Topological dimension of singular-hyperbolic attractors

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Abstract

An *attractor* is a transitive set of a flow to which all positive orbit close to it converges. An attractor is *singular-hyperbolic* if it has singularities (all hyperbolic) and is partially hyperbolic with volume expanding central direction [16]. The geometric Lorenz attractor [6] is an example of a singular-hyperbolic attractor with topological dimension ≥ 2. We shall prove that *all* singular-hyperbolic attractors on compact 3-manifolds have topological dimension ≥ 2. The proof uses the methods in [15].

1 Introduction

This paper is concerned with the topological dimension of attractors for flows on compact manifolds. By *attractor* we mean a transitive set of the flow to which all positive orbit close to it converges. The attractors under consideration will be *singular-hyperbolic* in the sence that they have singularities (all hyperbolic) and are partially hyperbolic with volume expanding central direction [16]. In particular, the singular-hyperbolic attractors are volume hyperbolic sets as defined in [1]. The geometric Lorenz attractor is an example of a singular-hyperbolic attractor with topological dimension ≥ 2. We shall prove that *all* singular-hyperbolic attractors on compact 3-manifolds have topological dimension ≥ 2. The proof uses the methods developed in [15]. Let us state our result in a precise way.

Hereafter X will be a C^1 vector field on a compact manifold M. The flow of X is denoted by X_t, $t \in \mathbb{R}$. Given $p \in M$ we define $\omega(p) = \omega_X(p)$, the ω-limit set of p, as the accumulation point set of the positive orbit of p. The *α-limit set* of p

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is the set $\alpha(p) = \alpha_X(p) = \omega_{-X}(p)$. A compact invariant set Λ of X is *transitive* or *attracting* depending on whether $\Lambda = \omega(p)$ or $\cap_{t>0} X_t(U)$ for some $p \in \Lambda$ or some compact neighborhood U of Λ respectively. An *attractor* is a transitive attracting sets. A closed orbit of X is either periodic or singular. A singularity of X is *hyperbolic* if none of its eigenvalues have zero real part.

A compact invariant set Λ of X is *partially hyperbolic* [8] if there are an invariant splitting $T\Lambda = E^s \oplus E^c$ and positive constants K, λ such that:

1. E^s *is contracting*, namely

$$|| DX_t/E^s_x || \leq Ke^{-\lambda t}, \quad \forall x \in \Lambda,\ \forall t > 0.$$

2. E^s *dominates* E^c, namely

$$|| DX_t/E^s_x || \cdot || DX_{-t}/E^c_{X_t(x)} || \leq Ke^{-\lambda t}, \quad \forall x \in \Lambda,\ \forall t > 0.$$

The central direction E^c of Λ is said to be *volume expanding* if the additional condition

$$| Det(DX_t/E^c_x) | \leq Ke^{-\lambda t}$$

holds $\forall x \in \Lambda$, $\forall t > 0$ where $Det(\cdot)$ means the jacobian. The above splitting $E^s \oplus E^c$ will be refered to as a (K, λ)-*splitting* in the Appendix.

Definition 1.1. *([16]) An attractor is* singular-hyperbolic *if it has singularities (all hyperbolic) and is partially hyperbolic with volume expanding central direction.*

Definition 1.2. *([9]) The* topological dimension *of a space E is either -1 (if $E = \emptyset$) or the last integer k for which every point has arbitrarily small neighborhoods whose boundaries have dimension less than k.*

The relation between dynamics and topological dimension was considered for hyperbolic systems [7, 18, 2, 3]; for expansive systems [10, 13]; and for singular-hyperbolic systems [14]. The result below generalizes to singular-hyperbolic attractors a well known property of both hyperbolic strange attractors and geometric Lorenz attractors.

Theorem A . *Singular-hyperbolic attractors on compact 3-manifolds have topological dimension ≥ 2.*

The idea of the proof is the following. Let Λ be a singular-hyperbolic attractor of a flow X on a compact 3-manifold M. It follows from [16] that all the singularities $\sigma \in \Lambda$ are *Lorenz-like*, namely the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of σ are real and satisfy $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$. The flow nearby σ can be described using the Grobman-Hartman Theorem [5]. In particular, a Lorenz-like singularity exhibits two *singular cross-sections* S^t, S^b and two *singular curves* l^t, l^b ([15]). A *singular cross section* of Λ is by definition a disjoint collection of singular cross sections S^t, S^b (as σ runs over all the singularities of Λ) whose horizontal boundaries does not intersect Λ. The *singular curve* of S is the union l of the respective singular curves l^t, l^b. A *singular partition* of Λ will be a compact neighborhood O of $\Lambda \cap l$ in S, for some singular cross section S of Λ, such that $\Lambda \cap l$ does not intersect the boundary of O and every regular orbit of Λ intersect O. The *size* of the singular partition O is the minimal $\epsilon > 0$ such that there is an invariant cone field in O (for the return map $\Pi : \mathrm{Dom}(\Pi) \subset O \to O$) on which the derivative of Π has expansion rate bigger than ϵ^{-1}. In Proposition 2.7 we prove that one-dimensional singular-hyperbolic attractors on compact 3-manifolds have singular partition with arbitrarily small size. The proof of this proposition uses the Lemmas 7.5 and 7.6 in [15]. These lemmas will be proved in the Appendix for the sake of completeness. In Theorem 2.9 we shall prove that singular-hyperbolic attractors Λ on compact 3-manifolds cannot have singular partitions with arbitrarily small size. Theorem A will follow from Proposition 2.7 and Theorem 2.9.

2 Proof

We start with some definitions. Hereafter Λ is a singular-hyperbolic attractor of a C^1 flow X on a compact 3-manifold M. Since Lorenz-like singularities σ are hyperbolic they are equipped with three invariant manifolds $W^s_X(\sigma), W^u_X(\sigma), W^{ss}_X(\sigma)$ each one tangent at σ to the eigenspace corresponding to $\{\lambda_2, \lambda_3\}, \{\lambda_1\}, \{\lambda_2\}$ respectively. It follows from [16] that every singularity σ of X in Λ is Lorenz-like and satisfies $\Lambda \cap W^{ss}_X(\sigma) = \{\sigma\}$. The classical Grobman-Hartman Theorem [5] gives the description of the flow nearby σ. This is done at Figure 1. Note that $W^{ss}_X(\sigma)$ separates $W^s_X(\sigma)$ in two connected components denoted the top and the bottom respectively. In one of these components, say the top one, we consider a cross-section $S^t = S^t_\sigma$ together with a curve $l^t = l^t_\sigma$ as in Figure 1. Similarly we consider a cross-section $S^b = S^b_\sigma$ and a curve $l^b = l^b_\sigma$ located in the bottom component of $W^s_X(\sigma)$. See Figure 1. Both S^* (for $* = 1, 2$) are homeomorphic to $[0,1] \times [0,1]$. S^* can be chosen in a way that l^* is contained in $W^s_X(\sigma) \setminus W^{ss}_X(\sigma)$. The positive flow lines of X starting at $S^t \cup S^b \setminus (l^t \cup l^b)$ exit a small neighborhood of σ passing through the cusp region as indicated in Figure 1. The positive orbits starting at $l^t \cup l^b$ goes directly to σ. We note that the boundary of S^* is formed by four curves, two of them transverse to l^* and two of them parallel to l^*. The union of the curves in the boundary of S^* which are parallel (resp. transverse) to l^* is denoted by $\partial^v S^*$ (resp. $\partial^h S^*$). The interior (as a submanifold) of S^* is denoted by $Int(S^*)$.

Remark 2.1. *An immediate consequence of $\Lambda \cap W_X^{ss}(\sigma) = \{\sigma\}$ is the following. Let σ be a singularity of X in Λ. Then there are cross-sections S^t, S^b as above arbitrarily close to σ such that $\Lambda \cap \partial^h S^* = \emptyset$ $(* = t, b)$. Since the two boundary points of l^* are in $\partial^h S^*$ we have that $\Lambda \cap l^* \subset Int(S^*)$.*

Definition 2.2. *We shall call the cross sections S^t, S^b as* singular cross sections *associated to σ. The curves l^t, l^b are called* singular curves *of S^t, S^b respectively. A* singular cross section *of Λ is a finite disjoint collection $\{S^t_\sigma, S^b_\sigma : \sigma$ is a singularity of X in $\Lambda\}$ satisfying $\Lambda \cap \partial^h S = \emptyset$. The* singular curve *of S is the associated collection of singular curves $l = \{l^t_\sigma, l^b_\sigma : \sigma$ is a singularity of X in $\Lambda\}$.*

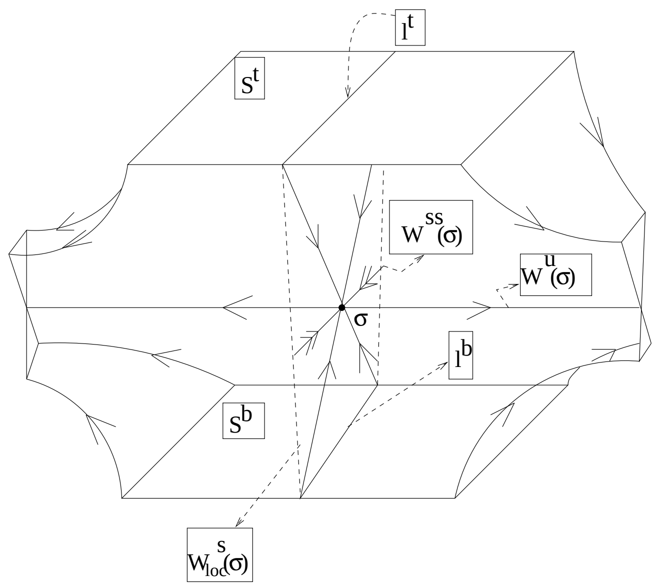

Figure 1: Lorenz-like singularity.

Hereafter we denote by $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ the singular-hyperbolic splitting of Λ. The contracting direction E^s is one-dimensional and contracting. So, E^s_Λ can be extended to an invariant contracting splitting $E^s_{U(\Lambda)}$ on a neighborhood $U(\Lambda)$ of Λ. The standard Invariant Manifold Theory [8] implies that $E^s_{U(\Lambda)}$ is tangent to a continuous foliation $\mathcal{F}$ on $U(\Lambda)$. If S is a singular cross-section contained in $U(\Lambda)$, we denote by $\mathcal{F}^S$ the foliation of S obtained projecting $\mathcal{F}$ into S along X. The space of leaves of $\mathcal{F}^S$ will be denoted by I^S. We extend E^c_Λ continuously to a subbundle $E^c_{U(\Lambda)}$ of $T_{U(\Lambda)}M$. *In what follows we fix such a neighborhood $U(\Lambda)$ of Λ.*

Remark 2.3. *It is possible to choose S arbitrarily close to the singularities of Λ in a way that l is a finite union of leaves of $\mathcal{F}^S$ and I^S is a finite disjoint union of compact intervals.*

The following lemma is a direct consequence of standard argument involving topological dimension. We prove it here for the sake of completeness.

Lemma 2.4. *Let S a singular cross-section and l be its associated singular curve. If Λ is one-dimensional, then there is a compact neighborhood O of $\Lambda \cap l$ in S whose boundary ∂O satisfies $\Lambda \cap \partial O = \emptyset$.*

Proof. Note that $\Lambda \cap \partial^h S = \emptyset$ since S is a singular cross-section. As noted in Remark 2.1 one has $\Lambda \cap l \subset Int(S)$. Fix $x \in \Lambda \cap l$. Then $x \in Int(S)$. Because Λ is one dimensional we have that $\Lambda \cap S$ is zero dimensional [9]. Then, by the definition of the topological dimension, one can find an open set S_x of $\Lambda \cap S$ containing x such that $\partial S_x = \emptyset$. Note that the topology in $\Lambda \cap S$ is the one induced by S. In follows that $S_x = (\Lambda \cap S) \cap O_x$ for some open set O_x of M. Since S is transversal to X we can choose O_x such that $\partial S_x = (\Lambda \cap S) \cap \partial O_x$ (for this we can use the Tubular Flow-Box Theorem [5]). It follows that

$$(\Lambda \cap S) \cap \partial O_x = \emptyset.$$

On the other hand, $\Lambda \cap l$ is compact in S and $\{S \cap O_x : x \in \Lambda \cap l\}$ is an open covering of $\Lambda \cap l$. It follows that there is a finite subcollection of $\{S \cap O_x : x \in \Lambda \cap l\}$ covering $\Lambda \cap l$. Denote by O the union of the closures (in S) of the elements of such a subcollection. It follows that O is a compact neighborhood of $\Lambda \cap l$ in S. Since O is a finite union of $S \cap O_x$'s satisfying $(\Lambda \cap S) \cap \partial O_x = \emptyset$ we have that $\Lambda \cap \partial O = \emptyset$. This proves the lemma. $\square$

Herefter O is a set contained in a singular cross section S. Clearly O defines a return map

$$\Pi : \mathrm{Dom}(\Pi) \subset O \to O$$

given by

$$\Pi(x) = X_{t(x)}(x),$$

where $\mathrm{Dom}(\cdot)$ denotes the domain and $t(\cdot)$ denotes the return time.

Remark 2.5. *Note that Π may be discontinuous in $\Pi^{-1}(\partial O)$. However if $x \in \Pi^{-1}(Int(O))$ then Π is C^1 in an open neighborhood of x contained in $Int(O)$. This is an immediate consequence of the Tubular Flow-Box Theorem.*

We denote by TO the tangent space of O relative S. If $x \in M$ we denote by $\angle(v_x, w_x)$ the tangent of the angle between $v_x, w_x \in T_x M$. If L_x is a linear subspace of $T_x M$, we define

$$\angle(v_x, L_x) = \inf_{w_x \in L_x} \angle(v_x, w_x).$$

Given $\alpha > 0$ we define the *cone*

$$C_\alpha(L_x) = \{v_x \in T_xM : \angle(v_x, L_x) \leq \alpha\}.$$

If $L : x \in Dom(L) \to L_x$ is a map and $\alpha > 0$ we define the *cone field*

$$\mathcal{C}_\alpha(L) = \{C_\alpha(L_x) : x \in Dom(L)\}.$$

The case $L = E^c$ will be interesting. The definition below is a minor modification of the corresponding definition in [15]. If $x \in M$ we denote $X_{I\!R}(x)$ the full orbit of x.

Definition 2.6. *A* singular partition *of Λ is a set O satisfying the following properties:*

1. *There is a singular cross-section S such that $O \subset Int(S)$ is a compact neighborhood of $\Lambda \cap l$.*
2. $\Lambda \cap \partial O = \emptyset$.
3. $\mathrm{Sing}_X(\Lambda) = \{q \in \Lambda : X_{I\!R}(q) \cap O = \emptyset\}$.

The size of O *is the minimal number $\epsilon > 0$ for which there is $\alpha > 0$ such that the cone field $\mathcal{C}_\alpha(E^c)$ satisfies :*

4. *If $x \in \mathrm{Dom}(\Pi)$, then*

$$D\Pi(x)\left(C_\alpha(E^c_x) \cap TO_x\right) \subset \mathrm{int}\left(C_{\frac{\alpha}{2}}(E^c_{\Pi(x)}) \cap TO_{\Pi(x)}\right).$$

5. *If $x \in \mathrm{Dom}(\Pi)$ and $v_x \in C_\alpha(E^c_x) \cap TO_x$, then*

$$\|D\Pi(x)(v_x)\| \geq \epsilon^{-1}\|v_x\|.$$

6. $\inf\{\angle(v_x, E^s_x) : x \in O, v_x \in C_\alpha(E^c_x) \cap TO_x\} > 0$.

The following proposition studies the existence of singular partition with arbitrarily small size for certain one-dimensional singular-hyperbolic sets. Its proof uses the methods developed in [15]. We let $\mathrm{Sing}_X(\Lambda) = \{\sigma_1, \cdots, \sigma_k\}$ be the set of singularities of X in Λ.

Proposition 2.7. *One-dimensional singular-hyperbolic attractors on compact* 3*-manifolds have singular partitions with arbitrarily small size.*

Proof. Let Λ be a singular-hyperbolic attractor of a C^1 flow X on a compact 3-manifold M. We shall assume that Λ has topological dimension 1. We shall prove that Λ has singular partition with arbitrarily small size $\epsilon > 0$. For this we proceed as follows. Since Λ has topological dimension 1 we have that Λ cannot contain hyperbolic sets (the unstable manifold of a hyperbolic set in Λ would be two-dimensional and contained in Λ). It follows that $\omega(x)$ cannot be hyperbolic for all $x \in \Lambda$. By [17] if $\mathcal{L} = \omega(x), \alpha(x)$ then

$$\mathcal{L} \cap Sing_X(\Lambda) \neq \emptyset, \quad \forall x \in \Lambda. \tag{1}$$

Choose $\alpha > 0$ such that

$$\inf\{\angle(v_x, E^s_x) : x \in U(\Lambda), v_x \in C_\alpha(E^c_x)\} > 0.$$

By [15, Lemma 7.5] (see Lemma 3.1) we can find a neighborhood $U_\alpha \subset U(\Lambda)$ of Λ and positive constants $T_\alpha, K_\alpha, \lambda_\alpha$ such that the following properties hold:

(P1). If $x \in U_\alpha$ and $t \geq T_\alpha$, then

$$DX_t(x)(C_\alpha(E^c_x)) \subset C_{\alpha/2}(E^c_{X_t(x)}).$$

(P2). If $x \in U_\alpha$ is regular, $X(x) \in C_\alpha(E^c_x)$, $t \geq T_\alpha$ and $v_x \in C_\alpha(E^c_x)$ is orthogonal to $X(x)$, then

$$\| P^t_x(v_x) \| \cdot \| X(X_t(x)) \| \geq K_\alpha e^{\lambda_\alpha t} \cdot \| v_x \| \cdot \| X(x) \|,$$

where P^t_x denotes the *Poincaré flow associated to* X (see [4, 15] or the Appendix).

Once we fix α and U_α we apply [15, Lemma 7.6] (see Lemma 3.2) to find, for every $i \in \{1, \cdots, k\}$, a pair of singular cross-sections $S^{*,0}_i$ associated to σ_i $(* = t, b)$ such that

$$X(x) \in C_\alpha(x), \quad \forall x \in S^{*,0}_i.$$

Define

$$S^0 = \cup_{i=1}^k (S^{t,0}_i \cup S^{b,0}_i).$$

It is clear that S^0 is a singular cross-section. We denote by l^0 the singular curve of S^0. Since S^0 is transversal to X one can find a constant $D > 0$ (depending on S^0) such that

$$\frac{\| X(x) \|}{\| X(y) \|} > D,$$

for all $x, y \in S^0$. We choose $T_\epsilon > T_\alpha$ large enough so that

$$K_\alpha e^{\lambda_\alpha t} \cdot D > \epsilon^{-1} \tag{2}$$

for all $t \geq T_\epsilon$.

For every $\delta > 0$ we consider a singular-cross section $S^\delta \subset S^0$ ($i = 1, \cdots, k$ and $* = t, b$) formed by small bands $S_i^{*,\delta}$ of diameter 2δ around the singular curve l_i^* of $S_i^{*,0}$. Note that the singular curve of S^δ is l^0 (the one of S^0) for all δ. Since Λ is one-dimensional Lemma 2.4 implies that $\forall x \in \Lambda \cap l^0$ there is a compact neighborhood $O = O^\delta \subset S^\delta$ of $\Lambda \cap l^0$ such that $\Lambda \cap \partial O = \emptyset$. Note that O is a singular partition of Λ. In fact, (1) and (2) of Definition 2.6 are obvious. And (3) of Definition 2.6 follows from Eq. (1) since O is a compact neighborhood of $\Lambda \cap l^0$.

Let us prove that if $\delta > 0$ small enough, then O has size ϵ. For this we need to prove that for δ small the cone field $\mathcal{C}_\alpha(E^c)$ satisfies the properties (4)-(6) of Definition 2.6. Let $\Pi : Dom(O) \subset O \to O$ be the return map induced by X in O. By definition $\Pi(x) = X_{t(x)}(x)$ where $t(x)$ is the return time of $x \in Dom(O) \subset O$ into O. To calculate $D\Pi(x)$ we can assume without loss of generality that S^0 is orthogonal to X. It follows that

$$D\Pi(x) = P_x^{t(x)}$$

for all $x \in Dom(\Pi)$. Shrinking δ one has $t(x) > T_\epsilon$ for all $x \in Dom(\Pi) \subset O$. This allows us to apply the properties (P1)-(P2) above. In fact, since $D\Pi(x) = P_x^{t(x)}$ one has

$$D\Pi(x)/C_\alpha(E_x^c) \cap TO_x = P_x^{t(x)}/C_\alpha(E_x^c) \cap TO_x.$$

Then, Definition 2.6-(4) follows from (P1). (P2) and Eq. (2) imply

$$\|D\Pi(x)(v_x)\| = \|P_x^{t(x)}(v_x)\| =$$

$$= \|P_x^{t(x)}(v_x)\| \cdot \|X(X_{t(x)}(x))\| \cdot \|X(X_{t(x)}(x))\|^{-1} \geq$$

$$\geq K_\alpha e^{\lambda_\alpha t(x)} \cdot \frac{\|X(x)\|}{\|X(X_{t(x)}(x))\|} \cdot \|v_x\| \geq K_\alpha e^{\lambda_\alpha t(x)} \cdot D \cdot \|v_x\| \geq \epsilon^{-1}\|v_x\|,$$

$\forall x \in \mathrm{Dom}(\Pi)$, $\forall v_x \in C_\alpha(E_x^c) \cap TO_x$ because $X(x) \in C_\alpha(E_x^c)$ $\forall x \in S^0$ and $t(x) > T_\epsilon$ ($\forall x \in \mathrm{Dom}(\Pi)$). This proves Definition 2.6-(5). Definition 2.6-(6) is a direct consequence of the choice of α. The result follows. $\square$

The following lemma will be used to prove Theorem 2.9. Recall that if S is a singular cross-section then S is endowed with a singular curve l. If $O \subset S$ then $\Pi : \mathrm{Dom}(\Pi) \subset O \to O$ denotes the return map associated to O.

Lemma 2.8. *Let Λ be a singular-hyperbolic attractor of a flow X on a compact* 3-*manifold. Let O be a singular partition of Λ. Then, there is an open neighborhood $O' \subset O$ of $\Lambda \cap O$ such that:*

1. $O' \setminus l \subset \mathrm{Dom}(\Pi)$.

2. Π *is* C^1 *in* $O' \setminus l$.

3. $\Pi(O' \setminus l) \subset O'$.

Proof. Because Λ is an attractor we have that the unstable manifold of any of its singularities is contained in Λ. In particular, every connected component of $W^u_X(\sigma_i) \setminus \{\sigma_i\}$ is contained in Λ $\forall i$. It follows from Definition 2.6-(3) that all such components intersect O. By Definition 2.6-(2) such intersections can occur only in $Int(O)$. This implies that there are small open bands, centered at the singular curves in l, whose union $V(l)$ satisfies $V(l) \setminus l \subset \Pi^{-1}(Int(O))$. As noted in Remark 2.5 we have that Π is C^1 in $V(l) \setminus l$. Again by Definition 2.6-(2)-(3) one has $(\Lambda \cap O) \setminus V(l) \subset \Pi^{-1}(Int(O))$. So, by Remark 2.5, since $(\Lambda \cap O) \setminus V(l)$ is compact, there is an open neighborhhod V of $(\Lambda \cap O) \setminus V(l)$ contained in $\mathrm{Dom}(\Pi)$ such that Π is C^1 in V. Observe that $V \cup V(l)$ is an open neighborhood of $\Lambda \cap O$ such that Π is C^1 in $(V \cup V(l)) \setminus l$. On the other hand, Λ is an attractor by assumption. Then, there is a neighborhood U^* such that $X_t(U^*) \subset U^*$, $\forall t > 0$. Clearly one can choose U^* to be *arbitrarily close to* Λ. In particular, $O' := O \cap U^*$ is contained in $V \cup V(l)$. It follows that $O' \setminus l \subset \mathrm{Dom}(\Pi)$ because $V \cup (V(l) \setminus l) \subset \mathrm{Dom}(\Pi)$. Because $X_t(U^*) \subset U^*$ for all $t > 0$ and the return time for the points in $\mathrm{Dom}(\Pi)$ is positive we conclude that $\Pi(O' \setminus l) \subset O'$. As Π is C^1 in $(V \cup V(l)) \setminus l$ and $O' \subset V \cup V(l)$ we conclude that Π is C^1 in $O' \setminus l$ This proves the result. ☐

Theorem 2.9. *Singular-hyperbolic attractors on compact* 3*-manifolds cannot have singular partitions with arbitrarily small size.*

Proof. Let Λ be a singular-hyperbolic attractor of a C^1 flow X on a compact 3-manifold M. By contradiction we assume that Λ has a singular partition O with arbitrarily small size $\epsilon > 0$. We fix $\epsilon \in (0, 1/2)$. We let O' be the open neighborhood obtained in Lemma 2.8 for O. Hereafter we say that a C^1 connected curve c in O is a C^u*-curve* if its tangent vector belongs to the cone field $C_\alpha(E^c)$ at Definition 2.6-(4). Definition 2.6-(4) implies that Π carries C^u curves in $\Pi^{-1}(Int(O))$ into C^u curves in O (see also Remark 2.5). Definition 2.6-(6) implies that a C^u curve in O intersects l in at most one point x_c. In that case x_c divides c in two connected components the largest one being denoted by c^+. Clearly if $L(\cdot)$ denotes the lenght, then

$$L(c^+) \geq (1/2)L(c).$$

Now, fix a C^u curve $c_1 \subset O' \setminus l$. Define $R = (2\epsilon)^{-1}$. The choice of ϵ implies $\epsilon^{-1} > R > 1$. Lemma 2.8-(1) implies $c_1 \subset \mathrm{Dom}(\Pi)$. Lemma 2.8-(2) implies that $c_2 = \Pi(c_1)$ is a C^u curve contained in O'. Definition 2.6-(5) implies $L(c_2) \geq \epsilon^{-1}L(c_1) \geq R \cdot L(c_1)$. Suppose we have constructed a sequence $c_1, c_2, \cdots, c_i$ of

C^u curves of O contained in O' satisfying $L(c_j) \geq R \cdot L(c_{j-1})$ for all $2 \leq j \leq i$. If $c_i \cap l = \emptyset$ we define $c_{i+1} = \Pi(c_i)$ and keep going. If $c_i \cap l \neq \emptyset$ we define $c_{i+1} = Closure(\Pi(c_i^+))$. In any case c_{i+1} is a C^u curve of O contained in O'. In the first case we have $L(c_{i+1}) \geq \epsilon^{-1} L(c_i) \geq R \cdot L(c_i)$. In the second case we have

$$L(c_{i+1}) = L(\Pi(c_i^+)) \geq \epsilon^{-1} L(c_i^+) \geq (\epsilon^{-1}/2) \cdot L(c_i) = R \cdot L(c_i).$$

In this way we can construct an infinite sequence $c_1, \cdots, c_i, c_{i+1}, \cdots$ of C^u curves of O in O' all of which satisfying $L(c_{i+1}) \geq R \cdot L(c_i)$. It follows that

$$L(c_i) \geq R^i \cdot L(c_1),$$

for all i. Since $l(c_1) > 0$ and $R = (2\epsilon)^{-1} > 1$ we conclude that

$$\lim_{i \to \infty} L(c_i) = \infty.$$

On the other hand, let S be the singular cross-section containing O given by Definition 2.6-(1). Let $\mathcal{F}^S$ be the projection of the stable manifold in $U(\Lambda)$ over S. As noted in Remark 2.3 the leave space I^S of $\mathcal{F}^S$ is a finite union of compact intervals. In particular I^S has finite diameter. Since $O' \subset O \subset Int(S)$ we have that all the curves c_i are contained in S. Since c_i is a C^u curve we have by Definition 2.6-(6) that c_i have positive angle with the leaves of $\mathcal{F}^S$ (note that these leaves are tangent to E^s). So, we can project c_i to obtain an infinite sequence of intervals in I^S. The lenght of these intervals goes to ∞ (as $i \to \infty$) since $L(c_i) \to \infty$ (as $i \to \infty$). This is a contradiction since I^S has finite diameter. This contradiction proves the result. □

Proof of Theorem A : Let Λ be a singular-hyperbolic attractor on a compact 3-manifold. If Λ has topological dimension < 2 then Λ would be one-dimensional because it has regular orbits [9]. It would follow from Proposition 2.7 that Λ has singular partitions with arbitrarily small size contradicting Theorem 2.9. The proof follows. □

3 Appendix

In this section we state (and prove) two technical lemmas which were used in the proof of Theorem A . These lemmas were proved in [15] and here we reproduce these proofs for the sake of completeness. Let us state some definitions and notations.

First we define the Linear Poincaré Flow [4]. Let X be a flow on a compact 3-manifold M. The Riemmanian Metric of M is denoted by $< \cdot, \cdot >$. If x is a regular point of X (i.e. $X(x) \neq 0$), we denote by $N_x = \{v_x \in T_xM :< v_x, X(x) >= 0\}$ the orthogonal complement of $X(x)$ in T_xM. Denote $O_x : T_xM \to N_x$ the orthogonal projection onto N_x. For every $t \in I\!R$ we define $P_x^t : N_x \to N_{X_t(x)}$ by

$$P_x^t = O_{X_t(x)} \circ DX_t(x).$$

It follows that $P = \{P_x^t : t \in I\!R, X(x) \neq 0\}$ satisfies the cocycle relation

$$P_x^{s+t} = P_{X_t(x)}^s \circ P_x^s,$$

for every $t, s \in I\!R$. The parametrized family P is called the *Linear Poincaré Flow* of X.

We denote by $\mathrm{vol}(v_x, w_x)$ the area of the parallelogram in T_xM generated by $v_x, w_x \in T_xM$. As M is a compact manifold, there is a constant $V \geq 1$ such that $V^{-1} \leq \mathrm{vol}(v_x, w_x) \leq V$, $\forall x \in M$, $\forall v_x, w_x \in T_xM$ satisfying $\|v_x\| = \|w_x\| = 1$ and $< v_x, w_x >= 0$. For simplicity we shall assume that $V = 1$. In other words,

$$\mathrm{vol}(v_x, w_x) = \|v_x\| \cdot \|w_x\|,$$

$\forall x \in M$, $\forall v_x, w_x \in T_xM$ with $< v_x, w_x >= 0$.

In addition,

$$\mathrm{vol}(v_x, X(x)) = \|O_x v_x\| \cdot \|X(x)\|,$$

$\forall x \in M$ regular, $\forall v_x \in T_xM$. In particular,

$$\mathrm{vol}(DX_t(x)v_x, X(X_t(x))) = \|P_x^t(v_x)\| \cdot \|X(X_t(x))\|, \tag{3}$$

$\forall x \in M$ regular, $\forall t \in I\!R$, $\forall v_x \in N_x$.

Recall that if Λ is a singular hyperbolic set of X with (K, λ)-splitting $T_\Lambda M = E_\Lambda^s \oplus E_\Lambda^c$, then

$$\mid Det(DX_t/E_x^c) \mid \geq Ke^{\lambda t},$$

$\forall x \in \Lambda$, $\forall t \geq 0$, where $Det(\cdot)$ denotes the jacobian. So, if Λ is a singular hyperbolic set as above one has

$$\mathrm{vol}(DX_t(x)v_x^c, DX_t(x)w_x^c) \geq Ke^{\lambda t}\, \mathrm{vol}(v_x^c, w_x^c),$$

$\forall x \in \Lambda$, $\forall t \geq 0$, $\forall v_x^c, w_x^c \in E_x^c$.

Remember that $U(\Lambda)$ denotes a neighborhood of Λ where the splitting $E_\Lambda^s \oplus E_\Lambda^c$ extends to $E_{U(\Lambda)}^s \oplus E_{U(\Lambda)}^c$.

Lemma 3.1. *Let Λ be a singular-hyperbolic attractor of a C^1 flow X on a compact 3-manifold M. Then for every $\alpha \in (0, 1]$ there are a neighborhood $U_\alpha \subset U(\Lambda)$ of Λ and constants $T_\alpha, K_\alpha, \lambda_\alpha > 0$ such that :*

1. *If $x \in U_\alpha$ and $t \geq T_\alpha$, then*

$$DX_t(x)(C_\alpha(E_x^c)) \subset C_{\frac{\alpha}{2}}(E_{X_t(x)}^c).$$

2. If $x \in U_\alpha$ is regular, $X(x) \in C_\alpha(E^c_x)$, $t \geq T_\alpha$ and $v_x \in C_\alpha(E_x) \cap N_x$, then

$$\|P^t_x(v_x)\| \cdot \|X(X_t(x))\| \geq K_\alpha e^{\lambda_\alpha t} \cdot \|v_x\| \cdot \|X(x)\|.$$

Proof. Let Λ and $\alpha \in (0,1]$ be as in the statement. As mentioned above $T_{U(\Lambda)}M = E^s_{U(\Lambda)} \oplus E^c_{U(\Lambda)}$ denotes the extension of the (K,λ)-splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ of Λ to a neighborhood $U(\Lambda)$ of Λ. Let π^s the projection of T_Λ on E^s_Λ, and π^c be the projection of T_Λ on E^c_Λ. Denote $v_x = v^s_x + v^c_x \in E^s_x \oplus E^c_x = T_xM$ $\forall x \in U(\Lambda)$, $\forall v_x \in T_xM$. In other words, $v^s_x = \pi^s(v_x)$ and $v^c_x = \pi^c(v_x)$

As E^s_Λ (K,λ)-dominates E^c_Λ we have that

$$\|DX_t(x)/E^s_x\| \leq K^{-1}e^{-\lambda t}m(DX_t(x)/E^c_x), \tag{4}$$

$\forall x \in \Lambda$, $\forall t \geq 0$.

Fix $R > 4$ such that

$$\frac{K}{R} < 1. \tag{5}$$

Choosing $T^1 = T^1_\alpha > 0$ large enough one has

$$\|DX_{T^1}(x)/E^s_x\| \leq \frac{K\alpha}{2R}m(DX_t(x)/E^c_x), \tag{6}$$

$\forall x \in \Lambda$, $\forall t \geq 0$.

Since $E^s_\Lambda \oplus E^c_\Lambda$ is invariant we have $\pi^{s(c)}_{X_t(x)} \circ DX_t(x) = DX_t(x) \circ \pi^{s(c)}_x$, and so

$$\angle(DX_t(x)v_x, E^c_{X_t(x)}) = \frac{\|DX_t(x)v^s_x\|}{\|DX_t(x)v^c_x\|}, \quad \forall x \in \Lambda \quad \forall t \geq 0. \tag{7}$$

Recall that $\angle$ denotes the tangent of the angle. The inequality (6) and the last equality imply

$$\angle(DX_{T^1}(x)v_x, E^c_{X_{T^1}(x)}) \leq \frac{K\alpha}{2R}, \quad \forall x \in \Lambda, \ \forall v_x \in C_\alpha(E^c_x).$$

So,

$$DX_{T^1}(x)(C_\alpha(E^c_x)) \subset C_{\frac{K\alpha}{2R}}(E^c_{X_{T^1}}(x)), \quad \forall x \in \Lambda.$$

Choose a neighborhood $U^1 = U^1_\alpha \subset U(\Lambda)$ of Λ sufficiently close to Λ such that

$$DX_{T^1}(x)(C_\alpha(E^c_x)) \subset C_{\frac{K\alpha}{R}}(E^c_{X_{T^1}}(x)), \quad \forall x \in U^1. \tag{8}$$

On the other hand, using (4) we get

$$\frac{\|DX_t(x)v^s_x\|}{\|v^s_x\|} \leq K^{-1}e^{-\lambda t}\frac{\|DX_t(x)v^c_x\|}{\|v^c_x\|}$$

and so,

$$\frac{\|DX_t(x)v_x^s\|}{\|DX_t(x)v_x^c\|} \leq K^{-1}e^{-\lambda t}\frac{\|v_x^s\|}{\|v_x^c\|} = K^{-1}e^{-\lambda t}\angle(v_x, E_x^c).$$

So, by (7), we get

$$\angle(DX_r(x)v_x, E^c_{X_r(x)}) \leq K^{-1}e^{-\lambda r}\angle(v_x, E_x^c) \leq K^{-1}\angle(v_x, E_x^c),$$

$\forall x \in \Lambda$, $\forall r \in [0, T^1]$, $\forall v_x \in T_xM$. This implies

$$DX_r(x)\left(C_{\frac{K\alpha}{R}}(E_x^c)\right) \subset C_{\frac{\alpha}{R}}(E^c_{X_r(x)}), \quad \forall x \in \Lambda \quad \forall r \in [0, T^1].$$

Choose a neighborhood $V^2 = V^2_\alpha \subset U^1$ of Λ sufficiently close to Λ such that

$$DX_r(x)\left(C_{\frac{K\alpha}{R}}(E_x^c)\right) \subset C_{\frac{2\alpha}{R}}(E^c_{X_r(x)}),\ \forall x \in V^2,\ \forall r \in [0, T^1]. \tag{9}$$

As Λ is an attractor there is a neighborhood $U^2 \subset V^2$ of Λ such that

$$X_t(U^2) \subset V^2, \quad \forall t \geq 0.$$

Now, let $x \in U^2$ and $t \geq T^1$ be given. Then, $t = nT^1 + r$ for some integer $n \geq 1$ and some $r \in [0, T^1]$. Thus,

$$DX_t(x)(C_\alpha(E_x^c)) = DX_{nT^1+r}(x)(C_\alpha(E_x^c)) =$$

$$= DX_r(X_{nT^1}(x))(DX_{nT^1}(x)(C_\alpha(E_x^c))). \tag{10}$$

Using (8) and (10) recursively, and that $\frac{K\alpha}{R} < \alpha$, and $n \geq 1$, we obtain

$$DX_{nT^1}(x)(C_\alpha(E_x^c)) = DX_{(n-1)T^1}(X_{T^1}(x))(DX_{T^1}(x)C_\alpha(E_x^c)) \subset$$

$$\subset DX_{(n-1)T^1}(X_{T^1}(x))\left(C_{\frac{K\alpha}{R}}(E^c_{X_{T^1}(x)})\right) \subset DX_{(n-1)T^1}(X_{T^1}(x))(C_\alpha(E^c_{X_{T^1}(x)})) \subset$$

$$\subset \cdots \subset DX_{T^1}(X_{(n-1)T^1}(x))(C_\alpha(E^c_{X_{(n-1)T^1}(x)})) \subset C_{\frac{K\alpha}{R}}(E^c_{X_{nT^1}(x)}).$$

Henceforth

$$DX_{nT^1}(x)(C_\alpha(E_x^c)) \subset C_{\frac{K\alpha}{R}}(E^c_{X_{nT^1}(x)}).$$

Applying $DX_r(X_{nT^1}(x))$ to both sides of the last expression, replacing in (10) and using (9) we obtain

$$DX_t(x)(C_\alpha(E_x^c)) \subset DX_r(X_{nT^1}(x))\left(C_{\frac{K\alpha}{R}}(E^c_{X_{nT^1}(x)})\right) \subset C_{\frac{2\alpha}{R}}(E^c_{X_t(x)}).$$

As $R > 4$ we have $\frac{2\alpha}{R} < \frac{\alpha}{2}$ and so

$$DX_t(x)(C_\alpha(E^c_x)) \subset C_{\frac{\alpha}{2}}(E^c_{X_t(x)}).$$

$\forall x \in U^2$, $\forall t \geq T^1$, proving (1) of Lemma 3.1.

Throughout we fix the neighborhood U^2 of Λ and the constant $T^1 > 0$ obtained above.

As E^c_Λ is (K, λ)-volume expanding we have

$$\mathrm{vol}(DX_t(x)v^c_x, DX_t(x)w^c_x) \geq Ke^{\lambda t}\,\mathrm{vol}(v^c_x, w^c_x),$$

$\forall x \in \Lambda$, $\forall t \geq 0$, $\forall v^c_x, w^c_x \in E^c_x$.

Clearly there is $L > 1$ such that

$$L^{-1} \cdot \mathrm{vol}(v^c_x, w^c_x) \leq \mathrm{vol}(v_x, w_x) \leq L \cdot \mathrm{vol}(v^c_x, w^c_x),$$

$\forall x \in \Lambda$, $\forall v_x, w_x \in C_\alpha(E^c_x)$, $\forall \alpha \in (0, 1]$. Applying the last two relations and the invariance of $E^s_\Lambda \oplus E^c_\Lambda$ we obtain

$$\mathrm{vol}(DX_t(x)v_x, DX_t(x)w_x) \geq L^{-1}\,\mathrm{vol}(DX_t(x)v^c_x, DX_t(x)w^c_x) \geq$$

$$\geq L^{-1}Ke^{\lambda t}\,\mathrm{vol}(v^c_x, w^c_x) \geq L^{-2}Ke^{\lambda t}\,\mathrm{vol}(v_x, w_x),$$

$\forall x \in \Lambda$, $\forall t \geq T^1$, $\forall v_x, w_x \in C_\alpha(E^c_x)$ (note that $DX_t(x)v_x, DX_t(x)w_x \in C_\alpha(E^c_{X_t(x)})$ since $t \geq T^1$).

Choose $S > 0$ large so that

$$\frac{S}{L^{-2}K} > 1. \tag{11}$$

It follows that there is $T^2 = T^2_\alpha > T^1$ such that

$$\mathrm{vol}(DX_{T^2}(x)v_x, DX_{T^2}(x)w_x) \geq \frac{2S}{L^{-2}K}\,\mathrm{vol}(v_x, w_x),$$

$\forall x \in \Lambda$, $\forall v_x, w_x \in C_\alpha(E^c_x)$. In particular,

$$\inf\{\mathrm{vol}(DX_{T^2}(x)v_x, DX_{T^2}(x)w_x) : x \in \Lambda,$$

$$v_x, w_x \in C_\alpha(E^c_x), \|v_x\| = \|w_x\| = 1, < v_x, w_x >= 0\} \geq \frac{2S}{L^{-2}K}.$$

Since Λ is compact there is a neighborhood $V^3 = V^3_\alpha \subset U^2$ of Λ so that

$$\inf\{\mathrm{vol}(DX_{T^2}(x)v_x, DX_{T^2}(x)w_x) : x \in U^3,$$

$$v_x, w_x \in C_\alpha(E^c_x), \|v_x\| = \|w_x\| = 1, < v_x, w_x >= 0\} \geq \frac{S}{L^{-2}K}.$$

Then,

$$\mathrm{vol}(DX_{T^2}(x)v_x, DX_{T^2}(x)w_x) \geq \frac{S}{L^{-2}K}\,\mathrm{vol}(v_x, w_x), \tag{12}$$

$\forall x \in U^3$, $\forall v_x, w_x \in C_\alpha(E^c_x)$ with $< v_x, w_x >= 0$. As Λ is an attractor there is a neighborhood $U^3 \subset V^3$ of Λ such that

$$X_t(U^3) \subset V^3, \quad \forall t \geq 0.$$

We have

$$\|P^{nT^2}_x v_x\|\;\|X(X_{nT^2}(x))\| = \|P^{T^2}_{X_{(n-1)T^2}(x)}(P^{(n-1)T^2}_x v_x\|\;\|X(X_{T^2}(X_{(n-1)T^2}(x))\|.$$

Call $z = X_{(n-1)T^2}(x)$, and $v_z = P^{(n-1)T^2}_x v_x$. From the last equality, using that $X(X_{nT^2}(x)) = DX_{nT^2}(x)(X(x))$, v_z is orthogonal to z, and combining (3) with (12) we get

$$\|P^{nT^2}_x v_x\|\;\|X(X_{nT^2}(x))\| = \|P^{T^2}_z v_z\|\;\|X(X_{T^2}(z))\| =$$

$$\mathrm{vol}(DX_{T^2}(z)v_z, X(X_{T^2}(z))) = \mathrm{vol}(DX_{T^2}(z)v_z, DX_{T^2}(z)(X(z)) \geq$$

$$\frac{S}{L^{-2}K}\,\mathrm{vol}(v_z, X(z)) = \frac{S}{L^{-2}K}\,\mathrm{vol}(P^{(n-1)T^2}_x v_x, X(X_{(n-1)T^2}(x))).$$

Thus,

$$\|P^{nT^2}_x v_x\| \cdot \|X(X_{nT^2}(x))\| \geq \left\{\frac{S}{L^{-2}K}\right\}^n \|v_x\| \cdot \|X(x)\|, \tag{13}$$

$\forall x \in U^3$ regular with $X(x) \in C_\alpha(E^c_x)$, $\forall n \in I\!N$, $\forall v_x \in C_\alpha(E^c_x) \cap N_x$ (recall that N_x denotes the orthogonal complement of $X(x)$ in T_xM).

On the other hand

$$\mathrm{vol}(DX_r(x)v_x, DX_r(x)w_x) \geq L^{-2}K \cdot \mathrm{vol}(v_x, w_x),$$

$\forall x \in \Lambda$, $\quad \forall r \in [0, T^2]$, $\quad \forall v_x, w_x \in C_\alpha(E^c_x)$.

As before there is a neighborhood $V^4 = V^4_\alpha \subset U^3$ of Λ such that

$$\mathrm{vol}(DX_r(x)v_x, DX_r(x)w_x) \geq \frac{L^{-2}K}{2}\,\mathrm{vol}(v_x, w_x), \tag{14}$$

$\forall x \in V^4$, $\forall v_x, w_x \in C_\alpha(E^c_x)$ with $< v_x, w_x >= 0$, $r \in [0, T^2]$. As Λ is an attractor there is a neighborhood $U^4 \subset V^4$ of Λ such that

$$X_t(U_4) \subset V^4, \quad \forall t \geq 0.$$

Now, let $x \in U^4$ regular with $X(x) \in C_\alpha(E^c_x)$, $t \geq T^2$ and $v_x \in C_\alpha(E^c_x) \cap N_x$. Then, $t = nT^2 + r$ for some integer $n \geq 1$ and some $r \in [0, T^2]$.

Applying (13), (14), and using (3) and (11) we obtain

$$\|P^t_x v_x\| \cdot \|X(X_t(x))\| = \|P^r_{X_{nT^2}(x)} P^{nT^2}_x v_x\| \cdot \|DX_r(X_{nT^2}(x))X(X_{nT^2}(x))\| =$$

$$= \mathrm{vol}\left(DX_r(X_{nT^2}(x))P^{nT^2}_x v_x, DX_r(X_{nT^2}(x))X(X_{nT^2}(x))\right) \geq$$

$$\geq \frac{L^{-2}K}{2}\,\mathrm{vol}(P^{nT^2}_x v_x, X(X_{nT^2}(x))) = \frac{L^{-2}K}{2}\|P^{nT^2}_x v_x\| \cdot \|X(X_{nT^2}(x))\| \geq$$

$$\geq \left(\frac{L^{-2}K}{2}\right) \cdot \left\{\frac{SN}{L^{-2}K}\right\}^n \cdot \|v_x\| \cdot \|X(x)\| =$$

$$= \left(\frac{L^{-2}K}{2}\right) \cdot \left(\frac{S}{L^{-2}K}\right)^{-\frac{r}{T^2}} \left\{\left(\frac{S}{L^{-2}K}\right)^{\frac{1}{T^2}}\right\}^t \cdot \|v_x\| \cdot \|X(x)\| \geq$$

$$\geq \left(\frac{L^{-4}K^2}{2S}\right) \cdot \left\{\left(\frac{S}{L^{-2}K}\right)^{\frac{1}{T^2}}\right\}^t \cdot \|v_x\| \cdot \|X(x)\|.$$

Thus, choosing $U_\alpha = U^4$, $T_\alpha = T^2$, $K_\alpha = \frac{L^{-4}K^2}{2S}$ and $\lambda_\alpha = \ln\left(\frac{S}{L^{-2}K}\right)^{\frac{1}{T^2}} > 0$ we obtain (2) of Lemma 3.1. $\square$

Lemma 3.2. *Let Λ a singular-hyperbolic attractor of X, $\alpha > 0$ and $U_\alpha \subset U(\Lambda)$ be a neighborhood of Λ. Let $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ the splitting of Λ and $E^c_{U_\alpha}$ be a continuous extension of E^c_Λ to U_α. Then, for every singularity σ of X in Λ there are singular cross-sections S^t, S^b associated to σ such that $S^t \cup S^b \subset U_\alpha$,*

$$\Lambda \cap (\partial^h S^t \cup \partial^h S^b) = \emptyset, \quad \textit{and} \quad X(x) \in C_\alpha(E^c_x),$$

$\forall x \in S^t \cup S^b$.

Proof. Let Λ, α be as in the statement. By Remark 2.1 applied to $U = U_\alpha$ there are singular cross-sections S^t_0, S^b_0 associated to σ such that $S^t_0, S^b_0 \subset U(\Lambda)$ and $\Lambda \cap (\partial^h S^t_0 \cup \partial^h S^b_0) = \emptyset$. Recall that $U(\Lambda)$ is the neighborhood of Λ where the (K, λ)-splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ has an extension to $T_{U(\Lambda)}M = E^s_{U(\Lambda)} \oplus E^c_{U(\Lambda)}$, with $E^s_{U(\Lambda)}$ invariant and contracting. We denote by l^t_0, l^b_0 the singular curves of S^t_0, S^b_0 respectively.

Choose two sequences of singular cross-sections $S^t_n \subset S^t_0, S^b_n \subset S^b_0$ associated to σ satisfying

(a) $\Lambda \cap (\partial^h S^t_n \cup \partial^h S^b_n) = \emptyset$;

(b) $\mathrm{diam}(S^t_n), \mathrm{diam}(S^b_n) \to 0$ as $n \to \infty$;

(c) the singular curves l_n^t, l_n^b of S_n^t, S_0^b satisfy $l_n^t = l_0^t$, $l_n^b = l_0^b$, $\forall n$.

The properties (b) and (c) imply that $\forall n$ large there is $T = T_n > 0$ such that

$$\angle(X(X_T(x)), E_\sigma^c) < \frac{\alpha}{2},$$

$\forall x \in S_n^t \cup S_n^b$. As $E_{U_\alpha}^c$ is a continuous extension of E_Λ^c, we have that $E_{X(X_T(x))}^c$ is close to E_σ^c, $\forall x \in S_n^t \cup S_n^b$. Then,

$$X_T(x) \in U_\alpha, \text{ and } X(X_T(x)) \in C_\alpha(E_{X(X_T(x))}^c),$$

$\forall n$ large, $\forall x \in S_n^t \cup S_n^b$.

By the Property (a) and the above relation we have that for n large enough $S^t = X_T(S_n^t)$ and $S^b = X_T(S_n^b)$ are singular cross-sections associated to σ satisfying the required properties. $\square$

References

[1] Bonatti, C.. *C^1-Generic Dynamics: Tame and Wild Behavior.* ICM 2002 (2002) **2**, 265-277.

[2] Bowen, R.. *Markov partitions and minimal sets for Axiom A diffeomorphisms.* Amer. J. Math. **92** (1970), 907-918.

[3] Bowen, R.. *Symbolic dynamics for hyperbolic flows.* Amer. J. Math. **95** (1973), 429-460.

[4] Doering C., I.. Persistently transitive vector fields on three- dimensional manifolds. In *Procs. on Dynamical Systems and Bifurcation Theory*, 1987.

[5] de Melo, W., Palis, J.. *Geometric Theory of Dynamical Systems-An Introduction.* Springer Verlag, Berlin, 1982.

[6] Guckenheimer, J., Williams, R., F.. *Structural stability of Lorenz attractors.* Inst. Hautes Études Sci. Publ. Math. **50** (1979), 59-72.

[7] Hirsch, M., W.. *On invariant subsets of hyperbolic sets.* Essays on Topology and Related Topics, Mémoires dédiés á Georges de Rham (1970), 126-135.

[8] Hirsch, M., W., Pugh, C., C., Shub, M.. *Invariant manifolds*, volume 583 of *Lect. Notes in Math.* Springer Verlag, Berlin, 1977.

[9] Hurewicz, W., Wallman, H.. *Dimension Theory.* Princeton University Press, Princeton, NJ, (1984).

[10] Keynes, H., B., Sears, M.. *Real-expansive flows and topological dimension.* Ergodic Theory Dynamical Systems **1** (1981), 179-195.

[11] Labarca, R., Pacifico, M., J.. *Stability of singular horseshoes.* Topology **25** (1986), 337-352.

[12] Mañé, R.. *An ergodic closing lemma.* Ann. of Math. (2) **116** (1982), 503-540.

[13] Mañé, R.. *Expansive homeomorphisms and topological dimension.* Trans. Amer. Math. Soc. **252** (1979), 313-319.

[14] Morales, C. A.. Singular-hyperbolic sets and topological dimension. Preprint (2002) to appear.

[15] Morales, C., A., Pacifico, M., J.. *A dichotomy for three-dimensional vector fields.* To appear in Ergodic Theory Dynamical Systems.

[16] Morales, C., A., Pacifico, M., J., Pujals, E., R.. *On C^1 robust singular transitive sets for three-dimensional flows.* C. R. Acad. Sci. Paris Sér. I Math. **326** (1998), 81-86.

[17] Morales, C., A., Pacifico, M., J., Pujals, E., R.. Singular hyperbolic systems. Proc. Amer. Math. Soc., 127, Number 11:3393–3401, 1999.

[18] Przytycki, F.. *Construction of invariant sets for Anosov diffeomorphisms and hyperbolic attractors.* Studia Math. **68** (1980), 199-213.

[19] Shilnikov, L., P., Turaev, D., V.. *An example of a wild strange attractor.* (Russian) Mat. Sb. **189** (1998), 137-160; translation in Sb. Math. **189** (1998), 291-314.

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