HOFER-ZEHNDER CAPACITY OF STANDARD COTANGENT BUNDLES

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Abstract. Let M be a compact manifold with an effective semi-free circle action whose fixed point set has trivial normal bundle. We prove that its cotangent bundle endowed with the canonical symplectic form has bounded Hofer-Zehnder sensitive capacity. We give several examples like the product of any compact manifold with $\mathbb{S}^n$ or a connected sum $\mathbb{CP}^n\#...\#\mathbb{CP}^n$.

1. Introduction

A remarkable fact in the modern development of Symplectic Topology was the introduction by H. Hofer and E. Zehnder of certain symplectic invariants relating the classical problem of the existence of periodic orbits on prescribed energy levels of Hamiltonian systems with symplectic rigidity phenomena [5]. It is defined in the following way:

Definition 1.1. Given a symplectic manifold (M,ω) define the Hofer-Zehnder capacity of M by

$$c_{\mathrm{HZ}}(M,\omega) = \sup_{H\in\mathcal{H}_a(M,\omega)} \max H,$$

where $\mathcal{H}_a(M,\omega)$ is the set of *admissible Hamiltonians* H defined on M, that is,

- $H \in \mathcal{H}(M) \subset C^\infty(M,\mathbb{R})$, where $\mathcal{H}(M)$ is the set of *pre-admissible Hamiltonians* on M, that is, H satisfies the following properties: $H \geq 0$, there exist an open set $V \subset M$ such that $H|_V \equiv 0$ and a compact set $K \subset M \setminus \partial M$ satisfying $H|_{M\setminus K} \equiv \max H$;
- every non-constant periodic orbit of the Hamiltonian vector field X_H has period greater than 1.

A symplectic manifold M has *bounded Hofer-Zehnder capacity* if $c_{\mathrm{HZ}}(U,\omega) < \infty$ for every open subset $U \subset M$ with compact closure. It is easy to prove that if M has bounded Hofer-Zehnder capacity then given any Hamiltonian $H : M \to \mathbb{R}$ with compact energy levels, there exists a dense subset $\Sigma \subset H(M)$ such that for every $e \in \Sigma$ the energy hypersurface $H^{-1}(e)$ has a periodic solution. In particular, the Weinstein conjecture is true for such manifolds.

Actually, if there exists a neighborhood of an energy hypersurface $H^{-1}(e_0)$ with finite Hofer-Zehnder capacity then there exists $\epsilon > 0$ and a dense subset $A \subset (e_0 - \epsilon, e_0 + \epsilon)$ such that $H^{-1}(e)$ has periodic orbits for every $e \in A$. This result was

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improved in [11], where it is shown that there are periodic orbits on $H^{-1}(e)$ for almost all $e \in (e_0 - \epsilon, e_0 + \epsilon)$ with respect to the Lebesgue measure.

We will consider here a refinement of the original Hofer-Zehnder capacity considering periodic orbits with homotopy class in a prescribed subset Γ of $\pi_1(M)$ [9, 10, 12]:

Definition 1.2. Given a symplectic manifold (M, ω) and a subset $\Gamma \subset \pi_1(M)$ define the *Hofer-Zehnder* Γ-*sensitive capacity* (or *Hofer-Zehnder* Γ-*semicapacity*) of M by

$$c^{\Gamma}_{\mathrm{HZ}}(M, \omega) = \sup_{H \in \mathcal{H}^{\Gamma}_a(M,\omega)} \max H,$$

where $\mathcal{H}^{\Gamma}_a(M, \omega)$ is the set of Γ-*admissible Hamiltonians* on M, that is,

- $H \in \mathcal{H}(M)$, that is, H is pre-admissible;
- every nonconstant periodic orbit of X_H whose homotopy class belongs to Γ has period greater than 1.

Obviously, $c_{\mathrm{HZ}}(M, \omega) = c^{\pi_1(M)}_{\mathrm{HZ}}(M, \omega) \leq c^{\Gamma}_{\mathrm{HZ}}(M, \omega)$ for every $\Gamma \subset \pi_1(M)$. Moreover, it is easy to see that if the Hofer-Zehnder Γ-sensitive capacity is bounded then the periodic orbits given by the discussion above have homotopy class in Γ.

Unfortunately, it is very hard, in general, to prove that a symplectic manifold has bounded Hofer-Zehnder capacity. For instance, for a particularly important class of symplectic manifolds, namely, standard cotangent bundles, very little is known. Actually, the most general class of known examples are cotangent bundles of manifolds endowed with free circle actions [10]. It generalizes previous results for product manifolds $M \times \mathbb{S}^1$ [4, 7, 8] and the torus $\mathbb{T}^n$ [5, 6].

The goal of this note is to give a larger class of manifolds for which the standard cotangent bundle has bounded Hofer-Zehnder sensitive capacity. Let us remember that a group action $G \times M \to M$ is *semi-free* if it is free outside the fixed point set and it is *effective* if for any $g \neq e$, there exists $x \in M$ such that $gx \neq x$, where $e \in G$ is the identity element.

Theorem 1.1. *Let M be a compact manifold with an effective semi-free circle action φ. Suppose that its fixed point set has trivial normal bundle. Then its standard cotangent bundle has bounded Hofer-Zehnder G_φ-sensitive capacity, where $G_\varphi \subset \pi_1(T^*M) = \pi_1(M)$ is the subgroup generated by the homotopy class of the orbits of φ.*

Remarks.

- It is a well know fact that each connected component of the fixed point set of a circle action is a smooth submanifold.
- Actually, what we need in the proof is the existence of a vector field on M nowhere tangent to the fixed point set. It obviously exists if the normal bundle is trivial.

There are a lot of examples of such manifolds. In particular, we have the following immediate corollaries:

Corollary 1.1 ([10])**.** *Let M be a compact manifold endowed with a free circle action φ. Then its standard cotangent bundle has bounded Hofer-Zehnder G_φ-sensitive capacity.*

Corollary 1.2. *Let M be a product of a compact manifold with the sphere $\mathbb{S}^n$ or a connected sum $\mathbb{CP}^n\#...\#\mathbb{CP}^n$. Then T^*M has bounded Hofer-Zehnder 0-sensitive capacity, that is, the periodic orbits are contractible.*

In fact, it is easy to see that both $\mathbb{S}^n$ and $\mathbb{CP}^n$ have semi-free circle actions with isolated fixed points. To get this action for the connected sum $\mathbb{CP}^n\#...\#\mathbb{CP}^n$, let p be a fixed point in $\mathbb{CP}^n$ and U a neighborhood of p such that we can identify each complex projective space by a map $\phi : \partial U \to \partial U$ equivariant with respect to the action. Then the connected sum also has a semi-free circle action with isolated fixed points. Notice that this argument works for any connected sum $M_1\#M_2$ such that both M_1 and M_2 have a circle action with isolated fixed points.

As an immediate consequence of the previous theorem, we have the following application on the existence of periodic orbits on prescribed energy levels of exact magnetic flows. For an introduction to magnetic flows see, for instance, [2, 3, 9].

Corollary 1.3. *Let M be a compact Riemannian manifold as in Theorem 1.1 and α a 1-form on M. Consider the Hamiltonian $H(q,p) = \|p - \alpha_q\|^2$ on T^*M. Then the Hamiltonian flow of H has periodic orbits with homotopy class in G_φ on almost all energy levels.*

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2. Proof of Theorem 1.1

Consider a Riemannian metric on M invariant by the circle action (it exists because the group is compact). Let TM be the tangent bundle endowed with the pullback ω_0 of the canonical symplectic form by the bundle isomorphism induced by the metric. Let ξ be the lifted circle action on TM given by $\xi_{-t}(x,v) = (\varphi_t(x), (d\varphi_t)_x v)$. This action is Hamiltonian since the metric is invariant by the action on M. Denote by F the fixed point set of φ.

Lemma 2.1. *Let $Y = (d/dt)|_{t=0}\varphi_t$ be the tangent vector field to φ. Then the derivative of Y restricted to the orthogonal complement of F is equivalent to a linear rotation with the same minimal period of φ.*

Proof. It is sufficient to prove the lemma for a circle action ζ on $\mathbb{R}^n$ with an isolated fixed point at the origin. Indeed, we can consider local coordinates on a neighborhood of $p \in F$ and restrict to the orthogonal complement of T_pF.

Consider a diffeomorphism g on $\mathbb{R}^n$ such that $g \circ \zeta \circ g^{-1}$ is a reparametrization of a linear rotation R with period equal to 1. We have that

$$Dg \circ Z \circ g^{-1} = fR,$$

where $Z = (d/dt)|_{t=0}\zeta_t$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function. Consequently,

$$(Dg \circ DZ \circ Dg^{-1})(0) = f(0)R.$$

Thus, we need to prove that $f(0) = 1/T$, where T is the (minimal) period of the action ζ.

Now, a straightforward computation shows that

$$g \circ \zeta_t \circ g^{-1}(v) = e^{R \int_0^t f(e^{Rs} v)\, ds} v,$$

for every $v \in \mathbb{R}^n$. Hence,

$$\int_0^T f(e^{Rs} v)\, ds = 1.$$

It implies that $f(0) = 1/T$. □

Consequently, the fixed point set of ξ is the tangent subbundle TF of the fixed point set F of φ. Note that each connected component of F has dimension strictly less than the dimension of M because the action is effective and the group is compact.

Lemma 2.2. *Let $U \subset TM$ be an open subset with compact closure. Then there exists an open subset with compact closure $\widetilde{U} \subset TM \setminus TF$ such that given a G-admissible Hamiltonian H on U there exists a G-admissible Hamiltonian $\widetilde{H}$ on $\widetilde{U}$ satisfying* $\max \widetilde{H} = \max H$.

Proof. Since F has trivial normal bundle, there exists an exact 1-form θ on M whose dual (gradient) vector field X satisfies $X(x) \in T_x^{\perp} F$ and $X(x) \neq 0$ for every $x \in F$. Moreover, since U has compact closure, there exists a constant $a > 0$ sufficiently great such that $TF + aX$ does not intersect U.

Remark. Note that the existence of θ already follows from the existence of a vector field Y on M nowhere tangent to F. Indeed, consider the submanifold S given by $\cup_{-\epsilon<t<\epsilon, x \in F} \exp_x(tY(x))$ for $\epsilon > 0$ sufficiently small. Let f be a smooth function on S such that $df(Y(x)) > 0$. Now, extend smoothly the function f to all of M and define $\theta = df$.

We can suppose, without loss of generality, that H is defined on M such that $H|_{M \setminus U} \equiv \max H$. Consider the Hamiltonian $\widetilde{H}(x,v) = H(x, v + aX)$. We have that $\widetilde{H}$ is also G-admissible because the translation by aX along the fiber is a symplectomorphism (since θ is closed) and homotopic to the identity. Notice that $\widetilde{H}$ is defined on $\widetilde{U} := U - aX$, which, by the choice of a, does not intersect TF. □

Now, we need the following result that corresponds to a particular case of a more general theorem proved in [10]:

Theorem 2.1 ([10]). *Let (M, ω) be a geometrically bounded exact symplectic manifold and $N \subset M$ an open subset that admits a free Hamiltonian circle action φ. Then, given any open subset $U \stackrel{i}{\hookrightarrow} N$ with compact closure,*

$$c_{\mathrm{HZ}}^{i_*^{-1} G_\varphi}(U, \omega) < \infty,$$

where $i_* : \pi_1(U) \to \pi_1(N)$ is the induced homomorphism on the fundamental group and $G_\varphi \subset \pi_1(N)$ is the subgroup generated by the orbits of the circle action.

We refer the reader to [1] for the definition and a discussion of the concept of geometrically bounded symplectic manifold. Here we only mention that it is a well know fact that standard cotangent bundles are geometrically bounded.

Consequently, combining the previous lemma and the theorem above, we conclude that

$$c_{\mathrm{HZ}}^{G_\varphi}(U,\omega_0) \le c_{\mathrm{HZ}}^{G_\varphi}(\widetilde{U},\omega_0) < \infty,$$

as desired.

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