ON FOCAL STABILITY IN DIMENSION TWO

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ABSTRACT. In [1] appears the Focal Stability Conjecture: the focal decomposition of the generic Riemann structure on a manifold M is stable under perturbations of the Riemann structure. In this paper, we prove the conjecture when M has dimension two, and there are no conjugate points.

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1. Introduction

Let M be a compact, smooth manifold of dimension m, and let $\mathcal{R} = \mathcal{R}^r$ be the space of C^r Riemann structures on M, equipped with the natural C^r topology, $2 \leq r \leq \infty$. Fix $p \in M$. The $\boldsymbol{k}^{\text{th}}$ **focal component** with respect to $g \in \mathcal{R}$ at p is

$$\sigma_k = \{v \in T_pM : \exists \text{ exactly } k \text{ vectors } v = v_1, \dots, v_k \in T_pM \text{ with } |v_1| = \cdots = |v_k| \text{ and } \exp(v_1) = \cdots = \exp(v_k)\},$$

where $1 \leq k \leq \infty$, and $|\ |$, exp refer to the Riemann structure g.

The focal decomposition $T_pM = \bigsqcup_k \sigma_k$ is said to be **focally stable** if a small perturbation of g has only a distant topological effect on $\bigsqcup \sigma_k$. Precisely, we require that given $\epsilon > 0$ and given a compact set $S \subset T_pM$, there is a neighborhood $\mathcal{U}$ of g in $\mathcal{R}$ and there are balls $B, B' \subset T_pM$ such that for each $g' \in \mathcal{U}$,

(a) $S \subset B \cap B'$.
(b) There is a homeomorphism $h : B \to B'$ that sends each $\sigma_k(g) \cap B$ onto $\sigma_k(g') \cap B'$.

Thus, the focal decomposition $\bigsqcup \sigma_k$ enjoys a kind of structural stability.

In [1] we investigated the concept of focal stability with an eye to proving the following **Focal Stability Conjecture**: the generic Riemann structure is focally stable. (Since $\mathcal{R}$ is an open subset of a complete metric space, genericity makes sense.) The main result of this paper concerns Riemann structures that have no conjugate points. It is most easily stated for the open set $\mathcal{N} \subset \mathcal{R}$ of Riemann structures on TM whose Gauss curvature is everywhere negative. See Section 7 for a discussion of the more general case that g has no conjugate points.

Theorem A. *For a compact manifold of dimension two, the generic Riemann structure $g \in \mathcal{N}$ is focally stable.*

A different sort of result is also given. It concens surfaces of constant negative curvature. Fix a compact smooth surface of genus $s \geq 2$, such as the bitorus, and let $\mathcal{H}$ denote the nonempty set of Riemann structures on M with curvature everywhere equal to -1. Since $\mathcal{H}$ is a clsoed subset of $\mathcal{R}$, genericity in $\mathcal{H}$ makes sense. Modulo isometric deformations $\mathcal{H}$ is the Teichmuller space τ_s.

Theorem B.

(a) *Fix $g \in \mathcal{H}$. For the generic $p \in M$, the focal decomposition of T_pM is stable with respect to perturbations of p in M.*
(b) *Fix $p \in M$. For the generic $g \in \mathcal{H}$, the focal decomposition of T_pM is stable with respect to perturbations of g within $\mathcal{H}$.*

2. Mediatrices

When the Riemann structure on M has non-positive curvature, there are no conjugate points and so $\exp_p : T_pM \to M$ is the universal covering space. Let $\bar{g}$ be the lift of g to T_pM, and let $\bar{d}$ be the corresponding metric on T_pM. The focal decomposition of T_pM can be described in terms of equidistance loci, called *mediatrices* by Peter Veerman in [2], as follows. A vector $v_1 \in \sigma_k$ has k "friends" – vectors $v_1, \ldots, v_k \in T_pM$ of equal length and equal exponential image. (A vector is always a friend of itself.) This means that there are exactly k points in $\exp_p^{-1}(p)$, one of which is the origin O_p of T_pM, and from which v_1 is equidistant with respect to the metric $\bar{d}$. See Figure 1.

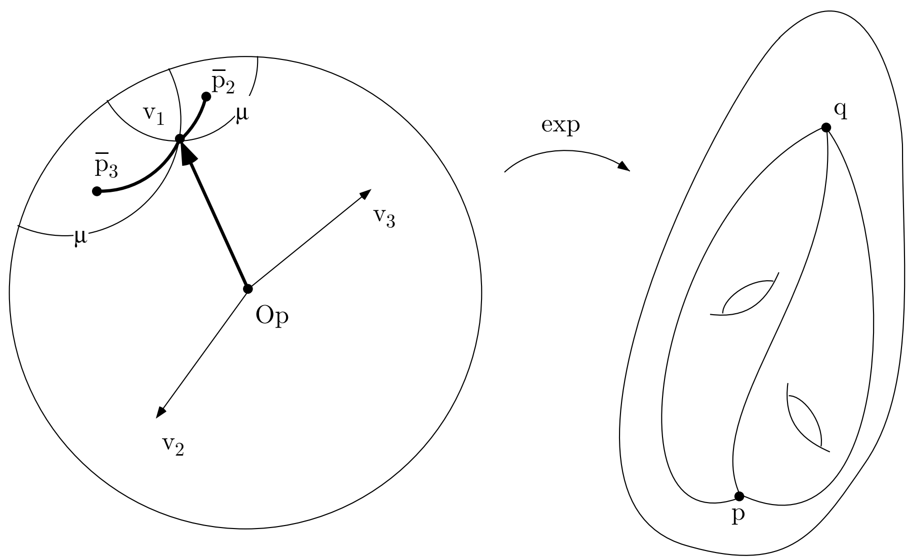

Figure 1. Mediatrices μ corresponding to the focal decomposition. The $\bar{d}$-distance from v_1 to $O_p, \bar{p}_2, \bar{p}_3$ is ℓ. The vectors v_1, v_2, v_3 have common exponential image q, while $O_p, \bar{p}_2, \bar{p}_3$ have exponential image p.

3. A Multitransversality Result

In [1], following Mather, we considered the multi-exponential map

$$\begin{aligned} E^k : V_p^k \times \mathcal{R} &\to (M \times \mathbb{R})^k \\ (v_1, \ldots, v_k, g) &\mapsto (\exp(v_1), |v_1|, \ldots \exp(v_k), |v_k|), \end{aligned}$$

where V_p^k is the set of k-tuples of distinct nonzero vectors in T_pM, and $\exp$, $|\ |$ refer to the Riemann structure g. The diagonal of $(M \times \mathbb{R})^k$ is

$$\Delta = \{(q, \ell, \ldots, q, \ell) : q \in M \text{ and } \ell \in \mathbb{R}\}.$$

Theorem 6.1 of [1] states that if $k \geq 3$ then E^k is transverse to Δ. Here we need also the case $k = 2$. Although the proof becomes easier if we use a negative curvature hypothesis, we give the proof in general, since we hope to use the theorem as tool when M has conjugate points.

Theorem 3.1. *$E^2 : V_p^2 \times \mathcal{R} \to (M \times \mathbb{R})^2$ is transverse to Δ.*

Proof. We give the proof in the case that M has dimension two, the main difference from the higher dimensional case being notational.

Lemma 6.3 of [1] states that, given $L > 0$, there is an open-dense set $\mathcal{G}(p, L) \subset \mathcal{R}$ such that for $g \in \mathcal{G}(p, L)$, there are at most a finite number of geodesic loops γ at p having length $\leq L$, and that

(a) γ is not a closed geodesic. (That is, the vectors tangent to γ at its beginning and end are distinct.)
(b) γ is "single" in the sense that it meets p only at its beginning and end, although other self-intersections are permitted.
(c) Under perturbation of g, γ evolves continuously: it does not disappear or bifurcate.

Although some of the geodesic loops γ may be self-conjugate in the sense that there is a transverse Jacobi field J along γ that vanishes at both ends of γ, a perturbation of g eliminates this feature. No such self-conjugacy can be created by a small perturbation of g, so we can restrict attention to Riemann structures in an open-dense subset $\mathcal{G}^*(p, L) \subset \mathcal{G}(p, L)$ that have no self-conjugate geodesic loops of length $\leq L$.

Let $P = (v_1, v_2, g) \in V_p^2 \times \mathcal{G}^*(p, L)$ have E^2-image $Q = (q, \ell, q, \ell) \in \Delta$. Let S be the sum $S = \mathrm{Image}(T_P E^2) + T_Q \Delta$. We must show

$$S = T_Q(M \times \mathbb{R})^2.$$

To do so we choose a basis of $T_Q(M \times \mathbb{R})^2$ as follows.

The natural inclusions

$$\begin{array}{cc} M \hookrightarrow (M \times \mathbb{R})^2 & \quad M \hookrightarrow (M \times \mathbb{R})^2 \\ z \mapsto (z, \ell, q, \ell) & \quad z \mapsto (q, \ell, z, \ell) \end{array}$$

induce isomorphisms

$$i_1 : T_q M \to T_q M \times \ell \times q \times \ell \qquad i_2 : T_q M \to q \times \ell \times T_q M \times \ell$$

into the tangent space $T_Q(M \times \mathbb{R})^2$.

We refer to the geodesics $t \mapsto \exp(tv_j)$ as γ_j, $j = 1, 2$, and to their terminal tangent vectors as $w_1 = \gamma_1'(1)$, $w_2 = \gamma_2'(1)$. The time parameter t is always restricted to $[0, 1]$. Choose vectors $u_1, u_2 \in T_q M$, normal to w_1, w_2. This gives bases $\{u_1, w_1\}$, $\{u_2, w_2\}$ of $T_q M$, which the inclusions convert to a basis $\{e_1, f_1, h_1, e_2, f_2, h_2\}$ of $T_Q(M\times\mathbb{R})^2$; namely

$$e_1 = i_1(u_1) \quad f_1 = i_1(w_1) \quad e_2 = i_2(u_2) \quad f_2 = i_2(w_2)$$

where h_1, h_2 are tangent to the appropriate factor $\mathbb{R}$ in $(M\times\mathbb{R})^2$.

Case 1. The geodesics γ_1, γ_2 are unequal pointsets. We will show that E^2 is submersive at P, i.e., that

$$\operatorname{Image} T_P E^2 = T_Q(M\times\mathbb{R})^2.$$

Because γ_1, γ_2 have the same length, neither contains the other, so there are "free spots" – points $z_1 \in \gamma_1 \setminus \gamma_2$ and $z_2 \in \gamma_2 \setminus \gamma_1$. (Note that even for the generic g, q may be conjugate to p along these geodesics.) See Figure 2.

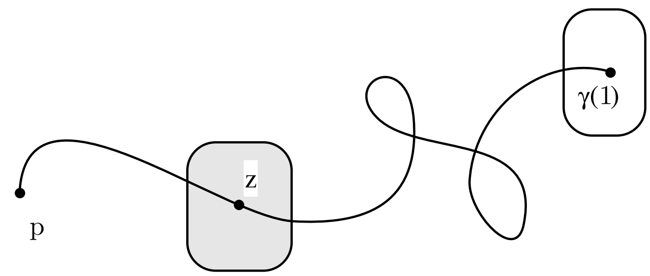

FIGURE 2. Varying g at a free spot z controls the endpoint $q = \gamma(1)$ of the geodesic γ.

Lemma 6.2 in [1] states that perturbation of g in the neighborhood of the free spots causes free and independent motion of the endpoints of γ_1, γ_2. Furthermore, perturbation of v_1 along itself makes $\ell_1 = |v_1|$ vary linearly; and yoked to this variation of ℓ_1, the endpoint $q_1 = \gamma_1(1)$ varies dependently along f_1. The corresponding facts hold for v_2, so we see that the image of $T_P E^2$ contains the vectors

$$e_1, f_1, f_1 + h_1, e_2, f_2, f_2 + h_2,$$

which is a basis for $T_Q(M\times\mathbb{R})^2$. This demonstrates that E^2 is submersive at P. Submersivity implies transversality.

Case 2. γ_1, γ_2 are equal as point sets – they are merely the same geodesic loop γ at p, traversed in opposite directions. See Figure 3. This implies that there are no free spots, so perturbation of the Riemann structure is futile.

Because γ is a geodesic loop, but not a closed geodesic, the terminal vectors w_1, w_2 are linearly independent. Since they have equal length and are perpendicular to u_1, u_2, the coefficients b, d in the expression

$$w_1 = au_2 + bw_2 \quad w_2 = cu_1 + dw_1.$$

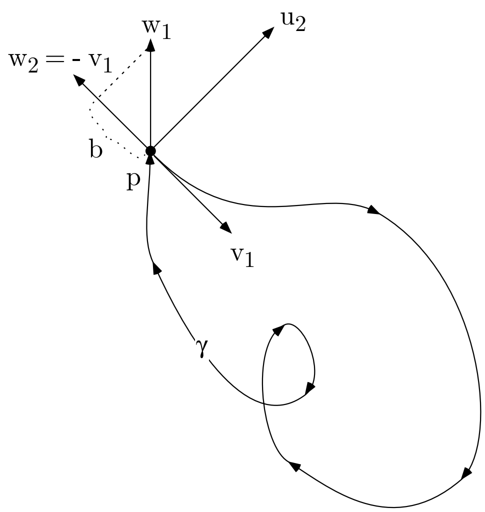

FIGURE 3. A geodesic loop.

satisfy

$$|b|, |d| < 1. \tag{1}$$

Because the loop γ is not self-conjugate, variation of v_1 perpendicular to itself produces free and independent variation of the endpoint $q_1 = \gamma_1(1)$ perpendicular to w_1. The same is true for v_2. Thus, the image of $T_P E^2$ contains the vectors e_1, e_2. As in Case 1, variation of v_1, v_2 along themselves gives vectors $f_1 + h_1, f_2 + h_2$ in the image of $T_P E^2$. Altogether, then, we have four vectors

$$e_1, f_1 + h_1, e_2, f_2 + h_2 \in T_P E^2.$$

The curves

$$\delta_1(t) = (\gamma_1(t), \ell, \gamma_1(t), \ell) \qquad \delta_2(t) = (\gamma_2(t), \ell, \gamma_2(t), \ell)$$

are contained in Δ, and hence $T_Q\Delta$ contains their tangents at $t = 1$, namely,

$$\delta_1'(1) = f_1 + i_2(au_2 + bw_2) = f_1 + ae_2 + bf_2$$
$$\delta_2'(1) = i_1(cu_1 + dw_1) + f_2 = f_2 + ce_1 + df_1.$$

The linear combination

$$\delta = \delta_1'(1) - b\delta_2'(1) = (1 - bd)f_1 + ae_2 - bce_1$$

of these vectors is tangent to Δ. By (1), this gives an explicit expression for $f_1 \in S = T_P E^2 + T_Q \Delta$ as

$$f_1 = \frac{1}{(1 - bd)}\Big(bce_1 - ae_2 + \delta\Big).$$

In the same way, f_2 belongs to S, and so do

$$h_1 = (f_1 + h_1) - f_1 \qquad h_2 = (f_2 + h_2) - f_2.$$

Since S contains the whole basis $\{e_1, \dots, h_2\}$, it equals $T_Q(M \times \mathbb{R})^2$, which completes the proof in Case 2. □

Corollary 3.2. *For the generic $g \in \mathcal{R}$, and for all $k \geq 1$, the multiexponential $E_g^k : V_p^k \to (M \times \mathbb{R})^k$ is transverse to the diagonal Δ. When M has dimension two, the pre-image of Δ is empty for $k \geq 4$, is a discrete set of points for $k = 3$, and is a smooth* 1*-manifold for $k = 2$.*

Proof. The Abraham Transversality Theorem asserts that if a smooth map

$$F : X \times \mathcal{A} \to Y \supset W$$

is transverse to W where $\mathcal{A}$ is a Banach manifold and X, Y, W are finite dimensional, then for all a in a resdual subset of $\mathcal{A}$, the map

$$F(a, \ ) : X \to Y \supset W$$

is transverse to W. In our case, $\mathcal{R}$ is an open set of a Banach space, and we know that

$$E^2 : \mathcal{R} \times V^2 \to (M \times \mathbb{R})^2 \supset \Delta$$

is transverse to Δ. Thus, for the generic $g \in \mathcal{R}$, E_g^2 is transverse to Δ.

When $k = 1$, transversality is trivial because the diagonal coincides with $M \times \mathbb{R}$, while for $k \geq 3$, transversality is proved in [1], Theorem 6.1.

Now assume that M has dimension two. The codimension of Δ in $(M \times \mathbb{R})^k$ is $3k - 3$, and the dimension of V_p^k is $2k$. Thus, if $k \geq 4$ then the codimension in the target exceeds the domain dimension, so transverse intersection implies empty intersection: $E_g^k(V_p^k) \cap \Delta = \emptyset$. Similarly, because transversality preserves codimension, the pre-image of Δ under E_g^3 is a discrete set of points in V_p^3, and the pre-image of Δ under E_g^2 is a 1-manifold in V_p^2. □

4. Focal Branches

Fix a $g \in \mathcal{R}$ and let

$$\nu^k = (E_g^k)^{-1}(\Delta) = \{(v, \dots, v_k) \in V_p^k : E_g^k(v_1, \dots, v_k) \in \Delta\}.$$

Clearly ν^k is invariant under permutation of the factors T_pM in V_p^k. Thus, if $\pi_j : V_p^k \to T_pM$ projects V_p^k onto the j^{th} factor,

$$\beta^k = \pi_j(\nu^k)$$

is independent of j. Furthermore, $\beta^2 \supset \beta^3 \supset \beta^4 \supset \dots$ and,

$$\sigma_1 = T_pM \setminus \beta^2 \quad \sigma_2 = \beta^2 \setminus \beta^3 \quad \dots \quad \sigma_k = \beta^k \setminus \beta^{k+1}.$$

Proposition 4.1. *When M has no conjugate points, the projection $\pi_j : \nu^k \to T_pM$ is a proper immersion onto a closed subset of T_pM.*

Proof. Properness means that the pre-image of a compact set is compact. Thus, from any given a sequence $(v_{1n}, \ldots, v_{kn})$ in ν^k such that for some fixed j, v_{jn} converges in T_pM as $n \to \infty$, we must extract a subsequence, convergent in ν^k.

When $k = 1$ the assertion is trivial since the projection is the identity map. Thus we assume $k \geq 2$.

Convergence of v_{jn}, say to $v_jmm \in T_pM$, implies that $|v_{jn}| \to |v_j| = \ell$. Since all the other v_{in} have the same length, there is a subsequence (unrelabeled) such that $(v_{1n}, \ldots, v_{kn}) \to (v_1, \ldots, v_k)$. Each v_i has length ℓ. Fix $i \neq j$. Then $v_{in} \neq v_{jn}$. Since $\exp_p(v_{in}) = \exp_p(v_{jn})$, the facts that $k \geq 2$ and that exp is a local diffeomorphism from a neighborhood of the origin in T_pM to a neighborhood of p in M implies that $\ell \neq 0$. Also, since there are no conjugate points, exp is a local diffeomorphism at v_j, which implies that $v_i \neq v_j$. Thus $(v_1, \ldots, v_k) \in \nu^k$, which completes the proof of properness.

A continuous proper map into a metric space necessarily has a closed range. Hence $\pi_j(\nu^k)$ is closed in T_pM.

To check that π_j is an immersion, we must show that the projection $v_j(t)$ of each nonsingular curve $(v_1(t), \ldots, v_k(t))$ in ν^k is nonsingular in T_pM. Fix a $t_0 \in (a, b)$ where (a, b) is the curve's domain of definition. For at least one i, $\dot{v}_i(t_0) \neq 0$. Thus, $v_i(t)$ is nonsingular at t_0. Since there are no conjugate points, $\exp(v_i(t))$ is also nonsingular at t_0. Since $(v_1(t), \ldots, v_k(t)) \in \nu^k$, $\exp(v_j(t)) = \exp(v_i(t))$ is also nonsingular at t_0. Therefore, $v_j(t)$ is nonsingular at t_0. □

5. Proof of Theorem A

We assume that M is a compact surface of genus ≥ 2, that $p \in M$ is fixed, and we denote by $\mathcal{N}$ the nonempty set of Riemann structures on TM having negative curvature. Clearly, $\mathcal{N}$ is an open subset of $\mathcal{R}$ and so it makes sense to speak of the generic $g \in \mathcal{N}$.

A Riemann structure with negative curvature has no conjugate points. Thus, according to Corollary 3.2 and Proposition 4.1, the focal decomposition of T_pM is quite simple for the generic $g \in \mathcal{N}$. Namely:

(a) For all $k \geq 4$, σ_k is empty.
(b) $\sigma_3 = \beta^3$ is a discrete subset of T_pM.
(c) $\sigma_2 = \beta^2 \setminus \beta^3$ and β^2 consists of a closed set of immersed curves in T_pM.

Furthermore, in any fixed compact subset of T_pM, properties (a), (b), (c) remain valid for all small perturbations of g.

Consider a vector $v_1 \in \beta^3$. It has two friends $v_2, v_3 \in \beta^3$ with equal length and equal exponential image. Thus $(v_1, v_2) \in \nu^2$, and there are nonsingular curves $v_1(t)$, $v_2(t)$ with

$$v_1(0) = v_1 \quad v_2(0) = v_2 \quad (v_1(t), v_2(t)) \in \nu^2.$$

Likewise there are nonsingular curves $v_1^*(t)$, $v_3(t)$ with

$$v_1^*(0) = v_1 \quad v_3(0) = v_3 \quad (v_1^*(t), v_3(t)) \in \nu^2.$$

We claim that $\dot{v}_1(0)$ and $\dot{v}_1^*(0)$ are linearly independent. Suppose not. Nonsingularity implies that there is a $c \neq 0$ such that

$$\dot{v}_1(0) = c\dot{v}_1^*(0).$$

At time $t = 0$ the curve $\exp(v_3(ct))$ has tangent

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \exp(v_3(ct)) &= T_{v_3} \exp_p(c\dot{v}_3(0)) = c\, T_{v_3} \exp_p(\dot{v}_3(0)) \\ &= c\, T_{v_1} \exp_p(\dot{v}_1^*(0)) = T_{v_1} \exp_p(\dot{v}_1(0)) \\ &= \frac{d}{dt}\Big|_{t=0} \exp(v_1(t)) \end{aligned}$$

Similarly,

$$\frac{d}{dt}\Big|_{t=0} |v_3(ct)| = \frac{d}{dt}\Big|_{t=0} |v_1(t)|.$$

Thus, at $t = 0$

$$t \mapsto E_g^3(v_1(t), v_2(t), v_3(ct))$$

is tangent to the diagonal $\Delta \subset (M \times \mathbb{R})^3$. The upshot is that the range of $T_{(v_1,v_2,v_3)}E_g^3$ contains a nonzero vector tangent to the diagonal in $(M \times \mathbb{R})^3$. This contradicts the fact that $E_g^3 : V_p^3 \to (M \times \mathbb{R})^3$ is transverse to Δ, since Δ has codimension 6, which is the same as the dimension of V_p^3.

Now we know that $\dot{v}_1(0)$ and $\dot{v}_1^*(0)$ are linearly independent. This means that in addition to properties (a) - (c), above, we have

(d) Branches of σ_2 meet transversally in pairs, they do so only at points of σ_3, and every point of σ_3 is such a crossing point,

where by a **branch** of σ_2 we mean the projection of an arc in ν^2. Since transversality is an open property, (d) also remains valid under perturbation of the Riemannn structure.

The remainder of the proof of focal stability follows the pattern of Theorem 5.1 in [1]. Fix a compact set $S \subset T_pM$. Then choose a disc B in T_pM that contains S. We know that the focal decomposition amounts to a smooth one-dimensional graph, namely $\Gamma = \sigma_2 \cup \sigma_3$, which has transverse crossings of multiplicity two. We adjust B so that its boundary is transverse to Γ. Let g' be a small perturbation of g and let Γ' be the corresponding graph. Since all aspects of the graph depend continuously on the Riemann structure, and all are transverse, if g' is sufficiently close to g, then there exists a homeomorphism from B to itself that sends $\Gamma \cap B$ to $\Gamma' \cap B$.

6. Proof of Theorem B

We assume that M is a compact surface of genus ≥ 2 and we denote by $\mathcal{H} \subset \mathcal{R}$ the nonempty set of Riemann structures whose curvature is identically equal to -1.

Fix $g \in \mathcal{H}$ and $p \in M$. As described in section 2, we can lift g to a Riemann structure $\bar{g}$ on T_pM and view the focal decomposition in terms of mediatrices for the corresponding metric $\bar{d}$. Since g has constant negative curvature, $\bar{d}$ is isometric to the Poincaré metric ρ on the unit disc $\mathbb{D}$, and mediatrices are ρ-geodesics. As such, mediatrices are circular arcs meeting $\partial\mathbb{D}$ perpendicularly. Thus, any two mediatrices meet one another transversally, and they do so at most once.

Let P be the lattice of pre-images of p in T_pM, and denote the corresponding mediatrix set as

$$\mu = \{v \in T_pM : \text{ for some } \bar{p} \in P \setminus \{0\} \text{ and } |v| = \rho(v, \bar{p})\}.$$

Fix a compact set $S \subset T_pM$. At most finitely many μ-mediatrices meet S. Choose a constant R and let B denote the compact disc

$$T_pM(R) = \{v \in T_pM : |v| \leq R\}.$$

When R is large, B contains S in its interior and we can adjust R so that ∂B is transverse to the μ-mediatrices. Let τ be the finite set of points in B at which the μ-mediatrices intersect one another. Thus

$$\tau = (\sigma_3 \cup \sigma_4 \cup \cdots \cup \sigma_\ell) \cap B$$

for some finite ℓ. If $v \in \sigma_k$, there are $k-1$ μ-mediatrices that pass through it. They are pairwise transverse. A small change of p preserves all transversalities in the disc of radius R; such a perturbation of the base point can not increase the multiplicity of a vector in τ, although it may lower it. (Here is where the argument uses the fact that the curvature is constant – mediatrices in the constant curvature case are always transverse to one another.) Thus, there is an open-dense set $U \subset M$ such that if $p \in U$ and p' is sufficiently near p then all multiplicities of the μ'-mediatrices in $B' = T_{p'}M(R)$ are the same as those in B. (By μ' we denote the mediatrices between the origin of $T_{p'}M$ and the other lifts of p' in $T_{p'}M$.) In other words, the graph of μ'-mediatrices in B' is homeomorphic to the graph of μ-mediatrices in B. Taking a sequence of compact sets S_n that exhausts T_pM leads to a sequence of such open dense sets U_n in M, and if $p \in \bigcap U_n$ then the focal decomposition in T_pM is stable with respect to perturbation of the base point p. This completes the proof of the first assertion in Theorem B.

The second assertion in Theorem B is proved in the same way. Again, mediatrix transversality implies that perturbation of g can only decrease intersection multiplicity, it cannot increase it. Thus, there is an open dense set in $\mathcal{U} \subset \mathcal{H}$ such that if $g \in \mathcal{U}$ and $g' \in \mathcal{H}$ is near enough to g then the μ'-mediatrix graph in B is homeomorphic to the μ-mediatrix graph in B. Again, choosing a sequence of compact sets S_n that exhausts T_pM leads to a sequence of open dense sets $\mathcal{U}_n$ in $\mathcal{H}$, and if $g \in \bigcap \mathcal{U}_n$ then the focal decomposition of T_pM is stable with respect to perturbation of g within $\mathcal{H}$.

Remark. Theorem B does not assert the multiplicity of the focal decomposition of the generic $g \in \mathcal{H}$ is at most three. We believe, however, that such an assertion is correct, and we can phrase our expectation as follows. If M has genus $s \geq 2$ then the Teichmuller space τ_s of hyperbolic structures on M amounts to $\mathcal{H}/\mathcal{D}$ where $\mathcal{D}$ denotes isometric deformations. It is a space smoothly parameterized by $6(s-1)$ real variables, and we expect that for a residual subset of these parameter values the corresponding hyperbolic structure has $\sigma_k = \emptyset$ for all $k \geq 4$. From this it would follow at once that the generic $g \in \mathcal{H}$ is focally stable with respect to variation of g in $\mathcal{R}$, not just with respect variation of g within $\mathcal{H}$, as in Theorem B.

7. Conjugate Points

A Riemann structure with non-negative curvature has no conjugate points, but the set $\mathcal{S}$ of such Riemann structures does not form an open subset of $\mathcal{R}$. For example, a flat Riemann structure on the torus has no conjugate points although it can be perturbed so that conjugate points appear. See [1], Proposition 4.8, where a bump is glued to a cylinder. Thus, the assertion of Theorem A′, below, should be viewed with caution, for a generic subset of $\mathcal{S}$ need not be dense in $\mathcal{S}$.

Theorem A′. *Focal stability (for a fixed $p \in M$) holds for the generic $g \in \mathcal{S}$.*

Proof. In the proof of Theorem A, we only used the assumption that $\exp_p$ is a local diffeomorphism, i.e., that there are no conjugate points, and the fact that the generic $g \in \mathcal{R}$ stably possess the transversality properties (a)-(d). $\square$

The next result shows that Theorem A′ has wider scope than Theorem A.

Proposition 7.1. *The interior of $\mathcal{S}$ is strictly larger than $\mathcal{N}$.*

Proof. Let M be the bitorus, or any surface with genus ≥ 2. Equip it with a Riemann structure of constant curvature -1. Cut out a small disc in M and replace it with a small polar cap having unit positive curvature. A smoothing collar is used to attach the cap. This gives a new Riemann structure g on M, $g \notin \mathcal{N}$. Any g-geodesic spends relatively little time in the polar cap or collar. Most of the time the geodesic travels through the part of the surface with curvature -1. Thus, there are no conjugate points, so g belongs to $\mathcal{S}$, and the same holds for all nearby Riemann structures. □

Remark. The question remains as to whether the generic Riemann structure on a surface has the focal stability property – i.e., whether we can do without the no conjugate point assumption. Much of what we proved above does hold when there are conjugate points, and also some generic properties of conjugate points are known. For example, in [3], Alan Weinstein announces that in dimension two, the singularities of the generic exponential map are either folds or cusps. These are the Whitney singularities for maps of the plane to itself. If, in addition to this, we knew how the fold and cusp singularities relate to the foliation of T_pM by circles of constant radius, then we could probably resolve the Focal Stability Conjecture for surfaces. In higher dimensions the singularities of the generic exponential map are much more complicated than in dimension two, cf. [3], which leads us to think that the Focal Stability Conjecture will be quite hard to resolve in full generality.

References

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