RESIDUES OF HOLOMORPHIC FOLIATIONS RELATIVE TO A GENERAL SUBMANIFOLD

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Abstract

Let $\mathcal{F}$ be a holomorphic foliation (possibly with singularities) on a nonsingular manifold M, and let V be a complex analytic subset of M. Usual residue theorems along V in the theory of complex foliations require that V be tangent to the foliation (i.e. union of leaves and singular points of V and $\mathcal{F}$): this is the case for instance for the blow-up of a nondicritical isolated singularity. In this paper, we will introduce residue theorems along subvarieties which are not necessarily tangent to the foliation, including the blow-up of the dicritical situation.

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1- Introduction and backgrounds

In the theory of complex foliations, residue theorems have proved to be essential for the analysis of their analytical properties. This is the case, for instance, of the Baum-Bott residue formula [3] and the Camacho-Sad residue theorem [9], in the study and classification of foliatons of complex compact surfaces by Brunella, [4], Mc Quillan [15] and Mendes [16]. This is also the case in the analysis of the trajectories of a holomorphic vector field around an isolated singular point, whose blow-up produces and invariant exceptional divisor ([6]). This is the so called nondicritical situation. Those residue theorems associated to foliations tangent to a submanifold have been generalized in various directions (cf. [11][12][13][14][8] ; a more complete bibliography is given in [17]). But they all required that the subvariety be tangent to the foliation, i.e. union of leaves and singularities.

In this paper, generalizing [5], we will introduce residue theorems along subvarieties which are not required to be tangent to the foliation. In particular, at least in section 3, we study the dicritical situation.

Let $\mathcal{F}$ be a holomorphic foliation (possibly singular) on a smooth complex manifold M without boundary, and let V be a complex analytic subset of M of pure complex dimension n (perhaps with singularities).

We shall study

- in section 2, the case where V is nonsingular, M is the total space of a holomorphic vector bundle $E \to V$ of rank k over V, the leaves of $\mathcal{F}$ have the same dimension n as V and are generically transverse to the fibers of E along V,

- in section 3, the case where V is a locally complete intersection : this implies in particular that the stable class of the normal bundle $N_0(V)$ to the regular part V_0 of V in M has a natural extension $N(V)$ to all of V ; we shall assume moreover that $\mathcal{F}$ is generically transverse to V and that there is a neighborhood of V isomorphic to some subbundle F of $N(V)$. An example of this ([5]) is the blow-up of an isolated dicritical singularity of a complex one-dimensional foliation in an open set of $\mathbb{C}^n$: this yields a 1-foliation on the total space of the tautological bundle over the divisor V of the blow-up (a complex $(n-1)$-projective space). The foliation is there assumed to be transverse to V, except at a codimension 2 algebraic subset $\Sigma \subset V$. The subvariety Σ contains in particular the singular points of the foliation.

Background on the normal sheaf to the foliation (cf. [3]) :

We shall assume that V is compact, and that $\mathcal{F}$ satisfies to the assumptions of Baum-Bott ([3]): the sheaf $T(\mathcal{F})$ tangent to the foliation (subsheaf of $\mathcal{O}_M$ modules and $\mathbb{C}$-Lie algebras of the sheaf $\mathcal{O}_M(TM)$ of germs of holomorphic vector fields on M which are tangent to $\mathcal{F}$) will be assumed to be coherent and "full" (which means that a vector field is a section of $T(\mathcal{F})$ if and only if its germ at every regular point of $\mathcal{F}$ is tangent to $\mathcal{F}$). The coherence of $T(\mathcal{F})$ implies in particular the coherence of the normal sheaf $\mathcal{N}(\mathcal{F}) = \mathcal{O}_M(TM)/T(\mathcal{F})$: thus, since

V is compact, there exists after [2] a neighborhood M' of V in M, and a $\mathcal{A}_{M'}$ locally free resolution of finite length $r \leq 2(n+k)$

$$(*) \qquad 0 \to \mathcal{A}_{M'}(E_r) \to \mathcal{A}_{M'}(E_{r-1}) \to \cdots \to \mathcal{A}_{M'}(E_0) \to \mathcal{A}_{M'} \otimes_{\mathcal{O}_{M'}} \mathcal{N}(\mathcal{F}) \to 0,$$

where

$\mathcal{N}(\mathcal{F})$ still denotes the restriction to M' of the normal sheaf to $\mathcal{F}$,

$\mathcal{O}_{M'}$ denotes the sheaf of holomorphic functions on M' ,

$\mathcal{A}_{M'}$ denotes the sheaf of real analytical $\mathbb{C}$-valued functions,

$E_j \to M'$ denotes some real-analytical complex vector bundle over M', and $\mathcal{A}_{M'}(E_j)$ the sheaf of the $\mathbb{R}$-analytical sections of its restriction to V.

Moreover, the element $N\mathcal{F}|_V = \sum_{i=0}^{r}(-1)^j E_j|_V$ in $K^0(V)$ does not depend on the resolution $(*)$ and induces on $V \cap M'_0$ the restriction of the stable class of the normal bundle $N_0\mathcal{F}$ to $\mathcal{F}$ in M'_0, where M'_0 denotes the set of regular points of $\mathcal{F}$ which are in M'.

We get in particular over M'_0 the exact sequence of vector bundles

$$(*_0) \qquad 0 \to E_r|_{M'_0} \to E_{r-1}|_{M'_0} \to \cdots \to E_0|_{M'_0} \to N_0\mathcal{F} \to 0.$$

Particular case of 1-dimensional foliations: For *dim* $\mathcal{F} = 1$, the foliation is defined by a morphism $h : \mathcal{L} \to TM$ of holomorphic vector bundles ($\mathcal{L}$ denoting some holomorphic line bundle). We have a $\mathcal{O}_M$ locally free resolution

$$0 \to \mathcal{O}_M(\mathcal{L}) \xrightarrow{h} \mathcal{O}_M(TM) \to \mathcal{N}(\mathcal{F}) \to 0$$

of $\mathcal{N}(\mathcal{F})$, and in this case it is not necessary to tensor by $\mathcal{A}_M$ (which is $\mathcal{O}_M$ flat according to [2]), so that $N\mathcal{F} = [TM - \mathcal{L}]$.

Background on the machinery for producing residues (cf. [11][13]) :

Let S be a closed subspace of V with connected components S_λ, let $\coprod U_\lambda$ be an open neighborhood of S in V, where the U_λ denote open neighborhoods of the S_λ's, such that $U_\lambda \cap U_\mu = \emptyset$ for $\lambda \neq \mu$.

The long exact sequence in cohomology associated to the pair $(V, V \setminus S)$, and the long exact sequence in homology associated to the pair (V, S) are related by Poincaré duality $[\mathcal{P}]$ and Alexander duality $[\mathcal{A}]$ in the following commutative diagram (called "the residue diagram"):

$$\begin{array}{ccccccccc}
\cdots & \to & H^*(V, V\setminus S) & \xrightarrow{j} & H^*(V) & \xrightarrow{\pi} & H^*(V\setminus S) & \to & \cdots \\
 & & \downarrow [\mathcal{A}] & & \downarrow [\mathcal{P}] & & & & \\
\cdots & \to & H_{2n-*}(S) & \xrightarrow{\iota} & H_{2n-*}(V) & \to & & \cdots &
\end{array}.$$

Thus, whenever we have some class $\varphi \in H^*(V)$ whose restriction $\pi(\varphi)$ to $H^*(V \setminus S)$ vanishes, this class lifts (canonically in many cases) to some $\varphi_0 \in H^*(V, V \setminus S)$, so that we obtain the residue formula

$$\sum_\lambda (\iota_\lambda)_*\big([\mathcal{A}]_\lambda(\varphi_0)\big) = \varphi \frown [V],$$

where we have written $[\mathcal{A}](\varphi_0) = \Big([\mathcal{A}]_\lambda(\varphi_0)\Big)_\lambda$ in $H_{2n-*}(S) = \sum_\lambda H_{2n-*}(S_\lambda)$.

The coefficients for the homology and cohomology depend on the coefficients for which we have the vanishing theorem $\pi(\varphi) = 0$. For instance, let (F_j) denote a family of complex vector bundles $(j = 1, \cdots, r)$ over V, and assume that there exists an exact sequence of bundles over $V \setminus S$

$$(*) \qquad 0 \to F_r|_{V\setminus S} \to F_{r-1}|_{V\setminus S} \to \cdots \to F_0|_{V\setminus S} \to 0.$$

The data of $(*)$ defines canonically an element ξ in the relative K-theory $K^0(V, V\setminus S)$ mapping onto $\sum_j (-1)^j F_j \in K^0(V)$ by the natural map $K^0(V, V \setminus S) \to K^0(V)$ (see [1] for instance). Thus, for any i, we have

$$\sum_\lambda (\iota_\lambda)_*\Big([\mathcal{A}]_\lambda\big(c_i(\xi)\big)\Big) = c_i\Big(\sum_j (-1)^j F_j\Big) \frown [V].$$

in the homology with coefficients in $\mathbb{Z}$.

But, on the other hand, if we need connections and Chern-Weil theory for proving this vanishing theorem, coefficients will be $\mathbb{R}$ or $\mathbb{C}$. In this case, we realize the diagram above by the following one (called "Mayer-Vietoris" diagram) at the chain and cochain level:

$$\begin{array}{ccccccccc}
0 & \to & MV^*(V, V \setminus S) & \xrightarrow{j} & MV^*(V) & \xrightarrow{\pi} & \Omega^*_{DR}(V \setminus S) & \to & 0 \\
 & & \downarrow \mathcal{A} & & \downarrow \mathcal{P} & & \downarrow & & \\
0 & \to & \bigoplus_\lambda \operatorname{Hom}\Big(\Omega^{2n-*}_{DR}(U_\lambda), \mathbb{C}\Big) & \xrightarrow{\iota} & \operatorname{Hom}\Big(MV^{2n-*}(V), \mathbb{C}\Big) & \to & Q & \to & 0\,,
\end{array}$$

with the notation explained now. Denoting by $(\Omega^*_{DR}(W), d_{DR})$ the de Rham differential graded algebra of differential forms on any non singular manifold W, we called Mayer-Vietoris complex the total Čech de Rham complex for the covering $\mathcal{U} = \big(V \setminus S, (U_\lambda)\big)$ of V, i.e:

$$MV^*(V) = \Omega^*_{DR}(V \setminus S) \oplus \Big[\bigoplus_\lambda \Big(\Omega^*_{DR}(U_\lambda) \oplus \Omega^{*-1}_{DR}(U_\lambda \setminus S_\lambda)\Big)\Big],$$

with the differential $D\big(\alpha_0, (\alpha_\lambda, \alpha_{0,\lambda})_\lambda\big) = \big(d_{DR}\alpha_0, (d_{DR}\alpha_\lambda, -d_{DR}\alpha_{0,\lambda} + \alpha_\lambda)_\lambda\big)$. It has a cohomology naturally isomorphic to the de Rham cohomology $H^*(V)$ with $\mathbb{C}$-coefficients. The subcomplex $MV^*(V, V \setminus S) = \bigoplus_\lambda \Big(\Omega^*_{DR}(U_\lambda) \oplus \Omega^{*-1}_{DR}(U_\lambda \setminus S_\lambda)\Big)$ (called "relative" Mayer-Vietoris complex) has a cohomology naturally isomorphic to the local cohomology of S with $\mathbb{C}$-coefficients $H^*(V, V \setminus S) = \oplus_\lambda H^*(U_\lambda, U_\lambda \setminus S_\lambda)$: it is in fact the kernel of the natural projection π of $MV^*(V)$ onto $\Omega^*_{DR}(V \setminus S)$, which realizes the restriction from V to $V \setminus S$ in cohomology. We have denoted by Q the quotient which makes the bottom line of the above diagram exact.

In this framework, the Poincaré duality $[\mathcal{P}] : H^*(V) \to H_{2n-*}(V)$ is induced by the map

$$\Big(\alpha_0, (\alpha_\lambda, \alpha_{0\lambda})_\lambda\Big) \overset{\mathcal{P}}{\mapsto} \Big[\Big(\beta_0, (\beta_\lambda, \beta_{0\lambda})_\lambda\Big) \mapsto \int_{\mathcal{T}_0} \alpha_0 \wedge \beta_0 + \sum_\lambda \Big(\int_{\mathcal{T}_\lambda} \alpha_\lambda \wedge \beta_\lambda - \int_{\partial \mathcal{T}_\lambda} \alpha_{0\lambda} \wedge \beta_\lambda\Big)\Big],$$

from $MV^*(V)$ into $\mathrm{Hom}\Big(MV^{2n-*}(V), \mathbb{C}\Big)$, where $H_{2n-*}(V, \mathbb{C})$ is computed as the homology of the complex $\mathrm{Hom}\Big(MV^{2n-*}(V), \mathbb{C}\Big)$, $\mathcal{T}_\lambda$ denotes any $2n$ dimensional compact manifold with boundary in U_λ containing S_λ in its interior, and $\mathcal{T}_0$ denotes the closure of the complement of $\bigcup_\lambda \mathcal{T}_\lambda$ in V. Similarly, the Alexander duality $[\mathcal{A}] : \oplus_\lambda H^*(U_\lambda, U_\lambda \setminus S_\lambda) \to \oplus_\lambda H_{2n-*}(S_\lambda)$ is induced by the map $\mathcal{A} : \Big(\Omega^*_{DR}(U_\lambda) \oplus \Omega^{*-1}_{DR}(U_\lambda \setminus S_\lambda)\Big)_\lambda \to \Big(Hom\Big(\Omega^{2n-*}_{DR}(U_\lambda), \mathbf{C}\Big)\Big)_\lambda$ defined by

$$(\alpha_\lambda, \alpha_{0\lambda}) \overset{\mathcal{A}_\lambda}{\mapsto} \Big[\beta \mapsto \int_{\mathcal{T}_\lambda} \alpha_\lambda \wedge \beta - \int_{\partial \mathcal{T}_\lambda} \alpha_{0\lambda} \wedge \beta\Big],$$

where $H_{2n-*}(S_\lambda, \mathbb{C})$ is computed as the homology of the complex $\mathrm{Hom}\Big(\Omega^{2n-*}_{DR}(U_\lambda), \mathbb{C}\Big)$.

Remark: This Mayer-Vietoris diagram can be used also even when V is a singular variety, by thickening V in the ambient manifold M (see [13] for details).

2- Case of $\mathcal{F}$ generically transverse to the fibers of E along V

We assume in this section that V is non singular, M is the total space of a holomorphic vector bundle $E \to V$ of rank k over V, the leaves of $\mathcal{F}$ have the same dimension n as V, and are generically transverse to the fibers of E along V. Let Σ be the set of points in V where $\mathcal{F}$ is either singular or tangent to a fiber of E. Let V_0 be the open set $V \setminus \Sigma$: we have then two natural isomorphisms $\Phi : TV_0 \overset{\cong}{\to} T\mathcal{F}|_{V_0}$ and $\Psi : N_0\mathcal{F}|_{V_0} \overset{\cong}{\to} E|_{V_0}$.

Remark : We assumed V to be non-singular and M to be the total space of some bundle over V only for simplification. We could in fact assume V to be a locally complete intersection in some M, or even only a coherent space (according to the terminology of [7]).

a) Localization of $[N\mathcal{F}|_V - E]$:

Combining $(*_0)$ with Ψ, we get an exact sequence of vector bundles

$$(*_1) \qquad 0 \to E_r|_{V_0} \to E_{r-1}|_{V_0} \to \cdots \to E_0|_{V_0} \to E|_{V_0} \to 0 \,.$$

Since all bundles occuring in $(*_1)$ are in fact defined over all of V, we get an element

$$\theta_1 = [E_r|_V, E_{r-1}|_V, \cdots, E_0|_V, E; (*_1)] \text{ in } K^0(V, V \setminus \Sigma) \,,$$

whose image in $K^0(V)$ by the natural map $K^0(V, V \setminus \Sigma) \to K^0(V)$ is equal to $[N\mathcal{F}|_V - E]$. Therefore, the Chern classes $c_j(\theta_1) \in H^{2j}(V, V \setminus \Sigma; \mathbb{Z})$ are natural lift of the Chern classes

$c_j([N\mathcal{F}|_V - E]) \in H^{2j}(V;\mathbb{Z})$. In other words, denoting by $\big(\Sigma_\lambda\big)_\lambda$ the family of the connected components of Σ and by $\mathrm{Res}_\Sigma(c_j,\theta_1) = \big(\mathrm{Res}_\lambda(c_j,\theta_1)\big)_\lambda$ the Alexander dual of $c_j(\theta_1)$ in the homology $H_{2(n-j)}(\Sigma;\mathbb{Z}) = \oplus_\lambda H_{2(n-j)}(\Sigma_\lambda;\mathbb{Z})$, we get the

Theorem 1 : *We have the following localization for the Chern classes of $[N\mathcal{F}|_V - E]$ with integral coefficients:*

$$\sum_\lambda (\iota_\lambda)_* \mathrm{Res}_\lambda(c_j,\theta_1) = c_j([N\mathcal{F}|_V - E]) \frown [V],$$

where $(\ .\ ) \frown [V]$ denotes the Poincaré duality, and $(\iota_\lambda)_* : H_{2(n-j)}(\Sigma_\lambda) \to H_{2(n-j)}(V)$ the map induced in homology by the natural inclusion $\iota_\lambda : \Sigma_\lambda \subset V$.

Case *dim* $\mathcal{F} = 1$**:** Assume that the foliation is defined by a morphism $h : \mathcal{L} \to TM$, as explained in the first section, V being a non-singular Riemann surface of genus g. We have then: $N\mathcal{F} = [TM - \mathcal{L}]$ and $TV = [TM|_V - E]$ in $K^0(V)$, so that $[N\mathcal{F}|_V - E] = [TV - \mathcal{L}|_V]$. Assume also that the Σ_λ's are isolated points m_λ of V. Let $(x, y_1, y_2, \cdots, y_k)$ be local coordinates near such a point m_λ, such that V is locally defined by the equations $y_\alpha = 0$ $(\alpha = 1, \cdots, k)$ and the fibers of E are defined by $x =$ constant. Let $\sigma_\mathcal{L}$ be a local trivialization of $\mathcal{L}$ near m_λ, and $v = h(\sigma_\mathcal{L})$ a vector field defining locally $\mathcal{F}$:

$$v = A(x,y)\frac{\partial}{\partial x} + \sum_\alpha B_\alpha(x,y)\frac{\partial}{\partial y_\alpha}.$$

Then, in the relative Mayer-Vietoris complex $\Omega^2_{DR}(U_\lambda) \oplus \Omega^1_{DR}(U_\lambda \setminus \{m_\lambda\})$, the component of $c_1(\theta_1)$ on $H^2(U_\lambda, U_\lambda \setminus \{m_\lambda\};\mathbb{R})$ is given by $(0, -\frac{dA}{A})$. In fact, if we define $\nabla^\mathcal{L}$ as the connection of type $(1,0)$ on $\mathcal{L}|_{U_\lambda}$ satisfying $\nabla^\mathcal{L}\sigma_\mathcal{L} \equiv 0$, and ∇^V as the connection of type $(1,0)$ on $TV|_{U_\lambda}$ satisfying $\nabla^V\frac{\partial}{\partial x} \equiv 0$, we have then $\Psi(\frac{\partial}{\partial x}) = \frac{\partial}{\partial x} + \sum_\alpha \frac{B_\alpha}{A}(x,y)\frac{\partial}{\partial y_\alpha}$, thus $\Psi(\frac{\partial}{\partial x}) = h(\frac{1}{A}\sigma_\mathcal{L})$. Therefore, the connection $\nabla'^\mathcal{L} = \Psi(\nabla^V)$ on $\mathcal{L}|_{U_\lambda \setminus \{m_\lambda\}}$ is the connection of type $(1,0)$ satisfying $\nabla'^\mathcal{L}\sigma_\mathcal{L} = \frac{dA}{A}\sigma_\mathcal{L}$. Hence, $c_1(\nabla'^\mathcal{L}, \nabla^\mathcal{L}) = -\frac{dA}{A}$, while $c_1(\nabla^\mathcal{L}) = 0$ and $c_1(\theta_1) = (0, -\frac{dA}{A})$.

Consequently, $\mathrm{Res}_\lambda(c_1,\theta_1)$ is the order of multiplicity of m_λ as a zero of $A(x,0)$. (It is also the index of the projection $A(x,y)\frac{\partial}{\partial x}$ of v onto TV parallel to E). We then obtain the

Corollary to theorem 1 : If *dim* $\mathcal{F} = 1$, we get, with the above notations, the formula

$$\frac{1}{2i\pi}\sum_\lambda \int_{\gamma_\lambda} \frac{dA}{A} = 2 - 2g - \Big(c_1(\mathcal{L}) \frown [V]\Big),$$

where γ_λ denotes a small circle in V_0 around m_λ.

b) Localization of the characteristic algebra $\mathrm{Chern}^*[E]$ **in dimension** $* > 2[\frac{n}{2}]$ **:**

Since E is a vector bundle, there is a canonical flat connection on the vector subbundle $\mathcal{V}E \to M$ of $TM \to M$ defined as the set of the tangent vectors to the fibers of E, in such a way that any holomorphic section σ of E extends naturally as a section $\tilde{\sigma}$ of $\mathcal{V}E$ (notice: $E = \mathcal{V}E|_V$). Since the transversality is an open condition, there exists also a tubular neighborhood M_0 of V_0 into M such that $\Psi : N_0\mathcal{F}|_{V_0} \overset{\cong}{\to} E|_{V_0}$ extends naturally as an isomorphism $\tilde{\Psi} : N_0\mathcal{F}|_{M_0} \overset{\cong}{\to} \mathcal{V}E|_{M_0}$.

Let $\pi : TM_0 \to N_0\mathcal{F}|_{M_0}$ be the natural projection: for any holomorphic section σ of E, there exists a holomorphic vector field Y_σ on M_0 such that $\tilde{\Psi}\big(\pi(Y_\sigma)\big) = \tilde{\sigma}$.

For any holomorphic vector field X tangent to V_0, denote by $\hat{\Phi}(X)$ some germ of vector field tangent to $\mathcal{F}$ extending $\Phi(X)$ near V_0. Let σ be as above, and define

$$\nabla^0_X \sigma = \Psi\big(\pi([\hat{\Phi}(X), Y_\sigma]|_{V_0}\big).$$

Lemma : *This definition does not depend on the choices of $\hat{\Phi}(X)$ extending $\Phi(X)$ and of Y_σ such that $\tilde{\Psi}\big(\pi(Y_\sigma)\big) = \tilde{\sigma}$.*

In fact, two vector fields Y^1_σ and Y^2_σ such that $\pi(Y^1_\sigma) = \pi(Y^2_\sigma)$ differ from a vector field tangent to $\mathcal{F}$. Since $\hat{\Phi}(X)$ is also tangent to $\mathcal{F}$, $\pi([\hat{\Phi}(X), Y^2_\sigma - Y^1_\sigma]) = 0$.

Moreover, if $(y_1, \cdots, y_k)$ are local coordinates, linear in each fiber, arising from a local trivialization of E, two extensions $\hat{\Phi}(X)_1$ and $\hat{\Phi}(X)_2$ of $\Phi(X)$ differ locally by a vector field of the shape $\sum_{\alpha=1}^k y_\alpha v_\alpha$, where v_α is tangent to $\mathcal{F}$. Thus, $[\hat{\Phi}(X)_2 - \hat{\Phi}(X)_1, \frac{\partial}{\partial y_\alpha}]$ has a restriction to V_0 equal to v_α and projects therefore by π onto 0.

Corollary (Vanishing theorem) : *The restriction $c_I(E)|_{V_0}$ of $c_I(E)$ to V_0 vanishes for any I such that $|I| > [\frac{n}{2}]$ in the cohomology with real coefficients, where $I = (i_1, i_2, \cdots, i_k)$ denotes a multi index of integers $i_j \geq 0$, $|I| = i_1 + 2i_2 + \cdots + ki_k$, and c_I denotes the Chern monomial $(c_1)^{i_1}.(c_2)^{i_2}.\cdots(c_k)^{i_k}$.*

Proof : Let ∇^0 be the canonical connection of type $(1,0)$ on $E|_{V_0}$ defined by

$\nabla^0_X \sigma = \Psi\big(\pi([\hat{\Phi}(X), Y_\sigma]|_{V_0}\big)$ for $X \in TV_0 = T^{(1,0)}(V_0)$,
and $\nabla^0_X \sigma = 0$ for $X \in T^{(0,1)}(V_0)$ when σ is holomorphic.

This is a holomorphic connection. Therefore, for any local coordinates $(x_1, x_2, \cdots, x_n)$ on V_0, the curvature form of ∇ has only terms in $dx_i \wedge dx_j$ (but none in $dx_i \wedge d\bar{x}_j$ or $d\bar{x}_i \wedge d\bar{x}_j$). Thus, any polynomial of degree $j > [\frac{n}{2}]$ with respect to the coefficients of the curvature must vanish, QED.

Theorem 2 :

(i) Let ∇^λ be any connection on $E|_{U_\lambda}$. For $|I| > [\frac{n}{2}]$, the element $\big(c_I(\nabla^\lambda), c_I(\nabla^0, \nabla^\lambda)\big)$ is a cocycle in the relative Mayer-Vietoris complex $\Omega^*_{DR}(U_\lambda) \oplus \Omega^{*-1}_{DR}(U_\lambda \setminus S_\lambda)$, and its cohomology class $[\big(c_I(\nabla^\lambda), c_I(\nabla^0, \nabla^\lambda)\big)] \in H^{2|I|}(U_\lambda, U_\lambda \setminus S_\lambda; \mathbb{C})$ does not depend on the choice of ∇^λ.

(ii) We have the formula

$$\sum_{\lambda}(\iota_\lambda)_*\mathrm{Res}_\lambda(c_I,E,V,\mathcal{F}) = c_I(E)\frown[V]$$

in the homology with coefficients in $\mathbb{Z}$, *where* $\mathrm{Res}_\lambda(c_I,E,V,\mathcal{F})$ *denotes the Alexander dual of* $[(c_I(\nabla^\lambda),c_I(\nabla^0,\nabla^\lambda))]$.

Proof: The fact that $\big(c_I(\nabla^\lambda),c_I(\nabla^0,\nabla^\lambda)\big)$ is a cocycle is obvious after the vanishing theorem above. If $\bar{\nabla}^\lambda$ is any other connection on $E|_{U_\lambda}$, then $\big(c_I(\bar{\nabla}^\lambda),c_I(\nabla^0,\bar{\nabla}^\lambda)\big)-\big(c_I(\nabla^\lambda),c_I(\nabla^0,\nabla^\lambda)\big)$ is equal to $D\big(c_I(\nabla^\lambda,\bar{\nabla}^\lambda),c_I(\nabla^0,\nabla^\lambda,\bar{\nabla}^\lambda)\big)$, thus is a coboundary in the relative Mayer Vietoris complex, hence part (i) of the theorem. Then, part (ii) results from the general residue machinery recalled above.

Example: case $n=k=1$, $c_I=c_1$ (cf. [5])

We assume that (x,y) are local coordinates near a singular point m_λ isolated in Σ, y being a linear coordinate in each fiber of E and V being defined locally by $y=0$: there is some local holomorphic non vanishing cross section σ_λ of E, such that (x,y) are the coordinates of the point $y\sigma_\lambda(x)$ in E, and we identify $\sigma_\lambda(x)$ with $(\frac{\partial}{\partial y})_{(x,0)}$. If $\mathcal{F}$ is locally defined by $v=A(x,y)\frac{\partial}{\partial x}+B(x,y)\frac{\partial}{\partial y}$, and if m_λ has coordinates $(0,0)$, then $A(x,0)\neq 0$ for $x\neq 0$. With respect to the above trivialization of $E|_{U_\lambda}$, the connection form ω_0 of ∇^0 is given, on $U_\lambda\setminus m_\lambda$, by $\omega_0=-\frac{\partial}{\partial y}|_{y=0}(\frac{B}{A})$. With respect to the same trivialization of $E|_{U_\lambda}$, we may define ∇^λ by the connection form $\omega_\lambda=0$. Then $\big(c_1(\nabla^\lambda),c_1(\nabla^0,\nabla^\lambda)\big)=\big(0,\omega_0\big)$. Thus $\mathrm{Res}_\lambda(c_1,E,V,\mathcal{F})$ is the usual Cauchy residue of $-\frac{\partial}{\partial y}|_{y=0}(\frac{B}{A})$ at m_λ. Applying the Theorem 2 above, we recover the theorem C of [5], generalizing the original case ([9]) where $\mathcal{F}$ was tangent to V:

Corollary to theorem 2 : For $n=k=1$, we get, with the notations above, the formula

$$\frac{-1}{2i\pi}\sum_\lambda\int_{\gamma_\lambda}\frac{\partial}{\partial y}\Big(\frac{B}{A}\Big)=c_1(E)\frown[V],$$

where γ_λ denotes a small circle in V_0 around m_λ.

3- Case of $\mathcal{F}$ generically transverse to V

We assume in this section that V is a a locally complete intersection, which implies in particular that the normal bundle $N_0(V)\to V_0$ to the regular part V_0 of V in M has a natural extension $N(V)\to V$ to all of V. We assume also that $\mathcal{F}$ is generically transverse to V_0. Let $\pi: TM|_{V_0}\to N_0(V)$ be the natural projection, V_0' be the open subset of V_0 where π is injective. We shall assume moreover that some vector subbundle $F\to V$ of $N(V)$, with rank equal to the dimension of the leaves, is given, such that, over V_0', π induces an isomorphism $T\mathcal{F}|_{V_0'}\to F|_{V_0'}$. Identifying $T\mathcal{F}|_{V_0'}$ and $F|_{V_0'}$ by this isomorphism, we get an exact sequence $0\to F|_{V_0'}\to TM|_{V_0'}\to N_0\mathcal{F}\to 0$. Then, choosing some (smooth) splitting

$0 \to N_0\mathcal{F}|_{V_0'} \to TM|_{V_0'} \to F|_{V_0'} \to 0$ of this exact sequence and combining with $(*_0)$, we get an exact sequence of vector bundles

$$(*_2) \qquad 0 \to E_r|_{V_0'} \to E_{r-1}|_{V_0'} \to \cdots \to E_0|_{V_0'} \to TM|_{V_0'} \to F|_{V_0'} \to 0,$$

Since all bundles occuring in $(*_2)$ are in fact defined over all of V, we get an element

$$\theta_2 = [E_r|_V, E_{r-1}|_V, \cdots, E_0|_V, TM|_V, F; (*_2)] \text{ in } K^0(V, V \setminus \Sigma'),$$

not depending on the chosen splitting (two of them being homotopic), and whose image in $K^0(V)$ by the natural map $K^0(V, V \setminus \Sigma') \to K^0(V)$ is equal to $[N\mathcal{F}|_V - TM|_V + F]$. Therefore, the Chern classes $c_j(\theta_2) \in H^{2j}(V, V \setminus \Sigma'; \mathbb{Z})$ are natural liftings of the Chern classes $c_j([N\mathcal{F}|_V - TM|_V + F]) \in H^{2j}(V; \mathbb{Z})$. In other words, denoting by $\left(\Sigma'_\lambda\right)_\lambda$ the family of the connected components of Σ' and by $\text{Res}_\Sigma(c_j, \theta_2) = \left(\text{Res}_\lambda(c_j, \theta_2)\right)_\lambda$ the Alexander dual of $c_j(\theta_2)$ in the homology $H_{2(n-j)}(\Sigma'; \mathbb{Z}) = \oplus_\lambda H_{2(n-j)}(\Sigma'_\lambda; \mathbb{Z})$,we obtain

Theorem 3 : *We have the following localization of the Chern classes of* $[N\mathcal{F}|_V - TV]$*:*

$$\sum_\lambda (\iota_\lambda)_* \text{Res}_\lambda(c_j, \theta_2) = c_j([N\mathcal{F}|_V - TM|_V + F]) \frown [V],$$

where $(\ .\ ) \frown [V]$ *denotes the Poincaré duality, and* $(\iota_\lambda)_* : H_{2(n-j)}(\Sigma'_\lambda) \to H_{2(n-j)}(V)$ *the map induced in homology by the natural inclusion* $\iota_\lambda : \Sigma'_\lambda \subset V$.

In particular, for *dim* $\mathcal{F} = 1$, the foliation is defined by a morphism $h : \mathcal{L} \to TM$ as explained in the first section. Then $[N\mathcal{F}|_V - TM|_V + F] = [F - \mathcal{L}|_V]$ in $K^0(V)$, and θ_2 is then defined by the isomorphism $\mathcal{L}|_{V_0'} \xrightarrow{\rho} F|_{V_0'}$ equal to the composition of $h : \mathcal{L} \to TM$ with the projection $TM|_V \to N(V)$ parallel to TV.

If we assume moreover that $n = 1$, that Σ'_λ is an isolated point m_λ and that h is given near m_λ by mapping a local holomorphic everywhere non-zero section $1_\mathcal{L}$ of $\mathcal{L}$ onto the vector field $A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}$ (same notations as in the example after theorem 2), then we get as a corollary

Corollary to theorem 3 : For *dim* $\mathcal{F} = 1$ and $n = 1$, we get, with the notations above, the formula

$$\frac{1}{2i\pi} \sum_\lambda \int_{\gamma_\lambda} \frac{dB}{B} = \Big(c_1(F) - c_1(\mathcal{L}|_V)\Big) \frown [V],$$

where γ_λ denotes a small circle in V_0' around m_λ.

Proof: This formula results from the Chern-Weil theory for K^0 in the framework of the Mayer-Vietoris complex (see for instance [13], [12], [10] or [17]). The residue is in fact given by the Alexander dual A_λ of the cocycle $\left(0, c_1\left(\nabla^F, \rho(\nabla^\mathcal{L})\right)\right)$ in $MV^2(U_\lambda, U_\lambda \setminus \{m_\lambda\})$, where ∇^F and $\nabla^\mathcal{L}$ denote trivial connections on F and $\mathcal{L}$ near m_λ satisfying respectively $\nabla^F \frac{\partial}{\partial y} = 0$

and $\nabla^{\mathcal{L}} 1_{\mathcal{L}} = 0$ respectively . The isomorphism ρ being given by $\rho(1_{\mathcal{L}}) = B\frac{\partial}{\partial y}$, we get $c_1\Big(\nabla^F, \rho(\nabla^{\mathcal{L}})\Big) = \frac{dB}{B}$, hence the formula III' (see [12] or [10] for further explanations).

Remark : When we have simultaneously $n = 1$ and $k = 1$, assume that $\Sigma \cup \Sigma'$ has only isolated points m_λ, and denote by $Z_\lambda(\mathcal{F})$ (resp. $P_\lambda(\mathcal{F})$) the order of m_λ as a zero (resp. a pole) of the meromorphic function $\frac{B}{A}$ near m_λ. Substracting then the formula in the corollary of theorem 3 from the formula in the corollary of theorem 1, we recover the theorem D of [5]:

$$\sum_\lambda Z_\lambda(\mathcal{F}) - \sum_\lambda P_\lambda(\mathcal{F}) = c_1(E) \frown [V] + 2g - 2.$$

Example : We consider the blow-up of a holomorphic vector field v in $\mathbb{C}^3$, at the origin $0 \in \mathbb{C}^3$ which is supposed to be an isolated dicritical singularity. We get a holomorphic 1-foliation $\mathcal{F}$ on the total space M of the tautological line bundle $L \to D$ over the divisor $D \cong \mathbb{CP}(2)$, and we assume it to be generically transverse to D. Let Γ be the tangency locus of $\mathcal{F}$ in D: this is an algebraic curve which is a locally complete intersection (hypersurface in D which is smooth): it has therefore a normal bundle $N_D(\Gamma)$ in D which is a subbundle of the normal bundle $N_M(\Gamma)$ in M. We shall assume that, along Γ, $\mathcal{F}$ is not only tangent to D but also transverse to Γ, except at some isolated points m_i where either $\mathcal{F}$ is tangent to Γ or is singular (Γ itself being possibly singular at such a point).

Let d be the order of v at 0, and k ($k \leq d$) be the degree of the algebraic curve Γ in D.

1) Applying Theorem 3 for $V = D$, $F = L$, $\mathcal{L}|_D = L^d$ and $\Sigma' = \Gamma$, the total Chern class $c([F - \mathcal{L}|_D)$ is equal to $\frac{1-\gamma}{1-d\gamma} = 1 + (d-1)\gamma + d(d-1)\gamma^2$, where γ denotes the Chern class $c_1(\check{\mathrm{L}}\ )$ of the bundle $\check{\mathrm{L}}$ dual to L. Thus, since Γ is connected,

$$\mathrm{Res}_\Gamma(c_2, \theta_2) = d(d-1), \text{ while } \mathrm{Res}_\Gamma(c_1, \theta_2) = (d-1)[\Gamma].$$

2) Applying now the corollary to theorem 3 for $V = \Gamma$, $F = (\check{\mathrm{L}})^k|_\Gamma$, $\mathcal{L}|_\Gamma = (L^d)|_\Gamma$ and $\Sigma' = \{m_i\}$, the total Chern class $c([F - \mathcal{L}|_\Gamma)$ is equal to $\frac{1+\gamma^k}{(1-d)\gamma} = 1 + (k+d)\gamma$, and

$$\sum_i \mathrm{Res}_{m_i}(c_1, \theta_2) = k(k+d).$$

Take for instance the blow-up of the holomorphic vector field

$$v = (xz + y^3)\frac{\partial}{\partial x} + (yz + x^3)\frac{\partial}{\partial y} + (z^2 + x^2 y)\frac{\partial}{\partial z}$$

at the point $0 \in \mathbb{C}^3$. Denote by $p : M \to \mathbb{C}^3$ the blow-up, and U_x, U_y and U_z the three open sets in M with respective coordinates (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) such that
$p(x_1, y_1, z_1)$ has coordinates $x = x_1$, $y = x_1 y_1$, $z = x_1 z_1$,
$p(x_2, y_2, z_2)$ has coordinates $x = y_2 x_2$, $y = y_2$, $z = y_2 z_2$,
$p(x_3, y_3, z_3)$ has coordinates $x = z_3 x_3$, $y = z_3 y_3$, $z = z_3$.

The foliation $\mathcal{F}$ is then defined
on U_x by the vector field

$$v_x = \left(z_1 + x_1(y_1)^3\right)\frac{\partial}{\partial x_1} + \left(1 - (y_1)^4\right)\frac{\partial}{\partial y_1} + \left(y_1 - z_1(y_1)^3\right)\frac{\partial}{\partial z_1},$$

on U_y by

$$v_y = \left(1 - (x_2)^4\right)\frac{\partial}{\partial x_2} + \left(z_2 + (x_2)^3 y_2\right)\frac{\partial}{\partial y_2} + \left((x_2)^2 - (x_2)^3 z_2\right)\frac{\partial}{\partial z_2},$$

and on U_z by

$$v_z = \left((y_3)^3 - (x_3)^3 y_3\right)\frac{\partial}{\partial x_3} + \left((x_3)^3 - (x_3)^2(y_3)^2\right)\frac{\partial}{\partial y_3} + \left(1 + z_3 y_3 (x_3)^2\right)\frac{\partial}{\partial z_3}.$$

The points in D with homogeneous coordinates [X,Y,Z] are thus given by $(0, y_1, z_1) = [1, y_1, z_1]$ in $U_x \cap D$, $(x_2, 0, z_2) = [x_2, 1, z_2]$ in $U_y \cap D$, and $(x_3, y_3, 0) = [x_3, y_3, 1]$ in $U_z \cap D$. Let H be the projective line $Z = 0$, through the points $m_X = [1, 0, 0]$ and $m_Y = [0, 1, 0]$. We observe that $\mathcal{F}$ is everywhere transverse to D except on H, and that it is everywhere tangent to D along H and transverse to H except at m_X and m_Y where it is tangent to H. Thus, $d = 2$ and $k = 1$.

For $V = D$, $F = L$, $\mathcal{L}|_D = L^2$ and $\Sigma' = H$, we get:
$c([F - \mathcal{L}|_D) = \frac{1-\gamma}{1-2\gamma} = 1 + \gamma + 2\gamma^2$, thus: $\mathrm{Res}_H(c_2, \theta_2) = 2$, and $\mathrm{Res}_H(c_1, \theta_2) = [H]$.

For $V = H$, $F = \check{\mathrm{L}}|_H$, $\mathcal{L}|_D = L^2|_H$ and $\Sigma' = \{m_X, m_Y\}$, we get:
$\mathrm{Res}_{m_X}(c_1, \theta_2) = 1$ since $v_x = y_1 \frac{\partial}{\partial z_1}$ (modulo TH) on $H \cap U_x$,
and $\mathrm{Res}_{m_Y}(c_1, \theta_2) = 2$ since $v_y = (x_2)^2 \frac{\partial}{\partial z_2}$ (modulo TH) on $H \cap U_y$.
Observe that the sum of these residues, equal to 3, is also equal to $c_1(\check{\mathrm{L}} - L^2) \frown [H]$, as predicted.

Remark : We could as well take for any n the blow-up $\mathcal{F}$ of a vector field v in $\mathbb{C}^n$ having an isolated dicritical singularity at the origin. We assume that there is a stratification $V_0 \subset V_1 \subset \cdots \subset V_{n-2} \subset V_{n-1} = D$ of the divisor $D \cong \mathbb{CP}(n-1)$ by locally complete intersections V_j of pure dimension j in M, with successive projections $N_M(V_j) \xrightarrow{p_j} N_M(V_{j+1})|_{V_j} \to 0$ of the normal bundles, with $\mathcal{F}$ transverse to V_{j+1} for any j off V_j and tangent to V_{j+1} (or singular) along V_j. This stratification generalizes the stratification $\{m_i\} \subset \Gamma \subset D$ seen above in the case $n = 3$. For each j we get residues of the previous kind taking for $F \to V$ the bundle $\mathrm{Ker}\, p_j \to V_j$. Notice that we always have $N_M(D) = L$ (the tautological line bundle over D), that V_{n-2} is an algebraic connected hypersurface of D whose degree k is at most equal to the order of the vector field v in $\mathbb{C}^n$ at the singular point, and that we have the exact sequence of vector bundles $0 \to \check{\mathrm{L}}^k \to N_M(V_{n-2}) \xrightarrow{p_{n-2}} N_M(V_{n-1})|_{V_{n-2}} \to 0$.

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