The Knot Group and the Fundamental Group of the Embedding 3-Manifold

Pedro Lopes *
Departamento de Matemática and Centro de Matemática e Aplicações
Instituto Superior Técnico
Av. Rovisco Pais
1049-001 Lisboa
Portugal
pelopes@math.ist.utl.pt

C. A. Morales
Instituto de Matemática
Universidade Federal do Rio de Janeiro
C. P. 68530
21945-970, RJ
Brazil
morales@impa.br

August 14, 2003

Abstract

We propose a conjecture concerning 3-manifolds (which implies the $\mathbb{Q}$ conjecture of Myers') and construct a countable sequence of examples which supports it.

1 Introduction

A well-known problem in 3-manifold topology is the description of those groups which realize as fundamental groups of 3-manifolds, 3-manifold groups, for short. We will now recall some facts and conjectures in this context relevant for this note.

An old result of Evans and Jaco states that every abelian subgroup of a compact, almost sufficiently large 3-manifold is finitely generated (cf. Corollary 3.3 on page 95 in [2]). In the same paper on page 95 it is remarked that no subgroup of any known compact 3-manifold group is isomorphic to a noncyclic subgroup of $\mathbb{Q}_+$, the additive group of rationals.

More recently, R. Myers set forth the $\mathbb{Q}$ conjecture (cf. Conjecture 4 on page 1511 in [5]):

Conjecture 1.1 *Every subgroup of an infinite compact 3-manifold group which embeds in $\mathbb{Q}_+$ is cyclic.*

The $\mathbb{Q}$ conjecture is important because should it be true it could be used to prove Thurston's Hyperbolization Conjecture (cf. Corollary 2 on page 1511 in [5]):

Conjecture 1.2 *Let M be a closed 3-manifold such that $\pi_1(M)$ is infinite and does not have any rank two free abelian subgroups. Then M is hyperbolic.*

*Temporary address/email: IMPA, Estrada D. Castorina 110, 22460-320 RJ, Brazil/pelopes@impa.br

Moreover, the $\mathbb{Q}$ conjecture can be seen as a reformulation of Evans-Jaco's remark since an abelian subgroup of a compact 3-manifold group is either finitely generated or embeds in $\mathbb{Q}_+$ (cf. [3]). We remark that it is not known whether every abelian subgroup of a compact 3-manifold group is finitely generated or not.

In this way we suggest the following approach. Consider an abelian subgroup A of the fundamental group $\pi_1(M)$ of a compact 3-manifold M. *Suppose that there is a simple closed curve L in M with irreducible exterior, $M \setminus L$ such that $\pi_1(M)$ embeds in $\pi_1(M \setminus L)$.* Then A is finitely generated by the result of Evans and Jaco mentioned before since $\pi_1(M \setminus L)$ is the fundamental group of a Haken manifold (it suffices to consider a tubular neighborhood of L in M). Now if A is finitely generated and embeds in $\mathbb{Q}_+$ then it is cyclic (cf. [4]). Hence we propose the following conjecture:

Conjecture 1.3 *For every compact 3-manifold M, there is a simple closed curve L with irreducible exterior such that $\pi_1(M)$ embeds in $\pi_1(M \setminus L)$.*

Clearly if this conjecture is true then the $\mathbb{Q}$ conjecture is also true, from the remarks above. In order to support this approach, we construct below a sequence of 3-manifolds, M_k, and a sequence of closed curves, L_k, such that $\pi_1(M_k)$ embeds in $\pi_1(M_k \setminus L_{k+1})$ for every $k \geq 0$.

1.1 Acknowledgements

The first author acknowledges support by the programme *Programa Operacional "Ciência, Tecnologia, Inovação"* (POCTI) of the *Fundação para a Ciência e a Tecnologia* (FCT), cofinanced by the European Community fund FEDER.

2 The Sequences

We construct a sequence of 3-manifolds, M_0, M_1, M_2, ... and a sequence of links, $L_1(\subset M_0)$, $L_2(\subset M_1)$, $L_3(\subset M_2)$, ... such that:

$$M_0 \supset M_0 \setminus L_1 \simeq M_1 \supset M_1 \setminus L_2 \simeq M_2 ...$$

and:

$$\pi_1(M_0) \hookrightarrow \pi_1(M_0 \setminus L_1) \cong \pi_1(M_1) \hookrightarrow \pi_1(M_1 \setminus L_2) \cong \pi_1(M_2) \hookrightarrow ...$$

where $\simeq$ denotes homotopical equivalence, $\cong$ denotes group isomorphism, and $\hookrightarrow$ denotes group monomorphism. In particular, we set $M_0 = S^3$ and choose a link L_1 in S^3 such that the first embedding at the level of the fundamental groups holds (trivial). Then we choose a link L_2 in $M_1 \cong M_0 \setminus L_1$ such that the second embedding at the level of the fundamental groups holds. And so on and so forth.

In 2.1 below we introduce the material necessary for constructing the sequences. In 2.2 we construct them.

2.1 The Background Material

In this subsection we introduce notation and develop the background material necessary for our construction. Detailed exposition of it can be found in [1].

The following is a presentation of the braid group on n strands

$$B_n \cong \langle\, \sigma_1, ..., \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1};\; \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| > 1 \,\rangle$$

where σ_i denotes the standard generator of the braid group (see figure 1)

There are other other ways of conceiving the braid group on n strands. The one that will interest us here is obtained by regarding it as a subgroup of the automorphism group of the free group on n generators, $Aut(F_n)$. The free group on n generators, $F_n \cong \langle x_1, ..., x_n \rangle$, is considered as the fundamental

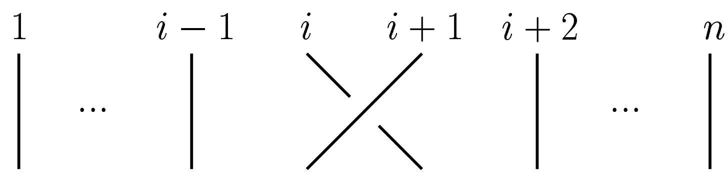

Figure 1: $\sigma_i \in B_n$

group of a disc with n holes, where the generators, x_i, are the usual loops around each hole (notation: $x_1, ..., x_n$). Each braid $\beta \in B_n$ gives rise to an element in $Aut(F_n)$ in the following way. For each $\sigma_i \in B_n$:

$$x_i\sigma_i = x_i x_{i+1} x_i, \qquad x_{i+1}\sigma_i = x_i, \qquad x_j\sigma_i = x_j \quad j \notin \{i, i+1\}$$

The extension to the homomorphism is unique allowing each $\beta \in B_n$ to be realized as an element of $Aut(F_n)$ - notice that the σ_i's act on the right. We will not carry through the characterization of B_n as a subgroup of $Aut(F_n)$ for it is beyond our needs in this note. We now recall how the fundamental group of the complement of a link in S^3 (link group, for short) can be computed using this characterization of B_n.

Any link L in S^3 is equivalent (Alexander's Theorem) to the closure of a braid $\beta \in B_n$ for some $n \in \mathbb{N}_2$ (the closure of a braid $\beta \in B_n$ will be denoted by $\hat{\beta}$). In this way

$$\pi_1(S^3 \setminus L) \cong \pi_1(S^3 \setminus \hat{\beta}) \cong \langle x_1, ..., x_n \mid x_i = x_i\beta, i = 1, ..., n\rangle$$

where β in the presentation above is regarded as an element of $Aut(F_n)$, as explained before.

2.2 An "Embedding" Sequence of Fundamental Groups

We now begin the construction of a sequence of 3-manifolds, M_0, M_1, M_2, ... and a sequence of links, L_1, L_2, L_3, ... such that each $\pi_1(M_{k-1})$ will embed in $\pi_1(M_{k-1} \setminus L_k)$, for $k = 1, 2, ...$ We set $M_0 = S^3$. At each step $k \in \mathbb{N}_1$ we set L_k equal to the Hopf link and obtain M_k by removing the current Hopf link, L_k, from the previous manifold, M_{k-1}. We remark that the Hopf link can be seen as the closure of $\sigma_1^2 \in B_2$ (see Fig. 1).

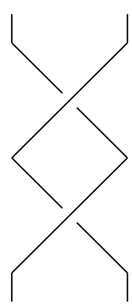

Figure 2: $\sigma_1^2 \in B_2$

Note that for $k \in \mathbb{N}_1$, the $(k+1)$-th Hopf link will be linked to the previous one (see Fig. 2 for $k = 1$). So $M_0 = S^3$. In step 1, let $L_1 = \widehat{\sigma_1^2}$ and remove σ_1^2 to obtain $M_1 = S^3 \setminus \widehat{\sigma_1^2}$. We will now compute the link group of L_1. According to the results mentioned above:

$$\pi_1(S^3 \setminus \widehat{\sigma_1^2}) \cong \langle x_1, x_2 \mid x_1 = x_1\sigma_1^2\rangle$$

and since $x_1\sigma_1^2 = x_1x_2x_1^{-1}\sigma_1 = x_1x_2x_1^{-1}x_1x_1x_2^{-1}x_1^{-1} = x_1x_2x_1x_2^{-1}x_1^{-1}$, we have:

$$\pi_1(S^3 \setminus \widehat{\sigma_1^2}) \cong \langle x_1, x_2 \mid [x_1, x_2] = 1 \rangle$$

thus we see that $\pi_1(S^3) \cong 1$ embeds in a trivial way in $\pi_1(S^3 - \widehat{\sigma}_1^2) \cong \langle x_1, x_2 \mid [x_1, x_2] = 1 \rangle$.

In step 2, we now link a Hopf link (full line) to the previous Hopf link (dim line) (see Fig. 2).

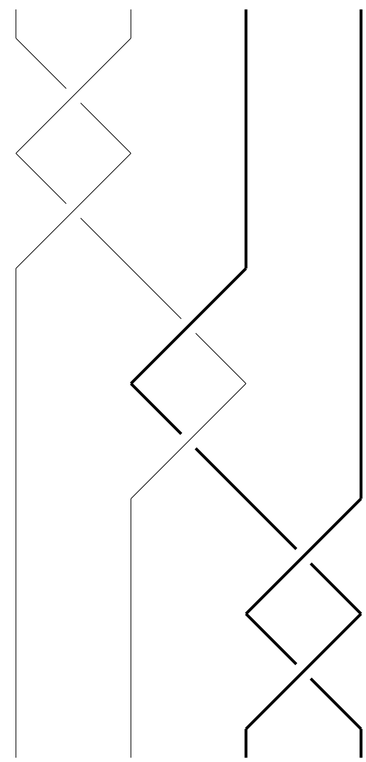

Figure 3: Hopf link linked to Hopf link (upon braid closure)

In this way, let L_2 be a Hopf link, again, and $M_2 = S^3 - L_1$. The knot group of L_2 is $\pi_1(M_1 \setminus L_2) \cong \pi_1(((S^3 \setminus L_1) \setminus L_2)) \cong \pi_1(S^3 \setminus \widehat{\sigma_1^2\sigma_2^2\sigma_3^2})$, and a presentation for the latter group is given by:

$$\pi_1(S^3 \setminus \widehat{\sigma_1^2\sigma_2^2\sigma_3^2}) \cong \langle x_1, x_2, x_3, x_4 \mid x_1 = x_1\sigma_1^2\sigma_2^2\sigma_3^2,\ \ x_2 = x_2\sigma_1^2\sigma_2^2\sigma_3^2,\ \ x_3 = x_3\sigma_1^2\sigma_2^2\sigma_3^2, \rangle$$

We will use the notation: $x * y := yxy^{-1}$ in order to simplify the calculations.

$$x_3 = x_3\sigma_1^2\sigma_2^2\sigma_3^2 = x_3\sigma_2^2\sigma_3^2 = x_2\sigma_2\sigma_3^2 = (x_3 * x_2)\sigma_3^2 = ((x_4 * x_3) * x_2)\sigma_3 = (x_3 * (x_4 * x_3)) * x_2$$

Analogously,

$$\begin{aligned} x_2 &= x_2\sigma_1^2\sigma_2^2\sigma_3^2 = x_1\sigma_1\sigma_2^2\sigma_3^2 = (x_2 * x_1)\sigma_2^2\sigma_3^2 = ((x_3 * x_2) * x_1)\sigma_2\sigma_3^2 = ((x_2 * (x_3 * x_2)) * x_1)\sigma_3^2 \\ &= ((x_2 * ((x_4 * x_3) * x_2)) * x_1)\sigma_3 = (x_2 * ((x_3 * (x_4 * x_3)) * x_2)) * x_1 \end{aligned}$$

and for x_1,

$$\begin{aligned} x_1 &= x_1\sigma_1^2\sigma_2^2\sigma_3^2 = (x_2 * x_1)\sigma_1\sigma_2^2\sigma_3^2 = (x_1 * (x_2 * x_1))\sigma_2^2\sigma_3^2 = (x_1 * ((x_3 * x_2) * x_1))\sigma_2\sigma_3^2 \\ &= (x_1 * ((x_2 * (x_3 * x_2)) * x_1))\sigma_3^2 = (x_1 * ((x_2 * ((x_4 * x_3) * x_2)) * x_1))\sigma_3 \\ &= x_1 * ((x_2 * ((x_3 * (x_4 * x_3)) * x_2)) * x_1) \end{aligned}$$

Replacing x_3 for $(x_3 * (x_4 * x_3)) * x_2$ on the second equation and x_2 for $(x_2 * ((x_3 * (x_4 * x_3)) * x_2)) * x_1$ on the first equation we have:

$$\begin{cases} x_3 = (x_3 * (x_4 * x_3)) * x_2 \\ x_2 = (x_2 * x_3) * x_1 \\ x_1 = x_1 * x_2 \end{cases}$$

So, from the last equation, x_1 and x_2 commute, which implies that x_2 and x_3 commute, which in turn implies that x_3 and x_4 commute:

$$\begin{cases} [x_1, x_2] = 1 \\ [x_2, x_3] = 1 \\ [x_3, x_4] = 1 \end{cases}$$

Thus a presentation of $\pi_1(M_{k-1} \setminus L_k)$, the knot group of L_2, is:

$$\langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = 1, \ [x_2, x_3] = 1, \ [x_3, x_4] = 1 \rangle$$

Moreover, consider the homomorphisms:

$$\begin{array}{rcccc} \phi_1 & : & \langle x_1, x_2 \mid [x_1, x_2] = 1 \rangle & \longrightarrow & \langle x_1', x_2', x_3', x_4' \mid [x_1', x_2'] = 1, \ [x_2', x_3'] = 1, \ [x_3', x_4'] = 1 \rangle \\ & & x_1 & \longmapsto & x_1' \\ & & x_2 & \longmapsto & x_2' \end{array}$$

and

$$\begin{array}{rcccc} \psi_1 & : & \langle x_1', x_2', x_3', x_4' \mid [x_1', x_2'] = 1, \ [x_2', x_3'] = 1, \ [x_3', x_4'] = 1 \rangle & \longrightarrow & \langle x_1, x_2 \mid [x_1, x_2] = 1 \rangle \\ & & x_1' & \longmapsto & x_1 \\ & & x_2' & \longmapsto & x_2 \\ & & x_3' & \longmapsto & 1 \\ & & x_4' & \longmapsto & 1 \end{array}$$

Clearly, $\psi_1 \circ \phi_1$ is a group isomorphism so ϕ_1 is a monomorphism, hence $\pi_1(M_1) \cong \langle x_1, x_2 \mid [x_1, x_2] = 1 \rangle$ embeds in $\langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = 1, \ [x_2, x_3] = 1, \ [x_3, x_4] = 1 \rangle$.

Proceeding as above, in the k-th step we will be removing a (linked) Hopf link from the M_{k-1} manifold (see Fig. 2 for the $k = 1$ case). Arguing as before, any presentation of the knot group is equivalent to a presentation of the fundamental group of the complement in S^3 of the closure of $\sigma_1^2 \sigma_2^2 ... \sigma_{2k-2}^2 \sigma_{2k-1}^2 \in B_{2k}$. The next proposition will yield the form of the relations for the presentation of $\pi_1(M_{k-1} \setminus L_k)$ (as before $x * y := yxy^{-1}$).

Proposition 2.1 *For each $k \in \mathbb{N}_1$, consider the set of expressions in the free group $F_{2k} \cong \langle x_1, ..., x_{2k} \rangle$:*

$$x_i \quad \sigma_1^2 \sigma_2^2 ... \sigma_{2k-2}^2 \sigma_{2k-1}^2, \quad i = 1, 2, ..., 2k-1$$

The calculations yield:

$$\left(x_i * \left(x_{i+1} * ... * \left(x_{2k-2} * \left((x_{2k-1} * (x_{2k} * x_{2k-1})) * x_{2k-2} \right) \right) * ... * x_i \right) \right) * x_{i-1}$$

for $2 \leq i \leq 2k-1$, and

$$x_1 * \left(\left(x_2 * \left(x_3 * ... * \left(x_{2k-2} * \left((x_{2k-1} * (x_{2k} * x_{2k-1})) * x_{2k-2} \right) \right) * ... * x_2 \right) \right) * x_1 \right)$$

for $i = 1$.

Proof: The proof will be by induction on k. Since particular cases have already been calculated (for $k = 1, 2$) we will just consider the induction step.

Consider the following subset from the $(k+1)$-th set of expressions:

$$x_j \quad (\sigma_1^2\sigma_2^2...\sigma_{2k-2}^2\sigma_{2k-1}^2)\sigma_{2k}^2\sigma_{2k+1}^2, \quad i = 1, 2, ..., 2k-1$$

Using the induction hypothesis we have:

$$\left(x_i * \left(x_{i+1} * ... * \left(x_{2k-2} * \left((x_{2k-1} * (x_{2k} * x_{2k-1})) * x_{2k-2}\right)\right) * ... * x_i\right)\right) * x_{i-1} \quad \sigma_{2k}^2\sigma_{2k+1}^2$$

for $2 \leq i \leq 2k-1$, and

$$x_1 * \left(\left(x_2 * \left(x_3 * ... * \left(x_{2k-2} * \left((x_{2k-1} * (x_{2k} * x_{2k-1})) * x_{2k-2}\right)\right) * ... * x_2\right)\right) * x_1\right) \quad \sigma_{2k}^2\sigma_{2k+1}^2$$

for $i = 1$.

Now the σ_{2k}^2 will act on x_{2k} replacing it by $x_{2k} * (x_{2k+1} * x_{2k})$; analogously the σ_{2k+1}^2 will act only on x_{2k+1} replacing it by $x_{2k+1} * (x_{2k+2} * x_{2k+1})$. In this way,

$$\left(x_i * \left(x_{i+1} * ... * \left(x_{2k} * \left((x_{2k+1} * (x_{2k+2} * x_{2k+1})) * x_{2k}\right)\right) * ... * x_i\right)\right) * x_{i-1}$$

for $2 \leq i \leq 2k-1$, and

$$x_1 * \left(\left(x_2 * \left(x_3 * ... * \left(x_{2k} * \left((x_{2k+1} * (x_{2k+2} * x_{2k+1})) * x_{2k}\right)\right) * ... * x_2\right)\right) * x_1\right)$$

for $i = 1$.

Now for the remaining two expressions:

$$\begin{cases} x_{2k} & (\sigma_1^2\sigma_2^2...\sigma_{2k-2}^2\sigma_{2k-1}^2)\sigma_{2k}^2\sigma_{2k+1}^2 \\ x_{2k+1} & (\sigma_1^2\sigma_2^2...\sigma_{2k-2}^2\sigma_{2k-1}^2)\sigma_{2k}^2\sigma_{2k+1}^2 \end{cases} \Leftrightarrow \begin{cases} x_{2k} & \sigma_{2k-1}^2\sigma_{2k}^2\sigma_{2k+1}^2 \\ x_{2k+1} & \sigma_{2k}^2\sigma_{2k+1}^2 \end{cases} \Leftrightarrow$$

$$\begin{cases} \left(x_{2k} * \left((x_{2k+1} * (x_{2k+2} * x_{2k+1})) * x_{2k}\right)\right) * x_{2k-1} \\ (x_{2k+1} * (x_{2k+2} * x_{2k+1})) * x_{2k} \end{cases}$$

which completes the proof. ∎

Proposition 2.2 *For each* $k \in \mathbb{N}_1$,

$$\pi_1(S^3 \setminus \widehat{\sigma_1^2...\sigma_{2k-1}^2}) \cong \langle x_1, ..., x_{2k} \mid [x_i, x_{i+1}] = 1, \ i = 1, ..., 2k-1 \rangle$$

Proof: By the previous proposition we know that, for any $k \in \mathbb{N}_1$,

$$x_i = x_i \ \sigma_1^2...\sigma_{2k-1}^2, \quad i = 1, ..., 2k-1 \qquad \Leftrightarrow$$

$$\begin{cases} x_i = \left(x_i * \left(x_{i+1} * ... * \left(x_{2k-2} * \left((x_{2k-1} * (x_{2k} * x_{2k-1})) * x_{2k-2}\right)\right) * ... * x_i\right)\right) * x_{i-1}, & 2 \leq i \leq 2k-1 \\ x_1 = x_1 * \left(\left(x_2 * \left(x_3 * ... * \left(x_{2k-2} * \left((x_{2k-1} * (x_{2k} * x_{2k-1})) * x_{2k-2}\right)\right) * ... * x_2\right)\right) * x_1\right), & i = 1 \end{cases}$$

We will now prove that this last system of relations is equivalent to $[x_i, x_{i+1}] = 1, \ i = 1, ..., 2k-1$, by induction on k. Since particular cases have already been calculated we just prove the induction step.

Consider the following subsystem from the (k+1)-th system of relations.

$$x_i = \left(x_i * \left(x_{i+1} * ... \left(x_{2k-1} * \left(\left(x_{2k} * \left((x_{2k+1} * (x_{2k+2} * x_{2k+1})) * x_{2k}\right)\right) * x_{2k-1}\right) ... * x_i\right)\right) * x_{i-1}$$

for $2 \leq i \leq 2k-1$, and

$$x_1 = x_1 * \left(\left(x_2 * \left(x_3 * ... \left(x_{2k-1} * \left(\left(x_{2k} * \left((x_{2k+1} * (x_{2k+2} * x_{2k+1})) * x_{2k}\right)\right) * x_{2k-1}\right) ... * x_2\right)\right) * x_1\right)$$

for $i = 1$.

Set $x'_{2k} = \left(\left(x_{2k} * \left((x_{2k+1} * (x_{2k+2} * x_{2k+1})) * x_{2k}\right)\right)$. With this substitution the system above reads:

$$x_i = \left(x_i * \left(x_{i+1} * ...(x_{2k-1} * (x'_{2k} * x_{2k-1}))... * x_i\right)\right) * x_{i-1}$$

for $2 \leq i \leq 2k-1$, and

$$x_1 = x_1 * \left(\left(x_2 * \left(x_3 * ...(x_{2k-1} * (x'_{2k} * x_{2k-1}))... * x_2\right)\right) * x_1\right)$$

for $i = 1$.

By the induction hypothesis, this is equivalent to $[x_i, x_{i+1}] = 1, \; i = 1, ..., 2k-2$ and $[x_{2k-1}, x'_{2k}] = 1$ for $i = 2k-1$. Let us now consider the relations corresponding to $i = 2k, 2k+1$ along with $[x_{2k-1}, x'_{2k}] = 1$, with x'_{2k} replaced by its expression in terms of x_{2k}.

$$\begin{cases} x_{2k+1} = (x_{2k+1} * (x_{2k+2} * x_{2k+1})) * x_{2k} \\ x_{2k} = \left(x_{2k} * \left((x_{2k+1}(*x_{2k+2} * x_{2k+1})) * x_{2k}\right)\right) * x_{2k-1} \\ x_{2k-1}\left(x_{2k} * \left((x_{2k+1}(*x_{2k+2} * x_{2k+1})) * x_{2k}\right)\right) = \left(x_{2k} * \left((x_{2k+1}(*x_{2k+2} * x_{2k+1})) * x_{2k}\right)\right)x_{2k-1} \end{cases} \Leftrightarrow$$

$$\begin{cases} x_{2k+1} = (x_{2k+1} * (x_{2k+2} * x_{2k+1})) * x_{2k} \\ x_{2k} = \left(x_{2k} * x_{2k+1}\right) * x_{2k-1} \\ x_{2k-1}(x_{2k} * x_{2k+1}) = (x_{2k} * x_{2k+1})x_{2k-1} \end{cases} \Leftrightarrow$$

$$\begin{cases} x_{2k+1} = x_{2k+1} * (x_{2k+2} * x_{2k+1}) \\ x_{2k} = x_{2k} * x_{2k+1} \\ x_{2k-1}(x_{2k} * x_{2k+1}) = (x_{2k} * x_{2k+1})x_{2k-1} \end{cases} \Leftrightarrow \begin{cases} [x_{2k+1}, x_{2k+2}] = 1 \\ [x_{2k}, x_{2k+1}] = 1 \\ [x_{2k-1}, x_{2k}] = 1 \end{cases}$$

which finishes the proof. ■

Theorem 2.1 *The construction above exhibits a sequence of links,* $\{L_k\}$*, and a sequence of manifolds,* $\{M_k\}$*, with* $L_k \subseteq M_{k-1}$*, such that* $\pi_1(M_k)$ *embeds in* $\pi_1(M_k \setminus L_{k+1})$*, for each* $k \in \mathbb{N}_0$*. Moreover, for each* $k \in \mathbb{N}_1$*, the embedding can be realized as follows:*

$$\begin{array}{cccc} \phi_k : \langle x_1, ..., x_{2k} \mid [x_i, x_{i+1}] = 1, \; i = 1, ..., 2k-1 \rangle & \longrightarrow & \langle x'_i, ..., x'_{2k+2} \mid [x'_i, x'_{i+1}] = 1, \; i = 1, ..., 2k+1 \rangle \\ x_i & \longmapsto & x'_i \\ & i = 1, ..., 2k & \end{array}$$

Proof: Consider the homomorphism:

$$\begin{array}{cccc} \psi_k : \langle x'_i, ..., x'_{2k+2} \mid [x'_i, x'_{i+1}] = 1, \; i = 1, ..., 2k+1 \rangle & \longrightarrow & \langle x_1, ..., x_{2k} \mid [x_i, x_{i+1}] = 1, \; i = 1, ..., 2k-1 \rangle \\ x'_i & \longmapsto & x_i \\ & i = 1, ..., 2k & \\ & x'_{2k+1} \longmapsto 1 & \\ & x'_{2k+2} \longmapsto 1 & \end{array}$$

Clearly, for any $k \in \mathbb{N}_1$, $\psi_k \circ \phi_k$ is a a group isomorphism which yields the result. ■

References

[1] J. Birman, *Braids, Links, and Mapping Class Groups*, Annals of Math. Studies, no. 82, Princeton University Press (1974)

[2] B. Evans, W. Jaco, *Varieties of groups and three-manifolds*, Topology **12** (1973), 83-97

[3] J. Hempel, 3-*Manifolds*, Annals of Math. Studies, no. 86, Princeton University Press (1976)

[4] I. Kaplansky, *Infinite Abelian Groups*, (revised edition) The University of Michigan Press, Ann Arbor, Mich. (1969)

[5] R. Myers, *Compactifying sufficiently regular covering spaces of compact* 3-*manifolds*, Proc. Amer. Math. Soc. **28** (2000), no. 5, 1507-1513