A higher order nonlinear Schrödinger equation with variable coefficients

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Abstract. We study the initial value problem (IVP) associated to a higher order nonlinear Schrödinger equation with variable coefficients. Under some regularity on its coefficients we establish local well-posedness for the IVP for data in $H^s(\mathbb{R})$, $s \geq 1/4$, improving previous results [22]. The main ingredient in our proof is an estimate of the maximal function associated to the linear solution similar to the sharp one obtained for linear solutions of the Schrödinger and Korteweg-de Vries equations.

1. Introduction

In this paper we will consider the following initial value problem (IVP),

$$\begin{cases} \partial_t u + i\alpha(t)\partial_x^2 u + \beta(t)\partial_x^3 u + i\gamma|u|^2 u + \delta|u|^2\partial_x u + \epsilon u^2 \partial_x \overline{u} = 0, \quad x, t \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

Here $u = u(x,t)$ is a complex valued function, $\alpha(t), \beta(t)$ are real functions defined in an interval $[-T_0, T_0]$, $\beta(t) \neq 0$ for all $t \in [-T_0, T_0]$ for some $T_0 > 0$, and γ, δ, ϵ are complex constants.

The equation above is a particular case of the most general one

$$i\partial_t u + a_3\,\partial_x^3 u + a_2(t)\partial_x^2 u = u(g * |u|^2) + i\partial_x(u(g * |u|^2)), \tag{1.2}$$

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This research was partially supported by CNP-q, Brazil.

which models the nonlinear propagation of pulses through optical fibers [23]. If we make a_2 constant and g the Dirac measure we obtain the equation introduced in ([6], [19]) by A. Hasegawa and Y. Kodama to study the propagation of pulses taking into account higher dispersion effects than those given by the Schrödinger equation.

We also notice than the equation in (1.1) reduces to well known dispersive models. For instance, by setting $\alpha = 1$, $\beta = \epsilon = \delta = 0$ in (1.1) we have the semilinear Schrödinger equation, i.e.,

$$\partial_t u + i\partial_x^2 u + i\gamma|u|^2 u = 0. \tag{1.3}$$

If we let $\beta = \gamma = 0$ and $\alpha = 1$ in (1.1) we obtain the derivative nonlinear Schrödinger equation

$$\partial_t u + i\partial_x^2 u + \delta|u|^2\partial_x u + \epsilon u^2\partial_x \overline{u} = 0. \tag{1.4}$$

Letting $\alpha = \gamma = \epsilon = 0$ in (1.1), the equation that arises is the complex modified Korteweg-de Vries (mKdV) equation,

$$\partial_t u + \beta\,\partial_x^3 u + \delta|u|^2\partial_x u = 0. \tag{1.5}$$

The initial value problem for the equations (1.3), (1.4) and (1.5) has been extensively studied in recent years, see for instance [1], [3], [5], [7], [8], [9], [10], [11], [12], [16], [13], [25], [24], [33], [30] and references therein.

Regarding the initial value problem (1.1), Laurey [22] recently showed that for data u_0 in $H^s(\mathbb{R})$, $s > 3/4$, and functions $\alpha(t)$ and $\beta(t)$ in $C^1([-T_0, T_0])$, $\beta \neq 0$, $T_0 > 0$, the IVP is locally well posed. Previously, Laurey in ([20], [21]) considered the equation in (1.1) with constant coefficients and proved local well-posedness of the IVP associated for data in $H^s(\mathbb{R})$, $s > 3/4$, and global well-posedness in $H^s(\mathbb{R})$, $s \geq 1$. In [26] Staffilani established local well-posedness for data in $H^s(\mathbb{R})$, $s \geq 1/4$, for the IVP with constant coefficients improving Laurey's result.

The aim of this paper is to show local well-posedness of the IVP (1.1) for data in $H^s(\mathbb{R})$, $s \geq 1/4$. This is done under some conditions on $\alpha(t)$ and $\beta(t)$. Our result is similar to the one in [26] for constants coefficients and it also improves the results in [22]. As we can see from the equation the extra difficulty we have in this case, besides the nonlinear terms and the loss of derivatives, is the fact that $\alpha(t)$ and $\beta(t)$ are variable however we will see that we can recover the same result as for the constant coefficients equation.

The main idea of the proof is to establish smoothing effects and maximal function estimates for solutions of the linear problem and then combine this with a fixed point argument to obtained the desired results. This idea was introduced by Kenig, Ponce and Vega to study the generalized Korteweg-de Vries equation

$$\partial_t u + \partial_x^3 u + u^p\partial_x u = 0, \quad p \in \mathbb{N} \tag{1.6}$$

(see [17]) and other equations, as well.

Looking carefully at the equation in (1.1) we can notice that its structure is similar to the complex mKdV equation. The leading order dispersion term in (1.1) is the one with three derivatives, so asking for $\beta \neq 0$ will guarantee that property for the time where this function is defined. On the other hand, the most difficult nonlinear term to deal with in (1.1) is the one similar to the mKdV equation. From this discussion we might expect to obtain a similar result for the IVP (1.1) as for the complex mKdV equation. In [17] Kenig, Ponce and Vega showed that the IVP associated to the real modified KdV equation is locally well-posed in $H^s(\mathbb{R})$, $s \geq 1/4$. The same proof applies to the complex case. In addition this result is sharp, see [17].

We shall observe that the best local well posedness result obtained for the IVP associated to the equation (1.3) is for data in $H^s(\mathbb{R})$, $s \geq 0$, (see [33]) which is also sharp (see [17]). On the other hand, the IVP associated for the derivative Schrödinger equation (1.4) is known to be locally well-posed for data in $H^s(\mathbb{R})$, $s \geq 1/2$. This result is also sharp (see [30], [1]).

Hence we will try to implement an approach similar to the one used to establish local well-posedness for the modified KdV equation. Therefore we consider the linear problem

$$\begin{cases} \partial_t u + i\alpha(t)\partial_x^2 u + \beta(t)\partial_x^3 u = 0, \quad x, t \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases} \tag{1.7}$$

whose solution is given by the evolution group $U(t)$ defined as

$$U(t)u_0(x) = \int_{\mathbb{R}} e^{i\phi(\xi,t)+ix\xi}\, \widehat{u}_0(\xi)\, d\xi \tag{1.8}$$

where

$$\phi(t,\xi) = a(t)\xi^2 + b(t)\xi^3$$

with

$$a(t) := \int_0^t \alpha(s)\, ds \quad \text{and} \quad b(t) := \int_0^t \beta(s)\, ds.$$

The main ingredients in our proof are the Kato smoothing effect ([11])

$$\|\partial_x U(t)u_0\|_{L_x^\infty L_T^2} \leq c\,(1+T)^{1/2}\|u_0\|_{L^2}, \tag{1.9}$$

the maximal function estimate ([18])

$$\|U(t)u_0\|_{L_x^4 L_T^\infty} \leq c\,(1+T)^{1/2}\|D^{1/4}u_0\|_{L^2}, \tag{1.10}$$

and the Strichartz estimate ([29])

$$\|U(t)u_0\|_{L_x^8 L_t^8} \leq c\,\|u_0\|_{L^2}. \tag{1.11}$$

Here appears the basic difference in our approach and Laurey's. We establish a maximal function estimate where less regularity to handle the maximal operator is required than that for the maximal function estimate

$$\|U(t)u_0\|_{L_x^2 L_T^\infty} \leq c(1+T)^{1/2}\|u_0\|_s, \tag{1.12}$$

where $s > 3/4$ and that was established in [22]. The maximal function estimate (1.10) is similar to the sharp one established by Kenig and Ruiz [18] and used in [16] and [26] to show local well-posedness for the modified Korteweg-de Vries equation and a higher order Schrödinger equation respectively.

Laurey [22] showed (1.9), using the Theorem T1 of David and Journé ([4]) under the hypotheses α, β in $C^1([-T_0,T_0])$. If we require a little more regularity in the coefficients, says $\alpha\in C^2([-T_0,T_0])$ and $\beta\in C^3([-T_0,T_0])$ we can obtain the same estimate without appealing to Theorem T1. See Theorem 2.1 below.

The Strichartz estimate (1.11) is useful to deal with the nonlinear term $|u|^2u$. It seems that the linear estimates used in [26] are not sufficient to control this term.

We also note that the approach used in [22] is similar to the one employed by Kenig, Ponce and Vega in [14] to establish well-posedness for the IVP associated to the Korteweg-de Vries equation. No fixed point argument was applied there.

Assuming $\alpha(t)\in C^2([-T_0,T_0])$ and $\beta(t)$ in $C^3([-T_0,T_0])$, $\beta(t)\neq 0$ we can give the statement of the main theorem.

Theorem 1.1. *Let $u_0\in H^s(\mathbb{R})$, $s\geq 1/4$, $\alpha(t)$ and $\beta(t)$ as above, then there exist $T=T(\|u_0\|_{H^{1/4}})>0$, $T<T_0$, and a unique solution of the IVP* (1.1) *such that*

$$u\in C([-T,T];H^s(\mathbb{R})), \tag{1.13}$$

$$\|\partial_x u\|_{L^\infty_x L^2_T}+\|D^s_x\partial_x u\|_{L^\infty_x L^2_T} < \infty, \tag{1.14}$$

$$\|D_x^{s-1/4}\partial_x u\|_{L^{20}_x L^{5/2}_T} < \infty, \tag{1.15}$$

$$\|u\|_{L^5_x L^{10}_T}+\|D^s_x u\|_{L^5_x L^{10}_T} < \infty, \tag{1.16}$$

$$\|u\|_{L^4_x L^\infty_T} < \infty, \tag{1.17}$$

$$\|u\|_{L^8_x L^8_T}+\|D^s_x u\|_{L^8_x L^8_T} < \infty. \tag{1.18}$$

Moreover for any $T'\in[-T,T]$ there exists a neighborhood $\mathcal{V}$ of $u_0\in H^s(\mathbb{R})$, $s\geq 1/4$, such that the map $\tilde u_0\mapsto \tilde u(t)$, from $\mathcal{V}$ into the class defined by (1.13)-(1.18), with T' instead of T is smooth.

Remark 1.2. *In the maximal function estimate* (1.10) *and the Strichartz estimate* (1.11) *we only require $\beta(t)\neq 0$ and $\alpha(t)$ in $C([-T_0,T_0])$.*

Remark 1.3. *Assuming $\alpha(t)\in C^2([-T_0,T_0])$ and $\beta(t)\neq 0$ in $C^3([-T_0,T_0])$ avoid the use of Theorem T1, otherwise we use* [22] *to obtain a similar estimate as in Theorem 1.1 with C^1 coefficients.*

Remark 1.4. *In an upcoming paper we will use the existence of solitary wave solutions to give an example proving that in the case of constant coefficients the local result is sharp. We do not know whether or not that example can be used to establish a similar result for the IVP* (1.1).

This paper is organized as follows: In section 2, we will prove the smoothing effect associated to the linear problem. The maximal function estimate will be established in section 3. We will establish the estimate (1.11) and some interpolated estimates in section 4. Finally, section 5 will be devoted to the proof of Theorem 1.1.

Before leaving this section we introduce some notation:

- The notation $\|f\|_{L^p_x L^q_T}$ means
$$\|f\|_{L^p_x L^q_T} = \Big(\int_{\mathbb{R}} \Big(\int_0^T |f(x,t)|^q \, dt \Big)^{p/q} dx \Big)^{1/p}. \tag{1.19}$$
If we integrate with respect to t on the whole line we use the notation $\|f\|_{L^p_x L^q_t}$. If the integration in (1.19) is interchanged we will write $\|f\|_{L^q_T L^p_x}$.
- For f we denote its Fourier transform as
$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix\xi} \, dx.$$
In most parts of this work we will omit the constant $1/\sqrt{2\pi}$.
- We denote the Riesz operators of order s by D^s_x which are defined via the Fourier transform as $D^s_x f = (|\xi|^s \widehat{f}(\xi))^\vee$.
- The notation $a \lesssim b$ means there exists a constant c such that $a \leq cb$.

2. Local Smoothing Effect

In this section we will prove the Kato smoothing effect presented by solutions of the linear problem (1.7). As we mentioned in the introduction we will require more regularity on the coefficients $\alpha(t)$ and $\beta(t)$ to give a simpler proof of this estimate. This will avoid the use of the theorem $T1$ of David and Journé ([4]) applied in the proof given in [22].

We suppose $\alpha(t) \in C^2([-T_0, T_0])$ and $\beta(t) \in C^3([-T_0, T_0])$, with $\beta(t) \neq 0$. Under these conditions we have.

Theorem 2.1. *Let $U(t)$ be the evolution operator defined in (1.8), and $u_0 \in \mathcal{S}(\mathbb{R})$. If $T \in (0, T_0/2)$, then*
$$\|\partial_x U(t) u_0(x)\|_{L^\infty_x L^2_T} \leq c(1+T)^{1/2} \|u_0\|_{L^2}, \tag{2.1}$$
where c is a constant independent of T and u_0.

Proof. Suppose $T_0 < 1$. Let $M > 0$, such that $|a^{(j)}(t)| \leq M$, $|b^{(j+1)}(t)| \leq M$, $j = 0, \dots, 3$ and $\varphi \in C^\infty_0([-T_0, T_0])$ with $\varphi = 1$ in $[-T, T]$, $0 \leq \varphi \leq 1$, $|\varphi^{(j)}(t)| \leq M$, $j = 1, \dots, 3$, and *supp* $\varphi = [-T_0/2 - T, T_0/2 + T]$. It is clear then that
$$\|\partial_x U(t) u_0(x)\|_{L^2_T} \leq \|\varphi(t) \partial_x U(t) u_0(x)\|_{L^2_t}.$$
Therefore if we show that
$$\sup_x \|\varphi(t) \partial_x U(t) u_0(x)\|_{L^2_t} \leq c(1+T)^{1/2} \|u_0\|_{L^2},$$

where $c = c(a, M)$, $a = a(T_0, \beta)$, $M = M(T_0, \alpha, \beta, \varphi)$, the estimate (2.1) will follow.

Using duality we have

$$\|\varphi(t)\partial_x U(t)u_0(x)\|_{L^2_t} = \sup_g \Big\{ \Big| \int_{\mathbb{R}} \varphi(t)\partial_x U(t)u_0(x)\overline{g(t)}dt \Big| : \|g\|_{L^2_t} \leq 1 \Big\}.$$

Using (1.8)

$$\partial_x U(t)u_0(x) = i \int_{\mathbb{R}} e^{i\phi(\xi,t)+ix\xi} \xi\, \widehat{u}_0(\xi)d\xi.$$

Applying the Fubini and Plancherel theorems it follows that

$$\begin{aligned} \Big| \int_{\mathbb{R}} \varphi(t)\partial_x U(t)u_0(x)\overline{g(t)}dt \Big| =& \Big| \int_{\mathbb{R}} u_0(\eta)\, \widehat{H}(\eta - x)\, d\eta \Big| \\ \leq& \|u_0\|_{L^2} \cdot \|H\|_{L^2}, \end{aligned}$$

where

$$H(\xi) = \xi \int_{\mathbb{R}} \varphi(t)\overline{g(t)} e^{i\phi(\xi,t)} dt.$$

Thus to prove (2.1) it is enough to show that

$$\|H\|_{L^2} \leq c(1+T)^{1/2}\|g\|_{L^2}.$$

Consider now the linear operator

$$Lg(\xi) = \xi \int_{\mathbb{R}} \varphi(t)g(t)e^{i\phi(\xi,t)}\, dt,$$

and its adjoint operator

$$L^* h(t) = \varphi(t) \int_{\mathbb{R}} e^{-i\phi(\xi,t)} \xi h(\xi) d\xi.$$

It follows then that

$$LL^* h(\xi) = \int_{\mathbb{R}} h(\eta)K(\xi,\eta)d\eta, \tag{2.2}$$

where

$$K(\xi,\eta) = \xi\eta \int_{\mathbb{R}} \varphi(t)^2\, e^{ib(t)(\xi^3-\eta^3)+ia(t)(\xi^2-\eta^2)}dt. \tag{2.3}$$

We finish the proof if we can show that LL^* is a bounded operator in L^2 with

$$\|LL^*\|_{L^2} \leq c\,(1+T).$$

To establish this inequality we need the following lemmas

Lemma 2.2 (Schur's lemma). *Let S be an integral operator, given by*

$$(Sf)(x) = \int_{\mathbb{R}} K(x,y)f(y)dy.$$

If the kernel K satisfies

$$\sup_x \int_{\mathbb{R}} |K(x,y)| dy \leq 1,$$
$$\sup_y \int_{\mathbb{R}} |K(x,y)| dx \leq 1.$$

Then $\|S\|_{L^2 \to L^2} \leq 1$.

Proof. See lemma in page 284 [28]. □

Lemma 2.3. *Let $K = K(\xi, \eta)$ be the kernel given by* (2.3)*, and φ as in Theorem 2.1. Then we have*

$$\sup_\eta \int_{\mathbb{R}} |K(\xi,\eta)|\, d\xi \leq c\,(1+T),$$

and

$$\sup_\xi \int_{\mathbb{R}} |K(\xi,\eta)|\, d\eta \leq c\,(1+T).$$

Proof. Let

$$J(\xi,\eta) := \int_{\mathbb{R}} \varphi(t)^2 e^{ib(t)(\xi^3-\eta^3)+ia(t)(\xi^2-\eta^2)}\, dt,$$

and define $k_3 := \xi^3 - \eta^3$ and $k_2 := \xi^2 - \eta^2$.

Since $\varphi(t)$ has compact support and $|b'(t)| = |\beta(t)| \geq \rho > 0$ an obvious integration by parts gives

$$\begin{aligned} J(\xi,\eta) &= \int_{\mathbb{R}} \varphi(t)^2 e^{ia(t)k_2} e^{ib(t)k_3}\, dt \\ &= \int_{\mathbb{R}} \psi(t,k_2) e^{ib(t)k_3}\, dt \\ &= -\int_{\mathbb{R}} \frac{d}{dt}\Big(\frac{\psi(t,k_2)}{ik_3 b'(t)}\Big) e^{ib(t)k_3}\, dt. \end{aligned} \tag{2.4}$$

Let M be a common bound for φ, a, b and its derivatives

$$|\varphi^{(j)}(t)| \leq M, \quad |a^{(j)}(t)| \leq M, \quad |b^{(j+1)}(t)| \leq M$$

for all $t \in [-1-T, 1+T]$, $j = 0, \cdots, 3$. Performing the derivative in (2.4) we obtain the estimate

$$|J(\xi,\eta)| \leq \frac{2(1+T)M}{\rho|k_3|}\Big(2 + \frac{M}{\rho} + M|k_2|\Big). \tag{2.5}$$

Integrating by parts 3 times (2.4) and estimating as in (2.5) we have

$$|J(\xi,\eta)| \leq \frac{1+T}{|k_3|^3}\Big(\sum_{j=0}^{3} e_j |k_2|^j\Big),$$

with e_j depending on M and a.

The lemma will follow if we show that

$$\sup_{\xi} I_j(\xi) < \infty, \quad j = 0, 1, \dots, 3. \tag{2.6}$$

where

$$I_j(\xi) := \int_{\mathbb{R}} \frac{|\xi\eta|\,|\xi^2 - \eta^2|^j}{1 + |\xi^3 - \eta^3|^3}\, d\eta, \quad j = 0, \dots, 3. \tag{2.7}$$

For $j = 0, \dots, 3$ we write

$$\begin{aligned} I_j &= \int_{|\xi-\eta|\le 1} \frac{|\xi\eta|\,|\xi^2 - \eta^2|^j}{1 + |\xi^3 - \eta^3|^3}\, d\eta + \int_{|\xi-\eta|>1} \frac{|\xi\eta|\,|\xi^2 - \eta^2|^j}{1 + |\xi^3 - \eta^3|^3}\, d\eta \\ &= I_j^1 + I_j^2. \end{aligned}$$

First we consider I_j^1,

If $|\xi| \le 1$ we have $I_j^1 \le 10$.

If $|\xi| > 1$ we perform the change of variables $v = \eta - \xi$, then

$$I_j^1 = \int_{|v|\le 1} \frac{|\xi|\,|v+\xi|\,|v+2\xi|^j\,|v|^j}{1 + |v|^3[(v + \frac{3}{2}\xi)^2 + \frac{3}{4}\xi^2]^3} dv. \tag{2.8}$$

Since $(v + \frac{3}{2}\xi)^2 + \frac{3}{4}\xi^2 \ge \xi^2$, the change of variables $x = v\xi^2$ implies

$$I_j^1 = \frac{1}{|\xi|^j} \int_{|x|\le \xi^2} \frac{|x/\xi^3 + 1|\,|x/\xi^3 + 2|^j\,|x|^j}{1 + |x|^3} dx. \tag{2.9}$$

On the other hand, for $|x| \le \xi^2$ and $|\xi| > 1$ we have that

$$\frac{|x|}{|\xi^3|} \le 1, \quad \text{and} \quad \frac{|x|^3}{|\xi|^3} \le \frac{|x|^2}{|\xi|}.$$

It follows from (2.9) that $I_j^1 \lesssim 1$ for $j = 0, \cdots, 3$.

Next we consider I_j^2.

If $|\xi| \le 1$, using the change of variables $v = \eta - \xi$ and the inequality $|v + \frac{3}{2}\xi| \le \frac{5}{2}\,|v|$ we obtain

$$I_j^2 \lesssim \int_{\mathbb{R}} \frac{1 + |v|^{2j+1}}{1 + (\frac{2}{5})^3 (v + \frac{3}{2}\,\xi)^9} dv \lesssim C$$

for $j = 0, \cdots, 3$.

If $|\xi| > 1$, the change of variables $\eta = \xi x$ yields

$$I_j^2 \le \frac{1}{|\xi|^{3-j}} \int_{\mathbb{R}} \frac{|x|\,|1 + x|^j}{[(x + 1/2)^2 + 3/4]^3}\, dx \lesssim C.$$

for $j = 0, \cdots, 3$. The estimate (2.6) then follows. $\square$

Going back to the proof of Theorem 1.1. The result in Lemma 2.3 allows us to apply Lemma 2.2. Thus recalling the definition in (2.2) it follows then that

$$\|LL^*\|_{L^2} \leq c\,(1+T).$$

This finishes the proof. □

3. Maximal Function Estimate

In this section our objective is to establish an estimate of the maximal function associated to the operator $U(t)$ defined in (1.8). This estimate is the key ingredient to obtain our local well-posedness result. This estimate is reminiscent of the sharp one obtained by Kenig and Ruiz [18] for the linear Schrödinger equation and then extended by Kenig, Ponce and Vega for the linear Korteweg-de Vries equation [14].

We begin by listing some technical lemmas useful in the proof of the main result in this section.

Lemma 3.1 (Van-der Corput). *Suppose φ is a real-valued and smooth function in $[a,b]$ and that $|\varphi^{(k)}(x)| \geq 1$ for all $x \in [a,b]$. Then*

$$\Big|\int_a^b \exp\{i\lambda\varphi(x)\}\,dx\Big| \leq c(k)\,\lambda^{-1/k}$$

holds when

i) $k \geq 2$, *or*

ii) $k = 1$ *and* $\varphi'(x)$ *is monotonic.*

The bound $c(k)$ is independent of φ, λ, a and b.

Proof. See Proposition 2, page 332 in [28]. □

Lemma 3.2. *Let f be a monotonic function and let $I = \int_a^b e^{i\phi(x)} f(x)dx$.*

i) *If* $\left|\frac{d\phi}{dx}\right| > \lambda > 0$ *in* $[a,b]$, $\frac{d\phi}{dx}$ *monotonic, then* $|I| \lesssim \frac{1}{\lambda}\max\limits_{[a,b]}|f(x)|$.

ii) *If* $\left|\frac{d^k\phi}{dx^k}\right| > \lambda > 0$, $k \geq 2$, *then* $|I| \lesssim \frac{1}{\lambda^{1/k}}\max\limits_{[a,b]}|f(x)|$.

Proof. The proof is a direct application of the Van-der Corput lemma. See Lemma 1 pag. 243 in [18]. □

Lemma 3.3. *Let $\ell > 0$, $\varphi(x) = b\,x^3 + a\,x^2 + x\eta$ and $I^{a,b}(\eta) = \int_{\mathbb{R}} e^{i\varphi(x)}\,\frac{1}{|x|^{1/2}}\,dx$. Then*

$$|I^{a,b}(\eta)| \leq \frac{c\,(1+\ell)}{|\eta|^{1/2}}, \tag{3.1}$$

for all $a, b \in \mathbb{R}$ such that $|ab^{-2/3}| \leq \ell$, where c is a constant independent of a, b and ℓ.

Proof. When $b = 0$ the inequality (3.1) was established by Kenig and Ruiz in [18]. For $a = 0$, Kenig, Ponce and Vega showed a similar result (see [16]).

We will assume $\ell \geq 1$. The case $\ell < 1$ follows as a consequence of the latter. Next we suppose $a, b \neq 0$ and $|ab^{-2/3}| \lesssim \ell$.

First we show (3.1) when $b = 1$, that is,

$$|I^{a,1}(\eta)| \leq \frac{c\,(1+\ell)}{|\eta|^{1/2}},$$

for every $a \in \mathbb{R}$ such that $|a| \leq \ell$.

We estimate $|I^{a,1}(\eta)|$ as

$$|I^{a,1}(\eta)| \leq \Big|\int_{|x|\leq 1} e^{i(x^3+ax^2+x\eta)}\,\frac{1}{|x|^{1/2}}\,dx\Big| + \Big|\int_{|x|\geq 1} e^{i(x^3+ax^2+x\eta)}\frac{1}{|x|^{1/2}}\,dx\Big| := I_1 + I_2.$$

Notice that in I_2 we have $|x|^{-1/2} \leq 1$. Therefore the Lemma 3.2 with $k = 3$ implies that

$$I_2 \lesssim 1,$$

and

$$I_1 \leq \int_{|x|\leq 1} \frac{1}{|x|^{1/2}}\,dx \lesssim 1.$$

Thus

$$|I^{a,1}(\eta)| \leq c_0, \quad c_0 \sim 1.$$

Hence, we shall assume that $|\eta| > 4\ell^2 \geq 4a^2$, since $|\eta| \leq 4\ell^2$ implies

$$|I^{a,1}(\eta)| \leq\; c_0 \leq \frac{2\ell\, c_0}{|\eta|^{1/2}}. \tag{3.2}$$

From now on we assume

$$|\eta|^{1/2} > 2|a|. \tag{3.3}$$

We split $I^{a,1}$ in three pieces, i.e.

$$\begin{aligned}|I^{a,1}| \leq \Big|\int\limits_{|x|\leq\frac{1}{|\eta|}} \frac{e^{i\varphi(x)}}{|x|^{1/2}}\,dx\Big| + \Big|\int\limits_{\frac{1}{|\eta|}\leq|x|\leq\frac{1}{3}|\eta|^{1/2}} \frac{e^{i\varphi(x)}}{|x|^{1/2}}\,dx\Big| + \Big|\int\limits_{|x|>\frac{1}{3}|\eta|^{1/2}} \frac{e^{i\varphi(x)}}{|x|^{1/2}}\,dx\Big| \\ := I_1 + I_2 + I_3.\end{aligned} \tag{3.4}$$

We first estimate I_1

$$I_1 \leq \int\limits_{|x|\leq\frac{1}{|\eta|}} \frac{dx}{|x|^{1/2}} = \frac{4}{|\eta|^{1/2}}.$$

The term I_2 can be estimated as follows. We first observe that where I_2 is defined the inequalities $|\eta|^{-1} \leq |x| \leq \frac{1}{3}|\eta|^{1/2}$ hold. Then using (3.3), we have $|3x^2 + 2ax| \leq \frac{2}{3}|\eta|$, and so

$$|\varphi'(x)| = |3x^2 + 2ax + \eta| \geq |\eta| - |3x^2 + 2ax| \geq \frac{1}{3}|\eta|.$$

From Lemma 3.2, with $f(x) = |x|^{-1/2} \leq |\eta|^{1/2}$ it follows that

$$I_2 \leq \frac{c}{|\eta|^{1/2}}.$$

Finally, we estimate I_3. Notice that in the domain of I_3 we have $|x|^{-1/2} \leq \sqrt{3}|\eta|^{-1/4}$. Since the inequality (3.3) also satisfies we obtain

$$|\varphi''(x)| = |6x + 2a| \geq 6|x| - 2|a| \geq |\eta|^{1/2}. \tag{3.5}$$

Again an application of Lemma 3.2, with $f(x) = |x|^{-1/2} \leq \sqrt{3}|\eta|^{-1/4}$, yields

$$I_3 \leq \frac{c\sqrt{3}}{|\eta|^{1/4}} \frac{1}{|\eta|^{1/4}} = \frac{c}{|\eta|^{1/2}}. \tag{3.6}$$

The case $b \neq 1$ can be reduced to $b = 1$ performing the change of variable $y = b^{1/3}x$. This completes the proof of the lemma. □

Now we can prove the main ingredient needed to prove our main result.

Theorem 3.4. *Let $U(t)$ be the evolution group defined in (1.8). If $T \in (0, T_0/2)$, then*

$$\|U(t)u_0\|_{L^4_x L^\infty_T} \leq c\,(1+T)^{1/2}\|u_0\|_{H^{1/4}}, \tag{3.7}$$

where c is a constant independent of T and u_0.

Proof. Let $T \in (0, T_0/2)$, and $\varphi \in C_0^\infty([-T_0, T_0])$, $\varphi(t) = 1$ in $[-T, T]$, $0 \leq \varphi \leq 1$ with $supp\,\varphi = [-\min\{1, T_0/2\} - T, \min\{1, T_0/2\} + T]$. Consider the operator

$$L(t)u_0(x) = \int_{\mathbb{R}} e^{ib(t)\xi^3 + ia(t)\xi^2 + ix\xi} \frac{1}{|\xi|^{1/4}} \widehat{u}_0(\xi)\, d\xi.$$

Oberve that

$$L(t)u_0(x) = D_x^{-1/4}\, U(t)u_0(x).$$

Therefore we will be done if we show the estimate

$$\|\varphi(t)L(t)u_0\|_{L^4_x L^\infty_t} \leq c\,(1+T)^{1/2}\|u_0\|_{L^2}, \tag{3.8}$$

where $c = c(a, M, T_0)$.

By a duality argument to prove (3.8) it is enough to show that

$$\Big\| \int_{\mathbb{R}} \varphi(t)\widetilde{L}(t)g(\cdot, t)\, dt \Big\|_{L^2_x} \leq c\,(1+T)^{1/2}\, \|g\|_{L^{4/3}_x L^1_t}, \tag{3.9}$$

where

$$\widetilde{L}(t)g(x,t)=\int_{\mathbb{R}} e^{-ib(t)\xi^3-ia(t)\xi^2+ix\xi}\frac{1}{|\xi|^{1/4}}\,\widehat{g}(\xi,t)\,d\xi.$$

Using the Tomas argument (see [32]) we obtain

$$\begin{aligned}\Big\|\int_{\mathbb{R}}\varphi(t)\widetilde{L}(t)g(\cdot,t)\,dt\Big\|^2_{L^2_x}&=\int_{\mathbb{R}^2}\overline{g(x,t)}\int_{\mathbb{R}}L(\tau,t)g(x,\tau)\,d\tau dt dx\\&\le\|g\|_{L^{4/3}_xL^1_t}\Big\|\int_{\mathbb{R}}L(\tau,t)g(\cdot,\tau)\,d\tau\Big\|_{L^4_xL^\infty_t},\end{aligned}\tag{3.10}$$

where

$$L(\tau,t)g(x,\tau)=\varphi(\tau)\varphi(t)\int_{\mathbb{R}}e^{i((b(t)-b(\tau))\xi^3+(a(t)-a(\tau))\xi^2+x\xi)}\,\frac{1}{|\xi|^{1/2}}\,\widehat{g}(\xi,\tau)\,d\xi.\tag{3.11}$$

Thus, to finish the proof it is enough to show

$$\Big\|\int_{\mathbb{R}}L(\tau,t)g(\cdot,\tau)\,d\tau\Big\|_{L^4_xL^\infty_t}\le c\,(1+T)^{1/2}\,\|g\|_{L^{4/3}_xL^1_t}.\tag{3.12}$$

Let

$$I_L^{t,\tau}(x)=\int_{\mathbb{R}}\varphi(\tau)\varphi(t)e^{i((b(t)-b(\tau))\xi^3+(a(t)-a(\tau))\xi^2+x\xi)}\,\frac{1}{|\xi|^{1/2}}\,d\xi.$$

Since $|b'(t)|=|\beta(t)|\ge\rho>0$, for all $t\in[-T_0,T_0]$ by the mean value theorem we have that

$$\frac{|a(t)-a(\tau)|}{|b(t)-b(\tau)|^{2/3}}\le\frac{M|t-\tau|}{\rho^{2/3}|t-\tau|^{2/3}}\lesssim\frac{M\,(1+T)^{1/3}}{\rho^{2/3}}$$

on the support of $\varphi(\tau)\varphi(t)$. Then the Fubini theorem and Lemma 3.3 yield

$$\begin{aligned}\Big|\int_{\mathbb{R}}L(\tau,t)g(x,\tau)\,d\tau\Big|&=\Big|\int_{\mathbb{R}}[g(\cdot,\tau)*I_L^{t,\tau}(\cdot)](\xi)\,d\tau\Big|\\&\le\;c\,(1+T)^{1/2}\int_{\mathbb{R}}\frac{\|g(\xi,\cdot)\|_{L^1_t}}{|x-\xi|^{1/2}}\,d\xi.\end{aligned}$$

Applying the Hardy-Littlewood-Sobolev theorem the desired estimate (3.12) follows. □

4. Some more linear estimates

In this section we first establish Strichartz estimates for solutions of the associated linear problem (1.7). Next we use the linear estimates proved so far to obtain linear estimates in interpolated spaces. We observe that the Strichartz estimates will allow to deal with the nonlinear term $|u|^2u$.

Assuming $\alpha(t)$ and $\beta(t)$ continuous and $\beta(t)\neq 0$ for all $t\in\mathbb{R}$. We will show that solutions of the linear problem (1.7) satisfy the next Strichartz' estimates.

Theorem 4.1. *Let* $u_0 \in L^2(\mathbb{R})$, $2 \le p \le \infty$, *and* q *such that* $\frac{2}{q} = \frac{1}{3} - \frac{2}{3p}$ *then*

$$\|U(t)u_0\|_{L^q_T L^p_x} \le c\,\|u_0\|_{L^2}, \tag{4.1}$$

and

$$\|\int_{\mathbb{R}} \widetilde{U}(\tau)g(\cdot,\tau)\,d\tau\|_{L^2} \le c\,\|g\|_{L^{q'}_T L^{p'}_x}. \tag{4.2}$$

where $U(t)$ *is defined as in* (1.8)

First we prove the following lemma.

Lemma 4.2. *Let* $u_0 \in \mathcal{S}$, $\phi(\xi) = b\xi^3 + a\xi^2$, $a, b \in \mathbb{R}$, $b \neq 0$ *and consider*

$$\mathcal{L}u_0(x) = \int_{\mathbb{R}} e^{i\phi(\xi)+ix\xi}\,\widehat{u}_0\,(\xi)\,d\xi = u_0 * I^{\phi}(x)$$

where

$$I^{\phi}(x) = \int_{\mathbb{R}} e^{i\phi(\xi)+ix\xi}d\xi. \tag{4.3}$$

Then for $0 \le \theta \le 1$, $\frac{1}{p} + \frac{1}{p'} = 1$ *and* $\frac{1}{p} = \frac{(1-\theta)}{2}$

$$\|\mathcal{L}u_0\|_{L^p} \le\; c\,|b|^{-\theta/3}\,\|u_0\|_{L^{p'}}, \tag{4.4}$$

where c *is a constant independent of* a *and* b.

Proof. To establish the estimate (4.4) we will use the Riesz-Thorin interpolation theorem.

The Plancherel theorem yields

$$\|\mathcal{L}u_0\|_{L^2} = \|u_0\|_{L^2}. \tag{4.5}$$

On the other hand, assuming that

$$|I^{\phi}(x)| \lesssim |b|^{-1/3}. \tag{4.6}$$

we will have

$$\|\mathcal{L}u_0\|_{L^\infty} \lesssim |b|^{-1/3}\,\|u_0\|_{L^1}. \tag{4.7}$$

Thus the inequality (4.4) follows using the Riesz-Thorin theorem and the estimates (4.7) and (4.5).

To finish the proof we will show (4.6). So setting $\xi = kz$, $k = \frac{a}{3b}$ in (4.3) we obtain

$$I^{\phi}(x) = k\int_{\mathbb{R}} e^{i(c_1\,z + c_2(z+1)^3 - c_2)}dz, \tag{4.8}$$

where $c_1 = kx - 3bk^3$ and $c_2 = k^3b$. Letting $c_2(z+1)^3 = \eta^3$ in (4.8), we have

$$\begin{aligned}|I^{\phi}(x)| &= |3b|^{-1/3}\Big|\int_{\mathbb{R}} e^{i(\eta c_1 (c_2)^{-1/3} + \eta^3)}\,d\eta\Big| \\ &\lesssim |b|^{-1/3}.\end{aligned}$$

This completes the proof of the lemma. $\square$

With the last result at hand we will prove Theorem 4.1. The argument of the proof follows similar ideas as the ones employed in the proof of Theorem 2.1.

Proof of Theorem 4.1. We first consider a function φ similar to the one used in the proof of Theorem 3.4.

Using duality it is enough to show that for any $g \in L_t^{q'} L_x^{p'}$,

$$\Big| \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(t) g(x,t)\, \overline{U(t)u_0(x)}\, dxdt \Big| \le c\, \|u_0\|_{L^2} \|g\|_{L_t^{q'} L_x^{p'}}.$$

Applying the Fubini theorem and the Cauchy-Schwarz inequality we obtain

$$\Big| \int_{\mathbb{R}^2} \varphi(t) g(x,t)\, \overline{U(t)u_0(x)}\, dxdt \Big| \le \|u_0\|_{L^2} \Big\| \int_{\mathbb{R}} \varphi(t)\, \widetilde{U}(t) g(\cdot,t) dt \Big\|_{L^2}.$$

Then to prove (4.1) it is sufficient to establish

$$\Big\| \int_{\mathbb{R}} \varphi(t)\, \widetilde{U}(t) g(\cdot,t)(x)\, dt \Big\|_{L_x^2} \le c \|g\|_{L_t^{q'} L_x^{p'}},$$

where

$$\widetilde{U}(t) g(x,t) = \int_{\mathbb{R}} e^{-ib(t)\xi^3 - ia(t)\xi^2 - ix\xi}\, \widehat{g}\,(\xi,t)\, d\xi.$$

We apply the Tomas argument (see [32]) to deduce that

$$\begin{aligned} \Big\| \int_{\mathbb{R}} \varphi(t) \widetilde{U}(t) g(\cdot,t)\, dt \Big\|_{L_x^2}^2 &= \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{g(x,t)} \int_{\mathbb{R}} U(\tau,t) g(x,\tau)\, d\tau dt dx \\ &\le \|g\|_{L_t^{q'} L_x^{p'}} \Big\| \int_{\mathbb{R}} U(\tau,t) g(\cdot,\tau)\, d\tau \Big\|_{L_t^q L_x^p}, \end{aligned}$$

where

$$U(\tau,t) g(x,\tau) = \varphi(\tau)\varphi(t) \int_{\mathbb{R}} e^{i((b(t)-b(\tau))\xi^3 + (a(t)-a(\tau))\xi^2 + ix\xi)}\, \widehat{g}\,(\xi,\tau)\, d\xi.$$

To finish the proof it is enough to show

$$\Big\| \int_{\mathbb{R}} U(\tau,t) g(\cdot,\tau)\, d\tau \Big\|_{L_t^q L_x^p} \le c\, \|g\|_{L_t^{q'} L_x^{p'}}.$$

For $t, \tau \in [-T_0, T_0]$, let

$$I^{t,\tau}(x) = \int_{\mathbb{R}} \varphi(\tau)\varphi(t)\, e^{i((b(t)-b(\tau))\xi^3 + (a(t)-a(\tau))\xi^2 + x\xi)}\, d\xi.$$

Since $|b'(t)| = |\beta(t)| \ge \rho > 0$, for all $t \in [-T_0, T_0]$ by the mean value theorem we have

$$|b(t) - b(\tau)| \ge \rho\, |t - \tau|.$$

Use of the Fubini theorem and Lemma 4.2 yields

$$
\begin{aligned}
\Big\| \int_{\mathbb{R}} U(\tau,t)g(\cdot,\tau)\,d\tau \Big\|_{L^q_t L^p_x} &\leq \Big\| \int_{\mathbb{R}} \|U(\tau,t)g(\cdot,\tau)\|_{L^p_x}\,d\tau \Big\|_{L^q_t} \\
&\leq \Big\| \int_{\mathbb{R}} \|\,[g(\cdot,\tau) * I^{t,\tau}(\cdot)\,]\|_{L^p_x}\,d\tau \Big\|_{L^q_t} \\
&\leq C \Big\| \int_{\mathbb{R}} \frac{\|g(\cdot,\tau)\|_{L^{p'}_x}}{|t-\tau|^{\theta/3}}\,d\tau \Big\|_{L^q_t} \\
&\leq C\|g\|_{L^{q'}_t L^{p'}_x}.
\end{aligned}
$$

where θ is as in Lemma 4.2 and in the last inequality we use the Hardy-Littlewood-Sobolev theorem. This completes the proof of the theorem. □

In particular, if $p = q = 8$ we have the following estimate for solutions of the linear problem (1.8).

Corollary 4.3. *If $u_0 \in L^2$, then*

$$
\|U(t)u_0\|_{L^8_t L^8_x} \leq c\,\|u_0\|_{L^2_x}. \tag{4.9}
$$

To complete the set of linear estimates we will use the estimates (2.1) and (3.7) to find some interpolated estimates.

Proposition 4.4. *If $u \in H^s$, $s \geq 1/4$, $v \in L^2$, $T \in (0, T_0/2)$ then*

$$
\|U(t)v\|_{L^5_x L^{10}_T} \leq C(1+T)^{1/2}\|v\|_{L^2}, \tag{4.10}
$$

$$
\|D_x U(t)u\|_{L^{20}_x L^{5/2}_T} \leq C(1+T)^{1/2}\|D_x^{1/4}\,u\|_{L^2}. \tag{4.11}
$$

Proof. We follow closely the argument in [14]. That is, we use the Stein interpolation theorem in [27] and estimates (2.1) and (3.7) to obtain (4.10). Similarly for the estimate (4.11). □

To finish this section we will give the main tool to deal with fractional derivatives of the nonlinear terms of the equation. That is the Leibniz rule for fractional derivatives.

Theorem 4.5. *Let $\alpha \in (0,1)$, α_1, $\alpha_2 \in [0,\alpha]$, $\alpha_1 + \alpha_2 = \alpha$. Let $p, p_1, p_2, q, q_1, q_2 \in (1,\infty)$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then*

$$
\|D_x^\alpha(fg) - fD_x^\alpha g - gD_x^\alpha f\|_{L^p_x L^q_T} \leq c\|D_x^{\alpha_1} f\|_{L^{p_1}_x L^{q_1}_T}\,\|D_x^{\alpha_2} g\|_{L^{p_2}_x L^{q_2}_T}. \tag{4.12}
$$

Moreover, for $\alpha_1 = 0$ the value $q_1 = \infty$ is allowed.

Proof. See Theorem $A.8$ in [16]. □

5. Proof of Theorem 1.1

Without loss of generality we will set $\gamma = \delta = \epsilon = 1$ in (1.1).

We denote by $u = \Phi(v) = \Phi_{u_0}(v)$ the solution of the IVP

$$\begin{cases} \partial_t u + i\alpha(t)\partial_x^2 u + \beta(t)\partial_x^3 u + F(v) = 0, \quad x, t \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases} \tag{5.1}$$

where

$$F(v) = i|v|^2 v + |v|^2 \partial_x v + v^2 \partial_x \bar{v} \tag{5.2}$$

is the nonlinear part of (1.1).

We then consider the equivalent integral form of (5.1)

$$u(t) = \Phi(v)(t) = \Phi_{u_0}(v)(t) = U(t)u_0 - \int_0^t U(\tau, t)F(v(\tau))d\tau, \tag{5.3}$$

where $U(t)$ is the evolution group defined in (1.8) and F is as above.

To prove local existence and uniqueness of the IVP (1.1) we will use the contraction mapping principle. We will use the operator defined in (5.3) in a suitable space. We will fix $s = 1/4$, the general case will follow since the derivatives in the considered norms grow linearly.

First we define the norms

$$\begin{aligned} \nu_1^T(u) &= \|u\|_{L_T^\infty L_x^2} + \|D_x^{1/4} u\|_{L_T^\infty L_x^2}, \\ \nu_2^T(u) &= \|\partial_x u\|_{L_x^\infty L_T^2} + \|D_x^{1/4}\partial_x u\|_{L_x^\infty L_T^2}, \\ \nu_3^T(u) &= \|D_x u\|_{L_x^{20} L_T^{5/2}}, \\ \nu_4^T(u) &= \|u\|_{L_x^5 L_T^{10}} + \|D_x^{1/4} u\|_{L_x^5 L_T^{10}}, \\ \nu_5^T(u) &= \|u\|_{L_x^4 L_T^\infty}, \\ \nu_6^T(u) &= \|u\|_{L_x^8 L_T^8} + \|D_x^{1/4} u\|_{L_x^8 L_T^8}, \end{aligned}$$

the set

$$Z_T = \{w \in C([-T,T]; H^{1/4}(\mathbb{R})) / \; |||w|||_T < \infty\}$$

and

$$Z_{a,T} = \{w \in C([-T,T]; H^{1/4}(\mathbb{R})) / \; |||w|||_T \le a\}, \tag{5.4}$$

where $|||w|||_T = \max\limits_{i=1,\dots,6} \nu_i^T(w)$.

Next we will show that there exist $a, T > 0$ such that $\Phi : Z_{a,T} \to Z_{a,T}$ with Φ a contraction. Thus the contraction mapping principle will guarantee the existence of a unique solution of (5.1) in $Z_{a,T}$. The uniqueness will allow us to extend the solution to the complete set Z_T and therefore we will obtain the result advertised in Theorem 1.1.

Using (2.1), (3.7), (4.9), (4.10) and (4.11) we see that the linear part of the integral equation (5.3) satisfies

$$\|U(t)u_0\|_T \leq C(1+T)^{1/2}\|u_0\|_{H^{1/4}}, \tag{5.5}$$

for $u_0 \in H^{1/4}$.

We now observe that for the nonlinear term $F(v)$ defined in (5.2) we have the following estimates after applying Minkowski's and Cauchy-Schwartz' inequalities.

$$\begin{aligned} \|\int_0^T U(\tau,t)\,F(v)(\tau)\,d\tau\|_T &\leq \int_0^T \|\,U(\tau,t)\,F(v)(\tau)\|_T\,d\tau \\ &\leq C\,(1+T)^{1/2}\int_0^T \|F(v)(\tau)\|_{H^{1/4}}\,d\tau \\ &\leq C\,(1+T)^{1/2}\,T^{1/2}\Big(\|F(v)(\tau)\|_{L^2_xL^2_T} + \|D_x^{1/4}\,F(v)(\tau)\|_{L^2_xL^2_T}\Big). \end{aligned} \tag{5.6}$$

Thus using (5.3), (5.5) and (5.6) we have

$$\begin{aligned} \|\Phi(v)\|_T &\leq C(1+T)^{1/2}\|u_0\|_{H^{1/4}} \\ &+C\,(1+T)^{1/2}\,T^{1/2}\,(\|F(v)\|_{L^2_xL^2_T} + \|D_x^{1/4}F(v)\|_{L^2_xL^2_T}). \end{aligned} \tag{5.7}$$

It will be enough to show how to estimate the last term in the inequality above since for the other term the argument is similar but simpler. Thus from (5.2) we get

$$\begin{aligned} \|D_x^{1/4}F(v)\|_{L^2_xL^2_T} &\leq \|D_x^{1/4}(v^2\partial_x\,\bar{v})\|_{L^2_xL^2_T} \\ &\quad + \|D_x^{1/4}(|v|^2\partial_x\,v)\|_{L^2_xL^2_T} + \|D_x^{1/4}(|v|^2v)\|_{L^2_xL^2_T}. \end{aligned} \tag{5.8}$$

The estimates for the first term is similar to the second one on the right hand of (5.8) so we will only estimate the last two terms.

Using the Leibniz rule for fractional derivatives (4.12) and Hölder's inequality we obtain

$$\begin{aligned} \|D_x^{1/4}(|v|^2\partial_xv)\|_{L^2_xL^2_T} &\leq \|D_x^{1/4}|v|^2\|_{L_x^{20/9}L_T^{10}}\|\partial_xv\|_{L_x^{20}L_T^{5/2}} + \|v\|^2_{L^4_xL^\infty_T}\|D_x^{1/4}\partial_x\,v\|_{L^\infty_xL^2_T} \\ &\leq c\,\|D_x^{1/4}\bar{v}\|_{L^5_xL^{10}_T}\|v\|_{L^4_xL^\infty_T}\|\partial_xv\|_{L_x^{20}L_T^{5/2}} + \|v\|^2_{L^4_xL^\infty_T}\|D_x^{1/4}\partial_x\,v\|_{L^\infty_xL^2_T} \\ &\leq c\,\|v\|_T^3. \end{aligned} \tag{5.9}$$

Applying the Leibniz rule (4.12) once again and Hölder's inequality we have

$$\begin{aligned} \|D_x^{1/4}(|v|^2v)\|_{L^2_xL^2_T} &\leq c\,\|v\|_{L^4_xL^\infty_T}\|D_x^{1/4}(|v|^2)\|_{L^4_xL^2_T} + c\|\,|v|^2D_x^{1/4}v\|_{L^2_xL^2_T} \\ &\leq c\,T^{1/4}\,\|v\|_{L^4_xL^\infty_T}\|D_x^{1/4}|v|^2\|_{L^4_xL^4_T} + \|v\|_{L^4_xL^\infty_T}\|\,\bar{v}D_x^{1/4}v\|_{L^4_xL^2_T} \\ &\leq c\,T^{1/4}\,\|v\|_{L^4_xL^\infty_T}\|v\|_{L^8_xL^8_T}\|D_x^{1/4}v\|_{L^8_xL^8_T} \\ &\leq c\,T^{1/4}\|v\|_T^3. \end{aligned} \tag{5.10}$$

The estimates (5.9) and (5.10) combined with (5.8) yield

$$\|\Phi(v)\|_T \le c\,(1+T)^{1/2}\|u_0\|_{H^{1/4}} + c\,T^{1/2}(1+T)^{3/4}\,\|v\|_T^3. \tag{5.11}$$

Taking $a = 2c\,(1+T)^{1/2}\|u_0\|_{H^{1/4}}$ and T such that

$$c\,T^{1/2}(1+T)^{3/4}a^2 \le 1/2, \tag{5.12}$$

it follows that $\Phi(v) \in Z_{a,T}$ for $v \in Z_{a,T}$.

Now we will see that $\Phi : Z_{a,T} \to Z_{a,T}$ is in fact a contraction. Let $v, w \in Z_{a,T}$ then we have

$$\|\Phi(v) - \Phi(w)\|_T \le c\,T^{1/2}(1+T)^{1/2}\|F(v) - F(w)\|_{L^2_T H^{1/4}}.$$

The previous argument shows that

$$\|F(v) - F(w)\|_{L^2_T H^{1/4}} \le c\,(1+T)^{1/4}(\|v\|_T + \|w\|_T)^2\|v - w\|_T.$$

Therefore

$$\|\Phi(v) - \Phi(w)\|_T \le c\,T^{1/2}(1+T)^{3/4}(\|v\|_T + \|w\|_T)^2\|v - w\|_T.$$

Hence from the choice of a and T in (5.12) it follows that Φ is a contraction.

To show that the map data–solution satisfies the requirement on the theorem we take $T_1 \in (0, T)$ and use the argument in the first part of the proof and the fact that Φ is a contraction to obtain the inequality

$$\begin{aligned}\|v - w\|_{T_1} = \|\Phi_{u_0}(v) - \Phi_{\tilde{u}_0}(w)\|_{T_1} &\le c\,(1+T_1)^{1/2}\|u_0 - \tilde{u}_0\|_{H^{1/4}} \\ &\quad + c\,T_1^{1/2}(1+T_1)^{3/4}(\|v\|_{T_1} + \|w\|_{T_1})^2\|v - w\|_{T_1},\end{aligned}$$

The choice of a and T in (5.12) allows us to deduce the result in $Z_{a,T}$. The general case will follow using a standard uniqueness argument. This completes the proof of Theorem 1.1. □

Remark 5.1. *If we consider a more general nonlinearity in (1.1), namely,*

$$\rho_1 u(g * |u|^2) + i\rho_2 u(g * (\partial_x |u|^2)) + i\rho_2(\partial_x u)(g * |u|^2),$$

where g is in a suitable space and $\rho_j, j = 1, 2, 3,$ are constants, the same proof above can be used to establish local well-posedness in $H^s(\mathbb{R})$ for $s \ge 1/4$ in this case.

In particular, for the nonlinearity considered in [22],

$$c_1\, u(g * |u|^2) + ic_2\, \partial_x[u(g * |u|^2)],$$

where $g = \delta + f$, i.e, the sum of the Dirac delta function and an integrable function f with c_1, $c_2 \in \mathbb{C}$ we can obtain a similar result as in Theorem 1.1. In fact, for $f \in L^1(\mathbb{R})$

it follows that,

$$\begin{aligned}
\|f * v\|_{L^p_x L^q_T} &\leq \|f\|_{L^1}\|v\|_{L^p_x L^q_T}\\
\|f * v\|_{L^q_T L^p_x} &\leq \|f\|_{L^1}\|v\|_{L^q_T L^p_x},\\
\|D_x^{s_1}\partial_x^{s_2}(f * v)\|_{L^p_x L^q_T} &\leq \|f\|_{L^1}\|D_x^{s_1}\partial_x^{s_2} v\|_{L^p_x L^q_T},\\
\|D_x^{s_1}\partial_x^{s_2}(f * v)\|_{L^q_T L^p_x} &\leq \|f\|_{L^1}\|D_x^{s_1}\partial_x^{s_2} v\|_{L^q_T L^p_x},
\end{aligned}$$

with $1 \leq p, q \leq \infty$, $s_1, s_2 \geq 0$, *since* $f = f(x)$ *does not depend on the variable* t *and*

$$D_x^{s_1}\partial_x^{s_2}(f * v) = (f * D_x^{s_1}\partial_x^{s_2} v), \quad \|f * v\|_{L^p} \leq \|f\|_{L^1}\|v\|_{L^p}.$$

Hence the estimates used in the proof of Theorem 1.1 for v *can be similarly used to estimate* $g * v$.

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