The marginal tariff approach without single-crossing

A. Araujo[a,b], H. Moreira[b], S. Vieira[a]

[a] *Instituto Nacional de Matemática Pura e Aplicada, Estrada Dona Castorina 110, Rio de Janeiro, Brasil*
[b] *Graduate School of Economics, Getúlio Vargas Foundation, Praia de Botafogo, 190, Rio de Janeiro, Brazil*

Abstract

This paper studies a one-dimensional nonlinear pricing model where the single-crossing condition does not hold. The solution for the problem is based on optimally splitting the set of the demanded quantities in subintervals that characterize the higher and lower demand groups. In each one of them, the monopolist uses the demand profile approach. At the same time, due to the lack of the single-crossing property, a new global incentive constraint has to be considered, which prevents the migration across groups. We then modify the demand profile approach accordingly to suit our problem and give a characterization of the solution.

Keywords: single-crossing, marginal tariff, demand profile approach, Spence and Mirrlees condition (SMC)

JEL Codes: D42, D82.

1. Introduction

This paper explores a situation in which the single-crossing condition may be relaxed in the context of the single-product nonlinear pricing problem. With some few exceptions, mechanism design literature has assumed that if all private information is captured by a single scalar variable, the private informed part preference satisfies the single-crossing condition.

Mussa and Rosen [1] is considered the seminal paper on nonlinear pricing for the provision of quality-differentiated goods. Goldman et al. [2] and Maskin and Riley [3] consider a more general model with nonlinear pricing over quantities to explore, among other things, the optimality of quantity discounts.[1] An alter-

Email addresses: aloisio@impa.br (A. Araujo), Humberto.Moreira@fgv.br (H. Moreira), srgvie@gmail.com (S. Vieira)

[1] Mussa and Rosen [1] did not use the mechanism design approach for solving their problem. On the other hand, the other two papers described their models in the class of adverse selection principal-agent problems. Goldman et al. [2] solve their problem through type-assignment functions (inverse functions of the direct mechanism), which will be used for a case treated here where the single-crossing condition does not hold.

native formulation for solving the monopolistic screening problem, introduced by Brown and Sibley [4] and then fully explored in Goldman et al. [2], is the demand profile approach.[2] An important result in this literature is that these approaches are equivalent and gives the optimal mechanism when the SMC holds.

According to Armstrong [6], "The main results of previous work on the single-product nonlinear pricing problem have been (i) discovering ways to solve for the optimal tariff; (ii) showing that in many cases the firm will wish to separate completely its customers, so that customers with different tastes buy different quantities; (iii) showing that those customers with the strongest preferences for the good are served efficiently, with others being served lesser quantities than would be efficient in a world with full information, and (iv) showing that in many cases it is optimal for the firm to offer quantity discounts, so that the marginal price for a unit of the good decreases with the total quantity purchased."

Building on Laffont et al. [7] framework, we assume that the parameters that define the linear demand curve (the intercept and the - absolute value of the - slope) are positively related. Or equivalently, the market size (the maximum demand for positive prices) is negatively related with the limit price that makes the demand vanish. Therefore, in order to extract rent and efficiently allocate the good units among customers, the monopolist faces two opposite incentive constraints. For low parameter (type) values, the market size is relatively large and the usual rent extraction and distortion trade-off works. That is, in order to sell more for customers who value most the good, the monopolist gives a price discount to prevent their deviation to quantities offered to customers who value relatively less the good. For high parameter (type) values, the market size is relatively small and the monopolist is more limited to sell large quantities. By reducing the quantity offered for these types he may pool them with low types. Incentive compatibility tells us that the marginal tariff should be the same across pooling types. Therefore, the monopolist has to adapt the rent extraction and distortion trade-off taking into accounting these pooling effects. This essentially gives us the optimal nonlinear tariff under the demand profile approach. Alternatively, using the parametric utility approach, Araujo and Moreira [8] derive the same solution. [3]

However, the solution obtained by the demand profile approach can be too restrictive and the monopolist can do even better. Suppose that the monopolist separates the quantity range in two intervals (groups): the low and high demand groups. He can do it by simply determining an entry fee such that if the type is high enough he pays this entry fee and is treated as a high demand type. Otherwise, the type stays in the low demand group. The procedure within each group is the same as in the demand profile approach. The only thing

[2] In a recent survey on multidimensional screening problem, [5] make distinction between this approach and the previous one, which they called parameter utility approach.

[3] The book by Wilson [9] is a comprehensive text on (nonlinear) pricing models, there is no explicit reference to models where the SMC is violated.

that has to be determined is the cut-off price such that the highest type in the high demand group (who demands the lowest quantity in this group) is indifferent to move to the low demand group. This mechanism strictly improves the monopolist profit by relaxing incentive compatibility of a bunch of types in the high demand group that was pooling with types in the low demand group. Therefore, it provides the monopolist more leeway to solve the rent extraction and distortion trade-off within each group. We implement some numerical examples and explicitly quantify the profit increase according to an exogenous parameter that captures the market size effect. Our computation indicates that the gain may be significant.

Our results when the single-crossing fails are then (i) the demand profile may be suboptimal and we propose a welfare enhancing approach that combines the demand profile approach within groups of customers and an endogenous division in high and low demand customers; (ii) complete separation of customers may not be possible: there may be discrete and continuous pooling at the optimal nonlinear pricing; (iii) the strongest type is not the highest type, which may not be efficiently served; (iv) the optimality of quantity discounts remains valid within each group of customers, however we have to add an 'entry fee' in the tariff structure to separate the groups of customers. These features teach us that when the single-crossing condition is violated new and interesting properties may arise.

The plan of the paper is the following. In Section 2 we set out a model for analyzing the single good nonlinear pricing without the single-crossing. In Section 3 we present the demand profile and show how we can extend it when the single-crossing does not hold and completely characterize the solution. In Section 4 we show which kind of solution emerges when we use the unmodified demand profile and in Section 5 we give the conclusions. All proofs are relegated to the Appendix.

2. Model

We use the principal-agent framework to analyze the monopolistic nonlinear pricing problem. The firm and the customer contract on a pair (q, t), with a quantity $q \in Q \subseteq \mathbb{R}_+$ and a tariff $t \in \mathbb{R}$. The firm is a profit maximizing monopolist who produces any quantity $q \in Q$ at a cost $C(q)$. The firm's payoff is $t - C(q)$. The customer's preference is represented by $v(q, \theta) - t$, where $\theta \in \Theta = [\underline{\theta}, \overline{\theta}] \subseteq \mathbb{R}$ is a type parameter observed only by him. The firm has a prior distribution over Θ given by $F(\theta)$ with a continuous density $f(\theta) > 0$. We assume that $v(q, \theta)$ is thrice differentiable and that $C(q)$ is differentiable.

Using the *'Revelation Principle'* the monopolist's problem can be stated as choosing a pair $(q, t) : \Theta \to \mathbb{R}_+ \times \mathbb{R}$ that solves

$$\max_{q(\cdot),t(\cdot)} \int_{\Theta} [t(\theta) - C(q(\theta))] f(\theta) d\theta, \tag{Π}$$

subject to the *individual-rationality* constraints

$$v(q(\theta), \theta) - t(\theta) \geq 0, \quad \forall \theta \in \Theta, \tag{IR}$$

and the *incentive compatibility* constraints

$$v(q(\theta),\theta) - t(\theta) \geq v(q(\theta'),\theta) - t(\theta'), \quad \forall \theta, \theta' \in \Theta. \tag{IC}$$

We say that a quantity assignment function $q : \Theta \to \mathbb{R}_+$ is implementable if there exists a tariff function $t : \Theta \to \mathbb{R}$ such that the pair (q, t) satisfies the incentive compatibility constraints. The single-crossing condition provides a simple characterization of implementability.

Definition 1 (Single-Crossing Condition)**.** *Consider a utility function $v \in C^2$. We say that v satisfies the single-crossing (SCC) or Spence-Mirrlees condition (SMC) when we have either*

$$\forall\ (q,\theta)\ \textit{in}\ Q \times \Theta : v_{q\theta} > 0, \tag{CS_+}$$

or

$$\forall\ (q,\theta)\ \textit{in}\ Q \times \Theta : v_{q\theta} < 0. \tag{CS_-}$$

When v satisfies the single-crossing condition, one can show that implementability is equivalent to the monotonicity of the quantity assignment function, with $q(\cdot)$ increasing under (CS_+) or decreasing under (CS_-).[4]

2.1. Relaxing the Single-Crossing Condition

Following Araujo and Moreira [8], we relax the single-crossing condition by assuming the existence of a decreasing curve q_0 dividing the (θ, q) plane in two single-crossing regions, with $v_{q\theta} > 0$ below q_0 and $v_{q\theta} < 0$ above. By an abuse of notation, we use CS_+ and CS_- respectively to represent these regions, as we can see in Fig. 1. Formally, we assume that:

A1. *The equation $v_{q\theta}(q,\theta) = 0$ defines implicitly a function $q_0(\theta)$ such that $v_{q\theta}(q,\theta) > 0$ when $q < q_0(\theta)$ (CS_+) and $v_{q\theta}(q,\theta) < 0$ when $q > q_0(\theta)$ (CS_-). Moreover, $v_{q\theta^2} < 0$ and $v_{q^2\theta} < 0$ holds on $\mathbb{R}_{++} \times \Theta$.*

Although A1 relaxes the single-crossing condition, it still imposes some structure on the space of implementable quantity assignment functions $q(\cdot)$. Now we are going to explore some of its consequences. We begin with the local monotonicity conditions.

Proposition 1 (Local Monotonicity Conditions)**.** *Let q be an implementable quantity assignment function continuous at θ_0. Then:*

(i) If $q(\theta_0) < q_0(\theta_0)$, then q is increasing at θ_0;

[4] The proof can be found in Rochet [10] and Fudenberg and Tirole [11], chapter 7.

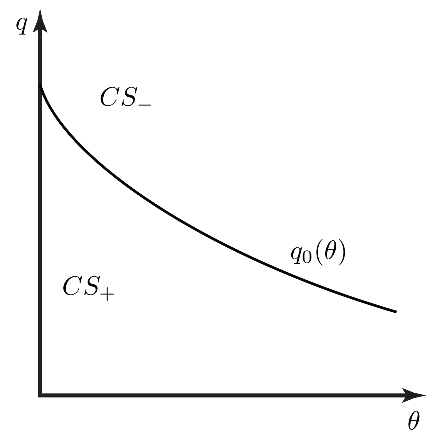

Figure 1: The curve $q_0(\theta)$ and the regions CS_+ and CS_-.

(ii) If $q(\theta_0) > q_0(\theta_0)$ *then* q *is decreasing at* θ_0.

In models where the single-crossing condition is satisfied, we know that the local monotonicity condition implies in implementability. However, under A1, these conditions given by Proposition 1 are no longer sufficient for implementability. Now, other constraints may be relevant for the problem. For instance, we have a marginal utility condition for types choosing the same quantity q. We use the *'Taxation Principle'*[5] to derive this new condition. This principle establishes that any pair $(q, t) : \Theta \to \mathbb{R}_+ \times \mathbb{R}$ satisfying the incentive compatibility constraints can be implemented by a nonlinear tariff $P : Q = q(\Theta) \to \mathbb{R}$, such that $P(q(\theta)) = t(\theta)$ for all $\theta \in \Theta$. From now on, we use the notation $P(q)$ for tariff and $p(q) = P'(q)$ for the marginal tariff. Using this tariff P, we can write the customer's problem as

$$\max_{q \in Q} v(q, \theta) - P(q).$$

Suppose that θ_1- and θ_2-type customers purchase the same quantity q. Then the first-order optimality condition of the customer's problem says that they must pay the same marginal tariff and get the same marginal utility.

Proposition 2 (Marginal Utility Condition)**.** *Suppose that* P *is differentiable at* $q \in q(\theta_1) \cap q(\theta_2)$ *an interior point of* $Q = q(\Theta)$. *Then:*

$$p(q) = v_q(q, \theta_1) = v_q(q, \theta_2). \tag{UC}$$

[5]This principle can be found in Guesnerie [12], Hammond [13] and Rochet [14].

2.2. Guiding Example

Let us introduce the guiding example for this paper. Laffont et al. [7] considered a problem where the monopolist is uncertain about the intercept and the slope of the customer's demand curve $p(q) = \alpha - \beta^2 q$. They were able to find the optimal contract when these two characteristics (α, β^2) are independently and uniformly distributed. On the other hand, when α and β have perfect positive correlation with $\alpha = \theta + 1$ and $\beta = b(\theta + 1)$, we can reduce the problem to a one-dimensional type problem, represented by the parameter θ that we may assume to be uniformly distributed on the closed interval $\Theta = [0, 1]$. The resulting utility function is

$$v(q, \theta) = (\theta + 1)q - b^2(\theta + 1)^2 \frac{q^2}{2}.$$

Thus, we have a family of utility functions indexed by the parameter b. When $b = 0$, this model is equivalent to Mussa and Rosen [1]. However, when $b > 0$, the utility function v does not satisfy the single-crossing condition. This is because the market size (i.e., the maximum demand with positive prices: $1/b^2(\theta + 1)$) is negatively related with the choke price (i.e, the lowest price where the demand vanishes: $\theta + 1$), as we can see in Fig. 2.[6] In particular, we can define for each

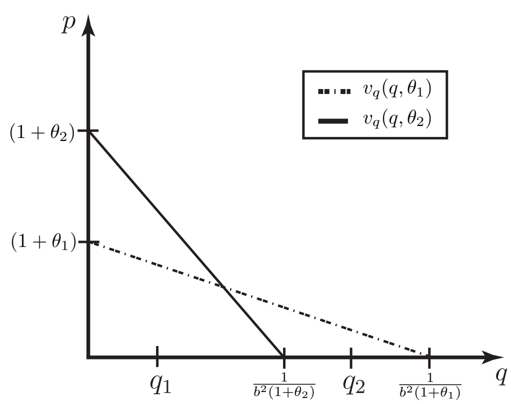

Figure 2: Inverse demand curves for θ_1 and θ_2 customers.

θ the function $q_0(\theta)$ as the solution of the equation $v_{q\theta}(q_0, \theta) = 0$:

$$q_0(\theta) = \frac{1}{2b^2(\theta + 1)}.$$

[6] Observe that in Figure 2 the inverse demand curves for customers θ_1 and θ_2 cross each other. Thus we have that for $q = q_1$, $v_q(q_1, \theta_1) < v_q(q_1, \theta_2)$ and for $q = q_2$, $v_q(q_2, \theta_1) > v_q(q_2, \theta_2)$.

Guiding Example. *We consider a customer with utility*

$$v(q,\theta) = (\theta+1)q - b^2(\theta+1)^2\frac{q^2}{2},$$

with $b \in (1/2, 8/10)$ *and* θ *uniformly distributed in* $[0,1]$ *and a monopolist with a quadratic cost function*

$$C(q) = \frac{q^2}{2}.$$

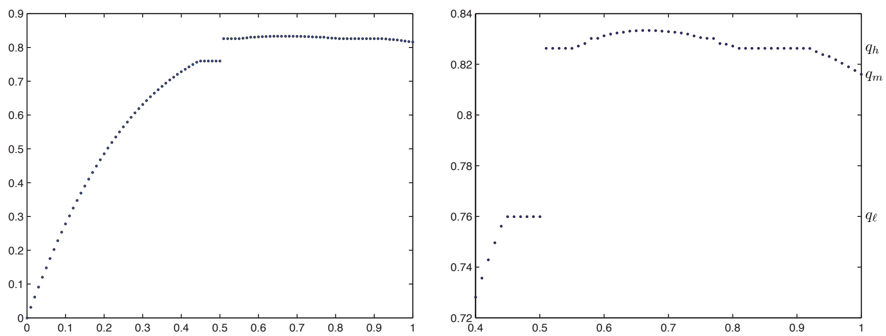

Figure 3: Numeric Solution for $b = \frac{6}{10}$.

As a first attempt to solve the monopolist's problem and to find the optimal quantity assignment function $q(\cdot)$ for the guiding example, we resorted to a numerical optimization package. We did a uniform grid with 101 points in the type set $\Theta = [0,1]$. Then we fed the computer software[7] with the discrete version of problem (Π). For this example, with $b = 6/10$, the numerical result for $q(\cdot)$ is depicted in Fig. 3[8]. This figure shows some interesting features of the optimal $q(\cdot)$. First, observe that it is non monotonic. Second, it has two bunching (flat) levels. Observe also, for the higher quantities, it has discrete pooling involving two types purchasing the same quantity. Finally, we can see a jump discontinuity separating high- and low-type customers and higher and lower purchased quantities. In Section 3, we will show how to characterize this solution, culminating with Fig. 6, where we can see the graph of both the numeric and the theoretical $q(\cdot)$ superposed.

This provides a motivation for the tariff we will propose in this paper. The strategy is to divide the type set in two groups: a low-type $[\underline{\theta}, \theta_d]$ and a high-type $[\theta_d, \overline{\theta}]$. For the first group, the monopolist offers quantities $q \in Q_1$ (low

[7]In this exercise we used AMPL and the KNITRO® software for numeric nonlinear optimization.

[8]On the right-hand side, we are zooming $q(\cdot)$ and only types in the interval $[4/10, 1]$ appear.

quantities) and for the second $q \in Q_2$ (high quantities). Moreover, we will consider a global incentive constraint that prevents a high-type customer from choosing a quantity $q \in Q_1$. We use then the demand profile approach[9] in each one of these groups, adding this global constraint, to find the required tariff.

2.3. The Direct Approach

Before introducing the demand profile, we are going to derive the monopolist's problem (Π) following Mussa and Rosen [1]. First, consider an incentive compatible pair $(q,t) : \Theta = [\underline{\theta}, \overline{\theta}] \to \mathbb{R}_+ \times \mathbb{R}$ and define the customer's informational rent as

$$V(\theta) = v(q(\theta), \theta) - t(\theta).$$

Then, using the envelope theorem from Milgrom and Segal [15] we get the marginal informational rent

$$V'(\theta) = v_\theta(q(\theta), \theta).$$

Next, we plug the informational rent $V(\cdot)$ into the objective function in (Π) and, after that, an integration by parts gives us the monopolist's expected profit

$$\int_\Theta \left[v(q,\theta) - C(q) - \frac{1 - F(\theta)}{f(\theta)} v_\theta(q,\theta) \right] f(\theta) d\theta - V(\underline{\theta}). \tag{1}$$

Assuming that $V(\cdot)$ is an increasing function we can set $V(\underline{\theta}) = 0$. Then, we define the virtual surplus by

$$g(q,\theta) = v(q,\theta) - C(q) - \frac{1 - F(\theta)}{f(\theta)} v_\theta(q,\theta). \tag{2}$$

The objective here is to find an implementable quantity assignment function $q(\cdot)$ that maximizes the monopolist's expected profit. We define a relaxed version of the monopolist's problem

$$\max_{q(\cdot)} \int_\Theta g(q(\theta), \theta) f(\theta) d\theta. \tag{Π_R}$$

The problem (Π_R) is called the monopolist's relaxed problem and its solution, represented by $q_R(\cdot)$, is the relaxed solution. Observe that when $q_R(\cdot)$ is implementable, it is the solution of the original problem (Π).

The relaxed problem (Π_R) is equivalent to the pointwise maximization problem:

$$q_R(\theta) = \arg\max_{q \in Q} g(q,\theta), \tag{3}$$

[9] Wilson Wilson [9] is a reference for the demand profile approach.

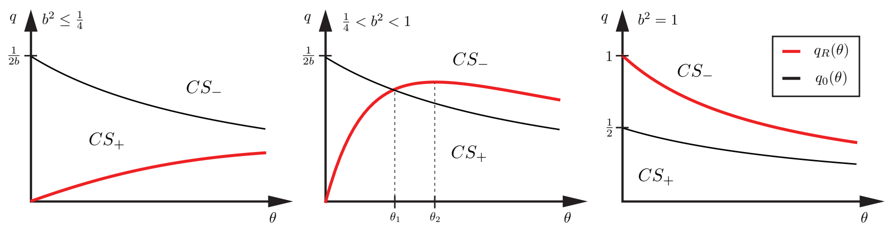

Figure 4: Three possibilities for the relaxed function $q_R(\theta)$.

for each $\theta \in \Theta$, and the Euler's equation for problem (Π_R) is just[10]

$$g_q(q, \theta) = 0, \tag{4}$$

which leads to the usual interpretation that the marginal benefit of quantity for a type θ, $v_q(q, \theta)f(\theta)$, must be equalized to the sum of its marginal cost, $C'(q)f(\theta)$, and the marginal rent left for the higher types, $v_{q\theta}(q, \theta)(1 - F(\theta))$.

For the virtual surplus, we are assuming the following:

A2. *The virtual surplus $g(\cdot, \theta)$ is concave for all $\theta \in \Theta$.*

This assumption assures that Euler's equation is a necessary and sufficient condition for optimality in (3). Considering our guiding example, the relaxed solution is given by

$$q_R(\theta) = \frac{2\theta}{1 + b\,(-1 + 2\theta + 3\theta^2)}. \tag{5}$$

When $b \in (1/2, 1)$, we observe in Figure 4 that the relaxed solution $q_R(\cdot)$ is a reverse U-shaped function that crosses $q_0(\cdot)$ increasing. Following Araujo and Moreira [8], we will focus in this case, assuming that:

A3. *The relaxed solution $q_R(\theta)$ has at most one peak (a global maximum) and increasing crosses $q_0(\theta)$. Moreover, $q_R(\underline{\theta}) < q_R(\overline{\theta})$.*

Notice that, when $b \in (1/2, 1)$, $q_R(\cdot)$ is not implementable. Indeed, we can find the points θ_1 and θ_2 as in Figure 4, such that, for all $\theta \in (\theta_1, \theta_2)$, the relaxed solution is increasing and $q_R(\theta) > q_0(\theta)$, contrary to Proposition 1.

Analyzing $q_R(\cdot)$ in light of Propositions 1 and 2, we see that even if an ironing procedure is performed to fix the monotonicity problem in CS_-, we still have to consider the constraint imposed by Proposition 2. Thus, to solve the monopolist's problem (Π), we need to use a method that takes into account the marginal utility condition (UC).

[10]We use the subindex q to represent the derivative with respect to the variable q.

3. The Demand Profile

The demand profile is a function that gives for each quantity $q \in Q$ the proportion of customers willing to pay the marginal tariff $p(q)$ for this unit. For all these customers the marginal utility from purchasing q exceeds the marginal tariff they pay for it, and so their marginal net benefit $v_q(q, \theta) - p(q)$ is positive. Hence, the demand profile function is formally defined by

$$N(p(q), q) = \Pr[v_q(q, \theta) \geq p(q)]. \tag{6}$$

The key observation regarding this function is that if the marginal tariff intersects each customer's demand function once and from below, then the demand profile $N(p(q), q)$ coincides with the fraction of customers purchasing q or more units of quantity.[11] As we will see below, the demand profile is useful for computing the monopolist's expected profit.

Now we are going to write the monopolist's problem (Π) using the demand profile. We adopt the notation $q_* = \inf\{q(\theta) : \theta \in \Theta\}$ and $q^* = \sup\{q(\theta) : \theta \in \Theta\}$ to denote the minimum and the maximum value of the quantity assignment function. Using the *'Taxation Principle'* and then the fundamental theorem of calculus, we have

$$\int_\Theta [P(q(\theta)) - C(q(\theta))] f(\theta) d\theta = P(q_*) - C(q_*) + \int_\Theta \int_{q_*}^{q(\theta)} [p(q) - C'(q)] dq f(\theta) d\theta. \tag{7}$$

Using Fubini's theorem we can write the right side of equation (7) as

$$P(q_*) - C(q_*) + \int_{q_*}^{q^*} [p(q) - C'(q)] \Pr[q \leq q(\theta)] dq. \tag{8}$$

The function defined by

$$\mathcal{N}(P(\cdot), q) = \Pr[q \leq q(\theta)]$$

represents the fraction of customers purchasing a quantity equal or greater than q. In general, this function depends on the entire tariff $P(\cdot)$ chosen by the monopolist. However, when for all customers the marginal tariff crosses the demand curve once and from below, $\mathcal{N}$ will depend only on the marginal tariff $p(q)$, and it will coincide with the demand profile function:

$$\mathcal{N}(P(\cdot), q) = N(p(q), q). \tag{DPA}$$

Under (DPA), we can rewrite expression (8) as

$$P(q_*) - C(q_*) + \int_{q_*}^{q^*} [p(q) - C'(q)] N(p(q), q) dq. \tag{9}$$

---

[11] Notice that under this condition a customer will buy q or more units of the good if and only if $v_q(q, \theta) - p(q) \geq 0$, at least when P is differentiable at q. Moreover, the monopolist will not offer quantities q at a marginal price $p(q)$ exceeding the maximum value of $v_q(q, \cdot)$.

In our model specification, by Assumption 1, the marginal utility function $v_q(q, \cdot)$ is concave for all $q > 0$ (it has the shape depicted in Fig. 5, increasing in CS_+ and decreasing in CS_-). Hence, for each positive p in the range of $v_q(q, \cdot)$ (i.e. $p \in v_q(q, \Theta)$), we can define its pseudo-inverses $\theta_s(p, q)$ and $\theta_b(p, q)$ as[12]

$$\theta_s(p, q) = \sup\{\theta \in \Theta : v_q(q, \theta) \geq p\} \text{ and } \theta_b(p, q) = \inf\{\theta \in \Theta : v_q(q, \theta) \geq p\}.$$

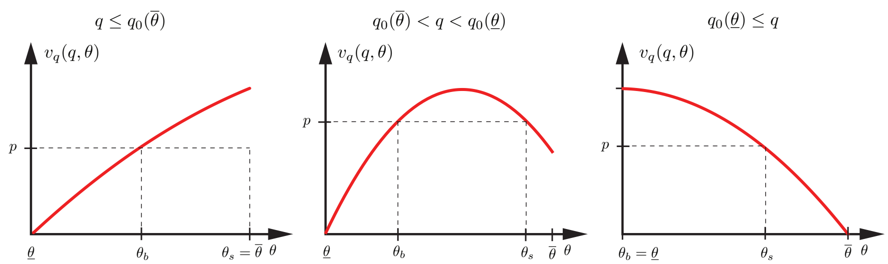

Figure 5: The pseudo-inverses θ_s and θ_b.

Using these pseudo-inverses the demand profile function is given by

$$N(p, q) = F(\theta_s(p, q)) - F(\theta_b(p, q)), \tag{10}$$

once types that have positive benefit purchasing q at marginal tariff p are precisely $\theta \in [\theta_b(p, q), \theta_s(p, q)]$. Under condition (DPA), these pseudo-inverses can be interpreted as type-assignments functions.[13]

Having Fig. 3 in mind, we examine what happens when exactly at θ_d-customer, $q(\cdot)$ has a jump with extremities q_ℓ and q_h. On one hand, assuming (DPA) implies in a tariff that makes θ_d-customer indifferent among all quantities in the interval $[q_\ell, q_h]$ (i.e. $[q_\ell, q_h] \subset q(\theta_d)$). This shows how restrictive this condition can be, as it imposes that the correspondence $q(\cdot)$ is convex valued. On the other hand, the existence of the left- and right- hand side limits of the quantity assignment function at θ_d and his indifference between these extremes points[14] are the only necessary conditions that must be satisfied.

[12]Sometimes, with some abuse of notation, we represent the pseudo-inverses by θ_s and θ_b only. We could have extended the definition of these pseudo-inverses to p outside the range of $v_q(q, \cdot)$ (i.e. $p \notin v_q(q, \Theta)$) using the convention $\theta_s(p, q) = \theta_b(p, q) = arg \max_\Theta v_q(q, \theta)$. However, the monopolist will not offer quantities q at a marginal price p greater than the maximum value $v_q(q, \theta)$ and so this is not a concern for us here.

[13]Notice that when condition (DPA) is satisfied, and $P(\cdot)$ is differentiable at q, then $p(q) = v_q(q, \theta_b)$ implies that q is an optimal choice for customer θ_b and $p(q) = v_q(q, \theta_s)$ implies that q is an optimal choice for customer θ_s.

[14]The existence of these limits is a consequence of the local monotonicity of the quantity assignment function shown in Proposition 1. The indifference property is a trivial consequence of the Maximum Theorem.

3.1. Extending the Demand Profile

In Section 4, we are going to characterize the optimal mechanism under (DPA). Here, instead of assuming (DPA), we are going to consider tariffs that make the θ_d-customer indifferent between quantities q_ℓ and q_h. This is equivalent to consider a tariff $P(q)$ satisfying the condition

$$v(q_\ell, \theta_d) - P(q_\ell) = v(q_h, \theta_d) - P(q_h), \tag{11}$$

or equivalently

$$\int_{q_\ell}^{q_h} [p(q) - v_q(q, \theta_d)]dq = 0. \tag{J}$$

Looking again at Fig. 3, we can see that some intermediate quantities are not offered by the monopolist. Indeed, define $q_m = q(\overline{\theta})$, then observe that customers are not purchasing $q \in (q_\ell, q_m)$. By simplicity, we will assume that the $\overline{\theta}$-customer is indifferent between q_m and q_ℓ[15],

$$v(q_m, \overline{\theta}) - P(q_m) = v(q_\ell, \overline{\theta}) - P(q_\ell). \tag{12}$$

Hence, the tariff is designed in such way that customers that demand the lowest quantity in the high demand group do not envy the highest quantity in the low demand group. This gives a natural separation of customers according to the interval they consume: low demand customers (with types below θ_d) and high demand customers (with types above θ_d).

Let us derive the monopolist's problem accordingly to allow for these more general tariffs. The main question is how we have to modify the function $\mathcal{N}(P(\cdot), q)$ in the interval $[q_\ell, q_h]$ for these tariffs. For this, notice that the only potential types that would demand quantities in this interval are above θ_d. Therefore, this demand function is given by

$$\tilde{N}(p(q), q) = F(\theta_s(p(q), q)) - F(\theta_d),$$

for all $q \in [q_\ell, q_h]$.[16] This corresponds to a more flexible tariff in this interval since we are relaxing the condition $p(q) = v_q(q, \theta_d)$ for all $q \in [q_\ell, q_h]$ imposed by (DPA) in the case of a jump at $q(\theta_d)$.

[15] In general, the $\overline{\theta}$-customer could strictly prefer q_m to q_ℓ. Let θ_1 solve $p(q_\ell-) = v_q(q_\ell, \theta_1)$. If $\theta_1 < \theta_d$ and $v(q_m, \overline{\theta}) - P(q_m) > v(q_\ell, \overline{\theta}) - P(q_\ell)$, then the assumption means that we can change q_ℓ in a way that we still have $\theta_1 < \theta_d$ but we get the equality $v(q_m, \overline{\theta}) - P(q_m) = v(q_\ell, \overline{\theta}) - P(q_\ell)$.

[16] Observe that when $q \in [q_\ell, q_h]$ the lower type-assignment function is always θ_d but $\theta_b(p(q), q)$ is not necessarily equal to θ_d. Remember that we defined (θ_b, θ_s) as the pseudo-inverses of $v_q(q, \cdot)$, while type-assignment functions are the (pseudo-)inverse of $q(\theta)$. Under the demand profile approach they coincide but now they may be different.

Taking into consideration the indifference condition (J), we can rewrite the modified monopolist problem as

$$\max_{P(\cdot)} \begin{cases} P(q_*) - C(q_*) & + \displaystyle\int_{q_*}^{q_\ell} [p(q) - C'(q)] N(p(q), q) dq \\ & + \displaystyle\int_{q_\ell}^{q_h} [p(q) - C'(q)] \tilde{N}(p(q), q) dq \\ & + \displaystyle\int_{q_h}^{q^*} [p(q) - C'(q)] N(p(q), q) dq \\ \\ \text{subject to (J).} & \end{cases} \tag{Π_{J}}$$

Notice that (Π_{J}) is an isoperimetric problem[17]. The technique for solving this problem is to append the constraint (J) to the objective function using a Lagrange multiplier λ. Then we proceed with a local optimality analysis - using the Euler's equation - to find the optimal marginal tariff $p(q)$ for each $q \in Q$. Thus, in problem (Π_{J}), the first-order necessary conditions for optimality are given by

$$N(p(q), q) + \frac{\partial N}{\partial p}(p(q), q)[p(q) - C'(q)] = 0, \tag{13}$$

when $q \in (q_*, q_\ell) \cup (q_h, q^*)$, and

$$\tilde{N}(p(q), q) + \frac{\partial \tilde{N}}{\partial p}(p(q), q)[p(q) - C'(q)] = \lambda, \tag{14}$$

when $q \in (q_m, q_h)$. The quantities $q \in (q_\ell, q_m)$ are not offered by the monopolist, so we do not have to consider them.

Next, we are going to use the virtual surplus $g(q, \theta)$ to get an equivalent but more detailed description of the first-order necessary condition. For example, let $q \in (q_*, q_\ell) \cup (q_h, q^*)$ and $p = p(q)$ and suppose that $v_q(q, \underline{\theta}) < p < v_q(q, \overline{\theta})$. Then $p = v_q(q, \theta_b)$ and, according to equation (10), the demand profile function is given by

$$N(p, q) = 1 - F(\theta_b(p, q)). \tag{15}$$

So the derivative of $N(p, q)$ with respect to p is

$$\frac{\partial N}{\partial p}(p, q) = -f(\theta_b) \frac{\partial \theta_b}{\partial p}(p, q) = -\frac{f(\theta_b)}{v_{q\theta}(q, \theta_b)}, \tag{16}$$

where in the last equality we used the inverse function theorem for the calculus of $\partial \theta_b / \partial p$.

[17] In Kamien and Schwartz [16] (Part I, Section 7), we can find a description of the problem and the technique used to solve it.

Now, plugging (15) and (16) into equation (13) and then multiplying the result by $-v_{q\theta}(q,\theta_b)$, we obtain

$$g_q(q,\theta_b)f(\theta_b) = \left(v_q(q,\theta_b) - C'(q) - \frac{1-F(\theta_b)}{f(\theta_b)} v_{q\theta}(q,\theta_b)\right) f(\theta_b) = 0. \tag{17}$$

Repeating the same reasoning used in the derivation of equation (17) but considering also other possibilities for $q \in Q$ and $p \geq 0$ in the range of $v_q(q,\cdot)$ we get the following:

Theorem 1. *The first-order necessary optimality conditions for $p = p(q)$ in problem (Π_{J}) are characterized,*

(a) for quantities $q \in (q_*, q_\ell) \cup (q_h, q^*)$ by

(i) if $v_q(q,\underline{\theta}) < p < v_q(q,\overline{\theta})$, then $p = v_q(q,\theta_b)$ and

$$g_q(q,\theta_b) = 0. \tag{18}$$

(ii) if $\max\{v_q(q,\underline{\theta}), v_q(q,\overline{\theta})\} < p < \max_{\theta\in\Theta} v_q(q,\theta)$, then $p = v_q(q,\theta_b) = v_q(q,\theta_s)$ and

$$\frac{g_q(q,\theta_b)}{v_{q\theta}(q,\theta_b)} f(\theta_b) = \frac{g_q(q,\theta_s)}{v_{q\theta}(q,\theta_s)} f(\theta_s). \tag{19}$$

(b) for quantities $q \in (q_m, q_h)$ by

(iii) if $v_q(q,\overline{\theta}) < p < v_q(q,\theta_d)$, then $p = v_q(q,\theta_s)$ and

$$\frac{g_q(q,\theta_s)}{v_{q\theta}(q,\theta_s)} f(\theta_s) = \lambda - (1 - F(\theta_d)). \tag{20}$$

Observe that equation (18) is the standard optimality condition in the literature of monopolistic screening. It states that at the optimum, the marginal virtual surplus is zero. Equation (19) is a little bit different, as it takes into account the (UC) condition. This equation was also derived by Araujo and Moreira [8] and has a straightforward interpretation. The trade-off between rent extraction and allocative distortion (i.e., $g_q(q,\cdot)$) measured in the customer's marginal rent unit (i.e., $v_{q\theta}(q,\cdot)$) must be the same for discrete pooled types. Finally, in equation (20), the Lagrange multiplier λ is present. It is the shadow price of the global constraint (J). If we define $\tilde{\lambda} = \lambda - (1 - F(\theta_d))$ then (20) is equivalent to

$$\frac{g_q(q,\theta_s)}{v_{q\theta}(q,\theta_s)} f(\theta_s) = \tilde{\lambda}. \tag{21}$$

i.e., the trade-off between rent extraction and allocative distortion (i.e., $g_q(q,\cdot)$) measured in the customer's marginal rent unit (i.e., $v_{q\theta}(q,\cdot)$) must be equal to the modified Lagrange multiplier $\tilde{\lambda}$, the difference between the shadow price of constraint (J) and the proportion of high taste customers (i.e. $1 - F(\theta_d)$).

For a fixed quartet q_ℓ, q_h, θ_d and λ, we use Theorem 1 to find $p(q)$ satisfying the first-order optimality condition for the each $q \in Q$. After that, we have to optimize the choice of q_ℓ, q_h and θ_d. In the next theorem, we derive the necessary first-order conditions for this optimization problem.

Theorem 2. *The first-order conditions for the problem of optimally choosing q_ℓ, q_h and θ_d are*

$$\text{(i)}\ \frac{\displaystyle\int_{I(q_\ell)} g_q(q_\ell,\theta) f(\theta) d\theta}{\displaystyle\int_{\theta_d}^{\bar{\theta}} v_{q\theta}(q_\ell,\theta) d\theta} = 1 - F(\theta_d) - \lambda; \tag{22}$$

$$\text{(ii)}\ \frac{\displaystyle\int_{I(q_h)} g_q(q_h,\theta) f(\theta) d\theta}{\displaystyle\int_{\theta_d}^{\theta_4} v_{q\theta}(q_h,\theta) d\theta} = \lambda - (1 - F(\theta_d)); \tag{23}$$

$$\text{(iii)}\ \frac{\displaystyle\int_{q_\ell}^{q_h} g_q(q,\theta_d) dq}{\displaystyle\int_{q_\ell}^{q_h} v_{q\theta}(q,\theta_d) dq} f(\theta_d) = \lambda - (1 - F(\theta_d)), \tag{24}$$

where λ is the Lagrange multiplier associated with constraint (J). The sets $I(q_\ell)$ and $I(q_h)$ are defined by $I(q_\ell) = [\theta_1, \theta_d] = \{\theta : q(\theta) = q_\ell\}$ and $I(q_h) = [\theta_d, \theta_2] \cup [\theta_3, \theta_4] = \{\theta : q(\theta) = q_h\}$, where $\theta_1 = \theta_b(p(q_\ell-), q_\ell)$, $\theta_2 = \theta_b(p(q_h+), q_h)$, $\theta_3 = \theta_s(p(q_h+), q_h)$ and $\theta_4 = \theta_s(p(q_h-), q_h)$.

Let us analyze the conditions derived in Theorem 2. Fixing λ and q_h, equation (22) says that a variation in q_ℓ must be compensated by a variation in θ_d. The reason is that a change in q_ℓ has an impact in the prices $P(q_\ell)$ and $P(q_h)$. Thus, we have to adjust θ_d to satisfy equation (11). For equation (23) we have a similar analysis, but now we have a variation in q_h. From the first two equations, we see that changes in q_ℓ and q_h will induce changes in the prices $P(q_\ell)$ and $P(q_h)$. This will be accommodated by changes in θ_d. Observe that we have two groups of quantities, $[q_*, q_\ell]$ and $[q_m, q^*]$ and with these changes in prices, customers that are close to θ_d may be encouraged to change from one group to the other.

We can eliminate the Lagrangian multiplier λ from the equations in Theorems 1 and 2.

Corollary 1. *After eliminating the Lagrangian multiplier λ from the equations* *in Theorems 1 and 2 we get*

$$(i)\ \frac{\displaystyle\int_{\{\theta:q(\theta)=q_\ell\}} g_q(q_\ell,\theta)f(\theta)d\theta}{\displaystyle\int_{\theta_d}^{\overline{\theta}} v_{q\theta}(q_\ell,\theta)d\theta} + \frac{\displaystyle\int_{\{\theta:q(\theta)=q_h\}} g_q(q_h,\theta)f(\theta)d\theta}{\displaystyle\int_{\theta_d}^{\theta_4} v_{q\theta}(q_h,\theta)d\theta} = 0 \tag{25}$$

$$(ii)\ \frac{\displaystyle\int_{q_\ell}^{q_h} g_q(q,\theta_d)dq}{\displaystyle\int_{q_\ell}^{q_h} v_{q\theta}(q,\theta_d)dq} f(\theta_d) = \frac{g_q(q,\theta_s)}{v_{q\theta}(q,\theta_s)} f(\theta_s),\ \textit{for}\ q\in(q_m,q_h). \tag{26}$$

Equation (25) is a combination of equations (22) and (23). It characterizes the bunching levels q_ℓ and q_h. Notice that it resembles the standard ironing condition[18]. Indeed, now we have two bunching levels q_ℓ and q_h and the constraint (J) to take into account. For each bunching level, if we correctly weight the average of the marginal virtual surplus then the terms will add up zero.

Equation (24) is the optimality condition for θ_d. Combining equations (20) and (24) we get (26). Notice that in the right-hand side of (26) we have the trade-off between rent extraction and allocative distortion measured in the customer's marginal rent unit. Therefore, we can interpret the left-hand side as an average measure (with $q\in[q_\ell,q_h]$) of the same kind for the θ_d-customer.

A final comment is about the interpretation of the solution characterized by Theorems 1 and 2. A way the monopolist can implement such solution is to design a system of two nonlinear pricing schedules, one for each group of customers, low and high demand groups, and an entry fee. In each the monopolist employes the usual demand profile approach. The entry fee is endogenously determined by the difference between the price charged to the customer who demands the highest quantity in the low demand group and the price charged to the customer who demands the lowest quantity in the high demand group. This mechanism strictly improves the monopolist profit because it relaxes incentive compatibility for a bunch of types in the high demand group that pool with types in the low demand group. This gives the monopolist more leeway to solve the rent extraction and distortion trade-off within each group.

3.2. Application

Guiding Example continued. *Now we apply the method to our guiding example with* $b = 6/10$.

[18] When we are under the single-crossing condition, and we need to use an ironing procedure to restore the monotonicity of $q(\cdot)$, this ironing is characterized by $\int_{\{\theta:q(\theta)=q\}} g_q(q,\theta)f(\theta)d\theta = 0$. That is, the average of the marginal virtual surplus is zero.

Let $p(q)$ satisfy the first-order conditions given by equations (13) and (14). Then, using Theorem 1 we can find the type-assignment functions $\psi_b(q)$ and $\psi_s(q) = \theta_s(p(q), q)$.[19] The result is

$$\psi_b(q) = \begin{cases} \theta_b(p(q),q) = -\frac{\sqrt{-351q^2-450q+625}+9q-25}{27q}, & \text{if } 0 \leq q \leq q_\ell, \\ \theta_d, & \text{if } q_\ell \leq q \leq q_h, \\ \theta_b(p(q),q) = \frac{-54q^2-5\sqrt{75q^2-108q^4}+75q}{54q^2}, & \text{if } q_h \leq q \leq \frac{5}{6}, \end{cases}$$

and

$$\psi_s(q) = \begin{cases} 1, & \text{if } 0 \leq q \leq q_m, \\ \frac{\sqrt{27q^2(3\mu^2-12\mu-13)+225q(\mu-2)+625}-9q(\mu+1)+25}{27q}, & \text{if } q_m \leq q \leq q_h, \\ \frac{-54q^2+5\sqrt{75q^2-108q^4}+75q}{54q^2}, & \text{if } q_h \leq q \leq \frac{5}{6}, \end{cases}$$

where $q_m = \frac{25(\mu+2)}{36\mu+61}$ and $\mu := -[\lambda - (1 - F(\theta_d))]$.

After that, we solve the four equations given by Theorem 2 and condition (J) to find q_ℓ, q_h, θ_d and λ:

$$q_\ell \approx 0.762548, \quad q_h \approx 0.825831, \quad \theta_d \approx 0.503686 \quad \text{and} \quad \lambda \approx 0.449729.$$

Finally, inverting $\psi_b(q)$ and $\psi_s(q)|_{[q_m, \frac{5}{6}]}$, we get the quantity assignment function

$$q(\theta) = \begin{cases} \frac{50\theta}{27\theta^2+18\theta+16} & \text{if } 0 \leq \theta \leq \theta_1, \\ \frac{50\theta_1}{27\theta_1^2+18\theta_1+16} & \text{if } \theta_1 \leq \theta \leq \theta_d, \\ \frac{25\left(9\theta_2+\sqrt{27\theta_2^2+54\theta_2-23}+9\right)}{6\left(27\theta_2^2+54\theta_2+52\right)} & \text{if } \theta_d \leq \theta \leq \theta_2 \text{ or if } \theta_3 \leq \theta \leq \theta_4, \\ \frac{25\left(\sqrt{27\theta^2+54\theta-23}+9\theta+9\right)}{6(27\theta^2+54\theta+52)} & \text{if } \theta_2 \leq \theta \leq \theta_3, \\ \frac{25(2\theta+\mu)}{27\theta^2+18\theta(\mu+1)+2(9\mu+8)} & \text{if } \theta_4 \leq \theta \leq 1, \end{cases} \tag{27}$$

where,

$$\theta_1 \approx 0.452637, \quad \theta_2 \approx 0.551809, \quad \theta_3 \approx 0.811806 \quad \text{and} \quad \theta_4 \approx 0.930762.$$

In Figure 6 we superpose the graph of $q(\theta)$, given by expression (27), and the numerical solution, already depicted in Fig. 3. We can see that the proposed method is capable of capturing all the nuances observed in Fig. 3. We leave the discussion about the implementability of this $q(\cdot)$ to the Appendix C.

The marginal tariff is given by

$$p(q) = \begin{cases} v_q(q, \psi_b(q)), & \text{if } 0 \leq q < q_l, \\ v_q(q, \psi_s(q)), & \text{if } q_m \leq q < q_h, \\ v_q(q, \psi_b(q)) = v_q(q, \psi_s(q)), & \text{if } q_h \leq q < q^* = \frac{5}{6}. \end{cases}$$

[19] The type-assignment functions $\psi_b(q)$ and $\psi_s(q)$ are the lower and the upper pseudo-inverses of the quantity assignment function $q(\cdot)$.

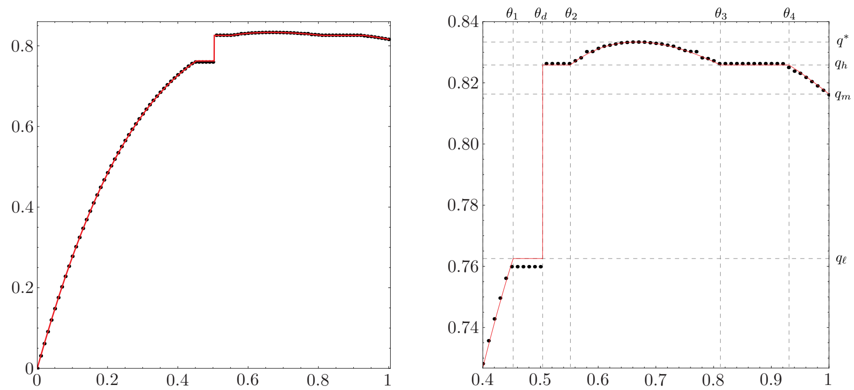

Figure 6: The quantity assignment function.

In Fig. 7 we present the graph of $p(q)$. For quantities $q \in (q_\ell, q_m)$ the dotted line represents $v_q(q, \overline{\theta})$.[20] Observe that at quantities q_ℓ and q_h the graph is discontinuous. These quantities corresponds to the bunching regions q_ℓ and q_h in the graph of $q(\cdot)$. The monopolist's expected profit can be computed using the objective function in (Π_J)[21]. The result is

$$\text{Profit} \approx 0.390005. \tag{28}$$

4. Imposing DPA Condition

In the previous section, we motivated how restrictive the condition (DPA) can be when we have a jump discontinuity in the quantity assignment function $q(\cdot)$. In this section, we do not relax the condition (DPA). Instead, we show how to use the demand profile approach and derive the same results found in Araujo and Moreira [8]. There we can find two classes of quantity assignment functions, the first one, with $q(\underline{\theta}) \leq q(\overline{\theta})$ and the second one, with $q(\underline{\theta}) > q(\overline{\theta})$.

We begin our analysis by observing expression (9), that gives the monopolist's expected profit under (DPA). We are going to pin down the expression for $P(q_*)$, which has a direct impact on the monopolist's expected profit. We assume that the customer's rent is increasing with type[22]. Hence, at the optimum,

[20] Although the monopolist do not offer quantities $q \in (q_\ell, q_m)$, it is possible to define a tariff for these quantities in a way that customers do not have incentive to change their purchases. For instance, if we set $P(q) = \infty$ in this interval, customers will not buy any q in this interval. In the Appendix, we show that for this example, we can set the marginal tariff $p(q) = v_q(q, \overline{\theta})$ for all $q \in (q_\ell, q_m)$. This tariff will not change the customer's behavior.

[21] We can use also expression (1) in this computation.

[22] This assumption is justified because the quantity assignment function $q(\cdot)$ solution of our problem will satisfy $v_\theta(q(\theta), \theta) \geq 0$. On the other hand, $v_\theta(q(\theta), \theta)$ is the derivative of the customer's rent, or informational rent.

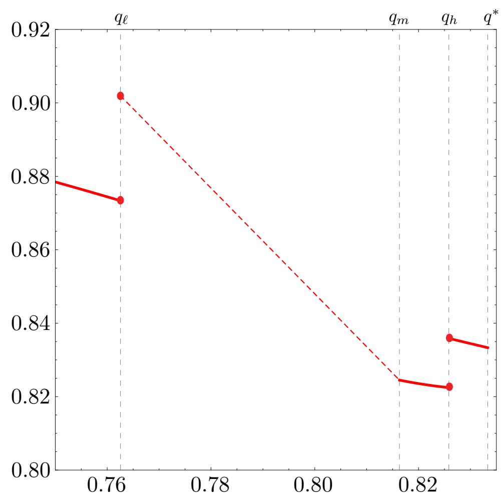

Figure 7: The marginal tariff $p(q)$.

we have

$$P(q(\underline{\theta})) = v(q(\underline{\theta}), \underline{\theta}). \tag{29}$$

Let us divide the set of quantity assignment functions into two subsets, depending on the quantity chosen by the boundary customers $\underline{\theta}$ and $\overline{\theta}$. The first one is formed by all $q(\cdot)$ such that $q(\underline{\theta}) \leq q(\overline{\theta})$ and the second one by all $q(\cdot)$ such that $q(\underline{\theta}) > q(\overline{\theta})$. In each one of these subsets we will have a different expression for $P(q_*)$. Indeed, when $q(\cdot)$ belongs to the first subset, $q_* = q(\underline{\theta})$ and equation (29) gives

$$P(q_*) = v(q(\underline{\theta}), \underline{\theta}). \tag{30}$$

On the other hand, when $q(\cdot)$ belongs to the second subset, we have $q_* = q(\overline{\theta})$. Then, equation (29) and the fundamental theorem of calculus imply that

$$P(q_*) = v(q(\underline{\theta}), \underline{\theta}) - \int_{q_*}^{q(\underline{\theta})} p(q)dq. \tag{31}$$

Since we have two different expressions for $P(q_*)$, these subsets will have separate treatments in the analysis that follows. In particular, we will find two distinct solutions that we will have to compare to determine the optimum under (DPA).

The first case, when $q(\cdot)$ is such that $q(\underline{\theta}) \leq q(\overline{\theta})$, will be presented here. In the end, we will compare the solution we get here with the one in Section 3. The second case will be relegated to the Appendix A, because it is not essential for this kind of comparison.

4.1. Case A ($q(\underline{\theta}) \leq q(\overline{\theta})$).

From expression (9), the monopolist's problem is then

$$\max_{P(\cdot)} P(q_*) - C(q_*) + \int_{q_*}^{q^*} [p(q) - C'(q)]N(p(q), q)dq, \tag{Π_{A}}$$

where $N(p, q)$ satisfies equation (DPA).

Notice that the basic difference between (Π_{A}) and (Π_{J}) is the absence of the (J) condition. Thus, using the same kind of arguments from Theorem 1, we can derive analogous conditions for problem (Π_{A}). We present the results in the next:

Theorem 3. *The first-order necessary optimality conditions for $p = p_U(q)$ in problem (Π_{A}) are characterized by*

(i) if $v_q(q, \underline{\theta}) < p < v_q(q, \overline{\theta})$, then $p = v_q(q, \theta_b)$ and

$$g_q(q, \theta_b) = 0; \tag{32}$$

(ii) if $\max\{v_q(q, \underline{\theta}), v_q(q, \overline{\theta})\} < p < \max_{\theta \in \Theta} v_q(q, \theta)$, then $p = v_q(q, \theta_b) = v_q(q, \theta_s)$ and

$$\frac{g_q(q, \theta_b)}{v_{q\theta}(q, \theta_b)} f(\theta_b) = \frac{g_q(q, \theta_s)}{v_{q\theta}(q, \theta_s)} f(\theta_s). \tag{33}$$

Theorem 3 gives us a candidate for the optimal marginal tariff, which we call $p_U(q)$. In Corollary 2 we apply Theorem 3 in our guiding example with $b \in (\frac{1}{2}, \frac{2}{\sqrt{5}})$. In Figure 8 we depict $\theta_b(p_U(q), q)$ and $\theta_s(p_U(q), q)$ according to three possible values for parameter b.

Corollary 2. *When $b \in (\frac{1}{2}, \frac{2}{\sqrt{5}})$, we can find $q_1(b), q_2^{\ell}(b)$ and $q_2^h(b)$ such that the optimal marginal tariff $p_U(q)$ satisfies*

$$\theta_b(p_U(q), q) = \begin{cases} \theta_b^1(q), & \textit{if } 0 \leq q < q_1(b), \\ \hat{\theta}_b(q), & \textit{if } q_1(b) \leq q < q_2^{\ell}(b), \\ \underline{\theta}, & \textit{if } q_2^{\ell}(b) \leq q < q_2^h(b), \\ \theta_b^2(q), & \textit{if } q_2^h(b) \leq q \leq \frac{1}{2b}, \end{cases}$$

where θ_b^1 is defined by equation (32) with $p = v_q(q, \theta_b^1)$, θ_b^2 by equation (33) with $p = v_q(q, \theta_b^2)$, and $\hat{\theta}_b$ implicitly by the solution of $v_q(q, \hat{\theta}_b) = v_q(q, \overline{\theta})$.[23]

[23]In the Appendix we give explicit expressions for θ_b^1 and θ_b^2. Besides, we may have $q_2^{\ell}(b) = q_2^h(b)$.

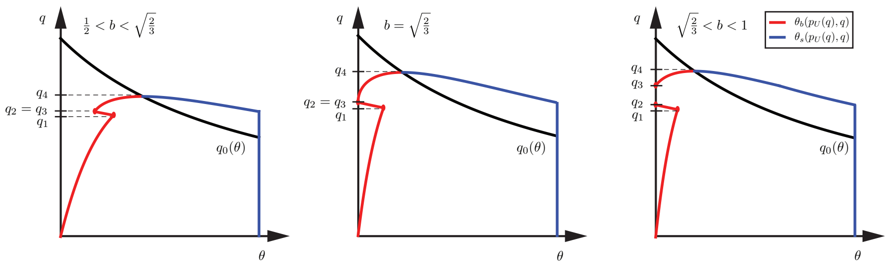

Figure 8: The pseudo-inverse corresponding to $p_U(q)$.

Now, we have to check whether this marginal tariff is in accordance to the demand profile approach or not, that is, whether $p_U(q)$ satisfies equation (DPA). Take a type where the vertical line $q = \theta$ crosses the pseudo-inverse $\theta_b(p_U(q), q)$ at three points in Figure 8. This corresponds to a situation where the marginal tariff crosses the demand curve exactly at these three points which is inconsistent with the demand profile approach. That is, this marginal tariff is not crossing their demand curve once and from below as required by the demand profile approach. The problem is that the lack of monotonicity of $\theta_b(p_U(q), q)$ prevents to associate a quantity assignment function to it in the interval $[q_1, q_4]$.

Following Goldman et al. [2] and Wilson [9], let us fix this problem by using a vertical ironing procedure, as illustrated in Figure 9. The idea is to bunch all quantities in the interval $[q_\ell, q_h]$ to the same type θ_d accordingly modifying the marginal tariff, now represented by $p(q)$. That is, we adjust the marginal tariff such that it coincides with the demand curve of θ_d-customer on the interval $[q_\ell, q_h]$. For quantities outside $[q_\ell, q_h]$, the marginal tariff remains the same. In particular, this new marginal tariff is such that $p(q) = v_q(q, \theta_d)$ and $\theta_b(p(q), q) = \theta_d$, for all $q \in [q_\ell, q_h]$.

The optimal ironing procedure consists in optimally finding q_ℓ, q_h and θ_d. Theorem 4 gives the optimality conditions for choosing θ_d, q_ℓ and q_h.

Theorem 4. *The optimal ironing is characterized by $p(q) = v_q(q, \theta_d)$ for all $q \in [q_\ell, q_h]$, where q_ℓ and q_h are the intersection of the line $q = \theta_d$ with $\theta_b(p_U(q), q)$ such that*

$$\int_{q_\ell}^{q_h} v_{q\theta}(q, \theta_d) \left. \frac{\partial [p - C'(q)] N(p, q)}{\partial p} \right|_{p = v_q(q, \theta_d)} dq = 0, \tag{34}$$

if θ_d is an interior point. If $\theta_d = \underline{\theta}$, we have a corner solution and the equality in equation (34) is replaced by the inequality '$\leq$'.

If one develops the calculus of the partial derivative in Theorem 4, then one would get a generalization of the condition derived in Theorem 3 (i) (see Araujo and Moreira [8] for details). Theorems 3 and 4 give the solution for problem (Π_{A}). The possible shapes of the optimal quantity assignment function $q(\cdot)$ are illustrated in Figure 9. Now it is easy to see that the associated marginal

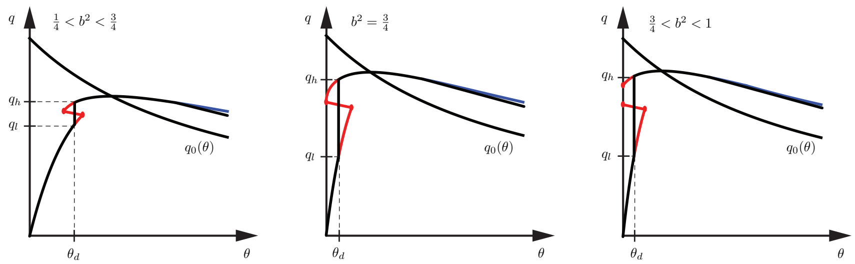

Figure 9: The vertical ironing.

tariff is in accordance to the demand profile approach since the type-assignment functions are monotonic.

The kind of solutions that appear in this section, depicted in Fig. 9 provides another motivation for the solution proposed at Section 3. Indeed, Theorem 4 introduces the ironing necessary to fix the non-monotonicity of the function $\theta_b(p_U(q), q)$ presented in Corollary 2. As we are assuming (DPA), this ironing will be very restrictive, as we have to impose

$$p(q) = v_q(q, \theta_d) \text{ for all } q \in [q_\ell, q_h]. \tag{35}$$

Thus, we can think of the method from Section 3 as an alternative way of fixing this non-monotonicity problem. There, instead of imposing (35), we have constraint (J) below:

$$\int_{q_\ell}^{q_h} [p(q) - v_q(q, \theta_d)] dq = 0. \tag{J}$$

We can think of (J) as a relaxed version of (35), because we are not imposing a pointwise equality and only an equality in average.

4.2. Application

Guiding Example continued. *Here we apply the demand profile approach to our guiding example with $b = 6/10$.*

Using Theorem 3 and Corollary 2 we can find the function

$$\theta_b(p_U(q), q) = \begin{cases} -\frac{\sqrt{-351q^2 - 450q + 625} + 9q - 25}{27q} & \text{if } 0 \le q \le q_1, \\ \frac{25}{9q} - 3 & \text{if } q_1 \le q \le q_2, \\ \frac{-54q^2 - 5\sqrt{75q^2 - 108q^4} + 75q}{54q^2} & \text{if } q_2 \le q \le \frac{5}{6}, \end{cases}$$

where q_1 and q_2 are given by[24]

$$q_1 = \frac{5}{123}\left(15 + 2\sqrt{5}\right) \text{ and } q_2 = \frac{25}{798}\left(18 + \sqrt{58}\right).$$

Observe that $\theta_b(p_U(q), q)$ is increasing in $[0, q_1] \cup [q_2, \frac{5}{6}]$ but decreasing in $[q_1, q_2]$. This nonmonotonicity implies that the marginal tariff $p_U(q)$ is not consistent with the demand profile approach because it does not satisfy equation (DPA). To fix this problem we appeal to an ironing procedure. Using Theorem 4 we get the new $p(q)$ and the following type-assignment function

$$\theta_b(p(q), q) = \begin{cases} -\frac{\sqrt{-351q^2-450q+625}+9q-25}{27q}, & \text{if } 0 \le q \le q_\ell, \\ \theta_d, & \text{if } q_\ell \le q \le q_h, \\ \frac{-54q^2-5\sqrt{75q^2-108q^4}+75q}{54q^2}, & \text{if } q_h \le q \le \frac{5}{6}, \end{cases}$$

where[25]

$$\theta_d \approx 0.483166, \quad q_\ell \approx 0.779297 \quad \text{and} \quad q_h \approx 0.810358.$$

For $\theta_s(p(q), q)$, we plug $p(q) = v_q(q, \theta_b(p(q), q))$ in the pseudo-inverse $\theta_s(\cdot, q)$ and we get

$$\theta_s(p(q), q) = \begin{cases} 1, & \text{if } 0 \le q \le \overline{q}, \\ -\theta_d + \frac{25}{9q} - 2, & \text{if } \overline{q} \le q \le q_h, \\ \frac{-54q^2+5\sqrt{75q^2-108q^4}+75q}{54q^2}, & \text{if } q_h \le q \le \frac{5}{6}, \end{cases}$$

where $\overline{q} \approx 0.792816$.

Inverting the type-assignment functions $\theta_b(p(q), q)$ and $\theta_s(p(q), q)|_{[\overline{q}, \frac{5}{6}]}$, we get the quantity assignment function

$$q(\theta) = \begin{cases} \frac{50\theta}{27\theta^2+18\theta+16}, & \text{if } \theta < \theta_d, \\ \frac{25\left(\sqrt{27\theta^2+54\theta-23}+9\theta+9\right)}{6(27\theta^2+54\theta+52)}, & \text{if } \theta_d \le \theta \le \hat{\theta}_d, \\ \frac{25}{9(\theta_d+\theta+2)}, & \text{if } \theta \ge \hat{\theta}_d, \end{cases} \tag{36}$$

where $\hat{\theta}_d = 0.944672$.

The marginal tariff is given by

$$p(q) = \begin{cases} v_q(q, \theta_b(p(q), q)), & \text{if } 0 \le q < q_l, \\ v_q(q, \theta_d), & \text{if } q_l \le q < q_h, \\ v_q(q, \theta_b(p(q), q)) = v_q(q, \theta_s(p(q), q)), & \text{if } q_h \le q < q^* = \frac{5}{6}. \end{cases}$$

[24] Their general expressions are presented in the proof of Corollary 2.

[25] In the Appendix we show that Theorem 4 is equivalent to Lemma 2. We use this Lemma to find q_ℓ, q_h and θ_d because it makes the computations easier.

The calculus of the monopolist's expected profit can be done using equation (9)[26]. The result is

$$\text{Profit} \approx 0.389962. \tag{37}$$

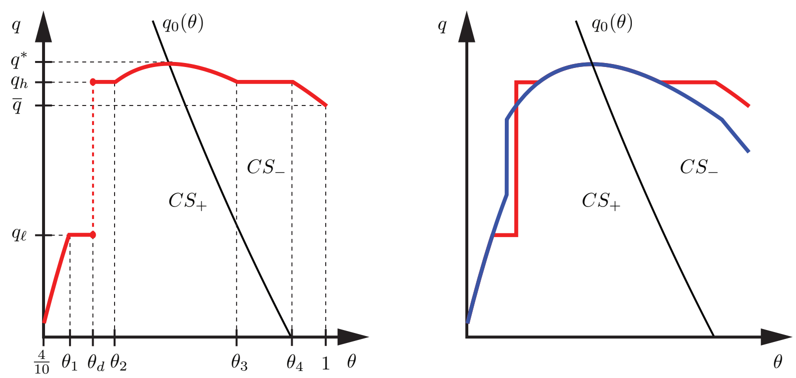

Figure 10: The quantity assignment functions.

In Figure 10 we have the graphs of the quantity assignment functions derived in this Example. On the left graph we have $q(\theta)$, given by expression (27), resulting from the method derived on Section 3. On the right graph, we repeat the left graph and superpose the graph of $q(\theta)$ given by expression (36), using the demand profile approach.[27]

5. Conclusion

In this paper we studied a one-dimensional nonlinear pricing model where the single-crossing condition fails to hold. First we presented a motivating numeric solution for a particular example. Then we wrote the problem in a more formal way, allowing us to capture the main features of its solution. We derived a mechanism that splits the set of the demanded quantity in two subintervals characterizing the high and low demand groups. Conditioning on in each demand group, the monopolist applies the demand profile approach. However, on the top of that, he has also to consider a global incentive constraint to avoid migration across groups. As a consequence we endogenize an entry fee tariff between these groups. This global constraint can be incorporated to the original problem as an isoperimetric constraint type. By doing so, we solved the problem using calculus of variation techniques.

[26] We can use also expression (1) in this computation.

[27] In the set of feasible parameters b, the maximum relative gain the monopolist can get between the solution obtained in Section 4 and the one obtained in Section 3 is approximately 1.3%.

In principle, the proposed method is capable to solve the same class of problems presented in Araujo and Moreira [8]. The solutions however, as we saw in Sections 3 and 4, differ. The reason is that to assume condition (DPA) can be restrictive. By relaxing it, the monopolist has more freedom to design tariffs as some global incentive compatibility constraints are relaxed. This results in a bigger expected profit as it was illustrated numerically. Therefore, in a non single-crossing world, this work gives another tool for the monopolist to discriminate among consumers.

Appendix A. Case B ($q(\underline{\theta}) > q(\overline{\theta})$).

We are going to continue the analysis of Section 4. Now we are going to consider $q(\cdot)$ such that $q(\underline{\theta}) > q(\overline{\theta})$.

Plugging equations (29) and (31) into the monopolist's expected profit given by expression (9) and using the notation $\hat{q} = q(\underline{\theta})$, the monopolist's problem can be stated now as

$$\max_{P(\cdot)} \begin{cases} P(\hat{q}) - C(\hat{q}) & + \displaystyle\int_{q_*}^{\hat{q}} [p(q) - C'(q)](N(p(q), q) - 1)dq \\ & + \displaystyle\int_{\hat{q}}^{q^*} [p(q) - C'(q)]N(p(q), q)dq. \end{cases} \tag{Π_{B}}$$

Notice that, differently from Case A, the quantity of $\underline{\theta}$-customer, $\hat{q}$, is an endogenous decision variable for the monopolist.[28] The tariff for this type is still determined by equation (29).

The strategy for solving this problem involves two steps. The first is to determine the necessary first-order optimality conditions for a given $\hat{q}$. The derivation is analogous to Theorem 1 with the difference that the objective function in problem (Π_{B}) has two integrals that have to be treated separately. The result is presented in the following:

Theorem 5. *The optimal marginal tariff $p = p(q)$ for problem (Π_{B}) is characterized by*

(i) if $q \in [q_*, \hat{q}]$ and $v_q(q, \overline{\theta}) < p < v_q(q, \underline{\theta})$, then $p = v_q(q, \theta_s)$ and

$$g_q(q, \theta_s) = 0; \tag{A.1}$$

(ii) if $q \in (\hat{q}, q^*]$ and $\max\{v_q(q, \underline{\theta}), v_q(q, \overline{\theta})\} < p < \max_{\theta \in \Theta} v_q(q, \theta)$, then $p = v_q(q, \theta_b) = v_q(q, \theta_s)$ and

$$\frac{g_q(q, \theta_b)}{v_{q\theta}(q, \theta_b)} f(\theta_b) = \frac{g_q(q, \theta_s)}{v_{q\theta}(q, \theta_s)} f(\theta_s). \tag{A.2}$$

The statement of Theorem 5 divides the analysis of the quantity spectrum $Q = [q_*, q^*]$ in the subintervals $[q_*, \hat{q}]$ and $[\hat{q}, q^*]$, for a given $\hat{q}$. For the interval $[q_*, \hat{q}]$ the marginal utility condition (UC) is not binding, which gives item (i). For the interval $[\hat{q}, q^*]$, condition (UC) is binding, which gives item (ii).

The first-order conditions derived in Theorems 3 and 5 are very related. Indeed, item (ii) is exactly the same and item (i) corresponds to the optimality condition of the decreasing part of $q_R(\theta)$.

[28] In Case A the optimal $\hat{q}$ corresponds to the lowest quantity ($\hat{q} = q_*$) and maximizes the function g defined in (2). In Case B, $\hat{q}$ is not necessary the lowest quantity ($\hat{q} \neq q_*$) and we have to determine its optimal transversality condition as we do in what follows.

The second step consists to optimally determine $\hat{q}$, which is the transition point between the solutions given in Theorem 5 (i) and (ii). This is equivalent to an optimal bunching procedure of the quantity assignment function. The next theorem provides the condition that the optimal bunching level $\hat{q}$ must satisfy.

Theorem 6. *The optimal $\hat{q}$ should satisfy*

$$v_q(\hat{q}, \underline{\theta}) - C'(\hat{q}) + (p_1 - C'(\hat{q}))(N(p_1, \hat{q}) - 1) - (p_2 - C'(\hat{q}))N(p_2, \hat{q}) = 0, \tag{A.3}$$

where $p_1 = p(\hat{q}_-)$ and $p_2 = p(\hat{q}_+)$ are determined by Theorem 5 (i) and (ii), respectively.

The equation (A.3) has a natural interpretation. From the expression of problem (Π_B), the first two terms give the marginal net benefit with type $\underline{\theta}$ who consumes $\hat{q}$ and the other two represent the marginal profit loss the monopolist suffers with other types of a marginal increase in $\hat{q}$.

As in Case A, Theorems 5 and 6 give us a recipe for computing the optimal marginal tariff $p(q)$. In this case we can easily associate a quantity assignment function to the optimal $\theta_s(p(q), q)$.[29]

Figure A.11 presents the optimal quantity assignment function $q(\theta)$. On the left graph, $\hat{q} < q^*$ and the optimal θ_s is characterized by equation (A.1), for all $q \in (q_*, \hat{q})$ and equation (A.2), for all $q \in (\hat{q}, q^*)$. On the right graph, $\hat{q} = q^*$ and θ_s is characterized by equation (A.1), for all $q \in (q_*, q^*)$.

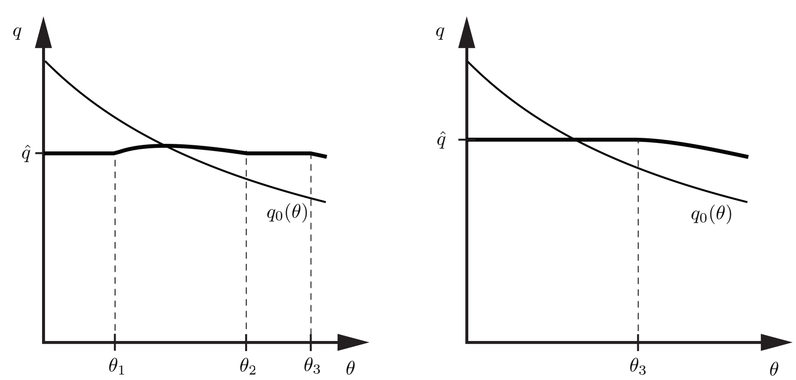

Figure A.11: The solution when $q(\underline{\theta}) > q(\overline{\theta})$.

[29]We can get the quantity assignment by inverting $\theta_b(p(q), q)$ and $\theta_s(p(q), q)$. The discontinuity in the type-assignment functions at $\hat{q}$ corresponds to the bunching of the quantity assignment function, shown in Figure A.11. Notice that the lack of monotonicity in Case A does not happen here, so that the consistency of the demand profile approach for Case B is automatically ensured.

Appendix B. Proofs

Proof of Proposition 1. (i) Consider the set

$$CS_+ = \{(\theta, q) \in \Theta \times \mathbb{R}_+ : q < q_0(\theta)\}.$$

When $(\theta_0, q(\theta_0)) \in CS_+$, using the continuity of $q(\cdot)$ at θ_0, we can build intervals $I = [\theta_0 - \delta, \theta_0 + \delta]$ and $J = [q(\theta_0) - \varepsilon, q(\theta_0) + \varepsilon]$ with $I \times J \subset CS_+$ such that $(\theta, q(\theta)) \in I \times J \subset CS_+$ for all $\theta \in I$. Then we can take $\theta_1 \in I$ and define:

$$\Delta(\theta_0, \theta_1) = [v(q(\theta_0), \theta_1) - v(q(\theta_0), \theta_0)] + [v(q(\theta_1), \theta_0) - v(q(\theta_1), \theta_1)].$$

Following Rochet [10], if $q(\cdot)$ is implementable, then we must have $\Delta(\theta_0, \theta_1) \leq 0$. We can write

$$\Delta(\theta_0, \theta_1) = -\int_{\theta_0}^{\theta_1} \int_{q(\theta_0)}^{q(\theta_1)} v_{q\theta}(q, \theta) dq d\theta.$$

The region of integration is a subset of $I \times J$ where the integrand $v_{q\theta}$ is always positive. This gives the following equivalence

$$\Delta(\theta_0, \theta_1) \leq 0 \Leftrightarrow (q(\theta_1) - q(\theta_0))(\theta_1 - \theta_0) \geq 0.$$

So we conclude that $q(\cdot)$ must be increasing in θ_0.

(ii) The proof is analogous to item (i).

□

Proof of Proposition 2. Let $P(\cdot)$ be the tariff resulting from the *'Taxation Principle'*. The customer's maximization problem is

$$\max_{q \in Q} v(q, \theta_i) - P(q), \text{ for } i = 1, 2.$$

If $P(\cdot)$ is differentiable at $q \in q(\theta_1) \cap q(\theta_2)$, then the first-order condition of the problem above is

$$P'(q) = v_q(q, \theta_i), \text{ for } i = 1, 2,$$

which results in

$$v_q(q, \theta_1) = v_q(q, \theta_2).$$

□

Proof of Theorem 1. For items *(i)* and *(ii)*, we derive the first-order necessary conditions of problem (L_A) below, considering the different possibilities for $p \geq 0$ in the range of $v_q(q, \cdot)$ (i.e. $p \in v_q(q, \Theta)$).

$$\begin{aligned} &\max_p \mathcal{L}_A(p,q) := (p - C'(q))N(p,q) \\ &\text{subject to } p \in v_q(q, \Theta). \end{aligned} \tag{L_A}$$

If $N(p, q)$ is differentiable at p, then the first-order condition for problem (L_A) is[30]

$$\frac{\partial \mathcal{L}_A}{\partial p} = N(p,q) + (p - C'(q))\frac{\partial N}{\partial p}(p,q) = 0. \tag{B.1}$$

But according to equation (10), the demand profile is

$$N(p,q) = F(\theta_s(p,q)) - F(\theta_b(p,q)),$$

and so its partial derivative is

$$\frac{\partial N}{\partial p}(p,q) = f(\theta_s(p,q))\frac{\partial \theta_s}{\partial p}(p,q) - f(\theta_b(p,q))\frac{\partial \theta_b}{\partial p}(p,q). \tag{B.2}$$

Now, plugging (B.2) into (B.1) gives

$$(p - C'(q))\left[f(\theta_s)\frac{\partial \theta_s}{\partial p} - f(\theta_b)\frac{\partial \theta_b}{\partial p}\right] + F(\theta_s) - F(\theta_b) = 0. \tag{B.3}$$

We use equation (B.3) to get items *(i)* and *(ii)*. We will consider two different possibilities for p in the range of $v_q(q, \cdot)$.

Remember that for item *(i)*, the proof was done in the text of Section 3. This item corresponds to $p \in (v_q(q, \underline{\theta}), v_q(q, \overline{\theta}))$.

For item *(ii)*, we suppose that the optimal p satisfies

$$\max\{v_q(q,\underline{\theta}), v_q(q,\overline{\theta})\} < p < \max_{\theta \in \Theta} v_q(q,\theta). \tag{B.4}$$

In this case, it follows by definition that $p = v_q(q, \theta_b) = v_q(q, \theta_s)$ and $\underline{\theta} < \theta_b < \theta_s < \overline{\theta}$, so $N(p, q) = F(\theta_s) - F(\theta_b)$. Using the inverse function theorem, we get

$$\frac{\partial \theta_b}{\partial p}(p,q) = \frac{1}{v_{q\theta}(q,\theta_b)} \text{ and } \frac{\partial \theta_s}{\partial p}(p,q) = \frac{1}{v_{q\theta}(q,\theta_s)}.$$

Plugging these two expressions into equation (B.3) we get

$$(p - C'(q))\left(\frac{f(\theta_s)}{v_{q\theta}(q,\theta_s)} - \frac{f(\theta_b)}{v_{q\theta}(q,\theta_b)}\right) + F(\theta_b) - 1 - (F(\theta_b) - 1) = 0.$$

[30] This is exactly equation (13).

Now, rearranging terms and remembering that $p = v_q(q, \theta_b) = v_q(q, \theta_s)$, we can write the equation above as

$$\frac{g_q(q, \theta_b)}{v_{q\theta}(q, \theta_b)} f(\theta_b) = \frac{g_q(q, \theta_s)}{v_{q\theta}(q, \theta_s)} f(\theta_s),$$

where g is defined in equation (2).

For item *(iii)*, we have to consider the problem

$$\begin{aligned} &\max_p \mathcal{L}_J(p, q) := (p - C'(q))\tilde{N}(p, q) - \lambda(p - v_q(q, \theta_d)) \\ &\text{subject to } p \in v_q(q, \Theta). \end{aligned} \tag{L_J}$$

If $\tilde{N}(p, q)$ is differentiable at p, then the first-order condition for problem (L_J) is

$$\frac{\partial \mathcal{L}_J}{\partial p} = \tilde{N}(p, q) + \frac{\partial \tilde{N}}{\partial p}(p, q)(p - C'(q)) - \lambda = 0, \tag{B.5}$$

This modified demand profile is given by

$$\tilde{N}(p, q) = F(\theta_s) - F(\theta_d). \tag{B.6}$$

Thus, the derivative with respect to p is

$$\frac{\partial \tilde{N}}{\partial p}(p, q) = f(\theta_s(p, q)) \frac{\partial \theta_s}{\partial p}(p, q) = \frac{f(\theta_s)}{v_{q\theta}(q, \theta_s)}. \tag{B.7}$$

Now, plugging (B.6) and (B.7) into (B.5) we get

$$(p - C'(q)) \frac{f(\theta_s)}{v_{q\theta}(q, \theta_s)} + F(\theta_s) - F(\theta_d) - \lambda = 0. \tag{B.8}$$

Multiplying equation (B.8) by $v_{q\theta}(q, \theta_s)$ and making $p = v_q(q, \theta_s)$, we get

$$\begin{aligned} (v_q(q, \theta_s) - C'(q))f(\theta_s) + (F(\theta_s) - F(\theta_d))v_{q\theta}(q, \theta_s) - \lambda v_{q\theta}(q, \theta_s) &= \\ (v_q(q, \theta_s) - C'(q))f(\theta_s) + \frac{F(\theta_s) - 1 + 1 - F(\theta_d)}{f(\theta_s)} v_{q\theta}(q, \theta_s) f(\theta_s) - \lambda v_{q\theta}(q, \theta_s) &= \\ (v_q(q, \theta_s) - C'(q) + \frac{F(\theta_s) - 1}{f(\theta_s)} v_{q\theta}(q, \theta_s))f(\theta_s) + (1 - F(\theta_d) - \lambda)v_{q\theta}(q, \theta_s) &= \\ g_q(q, \theta_s)f(\theta_s) + (1 - F(\theta_d) - \lambda)v_{q\theta}(q, \theta_s) &= 0. \end{aligned}$$

Therefore,

$$\frac{g_q(q, \theta_s)}{v_{q\theta}(q, \theta_s)} f(\theta_s) = \lambda - (1 - F(\theta_d)).$$

□

We use the next lemma will in the proof of Theorem 2.

Lemma 1. *The first-order conditions for the problem of optimally choosing q_ℓ and q_h are*

$$\text{(i)}\ \frac{(p_1 - C'(q_\ell))N(p_1, q_\ell) - (p_2 - C'(q_\ell))\tilde{N}(p_2, q_\ell)}{v_q(q_\ell, \theta_d) - p_2} = \lambda, \tag{B.9}$$

$$\text{(ii)}\ \frac{(p_4 - C'(q_h))N(p_4, q_h) - (p_3 - C'(q_h))\tilde{N}(p_3, q_h)}{v_q(q_h, \theta_d) - p_3} = \lambda, \tag{B.10}$$

where $p_1 = p(q_{\ell-}), p_2 = p(q_{\ell+}), p_3 = p(q_{h-}), p_4 = p(q_{h+})$

Proof of Lemma 1. When we append the constraint (J) with the Lagrangian multiplier λ to the objective function in problem (Π_{J}) we get a new objective function, depending on the choices of q_ℓ, q_h and θ_d,

$$\begin{aligned}\Pi(q_\ell, q_h, \theta_d) := P(q_*) - C(q_*) + &\int_{q_*}^{q_\ell} [p(q) - C'(q)]N(p(q), q)dq \\ &+ \int_{q_\ell}^{q_h} \{[p(q) - C'(q)]\tilde{N}(p(q), q) \\ &- \lambda[p(q) - v_q(q, \theta_d)]\}dq \\ &+ \int_{q_h}^{q^*} [p(q) - C'(q)]N(p(q), q)dq,\end{aligned}$$

where $N = F(\theta_s) - F(\theta_b)$ and $\tilde{N} = F(\theta_s) - F(\theta_d)$.

For item (i), differentiating $\Pi(q_\ell, q_h, \theta_d)$ with respect to q_ℓ, we get

$$[p(q_{\ell-}) - C'(q_\ell)]N(p(q_{\ell-}), q_\ell)$$

$$-\{[p(q_{\ell+}) - C'(q_\ell)]\tilde{N}(p(q_{\ell+}), q_\ell) - \lambda[p(q_{\ell+}) - v_q(q_\ell, \theta_d)]\} = 0. \tag{B.11}$$

Solving equation (B.11) for λ gives us the result.

For item (ii), differentiating $\Pi(q_\ell, q_h, \theta_d)$ with respect to q_h, we get

$$-[p(q_{h+}) - C'(q_h)]N(p(q_{h+}), q_h)$$

$$+\{[p(q_{h-}) - C'(q_h)]\tilde{N}(p(q_{h-}), q_h) - \lambda[p(q_{h-}) - v_q(q_h, \theta_d)]\} = 0. \tag{B.12}$$

Again, we just have to solve equation (B.12) for λ to get the result. ☐

Proof of Theorem 2. Defining

$$H(\theta, q) = (v_q(q, \theta) - C'(q))(1 - F(\theta)), \tag{B.13}$$

we get

$$H_\theta(q, \theta) = -g_q(q, \theta) f(\theta). \tag{B.14}$$

For item (i), by equation (12) we have $p_2 = v_q(q_\ell, \overline{\theta})$. Thus, using (B.13) we can write (B.9) as

$$H(\theta_1, q_\ell) - H(\theta_d, q_\ell) - (v_q(q_\ell, \overline{\theta}) - v_q(q_\ell, \theta_d))(1 - F(\theta_d)) = -\lambda \left(\int_{\theta_d}^{\overline{\theta}} v_{q\theta}(q_\ell, \theta) d\theta \right),$$

or

$$H(\theta_1, q_\ell) - H(\theta_d, q_\ell) = (1 - F(\theta_d) - \lambda) \left(\int_{\theta_d}^{\overline{\theta}} v_{q\theta}(q_\ell, \theta) d\theta \right).$$

Finally, using (B.14) we get

$$\int_{\theta_1}^{\theta_d} g_q(q_\ell, \theta) f(\theta) d\theta = -\int_{\theta_1}^{\theta_d} H_\theta(\theta, q_\ell) d\theta = (1 - F(\theta_d) - \lambda) \left(\int_{\theta_d}^{\overline{\theta}} v_{q\theta}(q_\ell, \theta) d\theta \right).$$

For item (ii), we rewrite equation (B.10) as

$$\begin{aligned} (v_q(q_h, \theta_4) - C'(q_h)) (F(\theta_4) - 1 + 1 - F(\theta_d)) + \\ -(v_q(q_h, \theta_3) - C'(q_h))(F(\theta_3) - 1 + 1 - F(\theta_2)) = \\ \lambda \int_{\theta_d}^{\theta_4} v_{q\theta}(q_h, \theta) d\theta, \end{aligned}$$

and using that $v_q(q_h, \theta_2) = v_q(q_h, \theta_3)$, we can write the equation above as

$$\begin{aligned} H(\theta_d, q_h) - H(\theta_2, q_h) + H(\theta_3, q_h) - H(\theta_4, q_h) = \\ (\lambda - (1 - F(\theta_d))) \int_{\theta_d}^{\theta_4} v_{q\theta}(q_h, \theta) d\theta. \end{aligned}$$

Finally, using (B.14) we get

$$\begin{aligned} \int_{\theta_d}^{\theta_2} g_q(q_h, \theta) f(\theta) d\theta + \int_{\theta_3}^{\theta_4} g_q(q_h, \theta) f(\theta) d\theta = \\ (\lambda - (1 - F(\theta_d))) \int_{\theta_d}^{\theta_4} v_{q\theta}(q_h, \theta) d\theta, \end{aligned}$$

and the result follows.

For item (iii), differentiating $\Pi(q_\ell, q_h, \theta_d)$ with respect to θ_d, we get[31]

$$\int_{q_\ell}^{q_h} \{-[p(q) - C'(q)]f(\theta_d) + \lambda v_{q\theta}(q, \theta_d)\}dq = 0, \tag{B.15}$$

and using the condition (J), we can write

$$\int_{q_\ell}^{q_h} p(q)dq = \int_{q_\ell}^{q_h} v_q(q, \theta_d)dq. \tag{B.16}$$

Plugging equation (B.16) into (B.15) and multiplying by (-1), we get

$$\int_{q_\ell}^{q_h} \{[v_q(q, \theta_d) - C'(q)]f(\theta_d) - \lambda v_{q\theta}(q, \theta_d)\}dq = 0. \tag{B.17}$$

For the integrand in equation (B.17) we have

$$\begin{aligned}[v_q(q, \theta_d) - C'(q)]f(\theta_d) - \lambda v_{q\theta}(q, \theta_d) = \\ \left[v_q(q, \theta_d) - C'(q) + \frac{F(\theta_d) - 1}{f(\theta_d)} v_{q\theta}(q, \theta_d)\right] f(\theta_d) + (1 - F(\theta_d) - \lambda)v_{q\theta}(q, \theta_d) = \\ g_q(q, \theta_d)f(\theta_d) + (1 - F(\theta_d) - \lambda)v_{q\theta}(q, \theta_d),\end{aligned}$$

and the result follows. □

Proof of Corollary 1. For item (i), we only have to add up equations (22) and (23).

For item (ii), we have to equate the left-hand sides of equations (20) and (24). □

Proof of Theorem 3. We just have to repeat the proof we did for Theorem 1*(a)*, items *(i)* and *(ii)*. □

Proof of Corollary 2. The proof consists in showing that the optimality conditions derived in Theorem 3 are also sufficient. For this purpose, we consider again the maximization problem (L_A), used in the proof of Theorem 1.

First we claim that we only need to consider quantities $q \in [0, \frac{1}{2b}]$. Indeed, when $q > \frac{1}{2b}$ we have $v_q(q, \theta) < C'(q)$ for all $\theta \in [0, 1]$. Thus $\mathcal{L}_A = (p - C'(q))N(p, q) < 0$ unless $N(p, q) = 0$. For this quantity q, it is optimal to choose p high enough such that no customer buys it, which is the same as not offering this q at all.

[31] $\Pi(q_\ell, q_h, \theta_d)$ was defined at the proof of Lemma 1.

The next step involves a more detailed analysis of the demand profile $N(p, q)$. We will consider $b \in (\frac{1}{2}, 1)$ and $q \in (0, \frac{1}{2b}]$. In this domain the marginal utility $v_q(q, \theta)$ is quadratic, concave and positive. Let us take $p \in v_q(q, \Theta)$ and then define $\phi_b(p, q)$ and $\phi_s(p, q)$ as the roots of equation $v_q(q, \cdot) = p$:[32]

$$\phi_b = \frac{1 - 2b^2 q - \sqrt{1 - 4b^2 pq}}{2b^2 q} \text{ and } \phi_s = \frac{1 - 2b^2 q + \sqrt{1 - 4b^2 pq}}{2b^2 q}.$$

Using ϕ_b and ϕ_s, we can write the demand profile as

$$N(p, q) = \min\{\phi_s(p, q), 1\} - \max\{\phi_b(p, q), 0\}. \tag{B.18}$$

As $p \in v_q(q, \Theta)$, we have three possibilities for $N(p, q)$, represented by functions N_1, N_2 and N_3, with

$$\begin{aligned} N_1(p, q) &= 1 - \phi_b(p, q), \\ N_2(p, q) &= \phi_s(p, q) - \phi_b(p, q), \\ N_3(p, q) &= \phi_s(p, q). \end{aligned}$$

To each one of these functions, we can associate a maximization problem defined by[33]

$$\begin{aligned} &\max_p \mathcal{L}_{A,i}(p, q) := (p - C'(q))N_i(p, q) \\ &\text{subject to } p \in [0, \tfrac{1}{4b^2 q}]. \end{aligned} \tag{$L_{A,i}$}$$

Our task now is to establish the concavity of the objective functions $\mathcal{L}_{A,i}(p, q)$ in the variable p. After some calculations, we can show that the signal of its second derivative with respect to p is equal to the signal of

$$b^2 q(3p + q) - 1.$$

Using that $p \leq \frac{1}{4b^2 q}$ and $q \leq \frac{1}{2b}$, we get

$$b^2 q(3p + q) - 1 \leq 0.$$

Therefore, $\mathcal{L}_{A,i}(\cdot, q)$ is concave and the first-order conditions are necessary and sufficient for solving problem ($L_{A,i}$). The characterization of the solution follows the same steps as we did for Theorem 1. Remembering the definition $g(q, \theta)$ and, as θ is uniformly distributed on $[0, 1]$, we have that $g(q, \theta) \equiv 0$ when $\theta \notin [0, 1]$. However, we are not restricting ϕ_a and ϕ_b to $[0, 1]$. Thus, we need to extend conveniently the function $g(q, \cdot)$ outside this interval. We adopt the following extension

$$g^e(q, \theta) = v(q, \theta) - C(q) - (1 - \theta)v_\theta(q, \theta).$$

[32] Notice that now we are not imposing that both ϕ_b and ϕ_s belong to $[0, 1]$.

[33] The upper boundary on p is just the maximum value of the quadratic and concave function $v_q(q, \theta)$, with $\theta \in \mathbb{R}$.

Using this extension g^e and repeating the same arguments of the proof of Theorem 1, we can characterize the interior optimum $p = p(q)$ by

$$\begin{aligned}
&\text{for } (L_{A,1}), \quad g^e_q(q, \phi_b(p,q)) = 0;\\
&\text{for } (L_{A,2}), \quad \frac{g^e_q(q,\phi_b(p,q))}{v_{q\theta}(q,\phi_b(p,q))} = \frac{g^e_q(q,\phi_s(p,q))}{v_{q\theta}(q,\phi_s(p,q))};\\
&\text{for } (L_{A,3}), \quad g^e_q(q,\phi_s(p,q)) + v_{q\theta}(q,\phi_s(p,q)) = 0.
\end{aligned}$$

Solving these equations, we can define $\theta^1_b(q), (\theta^2_b(q), \theta^2_s(q))$ and $\theta^3_s(q)$ by

$$\begin{aligned}
(L_{A,1}) \quad & \theta^1_b(q) := \phi_b(p(q),q) = -\frac{b^2q + \sqrt{4b^4q^2 - b^2q(3q+2)+1} - 1}{3b^2q},\\
(L_{A,2}) \quad & \theta^2_b(q) := \phi_b(p(q),q) = -\frac{6b^4q^2 - 3b^2q + \sqrt{3}\sqrt{-b^4q^2\left(4b^2q^2-1\right)}}{6b^4q^2},\\
& \theta^2_s(q) := \phi_s(p(q),q) = -\frac{6b^4q^2 - 3b^2q - \sqrt{3}\sqrt{-b^4q^2\left(4b^2q^2-1\right)}}{6b^4q^2}\\
(L_{A,3}) \quad & \theta^3_s(q) := \phi_s(p(q),q) = \frac{-2b^2q + \sqrt{b^4q^2 - b^2q(3q+1)+1} + 1}{3b^2q},
\end{aligned}$$

with $p(q) = v_q(q, \theta^1_b(q))$ in problem $(L_{A,1})$; $p(q) = v_q(q,\theta^2_b(q)) = v_q(q,\theta^2_s(q))$ in problem $(L_{A,2})$; $p(q) = v_q(q, \theta^3_s(q))$ in problem $(L_{A,3})$.

Now we use all the information about problem $(L_{A,i})$ in the analysis of problem (L_A). We have three cases to consider:

(a) $0 < q \leq q_0(1)$.

In this case, for all $\theta \in [0,1]$, $v_q(q,\theta) \leq v_q(q,1)$. Thus $\mathcal{L}_A(p,q) = \mathcal{L}_{A,1}(p,q)$ and the solution for problems $(L_{A,1})$ and (L_A) coincides, with $p = v_q(q, \theta^1_b(q))$.

(b) $q_0(1) < q \leq \min\{\frac{1}{2b}, q_0(\frac{1}{2})\}$.

Now, $v_q(q,0) \leq v_q(q,1)$, and

$$\mathcal{L}_A(p,q) = \begin{cases} \mathcal{L}_{A,1}(p,q), & \text{if } v_q(q,0) \leq p \leq v_q(q,1),\\ \mathcal{L}_{A,2}(p,q), & \text{if } v_q(q,1) \leq p \leq \frac{1}{4b^2q}. \end{cases}$$

Besides, one can show that if $q \in (q_0(0), \frac{1}{2b})$ (which is indeed the case), then $\theta^1_b(q) > \theta^2_b(q)$.

Let us implicitly define $\hat{\theta}_b(q)$ as the solution of the equation $v_q(q, \hat{\theta}_b(q)) = v_q(q,1)$. Investigating the local monotonicity of $\mathcal{L}_A(p,q)$ and considering the concavity of $\mathcal{L}_{A,1}(p,q)$ and $\mathcal{L}_{A,2}(p,q)$ we conclude that the optimal $p(q)$ satisfies:

(i) If $\theta^1_b(q) \leq \hat{\theta}_b(q)$, then $p(q) = v_q(q, \theta^1_b(q))$;
(ii) If $\theta^2_b(q) \geq \hat{\theta}_b(q)$, then $p(q) = v_q(q, \theta^2_b(q))$;

(iii) If $\theta_b^2(q) \leq \hat{\theta}_b(q) \leq \theta_b^1(q)$, then $p(q) = v_q(q, \hat{\theta}_b(q))$.

(c) $q_0(\frac{1}{2}) < q \leq \frac{1}{2b}$.

Finally, in this case we have $v_q(q,0) > v_q(q,1)$ and

$$\mathcal{L}_A(p,q) = \begin{cases} \mathcal{L}_{A,3}(p,q), & \text{if } v_q(q,1) \leq p \leq v_q(q,0), \\ \mathcal{L}_{A,2}(p,q), & \text{if } v_q(q,0) \leq p \leq \frac{1}{4b^2 q}. \end{cases}$$

Here we implicitly define $\hat{\theta}_s(q)$ as the solution of equation $v_q(q, \hat{\theta}_s(q)) = v_q(q,0)$. Comparing $\hat{\theta}_s(q)$ and $\theta_s^3(q)$, one can show that

$$\text{if } b \in \left(\frac{1}{2}, \frac{2}{\sqrt{5}}\right), \text{ then } \hat{\theta}_s(q) > \theta_s^3(q).$$

But when $p > v_q(q, \hat{\theta}_s(q)) = v_q(q,0)$, then $\mathcal{L}_A(p,q) = \mathcal{L}_{A,2}(p,q)$. Thus, analyzing the local monotonicity of $\mathcal{L}_A(p,q)$, we conclude that the optimal $p(q)$ satisfies:

(i) if $\theta_b^2(q) \geq 0$, then $p(q) = v_q(q, \theta_b^2(q))$;

(ii) if $\theta_b^2(q) < 0$, then $p(q) = v_q(q,0)$.

Equivalently, we can write $p(q) = v_q(q, \max\{\theta_b^2(q), 0\})$.

We are now in position to describe the function $\theta_b(p_U(q), q)$ using the local monotonicity analysis of problem $L_{A,i}(p,q)$ developed until now and the conclusions from items (a), (b) and (c). The result is

$$\theta_b(p_U(q), q) = \begin{cases} \theta_b^1(q), & \text{if } 0 \leq q \leq q_1(b), \\ \max\{\hat{\theta}_b(q), 0\}, & \text{if } q_1(b) \leq q \leq q_2(b), \\ \max\{\theta_b^2(q), 0\}, & \text{if } q_2(b) \leq q \leq \frac{1}{2b}. \end{cases}$$

The functions $q_1(b)$ and $q_2(b)$ are defined as the solutions of equations $\hat{\theta}_b(q_1) = \theta_b^1(q_1)$ and $\hat{\theta}_b(q_2) = \theta_b^2(q_2)$. Solving these equations we get

$$q_1(b) = \frac{5b^2 + \sqrt{5b^4 - b^2}}{20b^4 + b^2} \text{ and } q_2(b) = \frac{6b^2 + \sqrt{2}\sqrt{6b^4 - b^2}}{2\,(12b^4 + b^2)}.$$

Finally, for getting Corollary 2, we use $\theta_b(p_U(q), q)$ above. When $b \in \left(\frac{1}{2}, \sqrt{\frac{2}{3}}\right]$, we define $q_2^\ell(b) = q_2^h(b) = q_2(b)$. When $b \in \left(\sqrt{\frac{2}{3}}, \frac{2}{\sqrt{5}}\right)$ we get q_2^ℓ and q_2^h by solving $\hat{\theta}_b(q_2^\ell) = 0$ and $\theta_b^2(q_2^h) = 0$, which gives

$$q_2^\ell(b) = \frac{1}{3b^2} \text{ and } q_2^h(b) = \frac{3b^2 + \sqrt{b^2\,(3b^2 - 2)}}{6b^4 + 2b^2}.$$

□

Proof of Theorem 4. The ironing consists in changing the marginal tariff in the interval $q \in [q_\ell(\theta_d), q_h(\theta_d)]$ to $p_{\theta_d}(q) = v_q(q, \theta_d)$.

Considering the modification in the marginal tariff introduced by the ironing, the objective function in problem (Π_A) can be written as

$$\Pi(\theta_d) = P(q_*) - C(q_*) + \int_{q_*}^{q_\ell(\theta_d)} [p(q) - C'(q)]N(p(q), q)dq \tag{B.19}$$
$$+ \int_{q_\ell(\theta_d)}^{q_h(\theta_d)} [p_{\theta_d}(q) - C'(q)]N(p_{\theta_d}(q), q)dq$$
$$+ \int_{q_h(\theta_d)}^{q_*} [p(q) - C'(q)]N(p(q), q)dq.$$

Differentiating expression (B.19) with respect to θ_d we get the desired result.

To be precise, for $\theta_d > 0$, we can find $q_\ell(\theta_d)$ by using $q = q_\ell$ and $\theta_b = \theta_d$ in equation (18), which gives

$$q_\ell(\theta_d) = \frac{2\theta_d}{3b^2\theta_d^2 + 2b^2\theta_d - b^2 + 1}.$$

For $q_h(\theta_d)$, first we solve the equation $v_q(q_h, \theta_b) = v_q(q_h, \theta_s)$ for θ_s and then we plug $q = q_h$, $\theta_b = \theta_d$ and θ_s in equation (19). The result is

$$q_h(\theta_d) = \frac{3b^2\theta_d + \sqrt{b^2\left(3b^2\theta_d^2 + 6b^2\theta_d + 3b^2 - 2\right)} + 3b^2}{6b^4\theta_d^2 + 12b^4\theta_d + 6b^4 + 2b^2}.$$

□

Now we are going to derive an equivalent optimal ironing condition that is easier to use in the computation of *Case A* solutions.

Let's implicitly define the function $q_c(\theta, \theta_d)$ as the solution of

$$v_q(q_c(\theta, \theta_d), \theta_d) = v_q(q_c(\theta, \theta_d), \theta), \tag{B.20}$$

and define $\hat{\theta}_d$ as the solution of

$$v_q(q_h, \theta_d) = v_q(q_h, \hat{\theta}_d). \tag{B.21}$$

Lemma 2. *We can write the vertical ironing condition as*

$$g(q_h, \theta_d)f(\theta_d) - g(q_\ell, \theta_d)f(\theta_d) = \int_{\hat{\theta}_d}^{1} g_q(q_c(\theta, \theta_d), \theta)f(\theta)\frac{\partial q_c(\theta, \theta_d)}{\partial \theta_d}d\theta. \tag{B.22}$$

Proof of Lemma 2. Differentiating equation (B.20) with respect to θ_d and θ we get respectively

$$\frac{\partial q_c}{\partial \theta_d} = \frac{v_{q\theta}(q_c, \theta_d)}{v_{q^2}(q_c, \theta) - v_{q^2}(q_c, \theta_d)}, \tag{B.23}$$

and

$$\frac{\partial q_c}{\partial \theta} = \frac{v_{q\theta}(q_c,\theta)}{v_{q^2}(q_c,\theta_d) - v_{q^2}(q_c,\theta)}. \tag{B.24}$$

Let $\hat{q} = q_c(1,\theta_d)$, then for all $q \in (q_\ell, \hat{q})$ we have $\theta_s(p,q) = 1$ and $\frac{\partial \theta_s}{\partial p}(p,q) = 0$. Besides, $p = v_q(q,\theta_d)$ and $\theta_b(p,q) = \theta_d$ for all $q \in [q_\ell, q_h]$.

Thus, plugging equation (10) into (34) we get

$$\begin{aligned}
\int_{q_\ell}^{q_h} v_{q\theta}(q,\theta_d)[F(\theta_s(p,q)) - F(\theta_b(p,q)) \\
+(p - C'(q))(f(\theta_s)\frac{\partial \theta_s(p,q)}{\partial p} - f(\theta_b)\frac{\partial \theta_b(p,q)}{\partial p})]dq = \\
\int_{q_\ell}^{q_h} \left[F(\theta_s(p,q)) - 1 + (p - C'(q))f(\theta_s)\frac{\partial \theta_s(p,q)}{\partial p}\right] v_{q\theta}(q,\theta_d)dq \\
-\int_{q_\ell}^{q_h} \left[F(\theta_b(p,q)) - 1 + (p - C'(q))f(\theta_b)\frac{\partial \theta_b(p,q)}{\partial p}\right] v_{q\theta}(q,\theta_d)dq = \\
\int_{\hat{q}}^{q_h} \left[v_q(q,\theta_s) - C'(q) + \frac{F(\theta_s) - 1}{f(\theta_s)} v_{q\theta}(q,\theta_s)\right] \frac{f(\theta_s)}{v_{q\theta}(q,\theta_s)} v_{q\theta}(q,\theta_d)dq \\
-\int_{q_\ell}^{q_h} \left[v_q(q,\theta_b) - C'(q) + \frac{F(\theta_b) - 1}{f(\theta_b)} v_{q\theta}(q,\theta_b)\right] \frac{f(\theta_b)}{v_{q\theta}(q,\theta_b)} v_{q\theta}(q,\theta_d)dq = \\
\int_{\hat{q}}^{q_h} g_q(q,\theta_s)f(\theta_s)\frac{v_{q\theta}(q,\theta_d)}{v_{q\theta}(q,\theta_s)}dq - \int_{q_\ell}^{q_h} g_q(q,\theta_d)f(\theta_d)dq = 0.
\end{aligned}$$

That is,

$$g(q_h,\theta_d)f(\theta_d) - g(q_\ell,\theta_d)f(\theta_d) = \int_{\hat{q}}^{q_h} g_q(q,\theta_s)f(\theta_s)\frac{v_{q\theta}(q,\theta_d)}{v_{q\theta}(q,\theta_s)}dq. \tag{B.25}$$

By changing the variable from $q = q_c(\theta,\theta_d)$ to θ, we can rewrite the right-hand side equation (B.25) as

$$\begin{aligned}
\int_{\hat{q}}^{q_h} g_q(q,\theta_s)f(\theta_s)\frac{v_{q\theta}(q,\theta_d)}{v_{q\theta}(q,\theta_s)}dq = \int_{q_c(1,\theta_d)}^{q_c(\hat{\theta}_d,\theta_d)} g_q(q,\theta_s)f(\theta_s)\frac{v_{q\theta}(q,\theta_d)}{v_{q\theta}(q,\theta_s)}dq = \\
\int_1^{\hat{\theta}_d} g_q(q_c(\theta,\theta_d),\theta)f(\theta)\frac{v_{q\theta}(q_c,\theta_d)}{v_{q\theta}(q_c,\theta)}\frac{\partial q_c}{\partial \theta}d\theta = \\
\int_1^{\hat{\theta}_d} g_q(q_c(\theta,\theta_d),\theta)f(\theta)\frac{v_{q\theta}(q_c,\theta_d)}{v_{q^2}(q_c,\theta_d) - v_{q^2}(q_c,\theta)}d\theta = \\
\int_{\hat{\theta}_d}^1 g_q(q_c(\theta,\theta_d),\theta)f(\theta)\frac{\partial q_c}{\partial \theta_d}d\theta,
\end{aligned}$$

and the result is established. $\square$

Proof of Theorem 5. The proof is analogous to the proof of Theorem 1. For item (i), let us define the function

$$\mathcal{L}_{B,1}(p,q) = (p - C'(q))(N(p,q) - 1) + v_q(q,0) - C'(q),$$

and the correspondence

$$\Gamma_{B,1}(q) = [v_q(q,1), v_q(q,0)].$$

Then, for a fixed $\hat{q}$ and $q \in [q_*, \hat{q}]$, we have to solve

$$\begin{aligned} &\max_p \mathcal{L}_{B,1}(p,q) \\ &\text{subject to } p \in \Gamma_{B,1}(q). \end{aligned} \tag{$L_{B,1}$}$$

The first-order condition is

$$\frac{\partial \mathcal{L}_B}{\partial p} = N(p,q) - 1 + (p - C'(q))\frac{\partial N}{\partial p}(p,q) = 0. \tag{B.26}$$

Observe that if $p \in \Gamma_{B,1}(q)$, then $N(p,q) = F(\theta_s(p,q))$ and $\frac{\partial N}{\partial p}(p,q) = \frac{f(\theta_s(p,q))}{v_{q\theta}(q,\theta_s(p,q))}$. Plugging them into equation (B.26) we get

$$F(\theta_s) - 1 + (v_q(q,\theta_s) - C'(q))\frac{f(\theta_s)}{v_{q\theta}(q,\theta_s)} = 0. \tag{B.27}$$

Finally, multiplying equation (B.27) by $v_{q\theta}(q,\theta_s)$ we get $g_q(q,\theta_s) = 0$.

For item (ii), we define

$$\mathcal{L}_{B,2}(p,q) = (p - C'(q))N(p,q),$$

and the correspondence

$$\Gamma_{B,2}(q) = [v_q(q,0), \max_{\theta\in\Theta} v_q(q,\theta)].$$

Now, for a fixed $\hat{q}$ and $q \in [\hat{q}, q^*]$, the maximization problem is

$$\begin{aligned} &\max_p \mathcal{L}_{B,2}(p,q) \\ &\text{subject to } p \in \Gamma_{B,2}(q). \end{aligned} \tag{$L_{B,2}$}$$

Notice that when $p \in \Gamma_{B,2}(q)$, then $N(p,q) = F(\theta_s(p,q)) - F(\theta_b(p,q))$. Thus, the proof follows exactly as the one we did for Theorem 1 (ii).

□

Proof of Theorem 6. Let us define the function $\mathcal{L}_B(p,q)$ by

$$\mathcal{L}_B(p,q) = \begin{cases} \mathcal{L}_{B,1}(p,q), & \text{if } q_* \le q \le \hat{q}, \\ \mathcal{L}_{B,2}(p,q), & \text{if } \hat{q} < q \le q^*, \end{cases}$$

and the constraint correspondence

$$\Gamma(q) = \begin{cases} \Gamma_{B,1}(q) & \text{if } q_* \le q \le \hat{q}, \\ \Gamma_{B,2}(q), & \text{if } \hat{q} < q \le q^*. \end{cases}$$

Then, for a fixed $\hat{q}$, we have to solve

$$\begin{aligned} &\max_p \mathcal{L}_B(p, q) \\ &\text{subject to } p \in \Gamma(q). \end{aligned} \tag{L_B}$$

Let $p_i(q)$ be the solution of the maximization problem ($L_{B,i}$). We analyse the trade-off at $\hat{q}$ comparing $\mathcal{L}_{B,1}(p_1(q), q)$ with $\mathcal{L}_{B,1}(p_2(q), q)$. At the optimal $\hat{q} \in (q_*, q^*]$, we must have

$$\mathcal{L}_{B,1}(p_1(\hat{q}), \hat{q}) = \mathcal{L}_{B,2}(p_2(\hat{q}), \hat{q}),$$

otherwise, we could slightly move $\hat{q}$ up or down, depending on which $\mathcal{L}_{B,i}(p_i(\hat{q}), \hat{q})$ is bigger, and this would result in a positive increment in the overall value of $\mathcal{L}_B(p, q)$. Therefore, the result is established.

Alternatively, we may use the function $\Pi(\cdot)$ representing the dependence of the monopolist's expected profit on $\hat{q}$:

$$\begin{aligned} \Pi(\hat{q}) = v(\hat{q}, 0) - C(\hat{q}) &+ \int_{q_*}^{\hat{q}} [p(q) - C'(q)](N(P'(q), q) - 1) dq \\ &+ \int_{\hat{q}}^{q^*} [p(q) - C'(q)] N(P'(q), q) dq. \end{aligned}$$

Differentiating $\Pi(\hat{q})$ with respect to $\hat{q}$ gives the result.

Observe that we may have $\hat{q} = q^*$ in Theorem 6. Indeed, remember from the proof of Corollary 2 that if $q \ge \frac{1}{2b}$, then $\mathcal{L}_{B,2}(p, q) = (p - C'(q))N(p, q) < 0$ unless $N(p, q) = 0$, i.e. $\max_p \mathcal{L}_{B,2}(p, \hat{q}) = 0$. Thus, if $\hat{q} \ge \frac{1}{2b}$ then $\hat{q} = q^*$ and in this case, equation (A.3) takes the form

$$\mathcal{L}_{B,1}(p, \hat{q}) = v_q(\hat{q}, 0) - C'(\hat{q}) + (p_1 - C'(\hat{q}))N(p_1, \hat{q}) = 0.$$

Notice that in this Theorem we are not considering the possibility of exclusion. However, the result could be easily adapted to allow for it.

□

After establishing Theorems 5 and 6, we will show how to get the optimal marginal tariff $p(q)$ for our guiding example. The idea is that for a fixed $\hat{q}$, Theorem 5 gives pointwise optimality conditions for problem (L_B). Then Theorem 6 gives the recipe for finding the optimal transition point $\hat{q}$.

The analysis here is similar to the one we did for Corollary 2.[34] Our goal is to exhibit the optimal $p(q)$ for some values of b.

[34]We also re-use a few functions defined on the proof of this corollary.

Let us begin by defining the problem

$$\begin{aligned}&\max_p \mathcal{L}^e_{B,1}(p,q) := (p - C'(q))(\phi_s(p,q) - 1)\\&\text{subject to } p \in [0, \tfrac{1}{4b^2q}].\end{aligned} \tag{$L^e_{B,i}$}$$

Again, the signal of the second derivative of $\mathcal{L}^e_{B,1}(p,q)$ with respect to p is equal to the signal of

$$b^2q(3p+q) - 1. \tag{B.28}$$

Using that $p \leq \frac{1}{4b^2q}$ and restricting $q \leq \frac{1}{2b}$, we get

$$b^2q(3p+q) - 1 \leq 0.$$

So $\mathcal{L}_{B,1}(\cdot, q)$ is concave and the first-order conditions are necessary and sufficient for solving ($L^e_{B,i}$). We can characterize the interior optimum $p = p(q)$ by

$$g^e_q(q, \phi_s(p,q)) = 0.$$

The implicit solution of the above equation gives

$$\theta^1_s(q) := \phi_s(p(q), q) = \frac{1 - b^2q + \sqrt{4b^4q^2 - b^2q(3q+2) + 1}}{3b^2q}.$$

At $q = \frac{2}{4b^2+1}$ one can show that $\theta^1_s(q) = 1$. This will be the correct choice for q_*. Using Theorem 6, we can find b_* such that the optimal $\hat{q}$ is $\frac{2}{4b^2+1}$, i.e.

$$\mathcal{L}_{B,1}\left(p_1\left(\frac{2}{4b^2+1}\right), \frac{2}{4b^2+1}\right) = \mathcal{L}_{B,2}\left(p_2\left(\frac{2}{4b^2+1}\right), \frac{2}{4b^2+1}\right).$$

Numerically solving this equation we get $b_* = 0.764321$.

In summary, when $b \in (b_*, \frac{4}{5})$, using Theorem 6 we can find $\hat{q} \in (\frac{2}{4b^2+1}, \frac{1}{2b})$ such that the optimal marginal tariff satisfies

$$\theta_s(p(q), q) = \begin{cases} \theta^1_s(q), & \text{if } q_* = \frac{2}{4b^2+1} \leq q < \hat{q}, \\ \theta^2_s(q), & \text{if } \hat{q} < q \leq \frac{1}{2b} = q^*. \end{cases}$$

On the other hand, when $b \in [\frac{4}{5}, \frac{2}{\sqrt{5}})$ we will have $\hat{q} \geq \frac{1}{2b}$. Theorem 6 gives

$$\hat{q} = \frac{\sqrt{1-b^2}}{-2b^3 + \sqrt{1-b^2}b^2 + 2b}.$$

As $\hat{q} \geq \frac{1}{2b}$, we have that $q^* = \hat{q}$. Thus, the optimal $p(q)$ satisfies

$$\theta_s(p(q), q) = \theta^1_s(q), \text{ if } q_* = \frac{2}{4b^2+1} \leq q \leq \hat{q} = \frac{\sqrt{1-b^2}}{-2b^3 + \sqrt{1-b^2}b^2 + 2b} = q^*.$$

In this case, when $q \in (\frac{1}{2b}, \hat{q})$, we do not have the concavity of $\mathcal{L}^e_{B,1}(\cdot, q)$ anymore. However, the analysis of the signal of the second derivative of $\mathcal{L}^e_{B,1}(p, q)$ with respect to p, which is the same as the signal of expression (B.28), shows that it is increasing in p, assuming negative values for $p < p_c := \frac{1}{3b^2 q} - \frac{q}{3}$ and positive values when $p > p_c$. One can show that $p_c \in (v_q(q, \theta^1_s(q)), v_q(q, \theta^1_b(q)))$. Thus, $p = v_q(q, \theta^1_s(q))$ is a maximizer and $p = v_q(q, \theta^1_b(q))$ is a minimizer for problem ($L^e_{B,i}$).

Appendix C. Quantity Assignment Functions

In Section 4 we derived the optimal marginal tariff according to the demand profile approach. Now we are going to present the quantity assignment function $q(\theta)$ associated with this tariff. Observe that once we have an implementable $q(\cdot)$, we can also use expression (1) to compute the monopolist's expected profit, which is very convenient.

In *Case A*, Theorem 3 and 4 give $\theta_s(p(q), q)$ and $\theta_b(p(q), q)$. After that, we have to invert these functions to obtain

$$q(\theta) = \begin{cases} q_R(\theta) := -\frac{2\theta}{-3b^2\theta^2 - 2b^2\theta + b^2 - 1} & \text{if } 0 \le \theta < \theta_d, \\ q_{UC}(\theta) := \frac{3b^2\theta + 3b^2 + \sqrt{3b^4\theta^2 + 6b^4\theta + 3b^4 - 2b^2}}{2(3b^4\theta^2 + 6b^4\theta + 3b^4 + b^2)} & \text{if } \theta_d \le \theta \le \hat{\theta}_d, \\ q_C(\theta) := \frac{1}{b^2(\theta_d + \theta + 2)} & \text{if } \hat{\theta}_d \le \theta \le 1, \end{cases} \tag{C.1}$$

where $\hat{\theta}_d$ is implicitly defined by equation (B.21).

Using Lemma 2 we can find the optimal θ_d for all $b \in (\frac{1}{2}, \frac{2}{\sqrt{5}})$. We solved equation (B.25) numerically and the result is presented in the left hand side of Figure C.12.

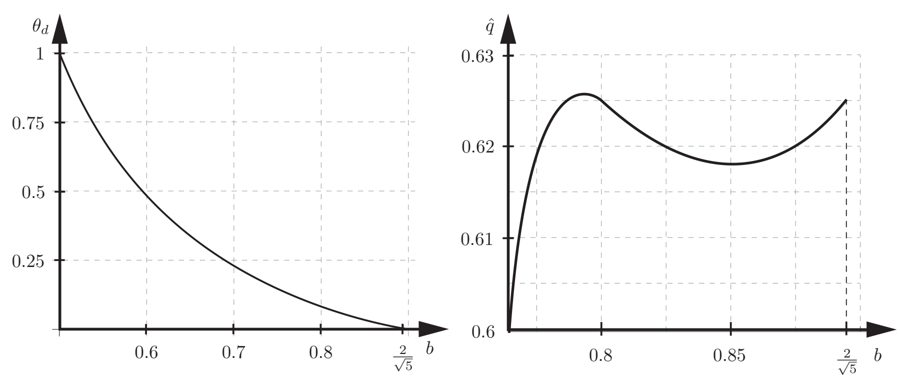

Figure C.12: The optimal θ_d and $\hat{q}$.

In *Case B*, let us consider the marginal tariff $p(q)$ emerging from Theorems 5 and 6. Again, we get the quantity assignment function $q(\theta)$ inverting $\theta_b(p(q), q)$ and $\theta_s(p(q), q)$. This procedure results in

$$q(\theta) = \begin{cases} \hat{q} & \text{if } 0 \leq \theta \leq \theta_1 \text{ and } \theta_2 \leq \theta \leq \theta_3, \\ q_{UC}(\theta) & \text{if } \theta_1 \leq \theta < \theta_2, \\ q_R(\theta) & \text{if } \theta_3 \leq \theta \leq 1, \end{cases} \tag{C.2}$$

where

$$\theta_1 = \theta_b(p_2, \hat{q}), \quad \theta_2 = \theta_s(p_2, \hat{q}) \quad \text{and} \quad \theta_3 = \theta_s(p_1, \hat{q}),$$

with $p_1 = p(\hat{q}_-)$ and $p_2 = p(\hat{q}_+)$ defined in Theorem 6.

In the right-hand side of Figure C.12 we present the optimal $\hat{q}$ that follows from Theorem 6 after solving numerically equation (A.3). We observe that if $b \geq \frac{8}{10}$, then $\theta_1 = \theta_2$ in expression (C.2). In this case, the quantity assignment function is

$$q(\theta) = \begin{cases} \hat{q} & \text{if } 0 \leq \theta \leq \theta_3, \\ q_R(\theta) & \text{if } \theta_3 \leq \theta \leq 1. \end{cases} \tag{C.3}$$

In this case we can find the exact expression for $\hat{q}$, and the result is

$$\hat{q} = \frac{\sqrt{1-b^2}}{-2b^3 + \sqrt{1-b^2}b^2 + 2b}. \tag{C.4}$$

Observe that the quantity assignment functions given by equations (C.2) and (C.3) are depicted in Figure A.11.

Finally, in Figure C.13 we can see the monopolist's expected profit using the demand profile approach. Notice that there is a point $b_{AB} \approx 0.8069$ such that for $b < b_{AB}$, the expected profit from *Case A* solution is greater. On the other hand, when $b > b_{AB}$ the profit from *Case B* solution is greater.

We now present the quantity assignment function that emerges from Theorems 1 and 2. Again, we need to invert the functions $\theta_s(p(q), q)$ and $\theta_b(p(q), q)$. This results in

$$q(\theta) = \begin{cases} q_R(\theta) = -\frac{2\theta}{-3b^2\theta^2 - 2b^2\theta + b^2 - 1} & \text{if } 0 \leq \theta \leq \theta_1, \\ q_\ell & \text{if } \theta_1 \leq \theta \leq \theta_d, \\ q_h & \text{if } \theta_d \leq \theta \leq \theta_2 \text{ or } \theta_3 \leq \theta \leq \theta_4, \\ q_{UC}(\theta) = \frac{3b^2\theta + 3b^2 + \sqrt{3b^4\theta^2 + 6b^4\theta + 3b^4 - 2b^2}}{2(3b^4\theta^2 + 6b^4\theta + 3b^4 + b^2)} & \text{if } \theta_2 \leq \theta \leq \theta_3, \\ q_J(\theta) := -\frac{\theta_d - 2\theta + \lambda - 1}{-2b^2(\theta+1)\theta_d + b^2(\theta+1)(3\theta - 2\lambda + 1) + 1} & \text{if } \theta_4 < \theta \leq 1, \end{cases}$$

where $\theta_1, \theta_2, \theta_3$ and θ_4 are defined implicitly by $q_R(\theta_1) = q_\ell$, $q_{UC}(\theta_2) = q_{UC}(\theta_3) = q_J(\theta_4) = q_h$. We may have $\theta_4 > 1$, and in this case q_J is excluded from the definition above. We may also have $q_h > \frac{1}{2b}$ and now, q_{UC} is excluded and we set $\theta_2 = \theta_3$.

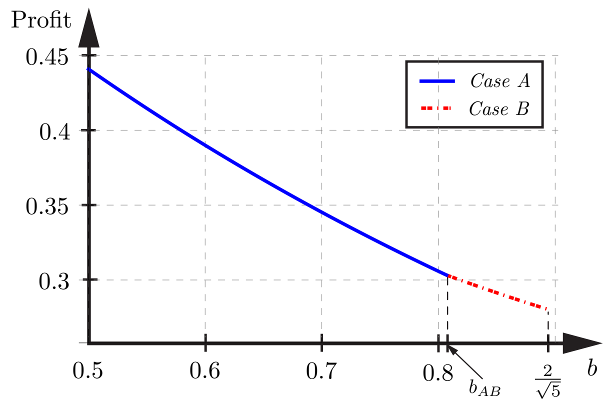

Figure C.13: The monopolist expected profit under the demand profile approach.

Observe that we have a natural division of the customers in two groups, $[0, \theta_d]$ and $[\theta_d, 1]$. The first one chooses $q \leq q_\ell$ and the second one chooses $q \geq \overline{q}$.

In each one of these groups we are applying the demand profile approach. The customer θ_d belongs to both groups and the role of condition (J) is to make this customer indifferent between choosing q_ℓ and q_h. When $b \in [1/2, \frac{2}{\sqrt{5}}]$ we numerically solved the equations from Theorem 2, and we find the monopolist's expected profit. In Figure C.14 we present the percentage gain for the monopolist when he chooses the marginal tariff proposed by this article. Observe that this gain is greater when b approaches b_{AB}.

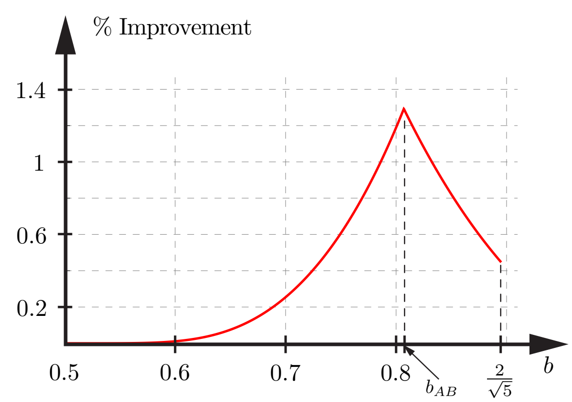

Figure C.14: Percentage Improvement.

After presenting the quantity assignment function, we will deal with its implementability. We need to check whether $q(\theta)$ satisfies

$$q(\theta) \in \arg\max_{q} v(q,\theta) - P(q). \tag{C.5}$$

where $P(q)$ is the tariff that we get by integrating the marginal tariff $p(q)$.

In Section 4, the marginal tariff needed to be consistent with the demand profile approach. Indeed, we had to check whether the consistency condition (DPA) was satisfied. This condition ensures the implementability of *Case A* and *Case B* solutions.

If we consider the groups $[\theta, \theta_d]$ and $[\theta_d, 1]$ separately, we can apply the same arguments used in Section 4 to show the implementability inside these groups.

The novelty is that we need to show that a customer from one group does not envy a customer from the other group. For this purpose, we are going to use the following Lemma, which appears in Araujo and Moreira [8].

Lemma 3. *The allocation rule $(q,t) : \Theta \to \mathbb{R}_+ \times \mathbb{R}$ is incentive compatible if and only if*

$$\Phi(\theta,\hat{\theta}) := \int_{\hat{\theta}}^{\theta}\int_{q(\hat{\theta})}^{q(\tilde{\theta})} v_{q\theta}(\tilde{q},\tilde{\theta})d\tilde{q}d\tilde{\theta} \geq 0, \quad \forall \theta, \hat{\theta} \in \Theta. \tag{C.6}$$

After finding the quantity assignment function $q(\theta)$, Lemma 3 gives a direct test for implementability. We have the following workflow, first we use Theorems 1 and 2 to get $\theta_s(p(q),q)$ and $\theta_b(p(q),q)$. After that, we invert these functions to get $q(\theta)$. Finally, we use Lemma 3 to verify the implementability of $q(\theta)$.[35]

We applied this workflow for all $b \in (\frac{1}{2}, \frac{2}{\sqrt{5}})$, and the percentage gain over Section 4 solution is depicted in Figure C.14.

As a final observation, we checked that the quantity assignment function lies in the region where the derivative $v_\theta \geq 0$. Thus, our assumption that the informational rent $V(0) = 0$ is correct.[36]

[35] In this last step, if necessary, we may have to numerically compute $\Phi(\theta,\hat{\theta})$.

[36] When we solve the equation $v_\theta(q_1,\theta) = 0$ for q_1, we get $q_1(\theta) = \frac{1}{b^2(\theta+1)}$. If the quantity assignment function satisfies $q(\theta) < q_1(\theta)$, then our assumption $V(0) = 0$ is justified. We did this check for all $q(\theta)$ when $b \in (\frac{1}{2}, \frac{2}{\sqrt{5}})$.

REFERENCES

[1] M. Mussa, S. Rosen, Monopoly and product quality, Journal of Economic Theory 18 (1978) 301–317.

[2] M. Goldman, H. Leland, D. Sibley, Optimal nonuniform prices, The Review of Economic Studies 51 (2) (1984) 305–319.

[3] E. Maskin, J. Riley, Monopoly with incomplete information, The RAND Journal of Economics 15 (2) (1984) 171–196.

[4] S. J. Brown, D. S. Sibley, The theory of public utility pricing, Cambridge Univ Pr, 1986.

[5] J.-C. Rochet, L. Stole, The economics of multidimensional screening, Advances in economics and econometrics: theory and applications, Eighth World Congress, Econometric Society, by Mathias Dewatripont, Lars Peter Hansen, Stephen J. Turnovsky (5) (2003) 392.

[6] M. Armstrong, Multiproduct Nonlinear Pricing, Econometrica: Journal of the Econometric Society 64 (1) (1996) 51–75.

[7] J.-J. Laffont, E. Maskin, J.-C. Rochet, Optimal nonlinear pricing with two dimensional characteristics, Information, Incentives, and Economic Mechanisms: Essays in Honor of Leonid Hurwicz by Theodore Groves, Roy Radner, and Stanley Reiter .

[8] A. Araujo, H. Moreira, Adverse selection problems without the Spence-Mirrlees condition, Journal of Economic Theory .

[9] R. Wilson, Nonlinear pricing, Oxford University Press, USA, 1997.

[10] J.-C. Rochet, A necessary and sufficient condition for rationalizability in a quasi-linear context, Journal of Mathematical Economics 16 (1987) 191–200.

[11] D. Fudenberg, J. Tirole, Game theory, 1993.

[12] R. Guesnerie, On taxation and incentives, 1981.

[13] P. Hammond, Straightforward individual incentive compatibility in large economies, The Review of Economic Studies 46 (2) (1979) 263–282.

[14] J.-C. Rochet, The taxation principle and multi-time Hamilton-Jacobi equations., Journal of Mathematical Economics 14 (1985) 113–128.

[15] P. Milgrom, I. Segal, Envelope theorems for arbitrary choice sets, Econometrica: Journal of the Econometric Society 70 (2) (2002) 583–601.

[16] M. Kamien, N. Schwartz, Dynamic optimization, the calculus of variations and optimal control in economics and management, Elsevier Science, 1991.