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# Homoclinic classes for generic $C^1$ vector fields

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*Abstract.* We prove that homoclinic classes for a *residual* set of $C^1$ vector fields $X$ on closed $n$-manifolds are maximal transitive, and depend continuously on periodic orbit data. In addition, $X$ does not exhibit cycles formed by homoclinic classes. We also prove that a homoclinic class of $X$ is isolated if and only if it is $\Omega$-isolated, and it is the intersection of its stable set with its unstable set. All these properties are well known for structural stable Axiom A vector fields.

## 1. *Introduction*

We show some properties of homoclinic classes for generic $C^1$ flows on closed $n$-manifolds. By homoclinic class we mean the closure of the transversal homoclinic points associated to a hyperbolic periodic orbit. So, homoclinic classes are transitive and the closure of its periodic orbits [**16, Chapter 2,** §8].

For structural stable Axiom A vector fields it is known that homoclinic classes are maximal transitive and depend continuously on the periodic orbit data. In addition, if $H$ is a homoclinic class of $X$ then it is saturated, that is, $H = W^s_X(H) \cap W^u_X(H)$, where $W^s_X(H)$ is the stable set of $H$ and $W^u_X(H)$ is the unstable set of $H$ [**2, pp. 371**]. Moreover, such vector fields do not exhibit cycles formed by homoclinic classes. In this paper we shall prove these properties for generic $C^1$ vector fields on closed $n$-manifolds $M$ (neither structural stability nor Axiom A is assumed). Furthermore, we prove that generically a homoclinic class is isolated if and only if it is isolated from the nonwandering set. In particular, all the mentioned properties hold for a dense set of $C^1$ vector fields on $M$. It is

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interesting to observe that neither structural stability nor Axiom A is dense in the space of $C^1$ vector fields on $M$, $\forall n \geq 3$.

To state our results in a precise way we use the following notation. $M$ is a compact boundaryless $n$-manifold, $\mathcal{X}^1(M)$ is the space of $C^1$ vector fields endowed with the $C^1$ topology. Given $X \in \mathcal{X}^1(M)$, $X_t$ denotes the flow induced by $X$. The $\omega$-limit set of $p$ is the set $\omega_X(p)$ of accumulation points of the positive orbit of $p$. The $\alpha$-limit set of $p$ is $\alpha_X(p) = \omega_{-X}(p)$, where $-X$ denotes the time-reversed flow of $X$. The nonwandering set $\Omega(X)$ of $X$ is the set of $p$ such that for every neighborhood $U$ of $p$ and $T > 0$ there is $t > T$ such that $X_t(U) \cap U \neq \emptyset$. Clearly $\Omega(X)$ is closed, nonempty and contains any $\omega$-limit ($\alpha$-limit) set. A compact invariant set $B$ of $X$ is *$\Omega$-isolated* if $\Omega(X) \setminus B$ is closed. $B$ is *isolated* if $B = \cap_{t \in \mathbb{R}} X_t(U)$ for some compact neighborhood $U$ of $B$ (in this case $U$ is called *isolating block*). We denote by $Per(X)$ the union of the periodic orbits of $X$ and $Crit(X)$ the set formed by the union of $Per(X)$ and the singularities of $X$.

A set is *transitive* for $X$ if it is the $\omega$-limit set of one of its orbits. A transitive set $\Lambda$ of $X$ is *maximal transitive* if it contains every transitive set $T$ of $X$ satisfying $\Lambda \cap T \neq \emptyset$. Note that a maximal transitive set is maximal with respect to the inclusion order. In [**4, 5**] it was asked whether every homoclinic class $H_f(p)$ of a generic diffeomorphism $f$ satisfies the property that if $T$ is a transitive set of $f$ and $p \in T$, then $T \subset H_f(p)$. In [**1**], M. C. Arnaud also considered homoclinic classes for $C^1$ diffeomorphisms on $M$, and in particular she gives a positive answer to this question [**1, Corollary 40**]. On the other hand, item (1) of Theorem A below states that generically any transitive set of a $C^1$ vector field intersecting the homoclinic class is included in it, and thus the diffeomorphism version of it extends this result of M. C. Arnaud.

If $\Lambda$ is a compact invariant set of $X$, we denote

$$W^s_X(\Lambda) = \{q \in M : dist(X_t(q), \Lambda) \to 0, t \to \infty\}$$

and

$$W^u_X(\Lambda) = \{q \in M : dist(X_t(q), \Lambda) \to 0, t \to -\infty\},$$

where $dist$ is the metric on $M$. These sets are called respectively the stable and unstable set of $\Lambda$. We shall denote $W^s_X(p) = W^s_X(\mathcal{O}_X(p))$ and $W^u_X(p) = W^u_X(\mathcal{O}_X(p))$ where $\mathcal{O}_X(p)$ is the orbit of $p$. We say that $\Lambda$ is *saturated* if $W^s_X(\Lambda) \cap W^u_X(\Lambda) = \Lambda$.

A *cycle* of $X$ is a finite set of compact invariant sets $\Lambda_0, \Lambda_1, \dots, \Lambda_n$ such that $\Lambda_n = \Lambda_0$, and $\Lambda_0, \Lambda_1, \dots, \Lambda_{n-1}$ are disjoint, and

$$(W^u_X(\Lambda_i) \setminus \Lambda_i) \cap (W^s_X(\Lambda_{i+1}) \setminus \Lambda_{i+1}) \neq \emptyset$$

for all $i = 0, ..., n-1$.

A compact invariant set $\Lambda$ of $X$ is *hyperbolic* if there is a continuous tangent bundle decomposition $E^s \oplus E^X \oplus E^u$ over $\Lambda$ such that $E^s$ is contracting, $E^u$ is expanding and $E^X$ denotes the direction of $X$. We say that $p \in Crit(X)$ is hyperbolic if $\mathcal{O}_X(p)$ is a hyperbolic set of $X$.

The Stable Manifold Theorem [**10**] asserts that $W^s_X(p)$ is an immersed manifold tangent to $E^s \oplus E^X$ for every $p$ in a hyperbolic set $\Lambda$ of $X$. Similarly for $W^u_X(p)$. This remark applies when $\Lambda = \mathcal{O}_X(p)$ for some $p \in Crit(X)$ hyperbolic. As already defined, the homoclinic class associated to a hyperbolic periodic orbit $p$ of $X$, $H_X(p)$, is the closure of the transversal intersection orbits in $W^s_X(p) \cap W^u_X(p)$.

We say that $X$ is *Axiom A* if $\Omega(X)$ is both hyperbolic and the closure of $Crit(X)$. The non wandering set of a nonsingular Axiom A flow splits in a finite disjoint union of homoclinic classes [**16, Chapter 0, pp 3**].

Another interesting property of homoclinic classes for Axiom A vector fields is their continuous dependence on the periodic orbit data, that is, the map $p \in Per(X) \to H_X(p)$ is upper-semicontinous.

In general, we say that a compact set sequence $\Lambda_n$ *accumulates* on a compact set $\Lambda$ if for every neighborhood $U$ of $\Lambda$ there is $n_0 > 0$ such that $\Lambda_n \subset U$ for all $n \geq n_0$. Note that this kind of accumulation is weaker than the usual Hausdorff metric accumulation.

If $\mathcal{Y}$ denotes a metric space, then $\mathcal{R} \subset \mathcal{Y}$ is *residual* in $\mathcal{Y}$ if $\mathcal{R}$ contains a countable intersection of open-dense subsets of $\mathcal{Y}$. Clearly a countable intersection of residual subsets of $\mathcal{Y}$ is a residual subset of $\mathcal{Y}$. For example, the set of Kupka-Smale vector fields $\mathcal{KS}^1(M)$ on $M$ is a residual subset of $\mathcal{X}^1(M)$ [**6, Chapter 3,** § **3**]. Recall that a vector field is Kupka-Smale if all its periodic orbits and singularities are hyperbolic and the invariant manifolds of such elements intersect transversally.

Theorem A. *The following properties hold for a residual subset of vector fields $X$ in $\mathcal{X}^1(M)$:*

(i) *The homoclinic classes of $X$ are maximal transitive sets of $X$. In particular, different homoclinic classes of $X$ are disjoint.*

(ii) *The homoclinic classes of $X$ are saturated.*

(iii) *The homoclinic classes of $X$ depends continuously on the periodic orbit data, that is, the map $p \in Per(X) \to H_X(p)$ is upper-semicontinous.*

(iv) *A homoclinic class of $X$ is isolated if and only if it is $\Omega$-isolated.*

(v) *The hyperbolic homoclinic classes of $X$ are isolated.*

(vi) *There is no cycle of $X$ formed by homoclinic classes of $X$.*

(vii) *$X$ has finitely many homoclinic classes if and only if the union of the homoclinic classes of $X$ is closed and every homoclinic class of $X$ is isolated.*

When $M$ has dimension three we obtain the following corollaries using Theorem A, [**13**], and [**14**]. Recall that an isolated set $\Lambda$ of a $C^r$ vector field $X$ is $C^r$ robust transitive ($r \geq 1$) if it exhibits an isolating block $U$ such that, for every vector field $Y$ $C^r$ close to $X$, $\cap_{t \in \mathbb{R}} Y_t(U)$ is both transitive and nontrivial for $Y$.

Corollary 1.1. *The properties below are equivalent for a residual set of nonsingular 3-dimensional $C^1$ vector fields $X$ and every nontrivial homoclinic class $H_X(p)$ of $X$:*

(i) *$H_X(p)$ is hyperbolic.*

(ii) *$H_X(p)$ is isolated.*
(iii) *$H_X(p)$ is $C^1$ robust transitive for $X$.*

COROLLARY 1.2. *The properties below are equivalent for a residual set of nonsingular 3-dimensional $C^1$ vector fields $X$:*
(i) *$X$ is Axiom A.*
(ii) *$X$ has finitely many homoclinic classes.*
(iii) *The union of the homoclinic classes of $X$ is closed and every homoclinic class of $X$ is isolated.*

The equivalence between the Items (1) and (2) of the above corollary follows from [**13, 14**]. It shows how difficult is to prove the genericity of vector fields exhibiting finitely many homoclinic classes. The equivalence between (2) and (3) follows from Theorem A-(7).

To prove Theorem A we show in Section 3 that homoclinic classes $H_X(p)$ for a residual set of $C^1$ vector fields $X$ satisfy $H_X(p) = \Lambda^+ \cap \Lambda^-$, where $\Lambda^+$ is Lyapunov stable for $X$ and $\Lambda^-$ is Lyapunov stable for $-X$. The main technical tool to prove such result is Lemma 3.6, a stronger version of Hayashi's $C^1$ Connecting Lemma [**8**], recently published in [**19, Theorem E, p.5214**] (see also [**1, 7, 9**]).

In Section 2 we study compact invariant sets $\Lambda$ of $X$ satisfying $\Lambda = \Lambda^+ \cap \Lambda^-$, where $\Lambda^\pm$ is Lyapunov stable for $\pm X$. The proof of Theorem A and Corollary 1.1 will be given in the final section.

Remark 1.3. *We observe that Theorem A is valid for a residual set of $C^1$ diffeomorphisms on any $n$-manifold $M$ by the usual method of suspension.*

We are thankful to S. Hayashi for useful conversations concerning the connecting lemma.

## 2. *Lyapunov stability lemmas*

In this section we shall establish useful properties of Lyapunov stable sets. A reference for Lyapunov stability theory is [**3**].

Recall we have denoted by $X_t$, $t \in I\!R$ the flow generated by $X \in \mathcal{X}^1(M)$. Given $A \subset M$ and $R \subset I\!R$ we set $X_R(A) = \{X_t(q) : (q,t) \in A \times R\}$. We denote $Cl(A)$ the closure of $A$, and $Int(A)$ the interior of $A$. If $\epsilon > 0$ and $q \in M$ we set $B_\epsilon(q)$ the $\epsilon$-ball centered at $q$.

A compact subset $A \subseteq M$ is *Lyapunov stable* for $X$ if for every open set $U$ containing $A$ there exists an open set $V$ containing $A$ such that $X_t(V) \subset U$ for every $t \geq 0$. Clearly a Lyapunov stable set is forward invariant.

The following lemma summarizes some classical properties of Lyapunov stable sets (see [**3, Chapter V**]).

LEMMA 2.1. *Let $\Lambda^+$ be a Lyapunov stable set of $X$. Then,*

(i) *If $x_n \in M$ and $t_n \geq 0$ satisfy $x_n \to x \in \Lambda^+$ and $X_{t_n}(x_n) \to y$, then $y \in \Lambda^+$;*
(ii) *$W^u_X(\Lambda^+) \subset \Lambda^+$;*
(iii) *if $\Gamma$ is a transitive set of $X$ and $\Gamma \cap \Lambda^+ \neq \emptyset$, then $\Gamma \subset \Lambda^+$.*

We are interested in invariant compact sets $\Lambda = \Lambda^+ \cap \Lambda^-$ of $X$, where $\Lambda^+$ is Lyapunov stable set for $X$ and $\Lambda^-$ is Lyapunov stable set for the reversed flow $-X$. We shall call such sets *neutral* for the sake of simplicity. As we shall see in the next section, homoclinic classes are neutral sets for generic $C^1$ vector fields on closed $n$-manifolds.

Elementary properties of neutral sets are given in the lemma below.

LEMMA 2.2. *Let $\Lambda$ be a neutral set of $X$. Then,*

(i) *$\Lambda$ is saturated;*
(ii) *$\Lambda$ is transitive for $X$ if and only if $\Lambda$ is maximal transitive for $X$. In particular, different transitive neutral sets of $X$ are disjoint.*

*Proof.* Let $\Lambda = \Lambda^+ \cap \Lambda^-$ with $\Lambda^\pm$ being Lyapunov stable for $\pm X$. Clearly $W^u_X(\Lambda) \subset \Lambda^+$ by Lemma 2.1-(2). Similarly, $W^s_X(\Lambda) \subset \Lambda^-$. Hence

$$W^u_X(\Lambda) \cap W^s_X(\Lambda) \subset \Lambda^+ \cap \Lambda^- = \Lambda.$$

Conversely, $\Lambda \subset W^u_X(\Lambda) \cap W^s_X(\Lambda)$ since $\Lambda$ is invariant. This proves (1).

Now, by Lemma 2.1-(3), if $\Gamma$ is a transitive set intersecting $\Lambda$, then $\Gamma \subset \Lambda^+$ and $\Gamma \subset \Lambda^-$. Thus, $\Gamma \subset \Lambda^+ \cap \Lambda^- = \Lambda$, and so, $\Lambda$ is maximal transitive. The converse is obvious. Different transitive neutral sets of $X$ are maximal transitive, and so, they are necessarily disjoint. This finishes the proof. $\square$

Note that a Smale horseshoe with a first tangency is an example of a maximal transitive set which is not neutral, see Proposition 2.6. This example also provides a hyperbolic homoclinic class which is not neutral (compare with Theorem 3.1).

PROPOSITION 2.3. *There is no cycle of $X$ formed by transitive neutral sets.*

*Proof.* By contradiction suppose that there exists a cycle $\Lambda_0, \ldots, \Lambda_n$ of $X$ such that every $\Lambda_i$ is a transitive neutral set of $X$. Recall $\Lambda_n = \Lambda_0$.

Set $\Lambda_i = \Lambda_i^+ \cap \Lambda_i^-$ where each $\Lambda_i^\pm$ is Lyapunov stable for $\pm X$. Choose

$$x_i \in (W^u_X(\Lambda_i) \setminus \Lambda_i) \cap (W^s_X(\Lambda_{i+1}) \setminus \Lambda_{i+1})$$

according to the definition.

We claim that $x_i \in \Lambda_0^-$ for every $i$. Indeed, as $W^s_X(\Lambda_0) \subset \Lambda_0^-$ one has $x_{n-1} \in \Lambda_0^-$. Assume by induction that $x_i \in \Lambda_0^-$ for some $i$. As $x_i \in W^u_X(\Lambda_i)$, the backward invariance of $\Lambda_0^-$ implies

$$\Lambda_0^- \cap \Lambda_i \supset \alpha_X(x_i) \neq \emptyset.$$

By Lemma 2.1-(3) one has $\Lambda_i \subset \Lambda_0^-$ since $\Lambda_i$ is transitive. In particular,

$W^s_X(\Lambda_i) \subset \Lambda_0^-$ by Lemma 2.1-(2) applied to $-X$. As $x_{i-1} \in W^s_X(\Lambda_i)$, one has $x_{i-1} \in \Lambda_0^-$. The claim follows by induction.

By the claim $x_0 \in \Lambda_0^-$. As $W^u_X(\Lambda_0) \subset \Lambda_0^+$ and $x_0 \in W^u_X(\Lambda_0)$ (by definition) one has $x_0 \in \Lambda_0^+ \cap \Lambda_0^- = \Lambda_0$. This contradicts $x_0 \in W^u_X(\Lambda_0) \setminus \Lambda_0$ and the proposition is proved. □

LEMMA 2.4. *If $\Lambda$ is neutral for $X$, then for every neighborhood $U$ of $\Lambda$ there exists a neighborhood $V \subset U$ of $\Lambda$ such that*

$$\Omega(X) \cap V \subset \cap_{t \in I\!R} X_t(U).$$

*Proof.* Let $U$ be a neighborhood of a neutral set $\Lambda$ of $X$. Choose $U' \subset Cl(U') \subset U$ with $U'$ being another neighborhood of $\Lambda$. We claim that there is a neighborhood $V \subset U'$ of $\Lambda$ so that:

(1) $t \geq 0$ and $p \in V \cap X_{-t}(V)$ implies $X_{[0,t]}(p) \subseteq U'$.

(2) $t \leq 0$ and $p \in V \cap X_t(V)$ implies $X_{[-t,0]}(p) \subseteq U'$.

Indeed, it were not true then there would exist a neighborhood $U$ of $\Lambda$ and sequences $p_n \to \Lambda$, $t_n > 0$ such that $X_{t_n}(p_n) \to \Lambda$ but $X_{[0,t_n]}(p_n) \not\subseteq U'$. Choose $q_n \in X_{[0,t_n]}(p_n) \setminus U'$. Write $q_n = X_{t'_n}(p_n)$ for some $t'_n \in [0, t_n]$ and assume that $q_n \to q$ for some $q \notin U'$. Let $\Lambda = \Lambda^+ \cap \Lambda^-$ with $\Lambda^\pm$ Lyapunov stable for $\pm X$. Since $\Lambda^+$ is Lyapunov stable for $X$ and $t'_n > 0$, Lemma 2.1-(1) implies $q \in \Lambda^+$. On the other hand, as we can write $q_n = X_{t'_n - t_n}(X_{t_n}(p_n))$ where $t'_n - t_n > 0$ and $X_{t_n}(p_n) \to \Lambda$ and using again Lemma 2.1-(1) we have that $q \in \Lambda^-$. This proves that $q \in \Lambda$, a contradiction since $q \notin U'$. This proves the claim.

Next we prove that $\Omega(X) \cap V \subseteq \cap_{t \in I\!R} X_t(U)$. Indeed, choose $q \in \Omega(X) \cap V$. By contradiction, we assume that there is $t_0 > 0$ (say) such that $X_{t_0}(q) \notin U$. Then, there is a ball $B_\epsilon(q) \subseteq V$ such that $X_{t_0}(B_\epsilon(q)) \cap Cl(U') = \emptyset$. As $q \in \Omega(X)$ there exists $t > t_0$ such that $B_\epsilon(q) \cap X_{-t}(B_\epsilon(q)) \neq \emptyset$. Pick $p \in B_\epsilon(q) \cap X_{-t}(B_\epsilon(q))$. By (1) above one has $X_{[0,t]}(p) \subseteq U'$ since $B_\epsilon(q) \subset V$. This contradicts $X_{t_0}(p) \in X_{t_0}(B_\epsilon(q))$ and $X_{t_0}(B_\epsilon(q)) \cap Cl(U') = \emptyset$. The proof is completed. □

A first consequence of the above lemma is the following corollary. Given compact subsets $A, B \subset M$ we denote $dist(A, B) = \inf\{dist(a, b); a \in A, b \in B\}$.

COROLLARY 2.5. *If $\Lambda$ is a neutral set of $X$ and $\Lambda_n$ is a sequence of transitive sets of $X$ such that $dist(\Lambda_n, \Lambda) \to 0$ as $n \to \infty$, then $\Lambda_n$ accumulates on $\Lambda$.*

*Proof.* Let $\Lambda_n$ and $\Lambda$ as in the statement. Fix a neighborhood $U$ of $\Lambda$ and let $V \subset U$ be the neighborhood of $\Lambda$ obtained by the previous lemma. As $dist(\Lambda_n, \Lambda) \to 0$ as $n \to \infty$ we have that $\Lambda_n \cap V \neq \emptyset$ for every $n$ large. Let $q_n$ the dense orbit of $\Lambda_n$. Clearly $q_n \in \Omega(X)$. We can assume that $q_n \in V$ for $n$ large, and so, $q_n \in \Omega(X) \cap V$. Then, $X_t(q_n) \in U$ for every $t$. In particular, $\Lambda_n = \omega_X(q_n) \subset Cl(U)$. This proves the corollary since $U$ is arbitrary. This finishes the proof. □

PROPOSITION 2.6. *A neutral set is isolated if and only if it is $\Omega$-isolated.*

*Proof.* We first claim that any saturated $\Omega$-isolated set $\Lambda$ of $X$ is isolated. Indeed, since $\Lambda$ is $\Omega$-isolated, there is $U \supset \Lambda$ open such that $Cl(U) \cap \Omega(X) = \Lambda$. This $U$ is an isolating block for $\Lambda$. For if $x \in \cap_{t\in I\!R} X_t(U)$, then $\omega_X(x) \cup \alpha_X(x) \subset Cl(U) \cap \Omega(X) \subset \Lambda$. So, $x \in W^s_X(\Lambda) \cap W^u_X(\Lambda) = \Lambda$. This proves that $\cap_{t\in I\!R} X_t(U) \subset \Lambda$. The opposite inclusion follows since $\Lambda$ is invariant. The claim follows.

To prove that invariant $\Omega$-isolated neutral set are isolated we use the above claim and Lemma 2.2-(1). To prove that isolated neutral sets are $\Omega$-isolated we use Lemma 2.4. □

PROPOSITION 2.7. *Transitive hyperbolic neutral sets are isolated.*

*Proof.* By Proposition 2.6 it is suffices to show that transitive neutral hyperbolic sets $\Lambda$ are $\Omega$-isolated.

Suppose by contradiction that $\Lambda$ is not $\Omega$-isolated. Then, there is a sequence $p_n \in \Omega(X) \setminus \Lambda$ converging to $p \in \Lambda$. Fix $U$ a neighborhood of $\Lambda$ and let $V$ be given in Lemma 2.4 for $U$. We can assume that $p_n \in V$ for every $n$. As $p_n$ is non wandering for $X$, for every $n$ there are sequences $q_i \in V \to p_n$ and $t_i > 0$ such that $X_{t_i}(q_i) \to p_n$ as $i \to \infty$. By (1) in the proof of Lemma 2.4 we have $X_{[0,t_i]}(q_i) \subset U$ for every $i$. So, we can construct a periodic pseudo orbit of $X$ arbitrarily close to $p_n$. By the Shadowing Lemma for Flows ([**11, Theorem 18.1.6, pp. 569**]) applied to the hyperbolic set $\Lambda$, such a periodic pseudo orbit can be shadowed by a periodic orbit. This proves that $p_n \in Cl(Per(X))$. As the neighborhood $U$ is arbitrary, we can assume that $p_n \in Per(X)$ for every $n$. Note that $\mathcal{O}_X(p_n)$ converges to $\Lambda$ by Corollary 2.5.

As $\Lambda$ is transitive we have that if $E^s \oplus E^X \oplus E^u$ denotes the corresponding hyperbolic splitting, then $dim(E^s) = s$ and $dim(E^u) = u$ are constant in $\Lambda$. Clearly neither $s = 0$ nor $u = 0$ since $\Lambda$ is not $\Omega$-isolated. As $\mathcal{O}_X(p_n)$ converges to $\Lambda$ both the local stable and unstable manifolds of $p_n$ have dimension $s$, $u$ respectively. Moreover, both invariant manifolds have uniform size as well. This implies that $W^u_X(p_n) \cap W^s_X(p) \neq \emptyset$ and $W^s_X(p_n) \cap W^u_X(p) \neq \emptyset$ for $n$ large. As $p_n \in Per(X)$ and $p \in \Lambda$, we conclude by the Inclination Lemma [**6**] that $p_n \in Cl(W^s_X(p) \cap W^u_X(p))$. As $p \in \Lambda$, $W^{s,u}_X(p) \subset W^{s,u}_X(\Lambda)$. So, $p_n \in Cl(W^s_X(\Lambda) \cap W^u_X(\Lambda))$. As $\Lambda$ is saturated, $W^s_X(\Lambda) \cap W^u_X(\Lambda) = \Lambda$ and hence $p_n \in Cl(\Lambda) = \Lambda$. But this is impossible since $p_n \in \Omega(X) \setminus \Lambda$ by assumption. This concludes the proof. □

Denote by $\mathcal{F}$ the collection of all isolated transitive neutral sets of $X$.

PROPOSITION 2.8. *A sub collection $\mathcal{F}'$ of $\mathcal{F}$ is finite if and only if $\cup_{\Lambda\in\mathcal{F}'}\Lambda$ is closed.*

*Proof.* Obviously $\cup_{\Lambda\in\mathcal{F}'}\Lambda$ is closed if $\mathcal{F}'$ is finite. Conversely, suppose that $\cup_{\Lambda\in\mathcal{F}'}\Lambda$

is closed. If $\mathcal{F}'$ were infinite then, it would exist sequence $\Lambda_n \in \mathcal{F}'$ of (different) sets accumulating some $\Lambda \in \mathcal{F}'$. By Corollary 2.5 we have $\Lambda_n \subseteq U$ for some isolating block $U$ of $\Lambda$ and $n$ large. And then, we would have that $\Lambda_n = \Lambda$ for $n$ large, a contradiction which completes the proof. □

## 3. *Homoclinic classes*

The main result of this section is

Theorem 3.1. *There is a residual subset $\mathcal{R}$ of $\mathcal{X}^1(M)$ such that every homoclinic class of every vector field in $\mathcal{R}$ is neutral.*

Corollary 3.2. *The following properties are equivalent for $X \in \mathcal{R}$ and every compact invariant set $\Lambda$ of $X$ :*
(i) *$\Lambda$ is a transitive neutral set with periodic orbits of $X$.*
(ii) *$\Lambda$ is a homoclinic class of $X$.*
(iii) *$\Lambda$ is a maximal transitive set with periodic orbits of $X$.*

*Proof.* That (2) implies (1) follows from Theorem 3.1. That (1) implies (3) follows from Lemma 2.4-2. Let us prove that (3) implies (2). If $\Lambda$ is as in (3) and $p \in Per(X) \cap \Lambda$, then $\Lambda \cap H_X(p) \neq \emptyset$. By Theorem 3.1 we can assume $H_X(p)$ is neutral, and so it is maximal transitive (using (1) $\longrightarrow$ (3)). As both $\Lambda$ and $H_X(p)$ are maximal transitive we conclude $\Lambda = H_X(p)$ and the proof follows. □

Corollary 3.3. *For $X \in \mathcal{R}$, a non singular compact isolated set of $X$ is neutral and transitive if and only if it is a homoclinic class.*

*Proof.* The converse follows from Theorem 3.1. To prove the direct, denote $\Lambda$ a transitive isolated neutral set of a generic $C^1$ vector field $X$. By Proposition 2.6 it follows that $\Lambda$ is also $\Omega$-isolated. Since $\Lambda$ is transitive we have $\Lambda \subset \Omega(X)$. Thus, by [**17**] it follows that $\Lambda = Cl(\Lambda \cap Per(X))$, and so, $\Lambda \cap Per(X) \neq \emptyset$. Then the conclusion follows from the previous corollary. □

The proof of Theorem 3.1 follows immediately from the two lemmas below.

Lemma 3.4. *There exists a residual set $\mathcal{R}$ of $\mathcal{X}^\infty(\mathcal{M})$ such that, for every $Y \in \mathcal{R}$ and $\sigma \in Crit(Y)$, $Cl(W^u_X(\sigma))$ is Lyapunov stable for $X$ and $Cl(W^s_X(\sigma))$ is Lyapunov stable for $-X$.*

Lemma 3.5. *There exists a residual set $\mathcal{R}$ in $\mathcal{X}^\infty(\mathcal{M})$ such that every $X \in \mathcal{R}$ satisfies*

$$H_X(p) = Cl(W^u_X(p)) \cap Cl(W^s_X(p))$$

*for all $p \in Per(X)$.*

Lemma 3.4 was proved in [**15, Theorem 6.1, p.372**] when $\sigma$ is a singularity and the same proof works when $\sigma$ is a periodic orbit. We shall give another proof of this lemma in the Appendix for completeness.

Before the proof of Lemma 3.5, let us introduce some notation. Recall that $M$ is a closed $n$-manifold, $n \geq 3$. We denote $2_c^M$ the space of all compact subsets of $M$ endowed with the Hausdorff topology. Recall that $\mathcal{KS}^1(M) \subset \mathcal{X}^1(M)$ denotes the set of Kupka-Smale $C^1$ vector fields on $M$.

Given $X \in \mathcal{X}^1(M)$ and $p \in Per(X)$ we denote $\Pi_X(p)$ the period of $p$. We set $\Pi_X(p) = 0$ if $p$ is a singularity of $X$.

If $T > 0$ we denote

$$Crit_T(X) = \{p \in Crit(X) : \Pi_X(p) < T\}.$$

If $p \in Crit(X)$ is hyperbolic, then there is a continuation $p(Y)$ of $p$ for $Y$ close enough to $X$ so that $p(X) = p$.

Note that if $X \in \mathcal{KS}^1(M)$ and $T > 0$, then

$$Crit_T(X) = \{p_1(X), \cdots, p_k(X)\}$$

is a finite set. Moreover,

$$Crit_T(Y) = \{p_1(Y), \cdots, p_k(Y)\}$$

for every $Y$ close enough to $X$.

Let $\mathcal{Y}$ be a metric space. A set-valued map

$$\Phi : \mathcal{Y} \to 2_c^M$$

is *lower semi-continuous* at $Y_0 \in \mathcal{Y}$ if for every open set $U \subset M$ one has $\Phi(Y_0) \cap U \neq \emptyset$ implies $\Phi(Y) \cap U \neq \emptyset$ for every $Y$ close to $Y_0$. Similarly, we say that $\Phi$ is *upper semi-continuous* at $Y_1 \in \mathcal{Y}$ if for every compact set $K \subset M$ one has $\Phi(Y_1) \cap K = \emptyset$ implies $\Phi(Y) \cap K = \emptyset$ for every $Y$ close to $Y_1$. We say that $\Phi$ is *lower semi-continuous* if it is lower semi-continuous at every $Y_0 \in \mathcal{Y}$. A well known result [**12, Corollary 1, p. 71**] asserts that if $\Phi : \mathcal{X}^1(M) \to 2_c^M$ is a lower semi-continuous map, then it is upper semi-continuous at every $Y$ in a residual subset of $\mathcal{X}^1(M)$.

The lemma below is the flow version of [**19, Theorem E, p. 5214**] (see also [**1, 7, 8, 9**]).

Lemma 3.6. *Let $Y \in \mathcal{X}^1(M)$ and $x \notin Crit(Y)$. For any $C^1$ neighborhood $\mathcal{U}$ of $Y$ there are $\rho > 1$, $L > 0$ and $\epsilon_0 > 0$ such that for any $0 < \epsilon \leq \epsilon_0$ and any two points $p, q \in M$ satisfying*

*(a) $p, q \notin B_\epsilon(Y_{[-L,0]}(x))$;*

*(b) $\mathcal{O}_Y^+(p) \cap B_{\epsilon/\rho}(x) \neq \emptyset$;*

*(c) $\mathcal{O}_Y^-(q) \cap B_{\epsilon/\rho}(x) \neq \emptyset$,*

*there is $Z \in \mathcal{U}$ such that $Z = Y$ off $B_\epsilon(Y_{[-L,0]}(x))$ and such that $q \in \mathcal{O}_Z^+(p)$.*

**Proof of Lemma 3.5:** Given $X \in \mathcal{X}^1(M)$ we denote by $Per_T(X)$ the set of periodic orbits of $X$ with period $< T$.

We first prove a local version of Lemma 3.5:

LEMMA 3.7. *If $X \in \mathcal{KS}^1(M)$ and $T > 0$ then there are a neighborhood $\mathcal{V}_{X,T} \ni X$ and a residual subset $\mathcal{P}_{X,T}$ of $\mathcal{V}_{X,T}$ such that if $Y \in \mathcal{P}_{X,T}$ and $p \in Per_T(Y)$ then $H_Y(p) = Cl(W^u_Y(p)) \cap Cl(W^s_Y(p))$.*

*Proof.* There is a neighborhood $\mathcal{V}_{X,T} \ni X$ such that

$$Per_T(Y) = \{\sigma_1(Y), \dots, \sigma_m(Y)\} \quad \forall \quad Y \in \mathcal{V}_{X,T}.$$

For each $1 \le i \le m$, let $\Psi_i : \mathcal{V}_{X,T} \ni Y \mapsto H_Y(\sigma_i(Y)) \in 2^M_c$. Note that $\Psi_i, \forall i$, is lower semi-continuous by the persistence of transverse homoclinic orbits. So, there is a residual subset $\mathcal{P}^i_{X,T}$ of $\mathcal{V}_{X,T}$ such that $\Psi_i$ is upper semi-continuous in $\mathcal{P}^i_{X,T}$. Set $\mathcal{P}_{X,T} = \mathcal{KS}^1(M) \cap (\cap \mathcal{P}^i_{X,T}) \cap \mathcal{R}$, where $\mathcal{R}$ is the residual set given in Lemma 3.4. Then $\mathcal{P}_{X,T}$ is residual in $\mathcal{V}_{X,T}$.

Let us prove that $\mathcal{P}_{X,T}$ satisfies the conclusion of the lemma. For this, let $\sigma \in Per_T(Y)$ for some $Y \in \mathcal{P}_{X,T}$. Then $\sigma = \sigma_i(Y)$ for some $i$, and so $\Psi_i(Y) = H_Y(\sigma)$.

Suppose, by contradiction, that $H_Y(\sigma) \neq Cl(W^u_Y(\sigma)) \cap Cl(W^s_Y(\sigma))$. Then there is $x \in Cl(W^u_Y(\sigma)) \cap Cl(W^s_Y(\sigma)) \setminus H_Y(\sigma)$.

We have either

(a) $x \notin Crit(Y)$ or

(b) $x \in Crit(Y)$.

It is enough to prove the lemma in case (a). Indeed, suppose that case (b) holds. As $Y$ is Kupka-Smale we have that $\mathcal{O}_Y(x)$ is hyperbolic. Clearly $\mathcal{O}_Y(x)$ is neither a sink or a source and so $W^s_Y(x) \setminus \mathcal{O}_Y(x) \neq \emptyset$ and $W^u_Y(x) \setminus \mathcal{O}_Y(x) \neq \emptyset$. Note that $Cl(W^u_Y(\sigma))$ is Lyapunov stable since $Y \in \mathcal{R}$. As $x \in Cl(W^u_Y(\sigma))$ we conclude that $W^u_Y(x) \subseteq Cl(W^u_Y(\sigma))$. As $x \in Cl(W^s_Y(\sigma))$, there is $x' \in Cl(W^s_Y(\sigma)) \cap (W^u_Y(x) \setminus \mathcal{O}(x))$ arbitrarily close to $x$ (for this use the Grobman-Hartman Theorem as in the proof of Lemma 3.4 in the Appendix). Obviously $x' \notin Crit(Y)$. If $x' \in H_Y(\sigma)$ we would have that $x \in H_Y(\sigma)$ since $\alpha_Y(x') = O_Y(x)$ contradicting $x \notin H_Y(\sigma)$. Henceforth $x' \in Cl(W^u_Y(\sigma)) \cap Cl(W^s_Y(\sigma)) \setminus H_Y(\sigma)$ and $x' \notin Crit(Y)$. Then we conclude as in case (a) replacing $x$ by $x'$.

Now we prove the lemma in case (a).

As $x \notin H_Y(\sigma)$, there is a compact neighborhood $K$ of $x$ such that $K \cap H_Y(\sigma) = \emptyset$. As $\Psi_i$ is upper semi-continuous at $Y$, there is a neighborhood $\mathcal{U}$ of $Y$ such that

$$K \cap H_Z(\sigma(Z)) = \emptyset, \tag{3.1}$$

for all $Z \in \mathcal{U}$.

Let $\rho, L, \epsilon_0$ be the constants in Lemma 3.6 for $Y \in \mathcal{X}^1(M)$, $x$, and $\mathcal{U}$ as above.

As $x \notin Crit(Y)$, $Y_{[-L,0]}(x) \cap \mathcal{O}_Y(\sigma) = \emptyset$. Then, there is $0 < \epsilon < \epsilon_0$ such that $\mathcal{O}_Y(\sigma) \cap B_\epsilon(Y_{[-L,0]}(x)) = \emptyset$ and $B_\epsilon(x) \subseteq K$.

Choose an open set $V$ containing $\mathcal{O}_Y(\sigma)$ such that $V \cap B_\epsilon(Y_{[-L,0]}(x)) = \emptyset$.

As $x \in Cl(W^u_Y(\sigma))$, one can choose $p \in W^u_Y(\sigma) \setminus \{\sigma\} \cap V$ such that

$$\mathcal{O}^+_Y(p) \cap B_{\epsilon/\rho}(x) \neq \emptyset.$$

Similarly, as $x \in Cl(W^s_Y(\sigma))$, one can choose $q \in W^s_Y(\sigma) \setminus \{\sigma\} \cap V$ such that

$$\mathcal{O}^-_Y(q) \cap B_{\epsilon/\rho}(x) \neq \emptyset.$$

We can assume that $\mathcal{O}^-_Y(p) \subset V$ and $\mathcal{O}^+_Y(q) \subset V$. Henceforth

$$(\mathcal{O}^-_Y(p) \cup \mathcal{O}^+_Y(q)) \cap B_\epsilon(Y_{[-L,0]}(x)) = \emptyset. \tag{3.2}$$

Observe that

$$q \notin \mathcal{O}^+_Y(p)$$

for, otherwise, $p$ would be a homoclinic orbit of $Y$ passing through $K$ contradicting (3.1).

By construction $\epsilon, p, q$ satisfy (b) and (c) of Lemma 3.6.

As $p, q \in V$ and $V \cap B_\epsilon(Y_{[-L,0]}(x)) = \emptyset$ we have that that $\epsilon, p, q$ also satisfy (a) of Lemma 3.6.

Then, by Lemma 3.6, there is $Z \in \mathcal{U}$ such that $Z = Y$ off $B_\epsilon(Y_{[-L,0]}(x))$ and $q \in \mathcal{O}^+_Z(p)$.

Clearly $\sigma(Z) = \sigma$ and by (3.2) we have $p \in W^u_Z(\sigma)$ and $q \in W^s_Z(\sigma)$ since $Z = Y$ off $B_\epsilon(Y_{[-L,0]}(x))$.

Hence $\mathcal{O} = \mathcal{O}_Z(p) = \mathcal{O}_Z(q)$ is a homoclinic orbit of $\sigma$.

As $q \notin \mathcal{O}^+_Y(p)$, we have that $\mathcal{O} \cap B_\epsilon(x) \neq \emptyset$.

Perturbing $Z$ we can assume that $\mathcal{O}$ is transverse, i.e. $\mathcal{O} \subseteq H_Z(\sigma)$.

As $\mathcal{O} \cap B_\epsilon(x) \neq \emptyset$ and $B_\epsilon(x) \subset K$ we would obtain $K \cap H_Z(\sigma(Z)) \neq \emptyset$ contradicting (3.1). This finishes the proof.

□

**Proof of Lemma 3.5:** Fix $T > 0$. For any $X \in \mathcal{KS}^1(M)$ consider $\mathcal{V}_{X,T}$ and $\mathcal{P}_{X,T}$ as in Lemma 3.7. Choose a sequence $X^n \in \mathcal{KS}^1(M)$ such that $\{X^n : n \in \mathbb{N}\}$ is dense in $\mathcal{X}^1(M)$ (recall that $\mathcal{X}^1(M)$ is a separable metric space). Denote $\mathcal{V}_{n,T} = \mathcal{V}_{X^n,T}$ and $\mathcal{P}_{n,T} = \mathcal{P}_{X^n,T}$.

Define

$$\mathcal{O}^T = \cup_n \mathcal{V}_{n,T} \quad \text{and} \quad \mathcal{P}^T = \cup_n \mathcal{P}_{n,T}.$$

Clearly $\mathcal{O}^T$ is open and dense in $\mathcal{X}^1(M)$.

We claim that $\mathcal{P}^T$ is residual in $\mathcal{O}^T$. Indeed, for any $n$ there is a sequence $D_{k,n,T}$, $k \in \mathbb{N}$, such that

$$\mathcal{P}_{n,T} = \cap_k D_{k,n,T},$$

and $D_{k,n,T}$ is open and dense in $\mathcal{V}_{n,T}$ for any $k$. As

$$\mathcal{P}^T = \cup_n \mathcal{P}_{n,T} = \cup_n(\cap_k D_{k,n,T}) = \cap_k(\cup_n D_{k,n,T})$$

and $\cup_n D_{k,n,T}$ is open and dense in $\cup_n \mathcal{V}_{n,T} = \mathcal{O}^T$ we concludethat $\mathcal{P}^T$ is residual in $\mathcal{O}^T$. This proves the claim.

In particular, $\mathcal{P}^T$ is residual in $\mathcal{X}^1(M)$ for every $T$. Set $\mathcal{P} = \cap_{N \in \mathbb{N}} \mathcal{P}^N$. It follows that $\mathcal{P}$ is residual in $\mathcal{X}^1(M)$. Choose $X \in \mathcal{P}$, $p \in Per(X)$ and $N_0 \in I\!N$ bigger than $\Pi_X(p) + 1$. By definition $X \in \mathcal{P}^{N_0}$, and so, $X \in \mathcal{P}_{X^n,N_0}$ for some $n$. As $N_0 > \Pi_X(p)$ we have $p \in Per_{N_0}(X)$. Then $H_X(p) = Cl(W^u_X(p)) \cap Cl(W^s_X(p))$ by Lemma 3.7 applied to $X^n$ and $T = N_0$. This completes the proof of the lemma. □

## 4. *Proof of Theorem A and Corollary* 1.*1*

**Proof of Theorem A:** By Theorem 3.1 we have that homoclinic classes for a residual subset of $C^1$ vector fields on closed manifolds are neutral sets. This leads us to apply the results in Section 2. Thus, Theorem A-(1) follows from Lemma 2.2-(2). Theorem A-(2) follows from Lemma 2.2-(1). Theorem A-(3) follows from Corollary 2.5. Theorem A-(4) follows from Proposition 2.6. Similarly, Theorem A-(5) follows from Proposition 2.7. Theorem A-(6) follows from Proposition 2.3.

To prove Theorem A-(7) we proceed as follows. If $X$ has finitely many homoclinic classes, then the union of the homoclinic classes of $X$ is obviously closed. By [**17**] it follows that $\Omega(X)$ is the union of the homoclinic classes of $X$ (recall that $X$ is $C^1$ generic). This implies that every homoclinic class of $X$ is $\Omega$-isolated, and so, they are isolated by Theorem A-(4). Conversely, suppose that the union of the homoclinic classes of $X$ is closed and that every homoclinic class of $X$ is isolated. Let $\mathcal{F}'$ be the collection of all homoclinic classes of $X$. As every homoclinic class of $X$ is isolated by assumption one has $\mathcal{F}' \subset \mathcal{F}$ (recall the notation in Proposition 2.8). We have that $\cup_{\Lambda \in \mathcal{F}'} \Lambda$ is closed by hypothesis. Then, Proposition 2.8 implies that $\mathcal{F}'$ is finite and the proof follows. □

**Proof of Corollary** 1.**1:** That (1) implies (2) follows from Theorem A-(5). To prove that (2) implies (1) we proceed as follows. If $H_X(p)$ is isolated, $H_X(p)$ is $\Omega$-isolated by Theorem A-(4). In particular, $H_X(p)$ is not in the closure of the sinks and sources of $X$ unless it is either a sink or a source of $X$ and we would be done. By [**13**], [**14**], as $M$ is 3-dimensional and $X$ is nonsingular and generic, one has that $H_X(p)$ is hyperbolic. That (1) implies (3) follows from the hyperbolic theory. Indeed, if $H_X(p)$ is hyperbolic, then $H_X(p)$ is isolated, transitive and hyperbolic. In other words, $H_X(p)$ is a basic set of $X$. Then, the conclusion follows from the structural stability of basic sets [**16**]. That (3) implies (1) follows from [**13**], [**18**] since $H_X(p)$ has no singularities (recall $X$ has no singularities by hypothesis). This finishes the proof. □

## *Appendix A. Proof of Lemma* 3.*4*.

Here we give a proof of Lemma 3.4 using Lemma 3.6. The proof we gave in [**15, Theorem 6.1, p.372**] uses a different version of the $C^1$ Closing Lemma rather than Lemma 3.6.

Clearly, Lemma 3.4 is a consequence of the following local lemma.

LEMMA A.1. *If $X \in \mathcal{KS}^1(M)$ and $T > 0$, then there is a neighborhood $\mathcal{U}_{X,T}$ of $X$ and a residual subset $\mathcal{R}_{X,T}$ of $\mathcal{U}_{X,T}$ such that if $Y \in \mathcal{R}_{X,T}$ and $p \in Crit_T(Y)$, then $Cl(W^u_Y(p))$ is Lyapunov stable for $Y$ and $Cl(W^s_Y(p))$ is Lyapunov stable for $-Y$.*

**Proof:** Recall that $Crit(Y) = \{p_1(Y), \ldots, p_k(Y)\}$ for every $Y$ in some neighborhood $\mathcal{U}_{X,T}$ of $X$, where $p_i(Y)$, $1 \le i \le k$, is either a periodic orbit or a singularity of $Y$.

For any $i \in \{1, \cdots, k\}$ we define $\Phi_i : \mathcal{U}_{X,T} \to 2^M_c$ by

$$\Phi_i(Y) = Cl(W^u_Y(p_i(Y)).$$

By the continuous dependence of unstable manifolds we have that $\Phi_i$ is a lower semi-continuous map, and so, $\Phi_i$ is also upper semi-continuous for every vector field in some residual subset $\mathcal{R}_i$ of $\mathcal{U}_{X,T}$. Set $\mathcal{R}_{X,T} = \mathcal{KS}^1(M) \cap (\cap_i \mathcal{R}_i)$. Then $\mathcal{R}_{X,T}$ is residual in $\mathcal{U}_{X,T}$. Let us prove that $\mathcal{R}_{X,T}$ satisfies the conclusion of the lemma.

Let $\sigma \in Crit_T(Y)$ for some $Y \in \mathcal{R}_{X,T}$. Then, $\sigma = p_i(Y)$ for some $i$, and so, $\Phi_i(Y) = Cl(W^u_Y(\sigma))$.

Suppose by contradiction that $Cl(W^u_Y(\sigma))$ is not Lyapunov stable for $Y$.

Then, there are an open set $U$ containing $Cl(W^u_Y(\sigma))$ and two sequences $x_n \to x \in Cl(W^u_Y(\sigma))$, $t_n \ge 0$ such that

$$Y_{t_n}(x_n) \notin U.$$

As in the proof of Lemma 3.5 we have either

(a) $x \notin Crit(Y)$ or

(b) $x \in Crit(Y)$.

Again it is enough to prove the lemma in case (a). Indeed, suppose that case (b) holds. As $Y$ is Kupka-Smale we have that $\mathcal{O}_Y(x)$ is hyperbolic. Clearly $\mathcal{O}_Y(x)$ is neither a sink or a source and so $W^s_Y(x) \setminus \mathcal{O}_Y(x) \ne \emptyset$ and $W^u_Y(x) \setminus \mathcal{O}_Y(x) \ne \emptyset$. Let $V \subset U$ be a small neighborhood of $x$ given by the Grobman-Hartman Theorem [**6**] such that $\partial(W^u_Y(x, V)) = D^u_Y(x)$ is a fundamental domain for $W^u_Y(x)$ (here $W^u_Y(x, V)$ denotes the connected component of $V \cap W^u_Y(x)$ containing $\mathcal{O}_X(x)$). Note that $D^u_Y(x) \subset W^u_Y(x) \setminus \mathcal{O}_Y(x)$. As $x_n \to x$, we can assume $x_n \in Int(V)$ for all $n$. As $Y_{t_n}(x_n) \notin U$, we have that $x_n \notin W^s_Y(\sigma)$. So, there is $s_n > 0$ such that $x'_n = Y_{s_n}(x_n) \in \partial V$ and $Y_s(x_n) \in Int(V)$ for $0 \le s < s_n$. Since $Y_{t_n}(x_n) \notin U$ we havee that $Y_{t_n}(x_n) \notin Cl(V)$ for all $n$. From this we conclude that $s_n < t_n$ for all $n$. On the other hand, as $x_n \to x$, passing to a subsequence if necessary, we can assume that $x'_n \to x'$ for some $x' \in D^u_Y(x) \subset W^u_Y(x) \setminus \mathcal{O}_Y(x)$.

Now we have the following claim.

Claim A.2. *$x' \in Cl(W^u_Y(\sigma))$.*

*Proof of the Claim:* As $x \in Cl(W^u_Y(\sigma))$, using the Connecting Lemma [**8**], there is $Z$ $C^1$ near $Y$ such that $W^u_Z(\sigma(Z)) \cap W^s_Z(x(Z)) \ne \emptyset$. In other words there is a saddle-connection between $\sigma(Z)$ and $x(Z)$. Breaking this saddle-connection as in the proof of Lemma 2.4 in [**6, p. 101**], using the Inclination Lemma [**6**], we can find $Z'$ $C^1$ close to $Z$ so that $W^u_{Z'}(\sigma(Z'))$ passes close to $x'$. This contradicts the upper-semicontinuity of $\Phi_i$ at $Y$. Thus, $x' \in Cl(W^u_Y(\sigma))$ and the Claim is proved. $\square$

Back to the proof of Lemma 3.4, as $x' \in D^u_Y(x)$ we have that $x' \notin Crit(Y)$. As $x'_n \to x'$ and $Y_{t_n - s_n}(x'_n) \notin U$ with $t_n - s_n > 0$, we conclude as in case (a) replacing $x$ by $x'$, $x_n$ by $x'_n$ and $t_n$ by $t_n - s_n$.

Now we prove the lemma in case (a).

As $Cl(W^u_Y(\sigma)) \subseteq U$ and $\Phi_i$ is upper semi-continuous, there is a $C^1$ neighborhood $\mathcal{U} \subset \mathcal{U}_{X,T}$ of $Y$ such that

$$Cl(W^u_Z(\sigma(Z))) \subseteq U, \tag{A 3}$$

for all $Z \in \mathcal{U}$.

Let $\rho, L, \epsilon_0$ as in Lemma 3.6 for $X = Y$, $x$, and $\mathcal{U}$ as above.

As $x \notin Crit(Y)$, $Y_{[-L,0]}(x) \cap \mathcal{O}_Y(\sigma) = \emptyset$.

As $x \in Cl(W^u_Y(\sigma))$, $Y_{[-L,0]}(x) \subset Cl(W^u_Y(\sigma))$ and so $Y_{[-L,0]}(x) \subset U$. Then, there is $0 < \epsilon \leq \epsilon_0$ such that $B_\epsilon(Y_{[-L,0]}(x)) \cap \mathcal{O}_Y(\sigma) = \emptyset$ and $B_\epsilon(Y_{[-L,0]}(x)) \subseteq U$.

Choose an open set $V$ containing $\mathcal{O}_Y(\sigma)$, $V \subset Cl(V) \subset U$, such that $V \cap B_\epsilon(Y_{[-L,0]}(x)) = \emptyset$.

For $n$ large, we have $x_n \in B_{\epsilon/\rho}(x)$ and we set $q = Y_{t_n}(x_n) \notin U$.

As $x_n \to x$ and $t_n > 0$ we have $\mathcal{O}^-_Y(q) \cap B_{\epsilon/\rho}(x) \neq \emptyset$.

As $x \in Cl(W^u_Y(\sigma))$, there is $p \in (W^u_Y(\sigma) \setminus \{\sigma\}) \cap V$ such that $\mathcal{O}^+_Y(p) \cap B_{\epsilon/\rho}(x) \neq \emptyset$ and $\mathcal{O}^-_Y(p) \subseteq V$.

By construction $\epsilon, p, q$ satisfy (b) and (c) of Lemma 3.6.

As $V \cap B_\epsilon(Y_{[-L,0]}(x)) = \emptyset$ and $q \notin U$, we have that $\epsilon, p, q$ also satisfy (a) of Lemma 3.6.

Then, by Lemma 3.6, there is $Z \in \mathcal{U}$ such that $Z = Y$ off $B_\epsilon(Y_{[-L,0]}(x))$ and $q \in \mathcal{O}^+_Z(p)$.

As $V \cap B_\epsilon(Y_{[-L,0]}(x)) = \emptyset$ and $\mathcal{O}^-_Y(p) \subset V$ we have

$$\mathcal{O}^-_Y(p) \cap B_\epsilon(Y_{[-L,0]}(x)) = \emptyset. \tag{A 4}$$

Now, (A 4), together with $V \cap B_\epsilon(Y_{[-L,0]}(x)) = \emptyset$ and $Z = Y$ off $B_\epsilon(Y_{[-L,0]}(x))$ imply that $\sigma(Z) = \sigma$ and $p \in W^u_Z(\sigma)$. As $q \notin U$ and $q \in W^u_Z(\sigma)$ (recall $p \in W^u_Z(\sigma)$ and $q \in \mathcal{O}^+_Z(p)$), we have a contradiction by (A 3). This finishes the proof. $\square$

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