

YET ANOTHER PROOF OF MARSTRAND'S THEOREM

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ABSTRACT. In a paper from 1954 Marstrand proved that if $K \subset \mathbb{R}^2$ is a Borel set with Hausdorff dimension greater than 1, then its one-dimensional projection has positive Lebesgue measure for almost-all directions. In this article, we give a combinatorial proof of this theorem, extending the techniques developed in our previous paper [9].

1. INTRODUCTION

If U is a subset of \mathbb{R}^n , the diameter of U is $|U| = \sup\{|x - y|; x, y \in U\}$ and, if \mathcal{U} is a family of subsets of \mathbb{R}^n , the *diameter* of \mathcal{U} is defined as

$$\|\mathcal{U}\| = \sup_{U \in \mathcal{U}} |U|.$$

Given $s > 0$, the *Hausdorff s -measure* of a set $K \subset \mathbb{R}^n$ is

$$m_s(K) = \lim_{\varepsilon \rightarrow 0} \left(\inf_{\substack{\mathcal{U} \text{ covers } K \\ \|\mathcal{U}\| < \varepsilon}} \sum_{U \in \mathcal{U}} |U|^s \right).$$

In particular, when $n = 1$, $m = m_1$ is the Lebesgue measure of Lebesgue measurable sets on \mathbb{R} . It is not difficult to show that there exists a unique $s_0 \geq 0$ for which $m_s(K) = \infty$ if $s < s_0$ and $m_s(K) = 0$ if $s > s_0$. We define the Hausdorff dimension of K as $\text{HD}(K) = s_0$. Also, for each $\theta \in \mathbb{R}$, let $v_\theta = (\cos \theta, \sin \theta)$, L_θ the line in \mathbb{R}^2 through the origin containing v_θ and $\text{proj}_\theta : \mathbb{R}^2 \rightarrow L_\theta$ the orthogonal projection. From now on, we'll restrict θ to the interval $[-\pi/2, \pi/2]$, because $L_\theta = L_{\theta+\pi}$.

In 1954, J. M. Marstrand [10] proved the following result on the fractal dimension of plane sets.

Theorem 1.1. *If $K \subset \mathbb{R}^2$ is a Borel set such that $\text{HD}(K) > 1$, then $m(\text{proj}_\theta(K)) > 0$ for m -almost every $\theta \in \mathbb{R}$.*

The proof is based on a qualitative characterization of the “bad” angles θ for which the result is not true. Specifically, Marstrand exhibits a Borel measurable function $f(x, \theta)$, $(x, \theta) \in \mathbb{R}^2 \times [-\pi/2, \pi/2]$, such that $f(x, \theta) = \infty$ for m_s -almost every $x \in K$, for every “bad” angle. In particular,

$$\int_K f(x, \theta) dm_s(x) = \infty. \tag{1.1}$$

On the other hand, using a version of Fubini's Theorem, he proves that

$$\int_{-\pi/2}^{\pi/2} d\theta \int_K f(x, \theta) dm_s(x) = 0$$

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which, in view of (1.1), implies that

$$m(\{\theta \in [-\pi/2, \pi/2]; m(\text{proj}_\theta(K)) = 0\}) = 0.$$

These results are based on the analysis of rectangular densities of points.

Many generalizations and simpler proofs have appeared since. One of them came in 1968 by R. Kaufman who gave a very short proof of Marstrand's Theorem using methods of potential theory. See [7] for his original proof and [12], [18] for further discussion.

In this article, we give a new proof of Theorem 1.1. Our proof makes a study on the fibers $K \cap \text{proj}_\theta^{-1}(v)$, $(\theta, v) \in \mathbb{R} \times L_\theta$, and relies on two facts:

(I) Transversality condition: given two squares on the plane, the Lebesgue measure of the set of angles for which their projections have nonempty intersection has an upper bound. See Subsection 3.2.

(II) After a regularization of K , (I) enables us to conclude that, except for a small set of angles $\theta \in \mathbb{R}$, the fibers $K \cap \text{proj}_\theta^{-1}(v)$ are not concentrated in a thin region. As a consequence, K projects into a set of positive Lebesgue measure.

The idea of (II) is based on the work [14] of the second author and was employed in [9] to develop a combinatorial proof of Theorem 1.1 when K is the product of two regular Cantor sets. In the present paper, we give a combinatorial proof of Theorem 1.1 without any restrictions on K . Compared to other proofs of Marstrand's Theorem, the new ideas here are the discretization of the argument and the use of dyadic covers, which allow the simplification of the method employed. These covers may be composed of sets with rather different scales and so a weighted sum is necessary to capture the Hausdorff s -measure of K .

The theory developed in [9] works whenever K is an *Ahlfors-David regular set*, namely when there are constants $a, b > 0$ such that

$$a \cdot r^d \leq m_s(K \cap B_r(x)) \leq b \cdot r^d, \quad \text{for any } x \in K \text{ and } 0 < r \leq 1.$$

Unfortunately, the general situation can not be reduced to this one, as proved by P. Mattila and P. Saaranen: in [13], they constructed a compact set of \mathbb{R} with positive Lebesgue measure such that it contains no nonempty Ahlfors-David subset.

We also show that the push-forward measure of the restriction of m_s to K , defined as $\mu_\theta = (\text{proj}_\theta)_*(m_s|_K)$, is absolutely continuous with respect to m , for m -almost every $\theta \in \mathbb{R}$, and its Radon-Nykodim derivative is square-integrable.

Theorem 1.2. *The measure μ_θ is absolutely continuous with respect to m and its Radon-Nykodim derivative is an L^2 function, for m -almost every $\theta \in \mathbb{R}$.*

Remark 1.3. Theorem 1.2, as in this work, follows from most proofs of Marstrand's theorem and, in particular, is not new as well.

Marstrand's Theorem is a classical result in Geometric Measure Theory. In particular, if $K = K_1 \times K_2$ is a cartesian product of two one-dimensional subsets of \mathbb{R} , Marstrand's theorem translates to " $m(K_1 + \lambda K_2) > 0$ for m -almost every $\lambda \in \mathbb{R}$ ". The investigation of such *arithmetic sums* $K_1 + \lambda K_2$ has been an active area of Mathematics, in special when K_1 and K_2 are dynamically defined Cantor sets. Although he did not know, M. Hall [5] proved, in 1947, that the Lagrange spectrum¹

¹The Lagrange spectrum is the set of best constants of rational approximations of irrational numbers. See [2] for the specific description.

contains a whole half line, by showing that the arithmetic sum $K(4) + K(4)$ of a certain Cantor set $K(4) \subset \mathbb{R}$ with itself contains $[6, \infty)$.

Marstrand's Theorem for product of Cantor sets is also fundamental in certain results of dynamical bifurcations, namely homoclinic bifurcations in surfaces. For instance, in [19] it is used to show that hyperbolicity is not prevalent in homoclinic bifurcations associated to horseshoes with Hausdorff dimension larger than one; in [16] it is used to prove that stable intersections of regular Cantor sets are dense in the region where the sum of their Hausdorff dimensions is larger than one; in [17] to show that, for homoclinic bifurcations associated to horseshoes with Hausdorff dimension larger than one, typically there are open sets of parameters with positive Lebesgue density at the initial bifurcation parameter corresponding to persistent homoclinic tangencies.

In the connection of these two applications, we point out that a formula for the Hausdorff dimension of $K_1 + K_2$, under mild assumptions of non-linear Cantor sets K_1 and K_2 , has been obtained by the second author in [14] and applied in [15] to prove that the Hausdorff dimension of the Lagrange spectrum increases continuously. In parallel to this non-linear setup, Y. Peres and P. Shmerkin proved the same phenomena happen to self-similar Cantor sets without algebraic resonance [20]. Finally, M. Hochman and P. Shmerkin extended and unified many results concerning projections of products of self-similar measures on regular Cantor sets [6].

The paper is organized as follows. In Section 2 we introduce the basic notations and definitions. Section 3 is devoted to the main calculations, including the transversality condition in Subsection 3.2 and the proof of existence of good dyadic covers in Subsection 3.3. Finally, in Section 4 we prove Theorems 1.1 and 1.2. We also collect final remarks in Section 5.

2. PRELIMINARIES

2.1. Notation. The distance in \mathbb{R}^2 will be denoted by $|\cdot|$. Let $B_r(x)$ denote the open ball of \mathbb{R}^2 centered in x with radius r . As in Section 1, the diameter of $U \subset \mathbb{R}^2$ is $|U| = \sup\{|x - y|; x, y \in U\}$ and, if \mathcal{U} is a family of subsets of \mathbb{R}^2 , the diameter of \mathcal{U} is defined as

$$\|\mathcal{U}\| = \sup_{U \in \mathcal{U}} |U|.$$

Given $s > 0$, the Hausdorff s -measure of a set $K \subset \mathbb{R}^2$ is $m_s(K)$ and its Hausdorff dimension is $\text{HD}(K)$. In this work, we assume K is contained in $[0, 1]^2$.

Definition 2.1. A Borel set $K \subset \mathbb{R}^2$ is an s -set if $\text{HD}(K) = s$ and $0 < m_s(K) < \infty$.

Let m be the Lebesgue measure of Lebesgue measurable sets on \mathbb{R} . For each $\theta \in \mathbb{R}$, let $v_\theta = (\cos \theta, \sin \theta)$, L_θ the line in \mathbb{R}^2 through the origin containing v_θ and $\text{proj}_\theta : \mathbb{R}^2 \rightarrow L_\theta$ the orthogonal projection onto L_θ .

A square $[a, a+l) \times [b, b+l) \subset \mathbb{R}^2$ will be denoted by Q and its center, the point $(a + l/2, b + l/2)$, by x .

We use Vinogradov notation to compare the asymptotic of functions. Let X be a set.

Definition 2.2. Let $f, g : X \rightarrow \mathbb{R}$ be two real-valued functions. We say $f \ll g$ if there is a constant $C > 0$ such that

$$|f(x)| \leq C \cdot |g(x)|, \quad \forall x \in X.$$

If $f \ll g$ and $g \ll f$, we write $f \asymp g$.

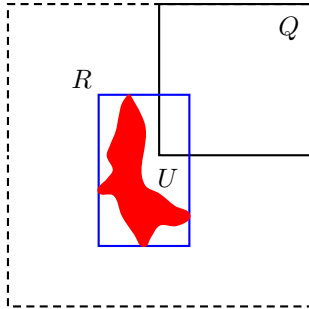
2.2. Dyadic squares. Let \mathcal{D}_0 be the family of unity squares of \mathbb{R}^2 congruent to $[0, 1)^2$ and with vertices in the lattice \mathbb{Z}^2 . Dilating this family by a factor of 2^{-i} , we obtain the family \mathcal{D}_i , $i \in \mathbb{Z}$.

Definition 2.3. Let \mathcal{D} denote the union of \mathcal{D}_i , $i \in \mathbb{Z}$. A *dyadic square* is any element $Q \in \mathcal{D}$.

The dyadic squares possess the following properties:

- (1) Every $x \in \mathbb{R}^2$ belongs to exactly one element of each family \mathcal{D}_i .
- (2) Two dyadic squares are either disjoint or one is contained in the other.
- (3) A dyadic square of \mathcal{D}_i is contained in exactly one dyadic square of \mathcal{D}_{i-1} and contains exactly four dyadic squares of \mathcal{D}_{i+1} .
- (4) Given any subset $U \subset \mathbb{R}^2$, there are four dyadic squares, each with side length at most $2 \cdot |U|$, whose union contains U .

(1) to (3) are direct. To prove (4), let R be smallest rectangle of \mathbb{R}^2 with sides parallel to the axis that contains \bar{U} . The sides of R have length at most $|U|$. Let $i \in \mathbb{Z}$ such that $2^{-i-1} < |U| \leq 2^{-i}$ and choose a dyadic square $Q \in \mathcal{D}_i$ that intersects R . If Q contains U , we're done. If not, Q and three of its neighbors cover U .



Definition 2.4. A *dyadic cover* of K is a finite subset $\mathcal{C} \subset \mathcal{D}$ of disjoint dyadic squares such that

$$K \subset \bigcup_{Q \in \mathcal{C}} Q.$$

First used by A.S. Besicovitch in his demonstration that closed sets of infinite m_s -measure contain subsets of positive but finite measure [1], dyadic covers were later employed by Marstrand to investigate the Hausdorff measure of cartesian products of sets [11].

Due to (4), for any family \mathcal{U} of subsets of \mathbb{R}^2 , there is a dyadic family \mathcal{C} such that

$$\bigcup_{Q \in \mathcal{C}} Q \supset \bigcup_{U \in \mathcal{U}} U \quad \text{and} \quad \sum_{Q \in \mathcal{C}} |Q|^s < 64 \cdot \sum_{U \in \mathcal{U}} |U|^s$$

and so, if K is an s -set, there exists a sequence $(\mathcal{C}_i)_{i \geq 1}$ of dyadic covers of K with diameters converging to zero such that

$$\sum_{Q \in \mathcal{C}_i} |Q|^s \asymp 1. \tag{2.1}$$

3. CALCULATIONS

Let $K \subset \mathbb{R}^2$ be a Borel set with Hausdorff dimension greater than one. From now on, we assume every cover of K is composed of dyadic squares of sides at most one. Before going into the calculations, we make the following reduction.

Lemma 3.1. *Let K be a Borel subset of \mathbb{R}^2 . Given $s < \text{HD}(K)$, there exists a compact s -set $K' \subset K$ such that*

$$m_s(K' \cap B_r(x)) \ll r^d, \quad x \in \mathbb{R}^2 \text{ and } 0 \leq r \leq 1.$$

In other words, there exists a constant $b > 0$ such that

$$m_s(K' \cap B_r(x)) \leq b \cdot r^d, \quad \text{for any } x \in \mathbb{R}^2 \text{ and } 0 \leq r \leq 1. \quad (3.1)$$

See Theorem 5.4 of [4] for a proof of the above lemma when K is closed and [3] for the general case. From now on, we assume K is a compact s -set, with $s > 1$, that satisfies (3.1).

Given a dyadic cover \mathcal{C} of K , let, for each $\theta \in [-\pi/2, \pi/2]$, $f_\theta^{\mathcal{C}} : L_\theta \rightarrow \mathbb{R}$ be the function defined by

$$f_\theta^{\mathcal{C}}(x) = \sum_{Q \in \mathcal{C}} \chi_{\text{proj}_\theta(Q)}(x) \cdot |Q|^{s-1},$$

where $\chi_{\text{proj}_\theta(Q)}$ denotes the characteristic function of the set $\text{proj}_\theta(Q)$. The reason we consider this function is that it captures the Hausdorff s -measure of K in the sense that

$$\begin{aligned} \int_{L_\theta} f_\theta^{\mathcal{C}}(x) dm(x) &= \sum_{Q \in \mathcal{C}} |Q|^{s-1} \cdot \int_{L_\theta} \chi_{\text{proj}_\theta(Q)}(x) dm(x) \\ &= \sum_{Q \in \mathcal{C}} |Q|^{s-1} \cdot m(\text{proj}_\theta(Q)) \end{aligned}$$

which, as $|Q|/2 \leq m(\text{proj}_\theta(Q)) \leq |Q|$, satisfies

$$\int_{L_\theta} f_\theta^{\mathcal{C}}(x) dm(x) \asymp \sum_{Q \in \mathcal{C}} |Q|^s.$$

If in addition \mathcal{C} satisfies (2.1), then

$$\int_{L_\theta} f_\theta^{\mathcal{C}}(x) dm(x) \asymp 1, \quad \forall \theta \in [-\pi/2, \pi/2]. \quad (3.2)$$

Denoting the union $\bigcup_{Q \in \mathcal{C}} Q$ by \mathcal{C} as well, an application of the Cauchy-Schwarz inequality gives that

$$m(\text{proj}_\theta(\mathcal{C})) \cdot \left(\int_{\text{proj}_\theta(\mathcal{C})} (f_\theta^{\mathcal{C}})^2 dm \right) \geq \left(\int_{\text{proj}_\theta(\mathcal{C})} f_\theta^{\mathcal{C}} dm \right)^2 \asymp 1.$$

The above inequality implies that if $(\mathcal{C}_i)_{i \geq 1}$ is a sequence of dyadic covers of K satisfying (2.1) with diameters converging to zero and the L^2 -norm of $f_\theta^{\mathcal{C}_i}$ is uniformly bounded, that is

$$\int_{\text{proj}_\theta(\mathcal{C}_i)} \left(f_\theta^{\mathcal{C}_i} \right)^2 dm \ll 1, \quad (3.3)$$

then

$$m(\text{proj}_\theta(K)) = \lim_{i \rightarrow \infty} m(\text{proj}_\theta(\mathcal{C}_i)) \gg 1$$

and so $\text{proj}_\theta(K)$ has positive Lebesgue measure, as wished. This conclusion will be obtained for m -almost every $\theta \in [-\pi/2, \pi/2]$ by showing that

$$I_i \doteq \int_{-\pi/2}^{\pi/2} d\theta \int_{L_\theta} \left(f_\theta^{\mathcal{C}_i} \right)^2 dm \ll 1. \quad (3.4)$$

3.1. Rewriting the integral I_i . For simplicity, let f denote $f_\theta^{\mathcal{C}_i}$. Then the interior integral of (3.4) becomes

$$\begin{aligned} \int_{L_\theta} f^2 dm &= \int_{L_\theta} \left(\sum_{Q \in \mathcal{C}_i} \chi_{\text{proj}_\theta(Q)} \cdot |Q|^{s-1} \right) \cdot \left(\sum_{\tilde{Q} \in \mathcal{C}_i} \chi_{\text{proj}_\theta(\tilde{Q})} \cdot |\tilde{Q}|^{s-1} \right) dm \\ &= \sum_{Q, \tilde{Q} \in \mathcal{C}_i} |Q|^{s-1} \cdot |\tilde{Q}|^{s-1} \cdot \int_{L_\theta} \chi_{\text{proj}_\theta(Q) \cap \text{proj}_\theta(\tilde{Q})} dm \\ &= \sum_{Q, \tilde{Q} \in \mathcal{C}_i} |Q|^{s-1} \cdot |\tilde{Q}|^{s-1} \cdot m(\text{proj}_\theta(Q) \cap \text{proj}_\theta(\tilde{Q})) \end{aligned}$$

and, using the inequalities

$$m(\text{proj}_\theta(Q) \cap \text{proj}_\theta(\tilde{Q})) \leq \min\{m(\text{proj}_\theta(Q)), m(\text{proj}_\theta(\tilde{Q}))\} \leq \min\{|Q|, |\tilde{Q}|\},$$

it follows that

$$\int_{L_\theta} f^2 dm \ll \sum_{Q, \tilde{Q} \in \mathcal{C}_i} |Q|^{s-1} \cdot |\tilde{Q}|^{s-1} \cdot \min\{|Q|, |\tilde{Q}|\}. \quad (3.5)$$

We now proceed to prove (3.4) by a double-counting argument. To this matter, consider, for each pair of squares $(Q, \tilde{Q}) \in \mathcal{C}_i \times \mathcal{C}_i$, the set

$$\Theta_{Q, \tilde{Q}} = \left\{ \theta \in [-\pi/2, \pi/2]; \text{proj}_\theta(Q) \cap \text{proj}_\theta(\tilde{Q}) \neq \emptyset \right\}.$$

Then

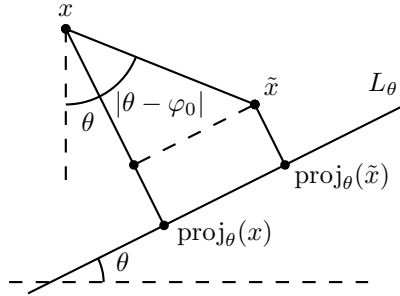
$$\begin{aligned} I_i &\ll \sum_{Q, \tilde{Q} \in \mathcal{C}_i} |Q|^{s-1} \cdot |\tilde{Q}|^{s-1} \cdot \min\{|Q|, |\tilde{Q}|\} \cdot \int_{-\pi/2}^{\pi/2} \chi_{\Theta_{Q, \tilde{Q}}}(\theta) d\theta \\ &= \sum_{Q, \tilde{Q} \in \mathcal{C}_i} |Q|^{s-1} \cdot |\tilde{Q}|^{s-1} \cdot \min\{|Q|, |\tilde{Q}|\} \cdot m(\Theta_{Q, \tilde{Q}}). \end{aligned} \quad (3.6)$$

3.2. Transversality condition. This subsection estimates the Lebesgue measure of $\Theta_{Q, \tilde{Q}}$.

Lemma 3.2. *If Q, \tilde{Q} are squares of \mathbb{R}^2 and $x, \tilde{x} \in \mathbb{R}^2$ are its centers, respectively, then*

$$m(\Theta_{Q, \tilde{Q}}) \leq 2\pi \cdot \frac{\max\{|Q|, |\tilde{Q}|\}}{|x - \tilde{x}|}.$$

Proof. Let $\theta \in \Theta_{Q, \tilde{Q}}$ and consider the figure.



Since $\text{proj}_\theta(Q)$ has diameter at most $|Q|$ (and the same happens to \tilde{Q}), we have $|\text{proj}_\theta(x) - \text{proj}_\theta(\tilde{x})| \leq 2 \cdot \max\{|Q|, |\tilde{Q}|\}$ and then, by elementary geometry,

$$\begin{aligned} \sin(|\theta - \varphi_0|) &= \frac{|\text{proj}_\theta(x) - \text{proj}_\theta(\tilde{x})|}{|x - \tilde{x}|} \\ &\leq 2 \cdot \frac{\max\{|Q|, |\tilde{Q}|\}}{|x - \tilde{x}|} \\ \implies |\theta - \varphi_0| &\leq \pi \cdot \frac{\max\{|Q|, |\tilde{Q}|\}}{|x - \tilde{x}|}, \end{aligned}$$

because $\sin^{-1} y \leq \pi y/2$. As φ_0 is fixed, the lemma is proved. \square

We point out that, although ingenious, Lemma 3.2 expresses the crucial property of transversality that makes the proof work, and all results related to Marstrand's Theorem use a similar idea in one way or another. See [21] where this transversality condition is also exploited.

By Lemma 3.2 and (3.6), we obtain

$$I_i \ll \sum_{Q, \tilde{Q} \in \mathcal{C}_i} |x - \tilde{x}|^{-1} \cdot |Q|^s \cdot |\tilde{Q}|^s. \quad (3.7)$$

3.3. Good covers. The last summand will be estimated by choosing appropriate dyadic covers \mathcal{C}_i . Let \mathcal{C} be an arbitrary dyadic cover of K . Remember K is an s -set satisfying (3.1).

Definition 3.3. The dyadic cover \mathcal{C} is *good* if

$$\sum_{\substack{\tilde{Q} \in \mathcal{C} \\ \tilde{Q} \subset Q}} |\tilde{Q}|^s < \max\{128b, 1\} \cdot |Q|^s, \quad \forall Q \in \mathcal{D}. \quad (3.8)$$

Any other constant depending only on K would work for the definition. The reason we chose this specific constant will become clear below, where we provide the existence of good dyadic covers.

Proposition 3.4. *Let $K \subset \mathbb{R}^2$ be a compact s -set satisfying (3.1). Then, for any $\delta > 0$, there exists a good dyadic cover of K with diameter less than δ .*

Proof. Let $i_0 \geq 1$ such that $2^{-i_0-1} < \|\mathcal{C}\| \leq 2^{-i_0}$. Begin with a finite cover \mathcal{U} of K with diameter less than $\delta/4$ such that

$$\sum_{\substack{U \in \mathcal{U} \\ U \subset Q}} |U|^s < 2 \cdot m_s(K \cap Q), \quad \forall Q \in \mathcal{D}_{i_0}.$$

Now, change \mathcal{U} by a dyadic cover \mathcal{C} according to property (4) of Subsection 2.1. \mathcal{C} has diameter at most δ and satisfies

$$\sum_{\substack{\tilde{Q} \in \mathcal{C} \\ \tilde{Q} \subset Q}} |\tilde{Q}|^s < 64 \cdot \sum_{\substack{U \in \mathcal{U} \\ U \subset Q}} |U|^s < 128 \cdot m_s(K \cap Q), \quad \forall Q \in \mathcal{D}_{i_0}.$$

By additivity, the same inequality happens for any $Q \in \bigcup_{0 \leq i \leq i_0} \mathcal{D}_i$ and so, as $m_s(K \cap Q) \leq b \cdot |Q|^s$, it follows that

$$\sum_{\substack{\tilde{Q} \in \mathcal{C} \\ \tilde{Q} \subset Q}} |\tilde{Q}|^s < 128b \cdot |Q|^s, \quad \forall Q \in \bigcup_{0 \leq i \leq i_0} \mathcal{D}_i, \quad (3.9)$$

that is, (3.8) holds for large scales. To control the small ones, apply the following operation: whenever $Q \in \bigcup_{i > i_0} \mathcal{D}_i$ is such that

$$\sum_{\substack{\tilde{Q} \in \mathcal{C} \\ \tilde{Q} \subset Q}} |\tilde{Q}|^s > |Q|^s,$$

we change \mathcal{C} by $\mathcal{C} \cup \{Q\} \setminus \{\tilde{Q} \in \mathcal{C}; \tilde{Q} \subset Q\}$. It is clear that such operation preserves the inequality (3.9) and so, after a finite number of steps, we end up with a good dyadic cover. \square

As the constant in (3.8) does not depend on δ , there is a sequence $(\mathcal{C}_i)_{i \geq 1}$ of good dyadic covers of K with diameters converging to zero such that

$$\sum_{\substack{\tilde{Q} \in \mathcal{C}_i \\ \tilde{Q} \subset Q}} |\tilde{Q}|^s \ll |Q|^s, \quad Q \in \mathcal{D} \text{ and } i \geq 1. \quad (3.10)$$

4. PROOF OF THEOREMS 1.1 AND 1.2

Proof of Theorem 1.1. Let $(\mathcal{C}_i)_{i \geq 1}$ be a sequence of good dyadic covers satisfying (3.10) such that $\|\mathcal{C}_i\| \rightarrow 0$. By (3.7),

$$\begin{aligned} I_i &\ll \sum_{Q, \tilde{Q} \in \mathcal{C}_i} |x - \tilde{x}|^{-1} \cdot |Q|^s \cdot |\tilde{Q}|^s \\ &= \sum_{Q \in \mathcal{C}_i} \sum_{j=0}^{\infty} \sum_{\substack{\tilde{Q} \in \mathcal{C}_i \\ 2^{-j-1} < |x - \tilde{x}| \leq 2^{-j}}} |x - \tilde{x}|^{-1} \cdot |Q|^s \cdot |\tilde{Q}|^s \\ &\leq \sum_{Q \in \mathcal{C}_i} \sum_{j=0}^{\infty} \sum_{\substack{\tilde{Q} \in \mathcal{C}_i \\ \tilde{Q} \subset B_{3 \cdot 2^{-j}}(x)}} |x - \tilde{x}|^{-1} \cdot |Q|^s \cdot |\tilde{Q}|^s \\ &\ll \sum_{Q \in \mathcal{C}_i} |Q|^s \sum_{j=0}^{\infty} 2^j \cdot (2^{-j})^s \\ &= \sum_{Q \in \mathcal{C}_i} |Q|^s \sum_{j=0}^{\infty} (2^j)^{1-s} \\ &\ll \sum_{Q \in \mathcal{C}_i} |Q|^s \\ &\ll 1, \end{aligned}$$

establishing (3.4). Define, for each $\varepsilon > 0$, the sets

$$G_\varepsilon^i = \left\{ \theta \in [-\pi/2, \pi/2]; \int_{L_\theta} \left(f_\theta^{C_i} \right)^2 dm < \varepsilon^{-1} \right\}, \quad i \geq 1.$$

Then $m([- \pi/2, \pi/2] \setminus G_\varepsilon^i) \ll \varepsilon$, and the same holds for the set

$$G_\varepsilon = \bigcap_{i \geq 1} \bigcup_{j=i}^{\infty} G_\varepsilon^j.$$

If $\theta \in G_\varepsilon$, then

$$m(\text{proj}_\theta(C_i)) \gg \varepsilon, \quad \text{for infinitely many } n,$$

which implies that $m(\text{proj}_\theta(K)) > 0$. Finally, the set $G = \bigcup_{i \geq 1} G_{1/i}$ satisfies $m([- \pi/2, \pi/2] \setminus G) = 0$ and $m(\text{proj}_\theta(K)) > 0$, for any $\theta \in G$. \square

A direct consequence is the

Corollary 4.1. *The measure $\mu_\theta = (\text{proj}_\theta)_*(m_s|_K)$ is absolutely continuous with respect to m , for m -almost every $\theta \in \mathbb{R}$.*

Proof. By Theorem 1.1, we have the implication

$$X \subset K, \quad m_s(X) > 0 \implies m(\text{proj}_\theta(X)) > 0, \quad m\text{-almost every } \theta \in \mathbb{R}, \quad (4.1)$$

which is sufficient for the required absolute continuity. Indeed, if $Y \subset L_\theta$ satisfies $m(Y) = 0$, then

$$\mu_\theta(Y) = m_s(X) = 0,$$

where $X = \text{proj}_\theta^{-1}(Y)$. Otherwise, by (4.1) we would have $m(Y) = m(\text{proj}_\theta(X)) > 0$, contradicting the assumption. \square

Let $f_\theta = d\mu_\theta/dm$. By the proof of Theorem 1.1, we have

$$\|f_\theta^{C_i}\|_{L^2} \ll 1, \quad m\text{-a.e. } \theta \in \mathbb{R}. \quad (4.2)$$

Proof of Theorem 2. Define, for each $\varepsilon > 0$, the function $f_{\theta,\varepsilon} : L_\theta \rightarrow \mathbb{R}$ by

$$f_{\theta,\varepsilon}(x) = \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} f_\theta(y) dm(y), \quad x \in L_\theta.$$

As f_θ is an L^1 -function, the Lebesgue differentiation theorem gives that $f_\theta(x) = \lim_{\varepsilon \rightarrow 0} f_{\theta,\varepsilon}(x)$ for m -almost every $x \in L_\theta$. If we manage to show that²

$$\|f_{\theta,\varepsilon}\|_{L^2} \ll 1, \quad m\text{-a.e. } \theta \in \mathbb{R}, \quad (4.3)$$

then Fatou's lemma establishes the theorem. To this matter, first observe that

$$\begin{aligned} f_{\theta,\varepsilon}(x) &= \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} f_\theta(y) dm(y) \\ &= \frac{1}{2\varepsilon} \cdot \mu_\theta([x-\varepsilon, x+\varepsilon]) \\ &= \frac{1}{2\varepsilon} \cdot m_s((\text{proj}_\theta)^{-1}([x-\varepsilon, x+\varepsilon]) \cap K). \end{aligned}$$

²We consider $\|f_{\theta,\varepsilon}\|_{L^2}$ as a function of $\varepsilon > 0$.

In order to estimate this last term, fix $\varepsilon > 0$ and let $i > 0$ such that $\mathcal{C} = \mathcal{C}_i$ has diameter less than ε . Then

$$\begin{aligned} m_s((\text{proj}_\theta)^{-1}([x - \varepsilon, x + \varepsilon]) \cap K) &\leq \sum_{\substack{Q \in \mathcal{C} \\ \text{proj}_\theta(Q) \subset [x - 2\varepsilon, x + 2\varepsilon]}} m_s(Q \cap K) \\ &\ll \sum_{\substack{Q \in \mathcal{C} \\ \text{proj}_\theta(Q) \subset [x - 2\varepsilon, x + 2\varepsilon]}} |Q|^s \\ &\asymp \sum_{\substack{Q \in \mathcal{C} \\ \text{proj}_\theta(Q) \subset [x - 2\varepsilon, x + 2\varepsilon]}} |Q|^{s-1} \cdot m(\text{proj}_\theta(Q)) \\ &\leq \int_{x-2\varepsilon}^{x+2\varepsilon} f_\theta^{\mathcal{C}}(y) dm(y), \end{aligned}$$

where in the second inequality we used (3.1). By the Cauchy-Schwarz inequality,

$$|f_{\theta,\varepsilon}(x)|^2 \ll \frac{1}{2\varepsilon} \int_{x-2\varepsilon}^{x+2\varepsilon} |f_\theta^{\mathcal{C}}(y)|^2 dm(y)$$

and so

$$\begin{aligned} \|f_{\theta,\varepsilon}\|_{L^2}^2 &\ll \int_{L_\theta} \frac{1}{2\varepsilon} \int_{x-2\varepsilon}^{x+2\varepsilon} |f_\theta^{\mathcal{C}}(y)|^2 dm(y) dm(x) \\ &\asymp \int_{L_\theta} |f_\theta^{\mathcal{C}}|^2 dm \\ &= \|f_\theta^{\mathcal{C}}\|_{L^2}^2 \end{aligned}$$

which, by (4.2), establishes (4.3). \square

5. CONCLUDING REMARKS

The good feature of the proof is that the discretization idea may be applied to other contexts. For example, we prove in [8] a Marstrand type theorem in an arithmetical context.

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