f}\mathrm{dVol} > \int_\Sigma (f^2 - |\sigma^-_{\mathrm{Min}}|) H_i \frac{1}{f}\mathrm{dVol}.$$

Next:

$$\frac{(1+\lambda)}{2}\int_\Sigma \|df\|_A^2 \frac{1}{f}\mathrm{dVol} \leqslant \frac{(1+\lambda)}{2}\int_\Sigma \|df\|^2 H_i \frac{1}{f^3}\mathrm{dVol}.$$

Likewise:

$$\frac{(1+2\lambda^{-1})}{4}\int_\Sigma \|T(J_0\cdot, \mathsf{N})\|_A^2 \frac{1}{f}\mathrm{dVol} \leqslant \frac{(1+2\lambda^{-1})}{4}\int_\Sigma \|T\|_O^2 H_i \frac{1}{f^3}\mathrm{dVol}.$$

Combining these relations yields:

$$\int_\Sigma \left(f^4 - |\sigma^-_{\mathrm{Min}}| f^2 - \frac{1}{4}\|T\|_O^2 - \left(\frac{1}{2\lambda}\|T\|_O^2 + \frac{(1+\lambda)}{2}\|df\|^2\right)\right) H_i \frac{1}{f^3}\mathrm{dVol}.$$

We claim that for an appropriate choice of $\lambda > 0$, the coefficient of H_i/f^3 is strictly positive. Indeed, by definition of K_0, since $f > K_0$:

$$A := (f^4 - |\sigma^-_{\mathrm{Min}}| f^2 - \frac{1}{4}\|T\|_O^2) > 0.$$

Denote:

$$C(\lambda) = \frac{1}{2}(\frac{1}{\lambda}\|T\|_O^2 + (1+\lambda)\|df\|^2).$$

The function $C(\lambda)$ takes its minimum value over the range $]0, +\infty[$ at the point $\lambda_0 := \|T\|_O/\|df\|$. At this point:

$$A - C(\lambda_0) = A - \|T\|_O\|df\| - \frac{1}{2}\|df\|^2.$$

It follows from the hypothesis on df in the definition of $\mathcal{O}$ that this quantity is strictly positive, and the assertion follows. We thus deduce from the compactness of K that there exists $\epsilon > 0$ which only depends on K such that for all $f \in K$:

$$A - C(\lambda_0) > \epsilon f^4.$$

Thus, for all $f \in K$:

$$\int_\Sigma f H_i \mathrm{dVol} \leqslant \frac{4\pi}{\epsilon} =: B.$$

This completes the proof. $\square$

5.6 Convergence and Compactness.

Let $M := M^{(n+1)}$ be an $(n+1)$-dimensional, oriented, Riemannian manifold. A complete **pointed immersed submanifold** is a pair $(S, p) := ((\Sigma, i), p)$ where $S := (\Sigma, i)$ is a complete, isometrically immersed submanifold and p is a point in Σ. Let $(S, p) := ((\Sigma, i), p)$ and $(S', p') := ((\Sigma', i'), p')$ be two pointed immersed submanifolds. For $R > 0$, we say that (S', p') is a **graph over** (S, p) **over radius** R if and only if there exists:

(i) a mapping $\alpha : B_R(p) \to \Sigma'$ which is a diffeomorphism onto its image; and

(ii) a smooth, normal vector field $X \in i^*TM$ over $B_R(p_0) \subseteq \Sigma$;

such that:

(i) $\alpha(p) = p'$; and

(ii) for all $p \in B_R(p)$:

$$(i' \circ \alpha)(p) = \mathrm{Exp}(X(p)).$$

We call the pair (α, X) a **graph reparametrisation** of (S', p') with respect to (S, p) over radius R. Let $(S_n, p_n)_{n\in\mathbb{N}} = ((\Sigma_n, i_n), p_n)_{n\in\mathbb{N}}$ be a sequence of pointed, immersed submanifolds in M. If $(S_0, p_0) = ((\Sigma_0, i_0), p_0)$ is another pointed, immersed submanifold, then we say that $(S_n, p_n)_{n\in\mathbb{N}}$ **converges** towards (S_0, p_0) if and only if for all $R > 0$, there exist $\nu \in \mathbb{N}$ such that:

(i) for all $n \geqslant \nu$, (S_n, p_n) is a graph over (S_0, p_0) over the radius R; and

(ii) if, for all $n \geqslant \nu$, (α_n, X_n) is a graph reparametrisation of (S_n, p_n) with respect to (S_0, p_0) over the radius R, then $(X_n)_{n\in\mathbb{N}}$ converges to 0 in the C^∞_{loc} sense over $B_R(p)$.

Remark: We see that if $(X_n)_{n\in\mathbb{N}}$ converges to 0 in the C^∞_{loc} sense over $B_R(p)$ for one choice of graph reparametrisations of $(S_n, p_n)_{n\in\mathbb{N}}$ with respect to (S_0, p_0), then it does so for every choice of graph reparametrisations.

We underline the following trivial but important consequence of this definition:

Proposition 5.9

Suppose that $(S_n, p_n)_{n\in\mathbb{N}}$ converges towards (S_0, p_0). If Σ_0 is compact, then, for sufficiently large n, Σ_n is diffeomorphic to Σ_0. Moreover, $(i_n)_{n\in\mathbb{N}}$ converges to i_0 in the C^∞ sense modulo reparametrisation.

Proof: Let R be the diameter of Σ_0. Choose $N > 0$ such that for all $n \geqslant N$, there exists a graph parametrisation (α_n, X_n) of (S_n, p_n) with respect to (S_0, p_0) over radius $2R$. For all $n \geqslant N$, α_n is a diffeomorphism onto its image and Σ_n is thus diffeomorphic to Σ_0. The first assertion follows. The second assertion is trivial, and this completes the proof. □

Suppose that M is Riemannian and oriented. Let $UM \subseteq TM$ be the bundle of unit vectors over M. The Riemannian structure on M induces a canonical Riemannian structure on UM (see [29] for details). Let $S = (\Sigma, i)$ be an oriented, immersed hypersurface in M. Let N be the outward pointing, unit normal over i. We denote $\hat{\imath} = \mathsf{N}$, and we define the **Gauss lifting** $\hat{S}$ of S by:

$$\hat{S} = (\Sigma, \hat{\imath}).$$

Suppose now that M is 3-dimensional. Let $(g_n)_{n\in\mathbb{N}}, g_0$ be metrics over M such that $(g_n)_{n\in\mathbb{N}}$ converges smoothly to g_0. In [14], Labourie obtains the following precompactness result:

Theorem 5.10, Labourie (1997)

Let $(S_n,p_n)_{n\in\mathbb{N}} = ((\Sigma_n,i_n),p_n)$ be a sequence of pointed, immersed surfaces in M such that, for all n, the Gauss Lifting $\hat{S}_n$ of S_n is complete. Suppose that there exist smooth, positive valued functions $(f_n)_{n\in\mathbb{N}}, f_0 : M \to]0,\infty[$ and a point $q_0 \in M$ such that:

(i) $(i_n(p_n))_{n\in\mathbb{N}}$ converges to q_0;

(ii) $(f_n)_{n\in\mathbb{N}}$ converges to f_0 in the C^∞_{loc} sense; and

(iii)for all $n \in \mathbb{N}$ and for all $p \in \Sigma_n$:

$$K_e(i_n)(p) = (f_n \circ i_n)(p).$$

Then there exists a complete, immersed surface $(S_0,p_0) = ((\Sigma_0,j_0),p_0) \in UM$ towards which $(\hat{S}_n,p_n)_{n\in\mathbb{N}}$ subconverges. Moreover, either:

(a) S_0 is a complete covering of the unit, normal, circle bundle over a complete geodesic; or

(b) S_0 is nowhere vertical. In other words, if $\pi : UM \to M$ is the canonical projection, then TS_0 is everywhere transverse to $\text{Ker}(\pi)$.

Remark: It follows in case (b) that $i_0 := \pi \circ j_0$ is an immersion. Importantly, however, $i_0^* g_0$ does not necessarily define a complete metric over Σ_0. Observe, nonetheless that, by definition of convergence, there exists a neighbourhood U of p_0 in Σ_0 and, for all n, a mapping $\alpha_n : U \to \Sigma_n$ which is a diffeomorphism onto its image such that:

(i) for all n, $\alpha_0(p_0) = p_n$; and

(ii) $(\hat{i}_n \circ \alpha_n)_{n\in\mathbb{N}}$ converges to j_0 in the C^∞ sense over U.

Thus, in particular, in case (b), $(i_n \circ \alpha_n)_{n\in\mathbb{N}}$ also converges to i_0 in the C^∞ sense over U and:

$$K_e(i_0)(p) = (f_0 \circ i_0)(p).$$

In the same spirit as in Theorem D of [15], we refine Theorem 5.10 as follows:

Theorem 5.11

Suppose that M is compact. Let $(f_n)_{n\in\mathbb{N}}, f_0 \in C^\infty(M,]0,\infty[)$ be positive functions such that $(f_n)_{n\in\mathbb{N}}$ converges to f_0 in the C^∞ sense. Let $(i_n)_{n\in\mathbb{N}} : \Sigma \to M$ be locally strictly convex immersions such that, for all n the extrinsic curvature of i_n is prescribed by f_n. For all n, let H_n be the mean curvature of i_n and suppose that:

(i) $f_0 > \sigma^-_{\text{Min}}(M)$; and

(ii) there exists $B > 0$ such that for all n:

$$\int_\Sigma f_n H_n \text{dVol} < B.$$

Then there exists a smooth immersion $i_0 : \Sigma \to M$ towards which $(i_n)_{n\in\mathbb{N}}$ subconverges in the C^∞ sense modulo reparametrisation.

Proof: For all n, let A_n be the shape operator of i_n. We first aim to show that the norm of A_n is uniformly bounded. Suppose the contrary. For all n, let $\mathsf{N}_n : \Sigma \to UM$ be the unit normal vector field over Σ compatible with the orientation and denote $\hat{\imath}_n = \mathsf{N}_n$. Let $p_n \in \Sigma$ be the point maximising A_n. Consider the sequence $(\Sigma, \hat{\imath}_m, p_m)_{m\in\mathbb{N}}$ of pointed, immersed submanifolds in UM. By Theorem 5.10 there exists a complete, immersed surface (Σ_0, j_0, p_0) in UM towards which $(\Sigma, \hat{\imath}_m, p_m)_{m\in\mathbb{N}}$ subconverges. We claim that j_0 is a covering of a circle bundle over a complete geodesic Γ. Indeed, suppose the contrary. Let $\pi : UM \to M$ be the canonical projection. Then $\pi \circ j_0$ is an immersion and there exists a sequences of neighbourhoods $(U_m)_{m\in\mathbb{N}}$ of $(p_m)_{m\in\mathbb{N}}$ in Σ such that the restrictions of the $(i_m)_{m\in\mathbb{N}}$ to $(U_m)_{m\in\mathbb{N}}$ subconverge to the restriction of $\pi \circ j_0$ in a neighbourhood of p_0. In particular $(A_n(p_n))_{n\in\mathbb{N}}$ converges to the shape operator of $\pi \circ j$ at p which is finite. This is absurd and the assertion follows.

Let HUM and VUM be the horizontal and vertical subbundles of TUM obtained using the Levi-Civita connexion of M. For any $X := X_p \in UM$, we identify HUM_X with TM_p and VUM_X with $\langle X\rangle^\perp \subseteq TM_p$ where $\langle X\rangle$ is the one dimensional subspace of TM_p generated by X. For all $n \in \mathbb{N}\cup\{0\}$, we define the metric $\hat{g}_n$ over UM such that for $(\alpha,\beta) \in HUM \oplus VUM = TUM$:

$$\hat{g}_n((\alpha,\beta),(\alpha,\beta)) = f_n^2\|\alpha\|^2 + \|\beta\|^2.$$

For all n, let $\widehat{\mathrm{dVol}}_n$ and dVol_n be the volume forms of the pull back of $\hat{g}_n$ through $\hat{\imath}_n$ and the pull back of the metric on M through i_n respectively. Since $K(i_n) = f_n \circ i_n$, we readily calculate:

$$\widehat{\mathrm{Vol}}_n = f_n H_n \mathrm{dVol}_n$$

Thus, for all n:

$$\mathrm{Vol}(\hat{\imath}_n^*\hat{g}_n) = \int_\Sigma \widehat{\mathrm{dVol}}_n = \int_\Sigma f_n H_n \mathrm{dVol}_n < B.$$

It thus follows from the mode of convergence used that:

$$\mathrm{Vol}(j_0^*\hat{g}_0) \leqslant \operatorname*{LimInf}_{n\to+\infty} \mathrm{Vol}(\hat{\imath}_n^*\hat{g}_n) \leqslant B.$$

and so $j_0^*\hat{g}_0$ has finite volume. Thus, since j_0 is a covering of the normal circle bundle over Γ, Γ is closed and j_0 is a covering of finite order. In particular, Σ_0 is compact and diffeomorphic to the torus $S^1 \times S^1$. However, since Σ_0 is compact, it follows from Proposition 5.9 that Σ_0 is also diffeomorphic to Σ. This is absurd since Σ is a sphere, and it follows that $\|A_m\|_{m\in\mathbb{N}}$ is uniformly bounded.

Choose $\epsilon > 0$ such that $f_0^2 - \sigma^-_{\mathrm{Min}}(M) > 2\epsilon$. For sufficently large m, $f_m^2 - \sigma^-_{\mathrm{Min}}(M) > \epsilon$. For all n, let K_n be the intrinsic curvature of the pull back through i_n of the metric on M. Then, for sufficiently large m:

$$K_m = f_m^2 - \sigma(T\Sigma_p) > f_m^2 - \sigma^-_{\mathrm{Min}}(M) > \epsilon.$$

We thus obtain uniform, positive lower bounds for K_m for m sufficiently large, and this yields a uniform upper bound for the intrinsic diameter of Σ with respect the pull back through i_m of the metric on M. It follows from the Arzela-Ascoli Theorem for immersed submanifolds (c.f. [26]) that there exists an immersion $i_0 : \Sigma \to M$ towards which $(i_m)_{m\in\mathbb{N}}$ subconverges in the C^∞ sense modulo reparametrisation. This completes the proof. □

Finally, we show that the hypotheses of Proposition 4.1 are satisfied:

Proposition 5.12

For any metric g on M, there exists $C > 0$ and $T > 0$ such that:

(i) $]T, +\infty[\subseteq \mathcal{O}$; and

(ii) if $t > T$ and $([i], t) \in \mathcal{Z}$, then $\mathrm{Diam}(\Sigma; i^*g) \leqslant Ct^{-1}$.

Proof: (i) is trivial. Likewise t sufficiently large, the intrinsice curvature of i^*g is bounded below by $t^2/4$, and (ii) follows trivially. □

We now prove Theorem 1.4:

Proof of Theorem 1.4: Since Σ is a 2-dimensional sphere, there exists no non-trivial diffeomorphisms of Σ having no fixed points, and so $\mathcal{I}$ only consists of simple immersions. By Theorem 5.11 and Proposition 5.8, $\pi : \mathcal{Z} \to \mathcal{O}$ is a proper map and so, by Theorems 2.10 and 3.14, $\mathrm{Deg}(\pi)$ is well defined. By Proposition 5.12, the hypotheses of Proposition 5.8 are satisfied, and it follows that $\mathrm{Deg}(\pi) = -\chi(M)$, where $\chi(M)$ is the Euler Characteristic of M. However, since M is odd-dimensional, $\chi(M) = 0$, and this completes the proof. □

A - Functional Analysis.

A.1 Immersions and Unparametrised Immersions.

Let $\Sigma := \Sigma^n$ and $M := M^{n+1}$ be compact, oriented manifolds of dimension n and $(n+1)$ respectively. Let $C^\infty(\Sigma, M)$ be the set of smooth mappings from Σ into M. We furnish $C^\infty(\Sigma, M)$ with the topology of smooth convergence. Let $C^\infty_{\mathrm{imm}}(\Sigma, M)$ be the open subset of $C^\infty(\Sigma, M)$ consisting of those mappings which are also immersions. Let $\mathrm{Diff}^\infty(\Sigma)$ be the group of smooth, orientation preserving diffeomorphisms of Σ. We furnish $\mathrm{Diff}^\infty(\Sigma)$ with the topology of C^∞ convergence. $\mathrm{Diff}^\infty(\Sigma)$ acts on $C^\infty_{\mathrm{imm}}(\Sigma, M)$ by composition and we define $\mathrm{Imm}(\Sigma, M)$, the space of **unparametrised immersions** from Σ into M, to be the quotient space of this action:

$$\mathrm{Imm}(\Sigma, M) = C^\infty_{\mathrm{imm}}(\Sigma, M)/\mathrm{Diff}^\infty(\Sigma).$$

We furnish $\mathrm{Imm}(\Sigma, M)$ with the quotient topology. For an element $i \in C^\infty_{\mathrm{imm}}(\Sigma, M)$ we denote its equivalence class in $\mathrm{Imm}(\Sigma, M)$ by $[i]$.

The group $\mathrm{Diff}^\infty(\Sigma)$ also acts on $C^\infty(\Sigma)$ by composition. We define the action of $\mathrm{Diff}^\infty(\Sigma)$ on the Cartesian product $C^\infty_{\mathrm{imm}}(\Sigma, M) \times C^\infty(\Sigma)$ by:

$$g \cdot (i, f) = (i \circ g, f \circ g).$$

We define $\mathrm{Smooth}(\Sigma, M)$ to be the quotient of $C^\infty_{\mathrm{imm}}(\Sigma, M) \times C^\infty(\Sigma)$ under this action:

$$\mathrm{Smooth}(\Sigma, M) = C^\infty_{\mathrm{imm}}(\Sigma, M) \times C^\infty(\Sigma)/\mathrm{Diff}^\infty(\Sigma).$$

We furnish $\mathrm{Smooth}(\Sigma, M)$ with the quotient topology. The projection onto the first factor, $\pi : C^\infty_{\mathrm{imm}}(\Sigma, M) \times C^\infty(M) \to C^\infty_{\mathrm{imm}}(\Sigma, M)$, quotients down to a projection $\pi : \mathrm{Smooth}(\Sigma, M) \to \mathrm{Imm}(\Sigma, M)$ which makes $\mathrm{Smooth}(\Sigma, M)$ into a topological vector bundle over $\mathrm{Imm}(\Sigma, M)$ with fibre $C^\infty(\Sigma)$. In the language of principal bundles, $C^\infty_{\mathrm{imm}}(\Sigma, M)$ is a principal $\mathrm{Diff}^\infty(\Sigma)$-bundle over $\mathrm{Imm}(\Sigma, M)$ and $\mathrm{Smooth}(\Sigma, M)$ is an associated bundle:

$$\mathrm{Smooth}(\Sigma, M) = C^\infty_{\mathrm{imm}}(\Sigma, M) \otimes_{\mathrm{Diff}^\infty(\Sigma)} C^\infty(\Sigma).$$

For a pair $(i, f) \in C^\infty_{\mathrm{imm}}(\Sigma, M) \times C^\infty(M)$, we denote its equivalence class in $\mathrm{Smooth}(\Sigma, M)$ by $[i, f]$. A functional $\mathcal{F} : \mathrm{Imm}(\Sigma, M) \to \mathrm{Smooth}(\Sigma, M)$ is said to be a **section** of $\mathrm{Smooth}(\Sigma, M)$ over $\mathrm{Imm}(\Sigma, M)$ if and only if $\pi \circ \mathcal{F} = \mathrm{Id}$.

Let X be a finite dimensional manifold. We say that a functional $\mathcal{F} : X \to C^\infty(\Sigma, M)$ is **strongly smooth** if and only if:

$$X \times \Sigma \to M; (x, p) \mapsto \mathcal{F}(x)(p)$$

is a smooth mapping. Let Y be a finite dimensional manifold. We say that a functional $\mathcal{G} : C^\infty(\Sigma, M) \to Y$ is **weakly smooth** if and only if for any strongly smooth functional, $\mathcal{F} : X \to C^\infty(\Sigma, M)$, the composition $\mathcal{G} \circ \mathcal{F}$ is smooth. We define strong smoothness (resp. weak smoothness) for functionals taking values in (resp. defined over) $C^\infty(\Sigma)$ and $C^\infty(M)$ in the same manner.

Let X be a finite dimensional manifold. We say that a functional $\mathcal{F} : X \to \mathrm{Imm}(\Sigma, M)$ is **strongly smooth** if and only if it lifts everywhere locally to a strongly smooth functional. Let Y be a finite dimensional manifold. As before, we say that a functional $\mathcal{G} : \mathrm{Imm}(\Sigma, M) \to Y$ is **weakly smooth** if and only if for any strongly smooth functional, $\mathcal{F} : X \to \mathrm{Imm}(\Sigma, M)$, the composition $\mathcal{G} \circ \mathcal{F}$ is smooth.

We say that a section $\mathcal{G} : \mathrm{Imm}(\Sigma, M) \to \mathrm{Smooth}(\Sigma, M)$ is **weakly smooth** if and only if for any for any strongly smooth functional, $\mathcal{F} : X \to \mathrm{Imm}(\Sigma, M)$, the composition $\mathcal{G} \circ \mathcal{F}$ is strongly smooth.

Choose $i \in C^\infty_{\mathrm{imm}}(\Sigma, M)$. Let N_i be the unit normal vector field over i compatible with the orientation. We define $\hat{\mathcal{E}}_i : C^\infty(\Sigma) \to C^\infty(\Sigma, M)$ by:

$$\hat{\mathcal{E}}_i(f)(p) = \mathrm{Exp}(f(p)\mathsf{N}_i(p)),$$

where $\mathrm{Exp} : TM \to M$ is the exponential map of M. Let $U_i \subseteq C^\infty(\Sigma)$ be a neighbourhood of the zero section whose image under $\hat{\mathcal{E}}_i$ consists only of immersions. We call such a triplet, $(i, U_i, \hat{\mathcal{E}}_i)$, a **graph slice** of $C^\infty_{\mathrm{imm}}(\Sigma, M)$.

Let $(i, U_i, \hat{\mathcal{E}}_i)$ be a graph slice of $C^\infty_{\mathrm{imm}}(\Sigma, M)$. Denote $\mathcal{E} = \pi \circ \hat{\mathcal{E}}_i$, where $\pi : C^\infty_{\mathrm{imm}}(\Sigma, M) \to \mathrm{Imm}(\Sigma, M)$ is the canonical projection. Reducing U_i if necessary, $\mathcal{E}$ defines a homeomorphism from U_i onto an open subset V_i of $\mathrm{Imm}(\Sigma, M)$. We call such a quadruplet, $(i, U_i, V_i, \mathcal{E}_i)$, a **graph chart** of $\mathrm{Imm}(\Sigma, M)$.

Remark: These charts provide $\mathrm{Imm}(\Sigma, M)$ with the structure of a tame Frechet orbifold (c.f. [9]) which is a manifold away from points corresponding to multiple covers. Using the Nash-Moser Theorem, the constructions of this section can be reformulated in this context. We have however chosen to work with the less sophisticated machinery of Banach orbifolds since the required hypotheses are slightly easier to prove. Care must be taken, however, since since $\mathrm{Imm}(\Sigma, M)$ cannot be extended to a smooth Banach orbifold, as we will see presently.

A.2 Strong Tangent Spaces, Differentiation and Linearisation.

Choose $i \in C^\infty_{\mathrm{imm}}(\Sigma, M)$. Let $\gamma, \eta :]-\epsilon, \epsilon[\to C^\infty_{\mathrm{imm}}(\Sigma, M)$ be strongly smooth functionals such that $\gamma(0) = \eta(0) = i$. Define the equivalence relation $\sim_i$ on such functionals such that $\gamma \sim_i \eta$ if and only if for all $p \in \Sigma$:

$$\partial_t \gamma(t)(p)|_{t=0} - \partial_t \eta(t)(p)|_{t=0} = 0.$$

We define the **strong tangent space**, $T_i C^\infty_{\mathrm{imm}}(\Sigma, M)$, to be the space of equivalence classes of strongly smooth functionals $\gamma :]-\epsilon, \epsilon[\to C^\infty(\Sigma, M)$ such that $\gamma(0) = i$. It is trivially a vector space. Given a strongly smooth functional $\gamma :]-\epsilon, \epsilon[\to C^\infty(\Sigma, M)$, we denote the class that it defines at 0 by $D\gamma_0$. For any strongly smooth functional $\mathcal{F} : X \to C^\infty(\Sigma, M)$ (resp. weakly smooth functional $\mathcal{G} : C^\infty(\Sigma, M) \to Y$), we define the **strong derivative**, $D\mathcal{F} : TX \to TC^\infty(\Sigma, M)$ (resp. **weak derivative** $D\mathcal{G} : TC^\infty(\Sigma, M) \to TY$), in the obvious manner. We say that a strongly smooth functional $\mathcal{F} : X \to C^\infty(\Sigma, M)$ is an **immersion** if and only if its strong derivative is everywhere injective. We say that it is an **embedding** if, in addition, it is injective. We define the strong tangent spaces, strong derivatives and weak derivatives for $C^\infty(\Sigma)$ and $C^\infty(M)$ in an analogous manner.

Choose $[i] \in \mathrm{Imm}(\Sigma, M)$. Let $\gamma, \eta :]-\epsilon, \epsilon[\to \mathrm{Imm}(\Sigma, M)$ be strongly smooth functionals such that $\gamma(0) = \eta(0) = [i]$. Define the equivalence relation $\sim_{[i]}$ on such functionals such that $\gamma \sim_{[i]} \eta$ if and only if for all lifts $\hat{\gamma}, \hat{\eta} :]-\epsilon, \epsilon[\to \mathrm{Imm}(\Sigma, M)$ of γ and η respectively such that $\hat{\gamma}(0) = \hat{\eta}(0) = i$, and for all $p \in \Sigma$, there exists $X_p \in T_p\Sigma$ such that:

$$\partial_t \hat{\gamma}(t)(p)|_{t=0} - \partial_t \hat{\eta}(t)(p)|_{t=0} = Ti_p \cdot X_p.$$

In other words, the difference between the strong derivatives of the lifts is tangent to i. We define the **strong tangent space** $T_{[i]}\mathrm{Imm}(\Sigma, M)$ to be the space of equivalence classes of strongly smooth functionals $\gamma :]-\epsilon, \epsilon[\to M$ such that $\gamma(0) = [i]$. As before, this is trivially a vector space. We define strong derivatives, weak derivatives, immersions and embeddings as before.

The standard identification of functions in $C^\infty(\Sigma)$ with infinitesimal normal deformations of immersions in $C^\infty_{\mathrm{imm}}(\Sigma, M)$ is described formally in the current context as a homomorphism $\hat{\mathcal{X}} : C^\infty_{\mathrm{imm}}(\Sigma, M) \times C^\infty(\Sigma) \to TC^\infty_{\mathrm{imm}}(\Sigma, M)$ as follows: choose $i \in C^\infty_{\mathrm{imm}}(\Sigma, M)$ and $f \in C^\infty(\Sigma)$. Let $(i, U_i, \hat{\mathcal{E}}_i)$ be the graph slice of $C^\infty_{\mathrm{imm}}(\Sigma, M)$ through i. Define $\hat{\gamma}(i, f) :]-\epsilon, \epsilon[\to C^\infty_{\mathrm{imm}}(\Sigma, M)$ by:

$$\hat{\gamma}(i, f)(t) = \hat{\mathcal{E}}_i(tf).$$

We define $\hat{\mathcal{X}}(i,f) \in TC^\infty_{\text{imm}}(\Sigma, M)$ by:

$$\hat{\mathcal{X}}(i,f) = D\hat{\gamma}(i,f)_0.$$

The homomorphism $\hat{\mathcal{X}}$ is equivariant in the following sense: for all $\phi \in \mathrm{Diff}^\infty(\Sigma)$:

$$\begin{array}{rll} & \hat{\gamma}(i \circ \phi, f \circ \phi)(t) & = (\hat{\mathcal{E}}_i(tf)) \circ \phi \\ \Rightarrow & \hat{\mathcal{X}}(i \circ \phi, f \circ \phi) & = \phi^* \hat{\mathcal{X}}(i,f). \end{array}$$

Thus, if $\pi : C^\infty_{\text{imm}}(\Sigma, M) \to \mathrm{Imm}(\Sigma, M)$ is the canonical projection, then, for all $\phi \in \mathrm{Diff}^\infty(\Sigma)$:

$$\pi_* \hat{\mathcal{X}}(i \circ \phi, f \circ \phi) = \pi_* \hat{\mathcal{X}}(i,f).$$

$\pi_* \hat{\mathcal{X}}$ thus quotients down to a homomorphism $\mathcal{X} : \mathrm{Smooth}(\Sigma, M) \to T\mathrm{Imm}(\Sigma, M)$ which is trivially bijective in each fibre.

We furnish $T\mathrm{Imm}(\Sigma, M)$ with the unique topology that makes $\mathcal{X}$ into a homeomorphism. We say that a section $\mathcal{F}$ of $T\mathrm{Imm}(\Sigma, M)$ is **weakly smooth** if and only if the composition $\mathcal{X}^{-1} \circ \mathcal{F}$ is.

Remark: In the sequel, we often identify a vector $[i,f] \in \mathrm{Smooth}(\Sigma, M)$ with its corresponding tangent vector $\mathcal{X}([i,f]) \in T\mathrm{Imm}(\Sigma, M)$.

Let $\pi : \mathrm{Smooth}(\Sigma, M) \to \mathrm{Imm}(\Sigma, M)$ be the canonical projection. We define the **strong vertical bundle**, $V\mathrm{Smooth}(\Sigma, M) \subseteq T\mathrm{Smooth}(\Sigma, M)$ by:

$$V\mathrm{Smooth}(\Sigma, M) = \mathrm{Ker}(D\pi).$$

Let $\mathcal{F} : \mathrm{Imm}(\Sigma, M) \to \mathrm{Smooth}(\Sigma, M)$ be a weakly smooth section. Interpreting graph charts as defining parallel transport up to first order, we define $\mathcal{L}\mathcal{F} : T\mathrm{Imm}(\Sigma, M) \to V\mathrm{Smooth}(\Sigma, M)$, the **linearisation** (covariant derivative) of $\mathcal{F}$ as follows: choose $[i,f] \in T\mathrm{Imm}(\Sigma, M)$. Let $\hat{\mathcal{F}} : C^\infty_{\text{imm}}(\Sigma, M) \to C^\infty(\Sigma)$ be the lift of $\mathcal{F}$ near i. Define $\hat{\gamma}(i,f)$ as before, and define $\hat{\gamma}_{\mathcal{F}}(i,f) :]-\epsilon, \epsilon[\to C^\infty(\Sigma)$ by:

$$\hat{\gamma}_{\mathcal{F}}(i,f) = \hat{\mathcal{F}} \circ \hat{\gamma}(i,f).$$

We define $\hat{\mathcal{L}}\hat{\mathcal{F}}(i,f) \in C^\infty(\Sigma)$ by:

$$\hat{\mathcal{L}}\hat{\mathcal{F}}(i,f) = D\hat{\gamma}_{\mathcal{F}}(i,f)_0.$$

$(0, \hat{\mathcal{L}}\hat{\mathcal{F}}(i,f))$ projects down to an element $\pi_* \hat{\mathcal{L}}\hat{\mathcal{F}}(i,f)$ of $V_{\mathcal{F}([i])}\mathrm{Smooth}(\Sigma, M)$. Moreover, $\hat{\mathcal{L}}\hat{\mathcal{F}}$ is equivariant in the following sense: for all $\phi \in \mathrm{Diff}^\infty(\Sigma)$:

$$\hat{\mathcal{L}}\hat{\mathcal{F}}(i \circ \phi, f \circ \phi) = \hat{\mathcal{L}}\hat{\mathcal{F}}(i,f) \circ \phi.$$

Thus, for all $(i', f') \in [i,f]$:

$$\pi_* \hat{\mathcal{L}}\hat{\mathcal{F}}(i', f') = \pi_* \hat{\mathcal{L}}\hat{\mathcal{F}}(i,f).$$

We thus define:

$$\mathcal{L}\mathcal{F}_{[i]} \cdot [i, f] = \pi_* \hat{\mathcal{L}}\hat{\mathcal{F}}(i, f).$$

If $\mathcal{Y}$ is a weakly smooth section of $T\mathrm{Imm}(\Sigma, M)$, we define $\mathcal{L}_{\mathcal{Y}}\mathcal{F}$ such that, for all $[i] \in \mathrm{Imm}(\Sigma, M)$:

$$(\mathcal{L}_{\mathcal{Y}}\mathcal{F})([i]) = \mathcal{L}\mathcal{F}_{[i]} \cdot \mathcal{Y}([i]),$$

where we identify $V_{\mathcal{F}([i])}\mathrm{Smooth}(\Sigma, M)$ with $\mathrm{Smooth}_{[i]}(\Sigma, M)$ in the canonical manner. $\mathcal{L}_{\mathcal{Y}}\mathcal{F}$ is a section of $\mathrm{Smooth}(\Sigma, M)$ over $\mathrm{Imm}(\Sigma, M)$ which is also weakly smooth.

Remark: Since $\mathcal{L}_{\mathcal{Y}}\mathcal{F}$ is weakly smooth, the process of linearisation may be iterated, and we may thus define higher order linearisations in the obvious manner.

A.3 General Bundles over the Source and Target Spaces.

We generalise the above discussion to sections of bundles over Σ and M. As in the finite dimensional case, the operation of linearisation satisfies the product and chain rules, making the following helpful in calculating linearisations. However, since linearisations can in general be calculated directly, it is not strictly necessary, and the reader uncomfortable with excessive formalism may skip to the next section if he so wishes.

Let E be a smooth, finite dimensional vector bundle over Σ. Let $\Gamma^\infty(E)$ be the set of smooth sections of E over Σ. Given a well defined pull back action of $\mathrm{Diff}^\infty(\Sigma)$ on $\Gamma^\infty(E)$, we define $\mathrm{Smooth}(\Sigma, M, E)$ by:

$$\mathrm{Smooth}(\Sigma, M, E) = C^\infty_{\mathrm{imm}}(\Sigma, M) \times \Gamma^\infty(\Sigma)/\mathrm{Diff}^\infty(\Sigma).$$

We furnish $\mathrm{Smooth}(\Sigma, M, E)$ with the quotient topology, and, as before, $\mathrm{Smooth}(\Sigma, M, E)$ defines a topological vector bundle over $\mathrm{Imm}(\Sigma, M)$ with fibre $\Gamma^\infty(E)$. We define strong and weak smoothness, the strong tangent space, strong and weak derivatives and linearisations of weakly smooth sections of $\mathrm{Smooth}(\Sigma, M, E)$ as before. In particular, if $\mathcal{Y}$ is a weakly smooth section of $T\mathrm{Imm}(\Sigma, M)$, then $\mathcal{L}_{\mathcal{Y}}\mathcal{F}$ is also a weakly smooth section of $\mathrm{Smooth}(\Sigma, M, E)$ and linearisation may be iterated.

Let F be a smooth, finite dimensional vector bundle over M. For $i \in C^\infty_{\mathrm{imm}}(\Sigma, M)$, let $\Gamma^\infty(F, i) \subseteq C^\infty(\Sigma, F)$ be the set of smooth sections of i^*F over Σ. This is the set of smooth mappings $\alpha : \Sigma \to F$ such that:

$$i = \pi \circ \alpha,$$

where $\pi : F \to M$ is the canonical projection. $\mathrm{Diff}^\infty(\Sigma)$ acts on $\Gamma^\infty(F, i)$ by pull back, and we obtain a topological vector bundle $\mathrm{Smooth}(\Sigma, M, F)$ over $\mathrm{Imm}(\Sigma, M)$ whose fibre over the point $[i]$ is $\Gamma^\infty(F, i)$. We define strong and weak smoothness, the strong tangent space, and strong and weak derivatives the same way as before.

We define linearisation as follows: let $\mathcal{F}$ be a weakly smooth section of $\mathrm{Smooth}(\Sigma, M, F)$ over $\mathrm{Imm}(\Sigma, M)$. Choose $[i.f] \in T\mathrm{Imm}(\Sigma, M)$. Let $\hat{\mathcal{F}} : C^\infty_{\mathrm{imm}}(\Sigma, M) \to C^\infty(\Sigma, F)$ be the lift of $\mathcal{F}$ near i. Define $\hat{\gamma}(i, f)$ as in the preceeding section. For $p, q \in M$ sufficiently close, let $\tau_{q,p} : F_p \to F_q$ be the parallel transport of F along the shortest geodesic of M joining p to q. Define $\hat{\gamma}_{\mathcal{F}(i,f)} :] - \epsilon, \epsilon[\to \Gamma^\infty(F, i)$ by:

$$\hat{\gamma}_{\mathcal{F}}(i, f)(t)(p) = \tau_{i(p),\hat{\gamma}(i,f)(t)(p)} \circ (\hat{\mathcal{F}} \circ \hat{\gamma}(i, f))(t)(p).$$

We define $\hat{\mathcal{L}}\hat{\mathcal{F}}(i, f) \in \Gamma^\infty(F, i)$ by:

$$\hat{\mathcal{L}}\hat{\mathcal{F}}(i, f) = D\hat{\gamma}_{\mathcal{F}}(i, f)_0.$$

$(0, \hat{\mathcal{L}}\hat{\mathcal{F}})$ projects down to an element, $\pi_*\hat{\mathcal{L}}\hat{\mathcal{F}}(i, f)$ of $V_{\mathcal{F}([i])}\mathrm{Smooth}(\Sigma, M, F)$. Moreover $\hat{\mathcal{L}}\hat{\mathcal{F}}$ is equivariant in the following sense: for all $\phi \in \mathrm{Diff}^\infty(\Sigma)$:

$$\hat{\mathcal{L}}\hat{\mathcal{F}}(i \circ \phi, f \circ \phi) = \hat{\mathcal{L}}\hat{\mathcal{F}}(i, f) \circ \phi.$$

Thus, for all $(i', f') \in [i, f]$:

$$\pi_*\hat{\mathcal{L}}\hat{\mathcal{F}}(i', f') = \pi_*\hat{\mathcal{L}}\hat{\mathcal{F}}(i, f).$$

We thus define:

$$\mathcal{L}\mathcal{F}_{[i]} \cdot [i, f] = \pi_*\hat{\mathcal{L}}\hat{\mathcal{F}}(i, f).$$

If $\mathcal{Y}$ is a weakly smooth section of $T\mathrm{Imm}(\Sigma, M)$, then $\mathcal{L}_{\mathcal{Y}}\mathcal{F}$ is also weakly smooth, and linearisation may be iterated.

We review the functionals that are used in the current paper, as well as their linearisations:

(i) **the exterior unit normal vector field:** given $i \in \mathrm{Imm}(\Sigma, M)$, $\mathsf{N}(i)$ is the outward pointing unit normal vector field over i. It is a smooth section of i^*TM and its linearisation is a first order differential operator given by:

$$\mathcal{L}\mathsf{N} \cdot [i, f] = [i, \nabla f] \in \mathrm{Smooth}(\Sigma, M, TM);$$

(ii) **the induced metric:** $g(i)$ is a smooth section of $T^*\Sigma \otimes T^*\Sigma$ and its linearisation is a zeroeth order differential operator given by:

$$\mathcal{L}g \cdot [i, f] = [i, 2fA_i] \in \mathrm{Smooth}(\Sigma, M, T^*\Sigma \otimes T^*\Sigma),$$

where A_i is the shape operator of i;

(iii) **the induced volume form:** $\mathrm{dVol}(i)$ is a smooth section of $\Lambda^n T^*\Sigma$ and its linearisation is a zeroeth order differential operator given by:

$$\mathcal{L}\mathrm{dVol} \cdot [i, f] = [i, fH_i\mathrm{dVol}_i] \in \mathrm{Smooth}(\Sigma, M, \Lambda^n T^*\Sigma),$$

where $H_i = \mathrm{Tr}(A_i)$ is the mean curvature of i;

(iv) **the shape operator:** $A(i)$ is a smooth section of $\mathrm{End}(T\Sigma)$ and its linearisation is a second order differential operator given by:

$$\mathcal{L}A \cdot [i, f] = [i, f(W_i - A_i^2) - \mathrm{Hess}(f)] \in \mathrm{Smooth}(\Sigma, M, \mathrm{End}(T\Sigma)),$$

where $W_i \in \Gamma(\mathrm{End}(T\Sigma))$ is given by:

$$W_i \cdot X = \overline{R}_{\mathsf{N}_i X}\mathsf{N}_i; \text{ and}$$

(v) **the curvature operator:** $K(i)$ is a smooth function over Σ and, by definition, its linearisation is a second order, elliptic, linear differential operator.

We finally consider functions and vector fields over the ambient space. These also define operators over $\mathrm{Imm}(\Sigma, M)$ by composition:

(vi) **composition by a smooth function:** given $\varphi \in C^\infty(M)$, $\varphi(i) := \varphi \circ i$ is a smooth function over Σ and its linearisation is a zeroeth order differential operator given by:

$$\mathcal{L}\varphi \cdot [i, f] = [i, \langle \nabla\varphi, \mathsf{N}_i\rangle f] \in \mathrm{Smooth}(\Sigma, M); \text{ and}$$

(vii) **composition by a smooth vector field:** given a smooth vector field, X, over M, $X(i) := i^*X$ is a smooth section of i^*TM and its linearisation is a zeroeth order differential operator given by:

$$\mathcal{L}X \cdot [i, f] = [i, f\nabla_{\mathsf{N}}X] \in \mathrm{Smooth}(\Sigma, M, TM).$$

A.4 Separable Banach Spaces.

Let M be a compact Riemannian manifold. Let $C^\infty(M)$ be the space of smooth real valued functions over M. Let $\mathbb{N}_0$ be the set of non-negative integers. For all $(k, \alpha) \in \mathbb{N}_0 \times]0, 1]$, let $C^{k,\alpha}(M)$ be the Banach space of real valued, $k + \alpha$ times Hölder differentiable functions. For all (k, α), let $\hat{C}^{k,\alpha}(M)$ to be the closure of $C^\infty(M)$ in $C^{k,\alpha}(M)$. $C^{k,\alpha}(M)$ and $\hat{C}^{k,\alpha}(M)$ are Banach spaces and $\hat{C}^{k,\alpha}(M)$ is separable. Observe that:

$$C^\infty(M) = \bigcap_{(k,\alpha)} C^{k,\alpha}(M) = \bigcap_{(k,\alpha)} \hat{C}^{k,\alpha}(M).$$

Let E_1, E_2 and F be Banach spaces and let $\mathcal{F} : E_1 \times E_2 \to F$ be a functional. For $m, n \in \mathbb{N}_0 \cup \{\infty\}$, we say that $\mathcal{F}$ is $C^{m,n}$ if and only if for all $i \leqslant m$, $j \leqslant n$, the partial derivative $D_1^i D_2^j \mathcal{F}$ exists and is continuous.

Lemma A.1

(i) For all (k, α):

$$C^{k,\alpha}(M) \times C^{k,\alpha}(M) \to C^{k,\alpha}(M); (f, g) \mapsto f + g$$

is a $C^{\infty,\infty}$ functional;

(ii) for all $k + \alpha$:

$$C^{k,\alpha}(M) \times C^{k,\alpha}(M) \to C^{k,\alpha}(M); (f, g) \mapsto fg$$

is a $C^{\infty,\infty}$ functional;

(iii) for all $k + \alpha > 1$ and for any smooth vector field, $X \in \Gamma(TM)$:

$$C^{k,\alpha}(M) \to C^{k-1,\alpha}(M); f \mapsto Xf$$

is a C^∞ functional; and

(iv) for all $m \in \mathbb{N}_0$ and for all $l + \beta > k + \alpha \geqslant 1$:

$$C^{l+m,\beta}(M) \times C^{k,\alpha}(M) \to C^{k,\alpha}(M); (f, g) \mapsto f \circ g$$

is a $C^{\infty,m}$ functional.

Let $\mathcal{F} : C^\infty(M) \to C^\infty(M)$ be a functional. We say that $\mathcal{F}$ is **smooth** if and only if there exists $r \in \mathbb{N}_0$ such that for all (k,α), $\mathcal{F}$ extends continuously to a smooth functional $\mathcal{F}^{k,\alpha}$ from $\hat{C}^{k+r,\alpha}(M)$ to $\hat{C}^{k,\alpha}(M)$. Since $C^\infty(M)$ is dense in $\hat{C}^{k,\alpha}(M)$ for all (k,α), the extension is unique when it exists. We call r the **order** of $\mathcal{F}$.

Proposition A.2

Let X and Y be finite dimensional manifolds.

(i) If $\mathcal{F} : X \to C^\infty(M)$ is smooth, then it is strongly smooth;

(ii) if $\mathcal{G} : C^\infty(M) \to Y$ is smooth, then it is weakly smooth; and

(iii)if $\mathcal{F} : C^\infty(M) \to C^\infty(M)$ is smooth, then it is weakly smooth.

Let $\mathcal{F} : C^\infty(M) \to C^\infty(M)$ be a smooth functional. Since, in particular, $\mathcal{F}$ is weakly smooth, we define the weak derivative, $D\mathcal{F} : TC^\infty(M) \to TC^\infty(M)$. We say that $\mathcal{F}$ is **elliptic** if and only if, for all $f \in C^\infty(M)$, $D\mathcal{F}_f : C^\infty(M) \to C^\infty(M)$ is an elliptic r'th order pseudo-differential operator. By classical elliptic theory and compactness of M (c.f. [8]) for all (k,α), $D\mathcal{F}_f^{k,\alpha} : C^{k+r,\alpha} \to C^{k,\alpha}$ is Fredholm. Moreover, if $f \in C^\infty(M)$:

$$\mathrm{Ker}(D\mathcal{F}_f^{k,\alpha}), \mathrm{Coker}(D\mathcal{F}_f^{k,\alpha}) \in C^\infty(M).$$

In particular, $\mathrm{Ind}(D\mathcal{F}_f^{k,\alpha})$, the Fredholm index of $D\mathcal{F}_f^{k,\alpha}$ is independant of (k,α). Since, by continuity, it is independant of f, we may speak of the **Fredholm index** of $\mathcal{F}$, and we denote it by $\mathrm{Ind}(\mathcal{F})$.

Proposition A.3

If $\mathcal{F}$ is elliptic, then for $f \in C^\infty(M)$, if $D\mathcal{F}_f : C^\infty(M) \to C^\infty(M)$ is surjective, then so is $D\mathcal{F}_g$ for all g sufficiently close to f.

Proof: Since $D\mathcal{F}_f$ is an elliptic pseudo-differential operator with smooth coefficients, so is its dual with respect to the L^2 norm over M. Thus, for all (k,α), $\mathrm{Coker}(D\mathcal{F}_f^{k,\alpha})$ is finite dimensional and consists only of smooth functions, and so, since $D\mathcal{F}_f$ is surjective, so is $D\mathcal{F}_f^{k,\alpha}$. Since surjectivity of Fredholm maps is an open property, there exists a neighbourhood, Ω of f in $C^{k,\alpha}(M)$ such that if $g \in \Omega$, $D\mathcal{F}_g^{k,\alpha}$ is surjective. We claim that for $g \in C^\infty(M) \cap \Omega$, $D\mathcal{F}_g$ is surjective. Indeed, choose $\phi \in C^\infty(M)$. There exists $\psi \in C^{k+r,\alpha}(M)$ such that $D\mathcal{F}_g^{k,\alpha} \cdot \psi = \phi$. By elliptic regularity, $\psi \in C^\infty(M)$, and:

$$D\mathcal{F}_g \cdot \psi = D\mathcal{F}_g^{k+r,\alpha} \cdot \psi = \phi.$$

The assertion follows, and this completes the proof. □

The following lemma is useful for extending the space of admissable data: let E and F be Banach spaces. Let $\mathcal{F} : E \times F \to E$ be a C^1 mapping which is Fredholm with respect to the first component. Define $Z \subseteq E \times F$ by:

$$Z = \mathcal{F}^{-1}(\{0\}).$$

Let $\pi_2 : Z \to F$ be the projection onto the second factor.

Proposition A.4

For all $(x,y) \in Z$, there exists a neighbourhood $U \times V$ of (x,y) in $E \times F$ such that the restriction of π_2 to $Z \cap (\overline{U} \times \overline{V})$ is proper.

Proof: Let $D_1\mathcal{F}$ and $D_2\mathcal{F}$ be the partial derivatives of $\mathcal{F}$ with respect to the first and second components respectively. By definition, $D_1\mathcal{F}_{(x,y)}$ is Fredholm. Let $W \subseteq E$ be the cokernel of $D_1\mathcal{F}_{(x,y)}$. Define $\hat{\mathcal{F}} : E \times F \times W \to E \times F$ by:

$$\hat{\mathcal{F}}(u,v,w) = (\mathcal{F}(u,v) + w, v).$$

$\hat{\mathcal{F}}$ is trivially C^1, and $D\hat{\mathcal{F}}$ is surjective at $(x,y,0)$. By the Implicit Function Theorem for differentiable functions between Banach Spaces, there exist neighbourhoods Ω_1, Ω_2 of $(0,y,0)$ and $(x,y,0)$ respectively in $E \times F \times W$ and a C^1 mapping $\Phi : \Omega_1 \to \Omega_2$ such that $\hat{\mathcal{F}} \circ \Phi$ coincides with projection onto the first and second factors. Let $U \times V$ be a neighbourhood of $(x,y) \in E \times F$ such that:

$$\overline{U} \times \overline{V} \times \{0\} \subseteq \Omega_2.$$

We claim that $U \times V$ is the desired neighbourhood. Indeed, let $(x_n, y_n)_{n\in\mathbb{N}} \in \overline{U} \times \overline{V}$ be such that $(x_n, y_n) \in Z$ for all n. Suppose that $(y_n)_{n\in\mathbb{N}}$ converges to $y_0 \in \overline{V}$. For all n, since $(x_n, y_n, 0) \in \Omega_2$, there exists $w_n \in W$ such that $\Phi(0, y_n, w_n) = (x_n, y_n, 0)$. Since W is finite dimensional, after extracting a subsequence, there exists $w_0 \in W$ towards which $(w_n)_{n\in\mathbb{N}}$ subconverges. Since Φ is a diffeomorphism onto its image, and since the closure of $U \times V \times \{0\}$ is contained in Ω_2, $(0, y_0, w_0) \in \Omega_1$. Define $x_0 \in \overline{U}$ by $\Phi(0, y_0, v_0) = (x_0, y_0, 0)$. $(x_n)_{n\in\mathbb{N}}$ subconverges to x_0, and the assertion follows. This completes the proof. □

A.5 Banach Manifolds.

Let E be a separable Banach space. A Banach manifold modelled on E is a separable, Hausdorff space, X, whose every point has a neighbourhood homeomorphic to an open subset of E such that the transition maps are smooth. Let Σ and M be smooth, compact, finite dimensional manifolds and let $C^\infty(\Sigma, M)$ be the space of smooth mappings from Σ into M. For all $(k,\alpha) \in \mathbb{N}_0 \times]0,1]$, let $C^{k,\alpha}(\Sigma, M)$ be the space of $C^{k,\alpha}$ mappings from Σ into M and let $\hat{C}^{k,\alpha}(\Sigma, M)$ be the closure of $C^\infty(\Sigma, M)$ in $C^{k,\alpha}(\Sigma, M)$. Observe that:

$$C^\infty(\Sigma, M) = \mathop{\cap}_{(k,\alpha)} C^{k,\alpha}(\Sigma, M) = \mathop{\cap}_{(k,\alpha)} \hat{C}^{k,\alpha}(\Sigma, M).$$

We show that, for all $k + \alpha \geqslant 1$, $\hat{C}^{k,\alpha}(\Sigma, M)$ is a Banach manifold (c.f. [13] for a detailed account of the case where $\Sigma = S^1$ is the circle). Since M may be embedded in $\mathbb{R}^N$ for some large N, $\hat{C}^{k,\alpha}(\Sigma, M)$ is contained in $\hat{C}^{k,\alpha}(\Sigma, \mathbb{R}^N)$ and is therefore separable. Let r be the injectivity radius of M. Choose $i \in C^\infty(\Sigma, M)$. Let $\hat{\Gamma}^{k,\alpha}_r(i^*TM)$ be the set of $\hat{C}^{k,\alpha}$ sections of i^*TM whose C^0 norm is less than r. Let $B^{k,\alpha}_r(i)$ be the set of all mappings in $\hat{C}^{k,\alpha}(\Sigma, M)$ whose C^0 distance to i is less than r. We define $\mathcal{E}_i : \hat{\Gamma}^{k,\alpha}_r(i^*TM) \to B^{k,\alpha}_r(i)$ by:

$$\mathcal{E}_i(X)(p) = \mathrm{Exp}_{i(p)}(X(p)).$$

Every element of $\hat{C}^{k,\alpha}(\Sigma, M)$ lies in $B^{k,\alpha}_r(i)$ for some $i \in C^\infty(\Sigma, M)$. Moreover, given $i_1, i_2 \in C^\infty(\Sigma, M)$:

$$\mathcal{E}_{i_2}^{-1} \circ \mathcal{E}_{i_1}(X)(p) = (\mathrm{Exp}^{-1}_{i_2(p)} \circ \mathrm{Exp}_{i_1(p)})(X(p)).$$

By Lemma A.1 these transition maps are smooth, and we conclude that $\hat{C}^{k,\alpha}(\Sigma, M)$ is a smooth Banach manifold.

Let $C^\infty_{\mathrm{imm}}(\Sigma, M)$ be the (open) subset of those maps in $C^\infty(\Sigma, M)$ which are also immersions. For all $k+\alpha > 1$, define $C^{k,\alpha}_{\mathrm{imm}}(\Sigma, M)$ and $\hat{C}^{k,\alpha}_{\mathrm{imm}}(\Sigma, M)$ in the obvious manner. Since it is an open subset of a Banach Manifold, $\hat{C}^{k,\alpha}_{\mathrm{imm}}(\Sigma, M)$ is also a Banach Manifold.

Let $\mathcal{F} : C^\infty_{\mathrm{imm}}(\Sigma, M) \to C^\infty(\Sigma)$ be a functional. We say that $\mathcal{F}$ is **smooth** of **order** r if and only if its restriction to every chart is smooth of order r. We say that $\mathcal{F}$ is **transversally elliptic** if and only if its restriction to every graph slice is elliptic.

Remark: This construction approximates $C^\infty_{\mathrm{imm}}(\Sigma, M)$ by Banach orbifolds. However, neither $\mathrm{Imm}(\Sigma, M)$ nor $\mathrm{Smooth}(\Sigma, M)$ can be approximated by Banach orbifolds in this manner since the composition operation is non-smooth (c.f. Lemma A.1, (iv)). We may bypass this by observing that $\mathrm{Imm}(\Sigma, M)$ is a Frechet orbifold (c.f. [9]), and re-expressing the constructions of this appendix using the Nash-Moser Theorem in place of the Implicit Function Theorem. Alternatively, we may continue to work within the Banach category by using the quotient differential structure, as in the case of weak and strong smoothness.

Let $\mathcal{F} : \mathrm{Imm}(\Sigma, M) \to \mathrm{Smooth}(\Sigma, M)$ be a section. We say that $\mathcal{F}$ is **smooth** of **order** r if and only its lift is smooth of order r. We say that $\mathcal{F}$ is **elliptic** if and only if its lift is transversally elliptic.

A.6 Smooth Bump Functionals and Lifting Charts.

As in the finite dimensional case, smooth bump functionals provide an important tool for constructing smooth functionals over $\mathrm{Imm}(\Sigma, M)$. Let $[i] \in \mathrm{Imm}(\Sigma, M)$ be a smooth immersion. Let $G_i \subseteq \mathrm{Diff}^\infty(\Sigma)$ be the subgroup of those immersions which preserve i. Thus, $\alpha \in G_i$ if and only if:

$$\alpha \circ i = \alpha.$$

Since i is an immersion, G_i is discrete. Since Σ is compact, G_0 is compact, and is therefore finite.

Let N_i be the unit normal vector field over i compatible with the orientation. We define $I : \Sigma \times \mathbb{R} \to M$ by:

$$I(p, t) = \mathrm{Exp}(t\mathsf{N}_i(p)),$$

G_i acts on $\Sigma \times \mathbb{R}$ in the obvious manner. Trivially, I is unchanged by pre-composition with elements of G_i. Since Σ is compact, there exists $\epsilon > 0$ such that the restriction of I to $\Sigma \times]-\epsilon, \epsilon[$ is an immersion. For $j \in C^\infty_{\mathrm{imm}}(\Sigma, M)$ such that $[j]$ is sufficiently close to $[i]$ in the C^0 sense, there exists an embedding $\tilde{\jmath} : \Sigma \to \Sigma \times]-\epsilon, \epsilon[$ such that:

$$I \circ \tilde{\jmath} = j.$$

Reducing ϵ further if necessary, $\tilde{\jmath}$ is unique up to post-composition with elements of G_i. We refer to the embedding, $\tilde{\jmath}$ as the **lift** of j with respect to i.

Lemma A.5

Let $K \subseteq \mathrm{Imm}(\Sigma, M)$ be compact. Let $U \subseteq \mathrm{Imm}(\Sigma, M)$ be a neighbourhood of K. There exists a smooth functional $\mathcal{F} : \mathrm{Imm}(\Sigma, M) \to \mathbb{R}$ such that:

(i) $\mathcal{F}$ is equal to 1 over K;

(ii) $\mathcal{F}$ is equal to 0 outside U; and

(iii) for any $i \in \mathrm{Imm}(\Sigma, M)$, $D\mathcal{F}_i$ is given by an integral operator.

Proof: Suppose first that K consists of a single point, $[i_0]$, where $i_0 \in C^\infty_{\mathrm{imm}}(\Sigma, M)$. Let $V \subseteq \mathrm{Imm}(\Sigma, M)$ be a neighbourhood of $[i_0]$ such that any $i \in C^\infty_{\mathrm{imm}}(\Sigma, M)$ whose class is in V may be lifted to an embedding, $\tilde{\imath}$ in $\Sigma \times]-\epsilon, \epsilon[$. Let $\hat{V} \subseteq C^\infty_{\mathrm{imm}}(\Sigma, M)$ be the set of all immersions whose class is in V.

Let $g_0 = i_0^*$ be the metric induced on Σ by the immersion i_0. Let $\pi : \Sigma \times]-\epsilon, \epsilon[\to \Sigma$ and $h : \Sigma \times]-\epsilon, \epsilon[\to]-\epsilon, \epsilon[$ be the projections onto the first and second factors respectively. For $k \in \mathbb{N}_0$, let $l \in \mathbb{N}_0$ be the lowest integer greater than $k + n/2$. Define the functional $\hat{\mathcal{F}}_k : \hat{V} \to \mathbb{R}$ by:

$$\hat{\mathcal{F}}_0(i) = \sum_{m=0}^{l} \int_\Sigma \|\nabla^m (h \circ \tilde{\imath})\|^2 \mathrm{dVol},$$

where ∇ is the covariant derivative of the metric $(\pi \circ \tilde{\imath})^* g_0$, $\| \cdot \|$ is its L^2 norm, and dVol is its volume form. $\hat{\mathcal{F}}_0(i)$ is trivially unchanged by post-composition of the lift $\tilde{\imath}$ of i with an element of G_{i_0}. By Lemma A.1, the functional $\hat{\mathcal{F}}_k$ is smooth. It is trivially equivariant under the action of $\mathrm{Diff}^\infty(\Sigma)$ on $C^\infty_{\mathrm{imm}}(\Sigma, M)$, and thus quotients to a smooth functional $\mathcal{F}_k : V \to \mathbb{R}$.

Let $(i_0, U_0, V_0, \mathcal{E}_0)$ be a graph chart of $\mathrm{Imm}(\Sigma, M)$ about i_0. We may suppose that $V = V_0$. Composing $\mathcal{F}_k$ with $\mathcal{E}_0$ yields:

$$(\mathcal{F}_k \circ \mathcal{E}_0)(f) = \sum_{m=1}^{l} \int_\Sigma \|\nabla^m f\|^2 \mathrm{dVol},$$

where ∇ is the covariant derivative of the metric g_0, $\| \cdot \|$ is its L^2 norm, and dVol is its volume form. By definition of the C^∞ topology, there exists $k \in \mathbb{N}_0$ such that if $i \in \mathrm{Imm}(\Sigma, M)$ is sufficiently close to i_0 in the C^k sense, then $i \in U$. Thus, by classical Sobolov Theory (c.f. [3]), for this value of k, there exists $\delta > 0$ such that:

$$\overline{\mathcal{F}_k^{-1}([0, \delta])} \subseteq U \cap V.$$

Let $\chi : [0, \infty[\to \mathbb{R}$ be a smooth function equal to 1 near 0 and equal to 0 over $[\delta, +\infty[$. We define the functional $\mathcal{F} : V \to \mathbb{R}$ by:

$$\mathcal{F}(i) = (\chi \circ \hat{\mathcal{F}}_k)(i).$$

We extend $\mathcal{F}$ to a smooth functional over $\mathrm{Imm}(\Sigma, M)$ by setting it equal to 0 on the complement of U. This functional satisfies properties (i) and (ii). Property (iii) follows after integrating by parts, and the general case follows by compactness. This completes the proof. □

Closely related to the construction of smooth bump functionals are lifting charts. These are also useful for defining smooth functions over $\mathrm{Imm}(\Sigma, M)$ by allowing us to represent any immersion as an embedding in some given manifold. We proceed as follows: we define the functional $\hat{\mathcal{I}} : C^\infty_{\mathrm{imm}}(\Sigma, \Sigma_\epsilon) \to C^\infty_{\mathrm{imm}}(\Sigma, M)$ by:

$$\hat{\mathcal{I}}(j)(p) = (I \circ j)(p).$$

By Lemma A.1, $\hat{\mathcal{I}}$ is smooth. Moreover, $\hat{\mathcal{I}}$ is trivially equivariant under the action of $\mathrm{Diff}^\infty(\Sigma)$.

We say that $p \in \Sigma$ is an **injective point** of i if and only if for all $q \neq p$:

$$i(q) \neq i(p).$$

Proposition A.6

If $[i]$ has a single injective point, then there exist a $\mathrm{Diff}^\infty(\Sigma)$-invariant neighbourhood $\hat{\mathcal{U}}$ of $[i]$ in $C^\infty_{\mathrm{imm}}(\Sigma, \Sigma_\epsilon)$ such that the restriction of $\mathcal{I}$ to $\mathcal{U}$ is a $\mathrm{Diff}^\infty(\Sigma)$-invariant diffeomorphism onto its image.

Proof: We identify i with its canonical lift in $C^\infty_{\mathrm{imm}}(\Sigma, \Sigma_\epsilon)$. $\hat{\mathcal{I}}$ is trivially a local diffeomorphism close to i. It thus suffices to show that $\hat{\mathcal{I}}$ is injective over a $\mathrm{Diff}(\Sigma)$-invariant neighbourhood of i. Let $p \in \Sigma$ be an injective point of i. There exists a neighbourhood V of p in Σ and $\epsilon > 0$ such that:

(i) the restriction of I to $V \times]-\epsilon, \epsilon[$ is injective; and

(ii) $i(V^c \times]-\epsilon, \epsilon) \cap i(V \times]-\epsilon, \epsilon[) = \emptyset$.

Let $\mathcal{W} \subseteq C^\infty(\Sigma)$ be a neighbourhood of 0 consisting of functions bounded above by ϵ in the C^0 norm. Let $\hat{\mathcal{U}} \subseteq C^\infty_{\mathrm{imm}}(\Sigma, \Sigma_\epsilon)$ be the set of those immersions which are reparametrisations of graphs over i of elements in $\mathcal{W}$. $\hat{\mathcal{U}}$ is trivially $\mathrm{Diff}^\infty(\Sigma)$ invariant. We claim that the restriction of $\hat{\mathcal{I}}$ to $\hat{\mathcal{U}}$ is injective. Indeed, suppose the contrary. Choose $j, j' \in C^\infty_{\mathrm{imm}}(\Sigma, \Sigma \times]-\epsilon, \epsilon[)$ such that $\hat{\mathcal{I}}(j) = \hat{\mathcal{I}}(j')$. In other words $I \circ j = I \circ j'$. The set over which j and j' coincide is closed. Since I is everywhere a local diffeomorphism, it is open. Since j is a graph of an element of $\mathcal{W}$, there exists $q \in \Sigma$ such that $j(q) \in V \times]-\epsilon, \epsilon[$. Thus:

$$(I \circ j')(q) = (I \circ j)(q) \in i(V \times]-\epsilon, \epsilon[)$$

Thus, by definition of V and ϵ:

$$j'(q) \in V \times]-\epsilon, \epsilon[.$$

Since the restriction of I to $V \times]-\epsilon, \epsilon[$ is injective, $j'(q) = j(q)$. It follows by connectedness that $j = j'$ and the assertion follows. This proves injectivity and we see that $\hat{\mathcal{U}}$ is the desired neighbourhood. □

We define $\hat{\mathcal{V}} \subseteq C^\infty_{\mathrm{imm}}(\Sigma, M)$ by $\hat{\mathcal{V}} = \hat{\mathcal{I}}(\hat{\mathcal{U}})$. $\hat{\mathcal{U}}$ and $\hat{\mathcal{V}}$ quotient down to open subsets $\mathcal{U}$ and $\mathcal{V}$ of $\mathrm{Imm}(\Sigma, \Sigma_\epsilon)$ and $\mathrm{Imm}(\Sigma, M)$ respectively, and $\hat{\mathcal{I}}$ quotients down to a functional $\mathcal{I} : \mathcal{U} \to \mathcal{V}$. By definition, this functional is smooth, and so is its inverse. We denote $\mathcal{L} = \mathcal{I}^{-1}$. We refer to the triplet $(\mathcal{L}, \mathcal{U}, \mathcal{V})$ as a **lifting chart** of $\mathrm{Imm}(\Sigma, M)$ about $[i]$, and we observe that for all $[j] \in \mathcal{V}$, $\mathcal{L}([j])$ is always an embedded submanifold of Σ_ϵ even though $[j]$ need not be and embedded submanifold of M.

A.7 The Sard-Smale Theorem.

We say that $X \subseteq C^\infty(M)$ is a **strongly smooth**, **embedded**, finite dimensional submanifold if and only if it is everywhere locally the image of a strongly smooth embedding. We define smooth, embedded, finite dimensional submanifolds of $C^\infty_{\mathrm{imm}}(\Sigma, M)$ and $\mathrm{Imm}(\Sigma, M)$ in an analogous manner.

Theorem A.7, Implicit Function Theorem

Let $\mathcal{F} : C^\infty(M) \to C^\infty(M)$ be a smooth, elliptic functional. If $D\mathcal{F}$ is surjective at every point of $\mathcal{F}^{-1}(\{0\})$, then $\mathcal{F}^{-1}(\{0\})$ is a strongly smooth, embedded, finite dimensional submanifold of $C^\infty(M)$ of dimension $\mathrm{Ind}(\mathcal{F})$.

Proof: Let r be the order of $\mathcal{F}$. For all (k, α), we denote by $\mathcal{F}^{k,\alpha} : \hat{C}^{k+r,\alpha}(M) \to \hat{C}^{k,\alpha}(M)$ the continuous extension of $\mathcal{F}$. By elliptic regularity:

$$(\mathcal{F}^{k,\alpha})^{-1}(\{0\}) \subseteq C^\infty(M),$$

and so:

$$(\mathcal{F}^{k,\alpha})^{-1}(\{0\}) = \mathcal{F}^{-1}(\{0\}).$$

As in the proof of Proposition A.3, $D\mathcal{F}^{k,\alpha}_f$ is surjective for all $f \in (\mathcal{F}^{k,\alpha})^{-1}(\{0\})$. Let n be the index of $\mathcal{F}$. Since $D\mathcal{F}^{k,\alpha}$ is Fredholm, by the Implicit Function Theorem for Banach manifolds, $(\mathcal{F}^{k,\alpha})^{-1}(\{0\})$ is a smooth submanifold of $\hat{C}^{k+r,\alpha}(M)$ of dimension $\mathrm{Ind}(D\mathcal{F}^{k,\alpha}) = \mathrm{Ind}(\mathcal{F})$.

We denote by $i^{k,\alpha}$ the canonical embedding of $\mathcal{F}^{-1}(\{0\})$ into $\hat{C}^{k,\alpha}(M)$. We claim that the differential structure defined over $\mathcal{F}^{-1}(\{0\})$ by pulling back the differential structure of $\hat{C}^{k,\alpha}(M)$ through $i^{k,\alpha}$ is independent of (k, α). Indeed, for $k' + \alpha' > k + \alpha$, the canonical embedding, $j^{(k,\alpha),(k',\alpha')}$, of $\hat{C}^{k,\alpha}(M)$ into $\hat{C}^{k',\alpha'}(M)$ is smooth. Since, trivially:

$$i^{k,\alpha} = j^{(k,\alpha),(k',\alpha')} \circ i^{k',\alpha'},$$

the assertion follows. In particular, $i^{k,\alpha}$ is smooth for all (k, α), and so, by definition, the canonical embedding, $i : \mathcal{F}^{-1}(\{0\}) \to C^\infty(M)$ is smooth. By Proposition A.2, $\mathcal{F}^{-1}(\{0\})$ is strongly smooth, and this completes the proof. $\square$

Let $\mathcal{F} : \mathrm{Imm}(\Sigma, M) \times C^\infty(M) \to \mathrm{Smooth}(\Sigma, M)$ be a family of sections of $\mathrm{Smooth}(\Sigma, M)$ over $\mathrm{Imm}(\Sigma, M)$. Suppose that $\mathcal{F}$ is smooth and Fredholm with respect to the first component, and weakly smooth with respect to the second. We consider the zero set:

$$\mathcal{Z} := \{[i, f] \mid \mathcal{F}([i, f]) = 0\},$$

and the canonical projection:

$$\pi : \mathcal{Z} \to C^\infty(M).$$

As before, we define $T\mathcal{Z}$, the **strong tangent space** of $\mathcal{Z}$ to be the space of equivalence classes of strongly smooth mappings from an open interval into $\mathcal{Z}$.

Proposition A.8

if $\mathcal{LF}$ is surjective at every point of $\mathcal{Z}$, then:

$$T\mathcal{Z} = \mathrm{Ker}(\mathcal{LF}).$$

In particular, $T\mathcal{Z}$ is a vector subspace of $T\mathrm{Imm}(\Sigma, M) \times TC^\infty(M)$.

Proof: Choose $([i], f) \in \mathcal{Z}$. Trivially:

$$T_{([i],f)}\mathcal{Z} \subseteq \mathrm{Ker}(\mathcal{LF}_{([i],f)}).$$

We aim to show that:

$$\mathrm{Ker}(\mathcal{LF}_{([i],f)}) \subseteq T_{([i],f)}\mathcal{F}.$$

Choose $(\alpha, \beta) \in \mathrm{Ker}(\mathcal{LF}_{([i],f)})$. Let $\mathcal{L}_1\mathcal{F}$ be the partial linearisation of $\mathcal{F}$ with respect to the first component. Since $\mathcal{L}_1\mathcal{F}_{([i],f)}$ is elliptic, its cokernel is finite dimensional. In other words, there exists a finite dimensional subspace $E \subseteq V_{\mathcal{F}([i],f)}\mathrm{Smooth}(\Sigma, M)$ such that:

$$V_{\mathcal{F}([i],f)}\mathrm{Smooth}(\Sigma, M) = \mathcal{L}_1\mathcal{F}_{[i]} \cdot T_{[i]}\mathrm{Imm}(\Sigma, M) + E.$$

Since $\mathcal{LF}$ is surjective, there exists a finite dimensional subspace $F \subseteq TC^\infty(M)$ such that:

$$\mathcal{LF}_{([i],f)} \cdot (T_{[i]}\mathrm{Imm}(\Sigma, M) \oplus F) = V_{\mathcal{F}([i],f)}\mathrm{Smooth}(\Sigma, M).$$

We assume, moreover, that $\beta \subseteq F$. Let $(i, U_i, V_i, \mathcal{E}_i)$ be a graph chart of $\mathrm{Imm}(\Sigma, M)$ about i. Define $\mathcal{G} : C^\infty(\Sigma) \times F \to \mathrm{Smooth}(\Sigma, M)$ by:

$$\mathcal{G}(g, h) = \mathcal{F}(\mathcal{E}_i(g), f + h).$$

Since $\mathcal{F}$ is weakly smooth, $\mathcal{G}$ is smooth. $\mathcal{G}$ is also elliptic and surjective at $(0, 0)$. Since surjectivity of elliptic functionals is an open property (c.f. Proposition A.3), $\mathcal{G}$ is surjective near $(0, 0)$. By the Implicit Function Theorem (Theorem A.7), $\mathcal{G}^{-1}(\{0\})$ is a strongly smooth, finite dimensional submanifold. We identify $\mathcal{G}^{-1}(\{0\})$ with its image under $\mathcal{E}_i$ in $\mathrm{Imm}(\Sigma, M) \times F \subseteq \mathrm{Imm}(\Sigma, M) \times C^\infty(M)$. This image is a subset of $\mathcal{Z}^{-1}(\{0\})$. Thus, in particular:

$$T_{([i],f)}\mathcal{G} \subseteq T_{([i],f)}\mathcal{Z}.$$

Since $(\alpha, \beta) \in T_{[i]}\mathrm{Imm}(\Sigma, M) \oplus F$ satisfies $\mathcal{LF}_{([i],f)} \cdot (\alpha, \beta) = 0$:

$$(\alpha, \beta) \in T_{([i],f)}\mathcal{G}.$$

This completes the proof. $\square$

Let $\mathcal{G} : X \to C^\infty(M)$ be a strongly smooth functional from a compact finite dimensional manifold into $C^\infty(M)$. Let $\mathcal{H} : \mathcal{Z} \to Y$ be a weakly smooth functional from $\mathcal{Z}$ into another finite dimensional manifold.

Definition A.9

For $x \in X$, we say that $\mathcal{G}$ is transverse to the restriction of π to $\mathcal{H}^{-1}(\{0\})$ at x if and only if, for all $([i], f) \in \mathcal{Z}$ such that:

(i) $\pi([i], f) = \mathcal{G}(x)$, and

(ii) $\mathcal{H}([i], f) = 0$,

and for all $\phi \in C^\infty(M)$, there exists $V \in T_pX$ and $(\alpha, \beta) \in T_{([i],f)}\mathcal{Z}$ such that:

(i) $\phi = D\mathcal{G}_x \cdot V + D\pi_{([i],f)} \cdot (\alpha, \beta)$, and

(ii) $D\mathcal{H}_{([i],f)} \cdot (\alpha, \beta) = 0$.

Given such a pair of functionals, we define $\mathcal{Z}(\mathcal{G}, \mathcal{H}) \subseteq \mathcal{Z} \times X$ by:

$$\mathcal{Z}(\mathcal{G}, \mathcal{H}) = \{([i], f, x) \mid \mathcal{G}(x) = \pi([i], f) \ \& \ \mathcal{H}([i], f) = 0\}.$$

The case where Y is 0 dimensional and H is trivial is of particular interest. We denote:

$$\mathcal{Z}(\mathcal{G}) := \mathcal{Z}(\mathcal{G}, 0).$$

We adapt the Implicit Function Theorem for Banach manifolds to our current setting:

Theorem A.10

Suppose that:

(i) $\mathcal{LF}$ is surjective at every point of $\mathcal{Z}$;

(ii) $D\mathcal{H}$ is surjective at every point of $\mathcal{Z}(\mathcal{G}, \mathcal{H})$; and

(iii) $\mathcal{G}$ is transverse to the restriction of π to $\mathcal{H}^{-1}(\{0\})$.

Then $\mathcal{Z}(\mathcal{G}, \mathcal{H})$ is a smooth, embedded, finite dimensional submanifold of $\mathcal{Z} \times X$. Moreover:

(i) the dimension of $\mathcal{Z}(\mathcal{G}, \mathcal{H})$ is equal to $\mathrm{Ind}(\mathcal{F}) + \mathrm{Dim}(X) - \mathrm{Dim}(Y)$; and

(ii) $\partial\mathcal{Z}(\mathcal{G}, \mathcal{H}) \subseteq \mathcal{Z} \times \partial X$.

Remark: It is important for our applications to note that $\mathcal{H}$ need only be defined over $\mathcal{Z}$ and need only be weakly smooth.

Proof: We first consider the case where Y is 0 dimensional and $\mathcal{H}$ is trivial. Choose $([i], f, x) \in \mathcal{Z}(\mathcal{G})$. Since the result is of a local nature, it suffices to prove it near $([i], f, x)$. Let $\hat{\mathcal{F}} : C^\infty_{\text{imm}}(\Sigma, M) \times C^\infty(M) \to C^\infty(\Sigma)$ be the lift of $\mathcal{F}$. Let $(i, U_i, \hat{\mathcal{E}}_i)$ be a graph slice of $C^\infty_{\text{imm}}(\Sigma, M)$ through i. Define $\hat{\mathcal{F}}_i : U_i \times C^\infty(M) \to C^\infty(\Sigma)$ by:

$$\hat{\mathcal{F}}_i(g, h) = \hat{\mathcal{F}}(\hat{\mathcal{E}}_i(g), h).$$

Define $\mathcal{I} : U_i \times X \to C^\infty(\Sigma)$ by:

$$\mathcal{I}(g, x) = \hat{\mathcal{F}}_i(g, \mathcal{G}(x)).$$

$\mathcal{I}$ is smooth. Moreover, $\mathcal{I}$ is elliptic with respect to the first component with index $\mathrm{Ind}(\mathcal{F})$. We claim that $D\mathcal{I}$ is surjective at $(0, x)$. Indeed, denote $g = \mathcal{G}(x)$ and choose $\phi \in C^\infty(\Sigma)$. By the hypothesis on $\mathcal{F}$, $D\hat{\mathcal{F}}_{i,(0,g)}$ is surjective and there exists $(\alpha, \beta) \in C^\infty(\Sigma) \times C^\infty(M)$ such that:

$$D_1\hat{\mathcal{F}}_{i,(0,g)} \cdot \alpha + D_2\hat{\mathcal{F}}_{i,(0,g)} \cdot \beta = \phi.$$

Since $\mathcal{G}$ is transverse to the restriction of π to $\mathcal{H}^{-1}(\{0\})$, in particular, it is transverse to π. There therefore exists $(\gamma, \delta) \in T_{(0,g)}\hat{\mathcal{F}}_i^{-1}(\{0\})$ and $U \in T_xX$ such that:

$$D\mathcal{G}_x \cdot U - \delta = \beta.$$

Thus:

$$(\alpha + \gamma, D\mathcal{G}_x \cdot U) = (\alpha + \gamma, \beta + \delta).$$

Since (γ, δ) is tangent to $\hat{\mathcal{F}}_i^{-1}(\{0\})$ at $(0, x)$:

$$\begin{aligned} & D_1\hat{\mathcal{F}}_{i,(0,g)} \cdot (\alpha + \gamma) + D_2\hat{\mathcal{F}}_{i,(0,g)} \cdot D\mathcal{G}_x \cdot U &&= \phi \\ \Rightarrow\quad & D\mathcal{I}_{(0,x)} \cdot (\alpha + \gamma, U) &&= \phi. \end{aligned}$$

The assertion now follows. Since surjectivity of elliptic mappings is an open property (c.f. Proposition A.3), $D\mathcal{I}$ is surjective in a neighbourhood of $(0, x)$. By the Implicit Function Theorem for Banach manifolds (Theorem A.7), $\mathcal{I}^{-1}(\{0\})$ is a smooth embedded submanifold of $U_i \times X$ near $(0, x)$ of finite dimension equal to $\mathrm{Ind}(\mathcal{F}) + \mathrm{Dim}(X)$. This proves (i) when $\mathrm{Dim}(Y) = 0$. (ii) follows trivially.

Consider the general case. Define $\mathcal{J} : \mathcal{Z}(\mathcal{G}) \to Y$ by:

$$\mathcal{J}([i], x) = \mathcal{H}([i], \mathcal{G}(x)).$$

Since $\mathcal{H}$ is weakly smooth, $\mathcal{J}$ is smooth. We claim that $D\mathcal{J}$ is surjective at every point of $\mathcal{H}^{-1}(\{0\})$. Indeed, choose $([i], x) \in \mathcal{J}^{-1}(\{0\})$ and $U \in T_0Y$. Denote $g = \mathcal{G}(x)$. Since $D\mathcal{H}$ is surjective at $([i], g)$, there exists $(\alpha, \beta) \in T_{([i],g)}\mathcal{Z}$ such that:

$$D_1\mathcal{H}_{([i],g)} \cdot \alpha + D_2\mathcal{H}_{([i],g)} \cdot \beta = \phi.$$

Since $\mathcal{G}$ is transverse to the restriction of π to $\mathcal{H}^{-1}(\{0\})$, there exists $(\gamma, \delta) \in T_{([i],g)}\mathcal{Z}$ and $U \in T_xX$ such that:

$$D_1\mathcal{H}_{([i],g)} \cdot \gamma + D_2\mathcal{H}_{([i],g)} \cdot \delta = 0,$$

and:

$$D\mathcal{G}_x \cdot U - \delta = \beta.$$

Thus:

$$\begin{aligned} & (\alpha + \gamma, D\mathcal{G}_x \cdot U) &&= (\alpha + \gamma, \beta + \delta) \\ \Rightarrow\quad & D_1\mathcal{H}_{([i],g)} \cdot (\alpha + \gamma) + D_2\mathcal{H}_{([i],g)} \cdot D\mathcal{G}_x \cdot U &&= \phi \\ \Rightarrow\quad & D\mathcal{J}_{([i],x)} \cdot (\alpha + \gamma, U) &&= \phi. \end{aligned}$$

However, as before:

$$D\mathcal{I}_{([i],x)} \cdot (\alpha + \gamma, U) = 0.$$

Thus $(\alpha+\gamma, U) \in T_{([i],x)}\mathcal{Z}(\mathcal{G})$ and the assertion follows. By the Implicit Function Theorem, $\mathcal{Z}(\mathcal{G}, \mathcal{H}) = \mathcal{J}^{-1}(\{0\})$ is a smooth, embedded submanifold of $\mathcal{Z}(\mathcal{G})$ of dimension:

$$\mathrm{Dim}(\mathcal{Z}(\mathcal{G})) - \mathrm{Dim}(Y) = \mathrm{Ind}(\mathcal{F}) + \mathrm{Dim}(X) - \mathrm{Dim}(Y).$$

This completes the proof. $\square$

Proposition A.11

Suppose that Y is of dimension 0, and $\mathcal{H}$ is trivial. With the same hypotheses as in Proposition A.10, if $([i], x) \in \mathcal{Z}(\mathcal{G})$ and if $\mathcal{L}_1\mathcal{F}$ is the partial linearisation of $\mathcal{F}$ with respect to $\mathrm{Imm}(\Sigma, M)$ at $([i], \mathcal{G}(x))$, then:

$$\mathrm{Ker}(\mathcal{L}_1\mathcal{F}_{[i]}) = T_{([i],x)}\mathcal{Z}(\mathcal{G}) \cap (T_{[i]}\mathrm{Imm}(\Sigma, M) \times \{0\}).$$

Proof: Denote $p = ([i], \mathcal{G}(x))$ and $\mathcal{I}(i, x) = \mathcal{F}(i, \mathcal{G}(x))$. For $f \in T_{[i]}\mathrm{Imm}(\Sigma, M)$:

$$\begin{aligned} \mathcal{L}_1\mathcal{F}_p \cdot f = 0 \quad &\Leftrightarrow \mathcal{L}_1\mathcal{F}_p \cdot f + \mathcal{L}_1\mathcal{F}_p \cdot D\mathcal{G}_x \cdot 0 = 0 \\ &\Leftrightarrow \mathcal{L}\mathcal{I}_{([i],x)} \cdot (f, 0) \qquad\qquad\quad = 0. \end{aligned}$$

Thus, by Proposition A.8, $\mathcal{L}_1\mathcal{F}_p \cdot f = 0$ if and only if $(f, 0) \in T_{([i],x)}\mathcal{Z}(\mathcal{G})$. $\square$

The following version of the Sard-Smale Theorem is best adapted to the current context:

Theorem A.12, Sard-Smale

Suppose that:

(i) π is a proper mapping; and

(ii) $\mathcal{L}\mathcal{F}$ is surjective at every point of $\mathcal{Z}$; and

(iii) $D\mathcal{H}$ is surjective at every point of $\mathcal{Z}(\mathcal{G}, \mathcal{H})$;

Let $X_0 \subseteq X$ be a closed subset such that $\mathcal{G}$ is transverse to the restriction of π to $\mathcal{H}^{-1}(\{0\})$ at every point of X_0. Then, there exists a strongly smooth functional $\mathcal{G}' : X \to C^\infty(M)$, as close to $\mathcal{G}$ as we wish, such that:

(i) $\mathcal{G}'$ is equal to $\mathcal{G}$ at every point of X_0; and

(ii) $\mathcal{G}$ is transverse to the restriction of π to $\mathcal{H}^{-1}(\{0\})$.

Proof: We first consider the case where Y is 0 dimensional and $\mathcal{H}$ is trivial. Choose $x \in X \setminus X_0$. Choose $([i], f) \in \mathcal{Z}$ such that $([i], x) \in \mathcal{Z}(\mathcal{G})$. Suppose that $D\pi_{([i],f)} \oplus D\mathcal{G}_x : T_{([i],f)}\mathcal{Z} \oplus T_xX \to C^\infty(M)$ is not surjective. Let $\mathcal{L}_1\mathcal{F}$ be the partial linearisation of $\mathcal{F}$ with respect to the first component. Since it is Fredholm, its cokernel is finite dimensional. There therefore exists a finite dimensional subspace $E_1 \subseteq V_{\mathcal{F}([i],f)}\mathrm{Smooth}(\Sigma, M)$ such that:

$$V_{\mathcal{F}([i],f)}\mathrm{Smooth}(\Sigma, M) = \mathcal{L}_1\mathcal{F}_{([i],f)} \cdot T_{[i]}\mathrm{Imm}(\Sigma, M) + E_1,$$

Since $\mathcal{LF}$ is surjective at the point $([i], f)$, there exists a finite dimensional subspace $E_2 \subseteq T_{[i]}\mathrm{Imm}(\Sigma, M) \oplus T_f C^\infty(M)$ such that:

$$\mathcal{LF}_{([i],f)} \cdot E_2 = E_1.$$

Let F be the projection of E_2 to $T_f C^\infty(M)$. The restriction of $\mathcal{LF}$ to $T_{[i]}\mathrm{Imm}(\Sigma, M) \oplus F$ is surjective. Since surjectivity of elliptic maps is an open property (c.f. Proposition A.3), this also holds throughout a neighbourhood of $([i], f)$ in $\mathrm{Imm}(\Sigma, M) \times C^\infty(M)$. Let $\chi \in C_0^\infty(X)$ be a smooth function equal to 1 near x and equal to 0 near X_0. Define $\mathcal{G}_1 : X \times F \to C^\infty(M)$ by:

$$\mathcal{G}_1(x, \gamma) = \mathcal{G}(x) + \chi(x)\gamma.$$

$\mathcal{G}_1$ is trivially strongly smooth and transverse at x to the restriction of π to a neighbourhood of $([i], f)$. Since π is proper, increasing the dimension of F if necessary, we may suppose that $\mathcal{G}_1$ is transverse to π at $(x, 0)$. Since π is proper, this holds for every point $(y, 0)$ near $(x, 0)$ in $X \times F$. Since X is compact, after increasing F yet further, we may assume that this holds for every point $(y, 0) \in X \times F$. Since X is compact and π is proper, there exists $\epsilon > 0$ such that this holds throughout $X \times B_\epsilon(0)$, where $B_\epsilon(0)$ is the ball of radius ϵ in F.

By the Implicit Function Theorem (Theorem A.10) $\mathcal{Z}(\mathcal{G}_1)$ is a smooth embedded submanifold of $\mathrm{Imm}(\Sigma, M) \times X \times B_\epsilon$ of dimension $\mathrm{Ind}(\mathcal{F}) + \mathrm{Dim}(X) + \mathrm{Dim}(F)$. Let $\pi_3 : \mathcal{Z}(\mathcal{G}_1) \to B_\epsilon(0)$ be the projection onto the third factor. Let $\gamma \in B_\epsilon(0)$ be a regular value of π_3. Define $\mathcal{G}_\gamma$ by:

$$\mathcal{G}_\gamma(x) = \mathcal{G}_1(x, \gamma).$$

We claim that $\mathcal{G}_\gamma$ is transverse to the restriction of π to $\mathcal{Z}$. Indeed, choose $([i], f, x)$ such that $([i], x, \gamma) \in \mathcal{Z}(\mathcal{G}_1)$. Choose $\phi \in C^\infty(M)$. By transversality of $\mathcal{G}_1$, there exists $(\alpha, \beta) \in T_{([i],f)}\mathcal{Z}$ and $(U, V) \in T_{(x,\gamma)}M \times F$ such that:

$$D\pi_{([i],f)} \cdot (\alpha, \beta) + D\mathcal{G}_{1,(x,\gamma)} \cdot (U, V) = \phi.$$

Since γ is a regular value of π_3, there exists $(\gamma, \delta) \in T_{([i],f)}\mathcal{Z}$ and $W \in T_x X$ such that $(\gamma, W, V) \in T_{([i],x,\gamma)}\mathcal{Z}(\mathcal{G}_1)$. In other words:

$$\begin{array}{rll} & \delta + D\mathcal{G}_{1,(x,g)} \cdot (W, V) & = 0 \\ \Leftrightarrow & D\pi_{([i],f)} \cdot (\gamma, \delta) + D\mathcal{G}_{1,(x,\gamma)} \cdot (W, V) & = 0. \end{array}$$

Thus:

$$\begin{array}{rll} & D\pi_{([i],f)} \cdot (\alpha - \gamma, \beta - \delta) + D\mathcal{G}_{1,(x,\gamma)} \cdot (U - W, 0) & = \phi \\ \Rightarrow & D\pi_{([i],f)} \cdot (\alpha - \gamma, \beta - \delta) + D\mathcal{G}_{\gamma,x} \cdot (U - W) & = \phi. \end{array}$$

The assertion follows. By the classical Sard's Theorem, the regular values of π_3 are dense in $B_\epsilon(0)$. Such a γ therefore exists, and may be chosen as close to 0 as we wish. Setting $\mathcal{G}' = \mathcal{G}_\gamma$, the result follows in the case where Y is of dimension 0 and $\mathcal{H}$ is trivial.

To prove the general case, we construct $\mathcal{G}_1 : X \times B_\epsilon(0) \to C^\infty(M)$ as before, but this time requiring that it be transverse to the restriction of π to $\mathcal{H}^{-1}(0)$. Define $\mathcal{I} : \mathcal{Z}(\mathcal{G}_1) \to Y$ by:

$$\mathcal{I}(i, f, x, \gamma) = \mathcal{H}(i, f).$$

Since $\mathcal{H}$ is weakly smooth, $\mathcal{I}$ is smooth. As in the proof of Proposition A.10, $D\mathcal{I}$ is surjective at every point of $\mathcal{I}^{-1}(\{0\})$. It follows by the Implicit Function Theorem for finite dimensional manifolds that $\mathcal{Z}(\mathcal{G}_1, \mathcal{H}) = \mathcal{I}^{-1}(\{0\})$ is a smooth, embedded submanifold of $\mathcal{Z}(\mathcal{G}_1)$ of codimension $\mathrm{Dim}(Y)$. Let $\pi_3 : \mathcal{Z}(\mathcal{G}, \mathcal{H}) \to B_\epsilon(0)$ be the projection onto the third factor. Let $\gamma \in B_\epsilon(0)$ be a regular value of π_3 and define $\mathcal{G}_\gamma$ by:

$$\mathcal{G}_\gamma(x) = \mathcal{G}_1(x, \gamma).$$

As before, $\mathcal{G}_\gamma$ is transverse to the restriction of π to $\mathcal{H}^{-1}(\{0\})$. By the classical Sard's Theorem, there exist regular values of π_3 as close to 0 as we wish, and setting $\mathcal{G}' = \mathcal{G}_\gamma$ yields the desired functional. $\square$

Remark: In fact, the function spaces considered in this paper can be approximated by Banach manifolds in the following manner: let $\mathcal{F} := \hat{K} : \mathrm{Imm}(\Sigma, M) \times C^\infty(M) \to \mathrm{Smooth}(\Sigma, M)$ be the curvature functional constructed in Section 2.1. For all (k, α), let $\mathcal{F}^{k,\alpha} : \mathrm{Imm}^{2,\alpha}(\Sigma, M) \times C^{k,\alpha}(M) \to \mathrm{Smooth}^{0,\alpha}(\Sigma, M)$ be the continuous extension of $\mathcal{F}$, and define $\mathcal{Z}^{k,\alpha}$ by:

$$\mathcal{Z}^{k,\alpha} = (\mathcal{F}^{k,\alpha})^{-1}(\{0\}).$$

Trivially:

$$\mathcal{Z} = \bigcap_{(k,\alpha)} \mathcal{Z}^{k,\alpha}.$$

Moreover, If $\hat{K}$ is elliptic, then, as in [33], for all $(k, \alpha) > 1$, $\mathcal{Z}^{k,\alpha}$ is a C^k Banach manifold modeled on $C^{k,\alpha}(M)$ and the canonical projection $\pi : \mathcal{Z}^{k,\alpha} \to C^{k,\alpha}(M)$ is a C^k Fredholm map of index 0. Our results may then be formulated in terms of the Sard-Smale Theorem for Banach manifolds. However, our current approach uses minimal deep functional analytis and also requires simpler hypotheses, which make its application, for example, in Sections 3.3 and 3.4 simpler than if we were to use Banach manifolds directly.

B - Bibliography.

[1] Aronszajn N., A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order, *J. Math. Pures Appl.*, **36**, (1957), 235–249

[2] Berger M., *A panoramic view of Riemannian geometry*, Springer-Verlag, Berlin, (2003)

[3] Brezis H., *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, (2011)

[4] Caffarelli L., Nirenberg L., Spruck J., Nonlinear second-order elliptic equations. V. The Dirichlet problem for Weingarten hypersurfaces, *Comm. Pure Appl. Math.*, **41**, (1988), no. 1, 47–70

[5] Colding T. H., De Lellis C., The min-max construction of minimal surfaces, *Surv. Differ. Geom.*, **VIII**, Int. Press, Somerville, MA, (2003), 75–107

[6] Elworthy K. D., Tromba A. J., Degree theory on Banach manifolds, in *Nonlinear Functional Analysis (Proc. Sympos. Pure Math.)*, Vol. **XVIII**, Part 1, Chicago, Ill., (1968), 86–94, Amer. Math. Soc., Providence, R.I.

[7] Espinar J. M., Rosenberg H., When strictly locally convex hypersurfaces are embedded, to appear in *Math. Zeit.*

[8] Friedlander F. G., *Introduction to the theory of distributions. Second edition. With additional material by M. Joshi.*, Cambridge University Press, Cambridge, (1998)

[9] Hamilton R. S., The inverse function theorem of Nash and Moser, *Bull. Amer. Math. Soc.*, **7**, (1982), no. 1, 65–222

[10] Hörmander L., *The analysis of linear partial differential operators. III. Pseudodifferential operators*, Grundlehren der Mathematischen Wissenschaften, 274, Springer-Verlag, Berlin, (1985)

[11] Huisken G., Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature, *Invent. Math.*, **84**, (1986), no. 3, 463–480

[12] Kato T., *Perturbation theory for linear operators*, Grundlehren der Mathematischen Wissenschaften, **132**, Springer-Verlag, Berlin, New York, (1976)

[13] Klingenberg W., *Lectures on closed geodesics*, Grundlehren der Mathematischen Wissenschaften, **230**, Springer-Verlag, Berlin, New York, (1978)

[14] Labourie F., Problèmes de Monge-Ampère, courbes holomorphes et laminations, *Geom. Funct. Anal.*, **7**, (1997), no. 3, 496–534

[15] Labourie F., Immersions isométriques elliptiques et courbes pseudo-holomorphes, *Geometry and topology of submanifolds (Marseille, 1987)*, 131–140, World Sci. Publ., Teaneck, NJ, (1989)

[16] Laurain P., Concentration of CMC surfaces in a 3-manifold, preprint

[17] Pacard F., Constant mean curvature hypersurfaces in riemannian manifolds., *Riv. Mat. Univ.*, **7**, (2005), no. 4, 141–162

[18] Pacard F., Xu X., Constant mean curvature spheres in Riemannian manifolds, *Manuscripta Math.*, **128**, (2009), no. 3, 275–295

[19] Robeday A., Masters Thesis, Univ. Paris VII

[20] Rosenberg H., Schneider M., Embedded constant curvature curves on convex surfaces, to appear in *Pac. J. Math.*

[21] Rosenberg H., Smith G., Degree theory of immersions in the presence of orbifold points, *in preparation*

[22] Schneider M., Closed magnetic geodesics on S^2, *J. Differential Geom.*, **87**, (2011), no. 2, 343–388

[23] Schneider M., Closed magnetic geodesics on closed hyperbolic Riemann surfaces, arXiv:1009.1723

[24] Smale S., An infinite dimensional version of Sard's theorem, *Amer. J. Math.*, **87**, (1965), 861–866

[25] Smith F. R., On the existence of embedded minimal 2-spheres in the 3-sphere, endowed with an arbitrary metric, Ph.D. thesis, Univ. Melbourne, Parkville, (1983)

[26] Smith G., An Arzela-Ascoli Theorem for Immersed Submanifolds, *Ann. Fac. Sci. Toulouse Math.*, **16**, no. 4, (2007), 817–866

[27] Smith G., Compactness results for immersions of prescribed Gaussian curvature I - analytic aspects, to appear in *Adv. Math.*

[28] Smith G., Constant curvature hyperspheres and the Euler Characteristic, arXiv:1103.3235

[29] Smith G., Special Lagrangian Curvature, to appear in *Math. Ann.*

[30] Smith G., The Non-Liner Dirichlet Problem in Hadamard Manifolds, arXiv:0908.3590

[31] Smith G., The Plateau Problem for General Curvature Functions, arXiv:1008.3545

[32] Tromba A. J., The Euler characteristic of vector fields on Banach manifolds and a globalization of Leray-Schauder degree, *Adv. in Math.*, **28**, (1978), no. 2, 148–173

[33] White B., The space of m-dimensional surfaces that are stationary for a parametric elliptic functional, *Indiana Univ. Math. J.*, **36**, (1987), no. 3, 567–602

[34] White B., Every three-sphere of positive Ricci curvature contains a minimal embedded torus, *Bull. Amer. Math. Soc.*, **21**, (1989), no. 1, 71–75

[35] White B., Existence of smooth embedded surfaces of prescribed genus that minimize parametric even elliptic functionals on 3-manifolds, *J. Differential Geom.*, **33**, (1991), no. 2, 413–443

[36] White B., The space of minimal submanifolds for varying Riemannian metrics, *Indiana Univ. Math. J.*, **40**, (1991), no. 1, 161–200

[37] Ye R., Foliation by constant mean curvature spheres, *Pacific J. Math.*, **147**, (1991), no. 2, 381–396