RADIALLY SYMMETRIC WEAK SOLUTIONS FOR A QUASILINEAR WAVE EQUATION IN TWO SPACE DIMENSIONS

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ABSTRACT. We prove the convergence of the radially symmetric solutions to the Cauchy problem for the viscoelasticity equations

$$\phi_{tt} - \Delta\phi - \operatorname{div}(\frac{1}{3}|\nabla\phi|^2\nabla\phi) = \varepsilon\Delta\phi_t,$$

as $\varepsilon \to 0$, with radially symmetric initial data $\phi^\varepsilon(x,0) = \phi_0^\varepsilon(r)$, $\phi_t^\varepsilon(x,0) = \phi_1^\varepsilon(r)$, $r = (x_1^2 + x_2^2)^{1/2}$, where $\phi_{0r}^\varepsilon \rightharpoonup \phi_{0r}$, $\phi_1^\varepsilon \rightharpoonup \phi_1$, to a weak solution of the Cauchy problem for the corresponding limit equation with $\varepsilon = 0$, and initial data $\phi(x,0) = \phi_0(r)$, $\phi_t(x,0) = \phi_1(r)$. Our analysis is based on energy estimates and the method of compensated compactness closely following D. Serre and J. Shearer (1993).

1. INTRODUCTION

We consider the question of obtaining globally defined weak solutions to the Cauchy problem for the quasilinear elasticity equation

$$\phi_{tt} - \Delta\phi - \operatorname{div}(\frac{1}{3}|\nabla\phi|^2\nabla\phi) = 0, \qquad x \in \mathbb{R}^2,\ t > 0, \tag{1.1}$$

$$\phi(x,0) = \phi_0(x), \quad \phi_t(x,0) = \phi_1(x), \qquad x \in \mathbb{R}^2, \tag{1.2}$$

when ϕ_0 and ϕ_1 are radially symmetric functions, that is,

$$\phi_0(x) = g_0(r), \qquad \phi_1(x) = g_1(r), \qquad r = (x_1^2 + x_2^2)^{1/2}, \tag{1.3}$$

satisfying

$$\int_0^\infty \{g_1(r)^2 + \Sigma(g_0'(r))\}\, r\, dr < +\infty, \qquad \Sigma(v) = \frac{1}{2}v^2 + \frac{1}{12}v^4. \tag{1.4}$$

We are concerned with the construction of a weak solution to (1.1),(1.2),(1.3) as limit of smooth solutions of the Cauchy problem for the viscoelasticity equations

$$\phi_{tt} - \Delta\phi - \operatorname{div}(\frac{1}{3}|\nabla\phi|^2\nabla\phi) = \varepsilon\Delta\phi_t, \qquad x \in \mathbb{R}^2,\ t > 0, \tag{1.5}$$

$$\phi(x,0) = \phi_0^\varepsilon(x) = g_0^\varepsilon(r), \quad \phi_t(x,0) = \phi_1^\varepsilon(x) = g_1^\varepsilon(r), \qquad x \in \mathbb{R}^2, \tag{1.6}$$

with

$$\phi_0^\varepsilon \in H^3(\mathbb{R}^2) \qquad \text{and} \qquad \phi_1^\varepsilon \in H^2(\mathbb{R}^2). \tag{1.7}$$

We first recall the following result proved in [5].

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Theorem 1.1. *Assume that ϕ_0^ε and ϕ_1^ε verify* (1.6) *and* (1.7). *Then, for each $\varepsilon > 0$, there exists a unique function*

$$\phi^\varepsilon \in C([0,+\infty[;H^3) \cap C^1([0,+\infty[;H^2) \cap C^2([0,+\infty[;L^2) \tag{1.8}$$

solution of the Cauchy problem (1.5),(1.6). *The solution ϕ^ε depends only on r and t.*

Remark 1.1. We take the oportunity to correct a small mistake in p. 539 of [5]: in the estimate in the two first lines H^2 (repectively H^1) must be replaced by H^3 (respectively H^2).

By well known trace properties of functions in Sobolev spaces, a function $\phi \in H^m(\mathbb{R}^2)$, $m \geq 1$, which is radially symmetric, belongs to $H^{m-1/2}(\mathbb{R}_+)$ as a function of r, where $\mathbb{R}_+ =]0,+\infty[$; we thank H. Beirão da Veiga for having pointed out that to us [3]. Hence, the solution ϕ^ε obtained in Theorem 1.1 verifies

$$\phi^\varepsilon(r,t) \in C([0,+\infty[;H^{5/2}(\mathbb{R}_+)) \cap C^1([0,+\infty[;H^{3/2}(\mathbb{R}_+)). \tag{1.9}$$

Passing to the variables r,t, equation (1.5) reads

$$\phi_{tt}^\varepsilon - \phi_{rr}^\varepsilon - \frac{1}{r}\phi_r^\varepsilon - (\phi_r^\varepsilon)^2\phi_{rr}^\varepsilon - \frac{1}{3}\frac{(\phi_r^\varepsilon)^3}{r} = \varepsilon(\phi_{trr}^\varepsilon + \frac{1}{r}\phi_{tr}^\varepsilon), \qquad (r,t) \in \mathbb{R}_+^2. \tag{1.10}$$

Setting, as usual,

$$\phi_r^\varepsilon = v^\varepsilon, \qquad \phi_t^\varepsilon = u^\varepsilon \tag{1.11}$$

we obtain the equivalent system

$$\begin{cases} v_t^\varepsilon - u_r^\varepsilon = 0, \\ u_t^\varepsilon - (v^\varepsilon + \frac{(v^\varepsilon)^3}{3})_r - \frac{1}{r}(v^\varepsilon + \frac{(v^\varepsilon)^3}{3}) = \varepsilon(u_{rr} + \frac{1}{r}u_r^\varepsilon), \end{cases} \qquad (r,t) \in \mathbb{R}_+^2, \tag{1.12}$$

with initial data

$$v^\varepsilon(r,0) = v_0^\varepsilon(r) := \frac{dg_0^\varepsilon(r)}{dr}, \qquad u^\varepsilon(r,0) = u_0^\varepsilon(r) := g_1^\varepsilon(r). \tag{1.13}$$

Notice that, by symmetry,

$$v^\varepsilon(0,t) = 0, \qquad t \geq 0. \tag{1.14}$$

Now, let us take $\varphi(r,t), \psi(r,t) \in C_0^\infty(\mathbb{R}^2)$, such that $\varphi(0,t) = 0$, for $t \geq 0$. Integrating in $\mathbb{R}_+^2$ (1.12) against $r\varphi$, and (1.13) against $r\psi$, and using integration by parts, we easily deduce

$$\int_0^\infty \int_{\mathbb{R}_+} \{v^\varepsilon \varphi_t - u^\varepsilon(\varphi_r + \frac{1}{r}\varphi)\}\, r\, dr\, dt + \int_{\mathbb{R}_+} v_0^\varepsilon \varphi(r,0)\, r\, dr = 0, \tag{1.15}$$

$$\int_0^\infty \int_{\mathbb{R}_+} \{u^\varepsilon \psi_t - (v^\varepsilon + \frac{(v^\varepsilon)^3}{3})\psi_r + \varepsilon u^\varepsilon(\psi_{rr} + \frac{1}{r}\psi_r)\}\, r\, dr\, dt + \int_{\mathbb{R}_+} u_0^\varepsilon \psi(r,0)\, r\, dr = 0. \tag{1.16}$$

We assume henceforth that the initial data v_0^ε, u_0^ε satisfy the following additional hypotheses:

$$\int_{\mathbb{R}_+} \{\frac{1}{2}(u_0^\varepsilon)^2 + \Sigma(v_0^\varepsilon)\}\, r\, dr \leq C, \qquad (v_0^\varepsilon, u_0^\varepsilon) \rightharpoonup (v_0, u_0), \text{ as } \varepsilon \to 0, \tag{1.17}$$

$$\varepsilon^2 \int_{\mathbb{R}_+} (v_{0r}^\varepsilon + \frac{1}{r}v_0^\varepsilon)^2\, r\, dr \leq C, \tag{1.18}$$

for some constant $C > 0$, where $\Sigma(v) = \frac{1}{2}v^2 + \frac{1}{12}v^4$ and the convergence in (1.17) is the one in $L^1_{loc}(\mathbb{R}_+)$ endowed with the weak topology induced by $C_0(\mathbb{R}_+)$ or, equivalently, in the distributions sense. Observe that (1.17) and (1.18) are verified in case u^ε and v^ε are standard mollifications of $u_0 = \phi_1$ and $v_0 = \phi_{0r}$ satisfying (1.4).

Let us consider the limit case of system (1.12), when $\varepsilon = 0$,

$$\begin{cases} v_t - u_r = 0, \\ u_t - \sigma(v)_r - \frac{1}{r}\sigma(v) = 0, \end{cases} \tag{1.19}$$

with initial and boundary condition given by

$$(v, u)(r, t)|_{t=0} = (v_0, u_0)(r), \tag{1.20}$$

$$v(r, t)|_{r=0} = 0. \tag{1.21}$$

Definition 1.1. We say that $(v, u) : \mathbb{R}^2_+ \to \mathbb{R}^2$, with $v \in L^3_{loc}(\overline{\mathbb{R}^2_+})$, $u \in L^1_{loc}(\overline{\mathbb{R}^2_+})$, is weak solution of (1.19)-(1.21) if

$$\int_0^\infty \int_{\mathbb{R}_+} \{v\varphi_t - u(\varphi_r + \frac{1}{r}\varphi)\}\, r\, dr\, dt + \int_{\mathbb{R}_+} v_0\varphi(r, 0)\, r\, dr = 0, \tag{1.22}$$

$$\int_0^\infty \int_{\mathbb{R}_+} \{u\psi_t - \sigma(v)\psi_r\}\, r\, dr\, dt + \int_{\mathbb{R}_+} u_0\psi(r, 0)\, r\, dr = 0, \tag{1.23}$$

for any $\varphi, \psi \in C_0^\infty(\mathbb{R}^2)$, such that $\varphi(0, t) = 0$, for $t \geq 0$.

Definition 1.2. A function $\phi(x, t) \in W^{1,1}_{loc}(\overline{\mathbb{R}^2 \times \mathbb{R}_+})$, such that $\nabla\phi \in L^3_{loc}(\overline{\mathbb{R}^2 \times \mathbb{R}_+})$, is a weak solution of (1.1)-(1.2) if

$$\int_0^\infty \int_{\mathbb{R}^2} \{\phi_t\zeta_t - (1 + |\nabla\phi|^2)\nabla\phi \cdot \nabla\zeta\}\, dx\, dt + \int_{\mathbb{R}^2} \phi_1(x)\zeta(x, 0)\, dx = 0, \tag{1.24}$$

$$\lim_{t\to 0} \int_{B(0;R)} |\phi(x, t) - \phi_0(x)|\, dx = 0, \tag{1.25}$$

for any $\zeta \in C_0^\infty(\mathbb{R}^3)$ and any $R > 0$, where $B(0; R)$ denotes the ball in $\mathbb{R}^2$ with center 0 and radius R.

It is an easy exercise to verify that if ϕ is a *radially symmetric* weak solution of (1.1)-(1.2), in the sense of Definition 1.2, with ϕ_0, ϕ_1 satisfying (1.3), then $(v, u)(r, t)$, with $v = \phi_r$, $u = \phi_t$, is a weak solution of (1.19)-(1.21), in the sense of Definition 1.1, where $v_0(r) = dg_0(r)/dr$ and $u_0(r) = g_1(r)$, with g_0, g_1 verifying (1.3), and, conversely, if $(v, u)(r, t)$ is a weak solution of (1.19)-(1.21) then

$$\phi(r, t) = g_0(r) + \int_0^t u(r, s)\, ds$$

is a radially symmetric weak solution of (1.1)-(1.3), with $g_0(r) = \int_0^r v_0(\tau)\, d\tau$ and $g_1(r) = u_0(r)$.

We now state our main result.

Theorem 1.2. *Let ϕ^ε be solutions of* (1.5)-(1.7)*, with initial data satisfying* (1.13), (1.17), (1.18). *Then there is a subsequence ϕ^{ε_k} which converges strongly in $W^{1,p}_{loc}(\overline{\mathbb{R}^2_+})$, $1 \leq p < 2$, to a global radially symmetric weak solution $\bar{\phi}$ of* (1.1)-(1.3)*, in the sense of Definition 1.1. Moreover, if $\frac{|\lambda|^q}{\Sigma(\lambda)} \to 0$ as $|\lambda| \to \infty$, then $v^{\varepsilon_k} = \phi_r^{\varepsilon_k}$ converges strongly in L^q_{loc}. In particular, there exists a radially symmetric weak solution of* (1.1)-(1.3)*, if the radially symmetric initial data ϕ_0, ϕ_1 satisfy* (1.4).

We refer to [7, 13, 6, 9, 16, 4] for other results using the framework of compensated compactness on one-dimensional correlated models.

The remaining of this paper is organized as follows. In section 2 we recall the first energy estimate and prove the important second energy estimate by adapting procedures going back to [8]. In section 3 we recall the theory of Young measures preparing the step for the application of compensated compactness, as became usual after the pioneering works [10, 15, 7]. In section 4 we outline the analysis of the support of the Young measures due to D. Serre and J. Shearer [13].

2. Energy Estimates

In this section we recall the first energy estimate, proved in [5], and prove the second energy estimate which, together with the first one, form the starting point for the application of compensated compactness described in the following sections.

Multiplying $(1.12)_1$ by $r\sigma(v^\varepsilon)$, with $\sigma(v) = \Sigma'(v) = v + \frac{1}{3}v^3$, $(1.12)_2$ by ru^ε, adding the resulting equations, integrating in $\mathbb{R}^2_+$ and using integration by parts (cf. [5]), we obtain the first energy estimate:

$$\int_{\mathbb{R}_+}\{\frac{1}{2}(u^\varepsilon(r,t))^2 + \Sigma(v^\varepsilon(r,t))\}\, r\,dr + \varepsilon\int_0^t\int_{\mathbb{R}_+}(u_r^\varepsilon)^2\, r\,dr\,dt \le C, \qquad t\ge 0, \tag{2.1}$$

with C as in (1.17). We now state and prove the second energy estimate.

Lemma 2.1. *Let* $(u^\varepsilon, v^\varepsilon)$ *the sequence of solutions to* (1.12),(1.13)*, defined by* (1.11)*. Then we have for* $t\ge 0$, $0<\varepsilon\le 1$,

$$\varepsilon\int_0^t\int_{\mathbb{R}_+}\{(1+v^{\varepsilon 2})v_r^{\varepsilon 2} + \frac{v^\varepsilon}{r^2}\sigma(v^\varepsilon) + (u_r^\varepsilon)^2\}\, r\,dr\,dt \le C_1, \tag{2.2}$$

with $C_1>0$ *independent of* ε *and* t.

Proof. For a function f defined on $\mathbb{R}_+$, let us denote

$$\delta_r f = f_r + \frac{1}{r}f.$$

Hence, $\delta_r v^\varepsilon = \Delta\phi^\varepsilon \in C^1([0,\infty[; L^2(\mathbb{R}^2)) \cap C([0,\infty[; H^1(\mathbb{R}^2))$, and so $\delta_r v^\varepsilon \in C^1([0,\infty[; L^2_r(\mathbb{R}^2)) \cap C([0,\infty[; H^1_r(\mathbb{R}^2))$, where the subscript r refers to the measure $r\,dr$. For the remaining of the proof we drop the superscript ε of $u^\varepsilon, v^\varepsilon$. From (1.12) we get

$$\begin{cases}\delta_r(u_r) = \delta_r(v_t) = (\delta_r v)_t,\\ u_t - \delta_r\sigma(v) = \varepsilon\delta_r u_r.\end{cases}$$

Multiplying the second equation above by $\delta_r v$, integrating in $\mathbb{R}_+\times(0,t)$ and using the first one, we deduce

$$\int_0^t\int_{\mathbb{R}_+}\{(\delta_r v)u_t - (\delta_r v)\delta_r\sigma(v)\}\, r\,dr\,dt = \varepsilon\int_{\mathbb{R}_+}\frac{(\delta_r v)^2}{2}\Big|_0^t\, r\,dr. \tag{2.3}$$

The integral of the first term in the left-hand side can be computed by

$$\begin{aligned}\int_0^t\int_{\mathbb{R}_+}(\delta_r v)u_t\, r\,dr\,dt &= \int_{\mathbb{R}_+}(\delta_r v)u\Big|_0^t\, r\,dr - \int_0^t\int_{\mathbb{R}_+}\delta_r(u_r)u\, r\,dr\,dt\\ &= \int_{\mathbb{R}_+}(\delta_r v)u\Big|_0^t\, r\,dr + \int_0^t\int_{\mathbb{R}_+}u_r^2\, r\,dr\,dt,\end{aligned} \tag{2.4}$$

while the integral of the second term in the left-hand side can be computed by

$$-\int_0^t \int_{\mathbb{R}_+} (\delta_r v)\delta_r \sigma(v)\, r\, dr\, dt = -\int_0^t \int_{\mathbb{R}_+} \{v_r \sigma(v)_r r\, dr\, dt + v_r\sigma(v) + \frac{1}{r} v\sigma(v)\}\, dr\, dt$$

$$= -\int_0^t \int_{\mathbb{R}_+} \{r(1+v^2)v_r^2 + \frac{1}{r} v\sigma(v)\}\, dr\, dt. \tag{2.5}$$

From (2.3), (2.4) and (2.5) we deduce

$$\varepsilon \int_0^t \int_{\mathbb{R}_+} \{(1+v^2)v_r^2 + \frac{1}{r^2} v\sigma(v)\}\, r\, dr\, dt$$

$$= \varepsilon \int_{\mathbb{R}_+} (\delta_r v) u|_0^t\, r\, dr + \varepsilon \int_0^t \int_{\mathbb{R}_+} u_r^2\, r\, dr\, dt - \varepsilon^2 \int_{\mathbb{R}_+} \frac{(\delta_r v)^2}{2}|_0^t\, r\, dr$$

$$= \varepsilon \int_{\mathbb{R}_+} (\delta_r v(t)) u(t)\, r\, dr - \varepsilon \int_{\mathbb{R}_+} (\delta_r v_0) u_0\, r\, dr + \varepsilon \int_0^t \int_{\mathbb{R}_+} u_r^2,\, r\, dr\, dt \tag{2.6}$$

$$- \varepsilon^2 \int_{\mathbb{R}_+} \frac{(\delta_r v(t))^2}{2}\, r\, dr + \varepsilon^2 \int_{\mathbb{R}_+} \frac{(\delta_r v_0)^2}{2}\, r\, dr$$

The first and second term on the right-hand of the last equation in (2.6) are estimated simply using Cauchy-Schwarz inequality

$$\varepsilon \int_{\mathbb{R}_+} (\delta_r v(t)) u(t) r\, dr \le \int_{\mathbb{R}_+} \{\frac{\varepsilon^2}{2} (\delta_r v(t))^2 + \frac{1}{2} u^2(t)\}\, r\, dr \tag{2.7}$$

$$- \varepsilon \int_{\mathbb{R}_+} (\delta_r v_0) u_0 r\, dr \le \int_{\mathbb{R}_+} \{\frac{\varepsilon^2}{2} (\delta_r v_0)^2 + \frac{1}{2} u_0^2\}\, r\, dr. \tag{2.8}$$

Finally, we deduce from (2.6), (2.7), (2.8) and (2.1),

$$\varepsilon \int_0^t \int_{\mathbb{R}_+} \{(1+v^2)v_r^2 + \frac{1}{r} v\sigma(v) + u_r^2\}\, r\, dr\, dt \le \frac{\varepsilon^2}{2} \int_{\mathbb{R}^2} (\delta_r v_0)^2\, r\, dr \tag{2.9}$$

$$+ 2\varepsilon \int_0^t \int_{\mathbb{R}_+} u_r^2\, r\, dr\, dt + \frac{1}{2} \int_{\mathbb{R}_+} \{u^2(t) + u_0^2\}\, r\, dr \le C_1, \tag{2.10}$$

where $C_1 > 0$ is independent of ε. □

3. Young Measures

In this section we recall well known fundamental facts about Young measures. We first define

$$\eta(u,v) := \frac{1}{2} u^2 + \Sigma(v), \qquad (u,v) \in \mathbb{R}^2. \tag{3.1}$$

If $\Omega \subseteq \mathbb{R}^N$ is a bounded open set, denote

$$L^\eta(\Omega) := \{(u,v)(x) \text{ Lebesgue measurable in } \Omega : \int_\Omega \eta(u,v)(x)\, dx < +\infty\}.$$

The first energy estimate (2.1) and assumption (1.17) tell us that the sequence $(u^\varepsilon, v^\varepsilon)(r(x), t)$ is uniformly bounded in $L^\eta(\mathbb{R}^2 \times [0,T])$, for any $T > 0$, where $r(x) = (x_1^2 + x_2^2)^{1/2}$.

We now state two simple lemmas which together give all the facts we need here about Young measures. Their statements are taken with slight adaptations from [1] and [14], respectively (see also, e.g., [15, 11, 2, 17]). We refer to [1] and [14], respectively, for the proofs. Let $\mathcal{L}^N$ denote the Lebesgue measure in $\mathbb{R}^N$.

Lemma 3.1. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set and $\{z^\varepsilon\} \subseteq [L^1(\Omega)]^M$. Then, there exist a subsequence $\{z^{\varepsilon_k}\}$ and a $\mathcal{L}^N$-measurable map ν_x defined in Ω such that*

(i) *for any $g \in C_c(\mathbb{R}^M)$ the $\{g \circ z^{\varepsilon_k}\}$ weakly* converges in $L^\infty(\Omega)$ to the function*
$$\bar{g}(x) := \int_{\mathbb{R}^M} g(y)\, d\nu_x(y);$$

(ii) *if $\{\|z^{\varepsilon_k}\|_{L^1(\Omega)}\}$ is bounded, then ν_x is a probability measure in $\mathbb{R}^M$ for $\mathcal{L}^N$-a.e. $x \in \Omega$ and*
$$\int_\Omega \int_{\mathbb{R}^M} |y|\, d\nu_x(y)\, dx \le \liminf_{k\to\infty} \|z^{\varepsilon_k}\|_1.$$

(iii) *if Ω and all z^ε are invariant for a certain symmetry group $\mathcal{G}$ of $\mathbb{R}^N$, that is, $z^\varepsilon(\gamma(x)) = z^\varepsilon(x)$, $\mathcal{L}^N$-a.e. $x \in \Omega$, for all $\gamma \in \mathcal{G}$, then ν_x is also invariant by $\mathcal{G}$, that is, $\nu_{\gamma(x)} = \nu_x$, $\mathcal{L}^N$-a.e. $x \in \Omega$, for all $\gamma \in \mathcal{G}$.*

Remark 3.1. The item (iii) is not stated in [1], but its proof is quite obvious, so we leave it to the reader.

Lemma 3.2. *Let $\eta : \mathbb{R}^M \to \mathbb{R}$ be a given non-negative convex function and suppose that the sequence $\{z^\varepsilon\}$ satisfies $\int_\Omega \eta(z^\varepsilon(x))\, dx < C$, for all ε, for some $C > 0$ independent of ε. Then*

(i) *for any $g \in C(\mathbb{R}^M)$ satisfying $g(z)/\eta(z) \to 0$ as $|z| \to \infty$, we have $\bar{g} \in L^1(\Omega)$, with $\bar{g}$ defined as above, and, as $k \to \infty$, $g(z^{\varepsilon_k}) \rightharpoonup \bar{g}$ in the distributions sense;*

(ii) *if $1/\eta(z) \to 0$ as $|z| \to \infty$ then ν_x is a probability measure for $\mathcal{L}^N$-a.e. $x \in \Omega$;*

(iii) *if, for some $i \in \{1, \dots, M\}$, $|z_i|^q/\eta(z) \to 0$ as $|z| \to \infty$, and the support of ν_x is a point for $\mathcal{L}^N$-a.e. $x \in \Omega$, then $z_i^{\varepsilon_k}$ converges strongly to $\bar{z}_i(x) = \int_{\mathbb{R}^M} z_i\, d\nu_x(z)$ in $L^q(\Omega)$.*

We apply the above lemmas with $\Omega = \{x \in \mathbb{R}^2 : |x| < R\} \times (0,T)$, for some $R, T > 0$, $z^\varepsilon = (u^\varepsilon, v^\varepsilon)$, η given by (3.1) and $\mathcal{G}$ is the group of symmetries in Ω whose elements have the form $\gamma = \Theta \times \mathrm{id}$, where Θ is any rotation of $\mathbb{R}^2$ and id is the identity map of $(0,T)$.

4. Compensated Compactness and Tartar's Commutation Relation

In this section we recall the analysis of the support of the Young measure, based on Tartar's commutation relation, due to D. Serre and J. Shearer [13].

Set $U = (v,u)$ and $F(U) = (-u, -\sigma(v))$. We recall that a smooth function $P(U)$ is said to be an entropy for (1.19), with associated etropy flux $Q(U)$, if we have

$$\nabla P(U) \nabla F(U) = \nabla Q(U). \tag{4.1}$$

As an example, it is easy to check that $\eta(U)$, given in (3.1), is an entropy for (1.19) with associated entropy flux $q(U) = u\sigma(v)$.

We also recall that (1.19) admits a standard pair of Riemann invariants

$$w_1 = u + z, \qquad w_2 = u - z, \qquad z = \int_0^v \sqrt{\sigma'(s)}\, ds. \tag{4.2}$$

The basic strategy of the application of compensated compactness to show the convergence of the viscosity solutions $(v^\varepsilon, u^\varepsilon)$, which verify (1.12)-(1.13), after the pioneering works of Murat [10], Tartar [15] and DiPerna [7], is to consider a sufficiently large class of entropy pair $(P(U), Q(U))$, for which one can prove that

$$\{P(U^\varepsilon)_t + Q(U^\varepsilon)_r\}_{\varepsilon>0} \text{ lies in a compact subset of } H^{-1}_{loc}(\mathbb{R}^2_+), \tag{4.3}$$

and then, through the div-curl lemma (see [10], [15]), to use Tartar's commutation relation between any two entropy pairs in that class, $(P_a(U), Q_a(U))$, $(P_b(U), Q_b(U))$,

$$\langle \nu, P_a Q_b - P_b Q_a\rangle = \langle \nu, P_a\rangle\langle \nu, Q_b\rangle - \langle \nu, P_b\rangle\langle \nu, Q_a\rangle, \tag{4.4}$$

and show that the only probability measure verifying (4.4), for any two pairs in the class, is the Dirac measure concentrated in a certain point $\bar U$. Multiplying (1.12) by ∇P we obtain

$$P(U^\varepsilon)_t + Q(U^\varepsilon)_r = \varepsilon(P_u u^\varepsilon_r)_r - \varepsilon(P_{uv} u^\varepsilon_r v^\varepsilon_r + P_{uu}(u^\varepsilon_r)^2) + P_u \frac{1}{r}\sigma(v^\varepsilon). \tag{4.5}$$

Following [12] and [14], [13] uses half-plane supported entropies, defined through a change of dependent variables indroduced in [14]

$$P = \frac{1}{2}(\sigma')^{-1/4}[\Phi + \Psi], \tag{4.6}$$

$$Q = \frac{1}{2}(\sigma')^{+1/4}[\Phi - \Psi], \tag{4.7}$$

considering the equation (4.1) in the variables w_1, w_2, Φ, Ψ,

$$\Phi_{w_1} = a\Phi, \tag{4.8}$$

$$\Psi_{w_2} = -a\Phi, \tag{4.9}$$

where $a = a(w_1 - w_2) = \sigma''(v(\frac{w_1-w_2}{2}))/8(\sigma'(v(\frac{w_1-w_2}{2}))^{3/2}$, choosing $(\bar w_1, \bar w_2)$, and prescribing Goursat data on the lines $w_1 = \bar w_1$, $w_2 = \bar w_2$,

$$\Phi(\bar w_1, w_2) = g(w_2), \tag{4.10}$$

$$\Psi(w_2, \bar w_2) = h(w_1). \tag{4.11}$$

For example, if we set $h \equiv 0$ and let g be supported in $w_2 > \bar w_2$, the corresponding pair (P, Q) is supported in the half-plane $w_2 > \bar w_2$. Actually, in this case we have

$$P(w_1, w_2) = \frac{1}{2}(\sigma')^{-1/4}\left[g(w_2) + \int_{\bar w_2}^{w_2} G(w_1, w_2, w) g(w)\, dw\right], \tag{4.12}$$

$$Q(w_1, w_2) = \frac{1}{2}(\sigma')^{+1/4}\left[g(w_2) + \int_{\bar w_2}^{w_2} H(w_1, w_2, w) g(w)\, dw\right], \tag{4.13}$$

where the kernels G, H depend on $(\bar w_1, \bar w_2)$. The representations (4.12), (4.13) are obtained through the integral operator $\mathcal{A}$, whose action on a function $f \in L^1_{loc}(\mathbb{R}^2)$ is defined as

$$(\mathcal{A}f)(w_1, w_2) = -\int_{\bar w_1}^{w_1}\int_{\bar w_2}^{w_2} a(\xi - w_2) a(\xi - \eta) f(\xi, \eta)\, d\eta\, d\xi. \tag{4.14}$$

In [14] it is proved that such half-plane supported entropy-entropy flux pairs satisfy

$$P_u,\ P_v/(\sigma')^{1/2},\ P_{uu},\ P_{uv}/(\sigma')^{1/2},\ P_{vv}/\sigma' \in L^\infty. \tag{4.15}$$

Using (4.15), one can prove (4.3), through Murat's Lemma, observing that the following three conditions are satisfied (cf. [14]):

(M1) $(P^\varepsilon, Q^\varepsilon)$ is uniformly bounded in $L^p_{loc}(\mathbb{R}^2_+)^2$, for some $p > 2$;
(M2) $\varepsilon(P^\varepsilon_u u^\varepsilon_r)_r$ is precompact in $H^{-1}_{loc}(\mathbb{R}^2_+)$;
(M3) $-\varepsilon(P_{uv}u^\varepsilon_r v^\varepsilon_r + P_{uu}(u^\varepsilon_r)^2) + P_u\frac{1}{r}\sigma(v^\varepsilon)$ is uniformly bounded in $L^1_{loc}(\mathbb{R}^2_+)$.

In [13] it is also considered a second class of entropies obtained by solving (4.8),(4.9) with continuous, compactly supported initial data on a noncharacteristic line $w_1 - w_2 = \xi_0$, i.e.,

$$\Phi(\xi_0 + w, w) = g(w), \qquad \Psi(\xi_0 + w, w) = h(w), \tag{4.16}$$

for some constant ξ_0. As observed in [13], the fact that this second class also satisfies (4.3) follows observing that if g, h are supported in (w_*, w^*) then $\Phi(w_1, w_2), \Psi(w_1, w_2)$ vanish in the quadrants $w_2 > w^*, w_1 > w^* + \xi_0$ and $w_2 < w_*, w_1 < w_* + \xi_0$ and coincide with solutions of Goursat problems like (4.8)-(4.11) in the quadrants $w_2 \geq w_*, w_1 \leq w^* + \xi_0$ and $w_2 \leq w^*, w_1 \geq w_* + \xi_0$.

As in [14], define

$$\begin{aligned} w_2^- &= \inf\{w_2 \in \mathbb{R} : \text{there is an entropy pair } (P,Q) \text{ with supp}\,(P,Q) \text{ in} \\ &\qquad \mathbb{R} \times (-\infty, w_2] \text{ and not both } \langle \nu, P\rangle, \langle \nu, Q\rangle \text{ are zero}\}, \\ w_2^+ &= \sup\{w_2 \in \mathbb{R} : \text{there is an entropy pair } (P,Q) \text{ with supp}\,(P,Q) \text{ in} \\ &\qquad \mathbb{R} \times [w_2, +\infty) \text{ and not both } \langle \nu, P\rangle, \langle \nu, Q\rangle \text{ are zero}\}, \end{aligned}$$

Analogously we define w_1^-, w_1^+. These numbers may take the values $\pm\infty$.

We now recall some lemmas from [13] leading to the desired conclusion that $\nu_{r,t}$ is a point mass for a.e. $(r,t) \in \mathbb{R}^2_+$.

For $\alpha_0 \in (\min\{w_2^-, w_2^+\}, \max\{w_2^-, w_2^+\})$ and $0 < \varepsilon_0 < \text{dist}(\alpha_0, \{w_2^-, w_2^+\})$, define $I = (\alpha_0 - \varepsilon_0, \alpha_0 + \varepsilon_0)$. In what follows we drop the subscript r, t of $\nu_{r,t}$.

Lemma 4.1 (cf. [12, 14, 13]). *For any $\bar\alpha_1, \bar\alpha_2 \in I$ and any two entropy pairs (P_a, Q_a), (P_b, Q_b), with supports satisfying*

$$\textit{either supp}\,(P_i, Q_i) \subseteq \mathbb{R} \times (-\infty, \bar\alpha_1], \ \textit{or supp}\,(P_i, Q_i) \subseteq [\bar\alpha_2, +\infty), \qquad i = a, b,$$

we have

$$\langle \nu, P_aQ_b - P_bQ_a\rangle = 0.$$

If w_2^- or w_2^+ is finite we may take $\varepsilon_0 = dist(\alpha_0, \{w_2^-, w_2^+\})$.

Denote $\alpha_1 = \alpha_0 - \varepsilon$, $\tilde\alpha_1 = \alpha_0 - \varepsilon/2$, $\alpha_2 = \alpha_0 + \varepsilon$ and $\tilde\alpha_2 = \alpha_0 + \varepsilon/2$, with $0 < \varepsilon \leq \varepsilon_0$. Let $g_i, \tilde g_i$, $i = 1, 2$, be continuously compactly supported functions with

$|g_i|, |\tilde g_i| < 2\varepsilon$, $i = 1, 2$, and satisfying:

$$g_1(w_2) = \begin{cases} w_2 - \alpha_1 & \text{for } \alpha_1 \le w_2 \le \alpha_2, \\ 0 & \text{for } w_2 \le \alpha_1 \text{ or } w_2 \ge \alpha_2 + \varepsilon, \end{cases}$$

$$\tilde g_1(w_2) = \begin{cases} w_2 - \tilde\alpha_1 & \text{for } \tilde\alpha_1 \le w_2 \le \alpha_2, \\ 0 & \text{for } w_2 \le \tilde\alpha_1 \text{ or } w_2 \ge \alpha_2 + \varepsilon, \end{cases}$$

$$g_2(w_2) = \begin{cases} \alpha_2 - w_2 & \text{for } \alpha_1 \le w_2 \le \alpha_2, \\ 0 & \text{for } w_2 \ge \alpha_2 \text{ or } w_2 \le \alpha_1 - \varepsilon, \end{cases}$$

$$\tilde g_2(w_2) = \begin{cases} \tilde\alpha_2 - w_2 & \text{for } \alpha_1 \le w_2 \le \tilde\alpha_2, \\ 0 & \text{for } w_2 \ge \tilde\alpha_2 \text{ or } w_2 \le \alpha_1 - \varepsilon, \end{cases}$$

We derive three couples of entropy pairs, $\{(P_1, Q_1), (P_2, Q_2)\}$, $\{(P_1, Q_1), (\tilde P_1, \tilde Q_1)\}$, $\{(P_2, Q_2), (\tilde P_2, \tilde Q_2)\}$, by using, to obtain each of these couples, two pairs of Goursat axes defined by the points $\{(\bar w_1, \alpha_1), (\bar w_1, \alpha_2)\}$, for the first couple, and using g_1, h and g_2, h with $h \equiv 0$ as respective Goursat initial data, proceeding analogously to obtain the other two couples of entropy pairs. By Lemma 4.1, we have

$$\langle \nu, P_1 Q_2 - P_2 Q_1 \rangle = 0, \tag{4.17}$$

$$\langle \nu, P_1 \tilde Q_1 - \tilde P_1 Q_1 \rangle = 0, \tag{4.18}$$

$$\langle \nu, P_2 \tilde Q_2 - \tilde P_2 Q_2 \rangle = 0. \tag{4.19}$$

The couples $\{(P_1, Q_1), (\tilde P_1, \tilde Q_1)\}$, $\{(P_2, Q_2), (\tilde P_2, \tilde Q_2)\}$ are used after we know that the support of ν is contained in a bounded rectangle $\mathcal{R} = [w_1^B, w_1^T] \times [w_2^B, w_2^T]$. In this case, we take $\alpha_2 = w_2^T$, for the first couple, and $\alpha_1 = w_2^B$, for the second one. It is easy to verify that (4.18) and (4.19) still hold with such limits in this case, for ε sufficiently small. Clearly, the quadratic form $P_1 Q_2 - P_2 Q_1$ is nonzero only for w_2 in the interval $[\alpha_1, \alpha_2]$, while, in case $\operatorname{supp}\nu$ is contained in a bounded rectangle $\mathcal{R}$, $(P_1 \tilde Q_1 - \tilde P_1 Q_1)|\mathcal{R}$ is nonzero only for $w_2 \in [\tilde\alpha_1, w_2^T]$, and $(P_2 \tilde Q_2 - \tilde P_2 Q_2)|\mathcal{R}$ is nonzero only for $w_2 \in [w_2^B, \tilde\alpha_2]$. Define $\Delta_\varepsilon = \Delta_\varepsilon(\alpha_0, w_2) = g_1(w_2) g_2(w_2)$ and $\tilde\Delta_{i\varepsilon} = \tilde\Delta_{i\varepsilon}(\alpha_0, w_2) = g_i(w_2)\tilde g_i(w_2)$, $i = 1, 2$.

Lemma 4.2 (cf. [13]). *For the couples of entropy pairs* $\{(P_1, Q_1), (P_2, Q_2)\}$, $\{(P_1, Q_1), (\tilde P_1, \tilde Q_1)\}$, $\{(P_2, Q_2), (\tilde P_2, \tilde Q_2)\}$ *defined above we have*

$$P_1 Q_2 - P_2 Q_1 = -\frac{1}{2}\varepsilon\Delta_\varepsilon a(w_1 - \alpha_0) + \varepsilon^2 \Delta_\varepsilon E(w_1, w_2, \alpha_0), \tag{4.20}$$

$$P_i \tilde Q_i - \tilde P_i Q_i = -\frac{1}{4}\varepsilon\tilde\Delta_{i\varepsilon} a(w_1 - \alpha_0) + \varepsilon^2 \tilde\Delta_{i\varepsilon} \tilde E_i(w_1, w_2, \alpha_0), \qquad i = 1, 2, \tag{4.21}$$

where the error terms $E, \tilde E_1, \tilde E_2$ *are bounded by a constant independent of* $w_1, w_2, \alpha_0, \varepsilon$.

Remark 4.1. The proof of (4.21) is not given in [13] but it follows by arguments analogous to those in the proof of (4.20) given in the proof of lemma 5 in [13].

The fact that ν is supported in a bounded rectangle is proved in lemmas 6 and 7 of [13].

Lemma 4.3 (cf. [13]). *The probability measure* ν *is supported in a bounded rectangle.*

Finally, [13] establishes that ν is in fact a point mass. We state this fact in the following lemma and outline a proof which is essentially the one in [13].

Lemma 4.4 (cf. [13]). *ν is a point mass.*

Proof. Let $\mathcal{R} = [w_1^B, w_1^T] \times [w_2^B, w_2^T]$ be the minimal rectangle with edges parallel to the coordinate axes and containing the support of ν. We must show that $\mathcal{R}$ reduces to a point. Let $\mathcal{L}$ denote the line $w_1 - w_2 = 0$, where $a = 0$. Suppose $\mathcal{L}$ does not intersect one of the edges of $\mathcal{R}$, say the one on the top which is included on the line $w_2 = w_2^T$, and let

$$w_2^* = \inf\{w_2 \in [w_2^B, w_2^T] \,:\, \mathcal{L} \cap \big([w_1^B, w_1^T] \times [w_2, w_2^T]\big) = \emptyset\}.$$

We will show that $\langle \nu, [w_1^B, w_1^T] \times (w_2^*, +\infty)\rangle = 0$, which contradicts the minimality of $\mathcal{R}$. We achieve this by showing that $D\pi_2\nu(\alpha_0) = 0$, for any $\alpha_0 \in (w_2^*, +\infty)$, where $\pi_2\nu$ is the projection of ν in the w_2-axis, defined by $\pi_2\nu((\alpha_1, \alpha_2)) = \nu(\mathbb{R} \times (\alpha_1, \alpha_2))$, and, for a measure μ defined on the line, $D\mu$ denotes the derivative with respect to the Lebesgue measure on the line. For $\alpha_0 \in (w_2^T, +\infty)$, we obviously have $D\pi_2\nu(\alpha_0) = 0$. For $\alpha_0 \in (w_2^*, w_2^T)$ this is proved by taking $\varepsilon < \operatorname{dist}(\alpha_0, \{w_2^*, w_2^T\})$ and the couple of entropy pairs (P_1, Q_1), (P_2, Q_2), and using (4.17) and the estimate (4.20), observing the fact that

$$\Delta_\varepsilon a(w_1 - \alpha_0) \geq C_0 \varepsilon^2, \qquad \text{for } (w_1, w_2) \in [w_1^B, w_1^T] \times [\alpha_0 - \varepsilon/2, \alpha_0 + \varepsilon/2],$$

for some constant $C_0 > 0$ independent of ε, and that E is supported in the strip $\alpha_0 - \varepsilon \leq w_2 \leq \alpha_0 + \varepsilon$ and is bounded by a constant independent of ε. We then obtain

$$\varepsilon^3 \pi_2\nu([\alpha_0 - \varepsilon/2, \alpha_0 + \varepsilon/2]) \leq C\varepsilon^4 \pi_2\nu([\alpha_0 - \varepsilon, \alpha_0 + \varepsilon]),$$

for some $C > 0$ independent of ε, which, dividing by ε^4 and taking lim sup when $\varepsilon \to 0$, gives

$$0 \leq \bar{D}\pi_2\nu(\alpha_0) \leq C\pi_2\nu(\alpha_0) \leq C,$$

where $\bar{D}\mu$ denotes the upper derivative with respect to the linear Lebesgue measure. Now, if $\pi_2\nu(\alpha_0) > 0$, then $\bar{D}\pi_2\nu(\alpha_0) = +\infty$, which contradicts the above inequality. Hence $\pi_2\nu(\alpha_0) = 0$ and, again by the above inequality, we have $\bar{D}\pi_2\nu(\alpha_0) = 0$, which proves the assertion also in this case. So, it only remains to prove that $\bar{D}\pi_2\nu(w_2^T) = 0$. To prove this we proceed as above, but now we use the couple of entropy pairs (P_1, Q_1), $(\tilde{P}_1, \tilde{Q}_1)$, with $\alpha_2 = w_2^T$. We then use the estimate (4.21), and (4.18) to arrive at the desired conclusion exactly as above.

We now consider the case when $\mathcal{L}$ intersects opposite vertices of the rectangle $\mathcal{R}$ which we denote as $\mathbf{w}^T = (w_1^T, w_2^T)$ and $\mathbf{w}^B = (w_1^B, w_2^B)$. This part of the proof follows very closely the one in [13]; we outline it here just for the sake of completeness. The main tool here is the weak* trace introduced and used by DiPerna in [7]. Again, consider the couple of entropies (P_1, Q_1), $(\tilde{P}_1, \tilde{Q}_1)$, with $\alpha_2 = w_2^T$, and let $\tilde{\delta}_{1\varepsilon}(w_2) = \varepsilon^{-3}\tilde{\Delta}_{1\varepsilon} = \varepsilon^{-3} g_1(w_2)\tilde{g}_1(w_2)$. Define the probability measures along the top edge of $\mathcal{R}$

$$\langle \mu_\varepsilon^T, f\rangle = \frac{\langle \nu, \tilde{\delta}_{1\varepsilon} f\rangle}{\langle \nu, \tilde{\delta}_{1\varepsilon}\rangle},$$

where $f = f(w_1)$ is any continuous function. By the weak* compactness of probability measures, there exists a subsequence ε_k and a probability measure μ^T defined on the top edge of $\mathcal{R}$ such that

$$\langle \mu^T, f\rangle = \lim_{k\to\infty} \langle \mu_{\varepsilon_k}^T, f\rangle.$$

We call μ^T a standard trace of ν on the top edge. Analogously, we define μ^B, μ^L and μ^R, standard traces of ν on the bottom, left and right edges, respectively; we denote the corresponding approximate delta functions by $\tilde\delta_{2\varepsilon}$, $\tilde\delta_{3\varepsilon}$ and $\tilde\delta_{4\varepsilon}$, respectively. The crucial property of standard traces is that they are point masses. More precisely, $\mu^T = \mu^R = \delta_{\mathbf{w}^T}$ and $\mu^B = \mu^L = \delta_{\mathbf{w}^B}$, where $\delta_{\mathbf{w}}$ denotes the Dirac measure concentrated on $\mathbf{w}$. To see this, set $f^T = a(w_1 - w_2^T)$ and observe that, by the estimate (4.21) and (4.18), we have

$$\begin{aligned}\langle \mu^T, f^T\rangle &= \lim_{k\to\infty} \langle \mu^T_{\varepsilon_k}, a\rangle + \varepsilon_k \langle \mu^T_{\varepsilon_k} \tilde E_1\rangle \\ &= \lim_{k\to\infty} \frac{\langle \nu, -4\varepsilon_k^{-4}(P_1\tilde Q_1 - \tilde P_1 Q_1)\rangle}{\langle \nu, \tilde\delta_{1\varepsilon_k}\rangle} = 0,\end{aligned}$$

and, since f^T is non-negative on $[w_1^B, w_1^T]$, vanishing only on w_1^T, we must have $\mu^T = \delta_{\mathbf{w}^\mathbf{T}}$. We prove the corresponding assertions for μ^B, μ^L and μ^R similarly.

Let us consider the points

$$\begin{aligned}(w_1^N, w_2^N) &= (\frac{1}{2}(w_1^B + w_1^T), w_2^T - 2\varepsilon), \qquad &(w_1^S, w_2^S) &= (\frac{1}{2}(w_1^B + w_1^T), w_2^B + 2\varepsilon),\\ (w_1^W, w_2^W) &= (w_1^B + 2\varepsilon', \frac{1}{2}(w_2^B + w_2^T)), \qquad &(w_1^E, w_2^E) &= (w_1^T - 2\varepsilon', \frac{1}{2}(w_2^B + w_2^T)),\end{aligned}$$

with $0 < \varepsilon, \varepsilon' \lll \frac{1}{2}(w_2^T - w_2^B) = \frac{1}{2}(w_1^T - w_1^B)$. We consider ortogonal axes centered in each of these points and construct four half plane supported entropy pairs, one family for each direction, north, south, east and west, which indicate the type of supporting half plane. As Goursat initial data, we set $g = g_\varepsilon^N := g_1$, in the line $w_1 = w_1^N$, with $\alpha_2 = w_2^T$, and $h = 0$, in the line $w_2 = w_2^N$, to define the pair (P_N, Q_N); $g = g_\varepsilon^S = g_2$, in the line $w_1 = w_1^S$, with $\alpha_1 = w_2^B$, and $h = 0$ in the line $w_2 = w_2^S$, to define the pair (P_S, Q_S); $g = 0$, in the line $w_1 = w_1^W$ and $h = h_{\varepsilon'}^W := \tilde\delta_{3\varepsilon'}$, in the line $w_2 = w_2^W$, with $\alpha_3 = w_1^B$, to define (P_W, Q_W); $g = 0$, in the line $w_1 = w_1^E$ and $h = h_{\varepsilon'}^E := \tilde\delta_{4\varepsilon'}$, in the line $w_2 = w_2^E$, with $\alpha_4 = w_1^T$, to define (P_E, Q_E). Recall that for (P_E, Q_E) we have

$$\begin{aligned}P_E(w_1, w_2) &= \frac{1}{2}(\sigma')^{-1/4}\left[h_{\varepsilon'}^E(w_1) + \int_{(w_1^T - \varepsilon')}^{w_1} G h_{\varepsilon'}^E \, dw\right],\\ Q_E(w_1, w_2) &= \frac{1}{2}(\sigma')^{+1/4}\left[h_{\varepsilon'}^E(w_1) + \int_{(w_1^T - \varepsilon')}^{w_1} H h_{\varepsilon'}^E \, dw\right],\end{aligned}$$

with G and H uniformly bounded in $\mathcal{R}$ by a constant independent of ε'. An analogous representation holds for (P_W, Q_W). Denote by μ^L and μ^R the standard weak* trace of ν in the left and right edge of $\mathcal{R}$, respectively. We then have

$$\begin{cases}\langle \mu^L, \frac{1}{2}(\sigma')^{-1/4} f\rangle = \lim_{\varepsilon'\to 0} \frac{\langle \nu, P_W f\rangle}{\langle \nu, h_{\varepsilon'}^W\rangle},\\ \langle \mu^L, \frac{1}{2}(\sigma')^{+1/4} f\rangle = \lim_{\varepsilon'\to 0} \frac{\langle \nu, Q_W f\rangle}{\langle \nu, h_{\varepsilon'}^W\rangle},\\ \langle \mu^R, \frac{1}{2}(\sigma')^{-1/4} f\rangle = \lim_{\varepsilon'\to 0} \frac{\langle \nu, P_E f\rangle}{\langle \nu, h_{\varepsilon'}^E\rangle},\\ \langle \mu^R, \frac{1}{2}(\sigma')^{+1/4} f\rangle = \lim_{\varepsilon'\to 0} \frac{\langle \nu, Q_E f\rangle}{\langle \nu, h_{\varepsilon'}^E\rangle}.\end{cases} \tag{4.22}$$

Now, as in [13], denoting $\langle\nu, P_N\rangle, \dots$ simply by $\langle P_N\rangle, \dots$, we observe that Tartar's relation implies

$$\begin{aligned}
&\langle P_N Q_E - P_E Q_N\rangle\langle P_S Q_W - P_W Q_S\rangle \\
&\quad - \langle P_N Q_W - P_W Q_N\rangle\langle P_S Q_E - P_E Q_S\rangle \\
&= -\langle P_N\rangle\langle Q_E\rangle\langle P_W\rangle\langle Q_S\rangle - \langle P_E\rangle\langle Q_N\rangle\langle P_S\rangle\langle Q_W\rangle \\
&\quad + \langle P_N\rangle\langle Q_W\rangle\langle P_E\rangle\langle Q_S\rangle + \langle P_W\rangle\langle Q_N\rangle\langle P_S\rangle\langle Q_E\rangle \\
&= \langle P_N Q_S - P_S Q_N\rangle\langle P_E Q_W - P_W Q_E\rangle \\
&= 0,
\end{aligned}$$

the latter equality because the supports of (P_N, Q_N) and (P_S, Q_S) do not intersect. Now, dividing the above equation by $\langle\nu, h^E_{\varepsilon'}\rangle\langle\nu, h^W_{\varepsilon'}\rangle$, making $\varepsilon' \to 0$ and using (4.22), we get

$$\langle\mu^R, (\sigma')^{1/4}P_N - (\sigma')^{-1/4}Q_N\rangle\langle\mu^L, (\sigma')^{1/4}P_S - (\sigma')^{-1/4}Q_S\rangle = 0, \tag{4.23}$$

where we have used the fact that integration against μ^R is evaluation at $\mathbf{w}^T$, integration against μ^L is evaluation at $\mathbf{w}^B$, $(P_N, Q_N)(\mathbf{w}^B) = (0,0)$ and $(P_S, Q_S)(\mathbf{w}^T) = (0,0)$.

Now, from the estimates carried out in [14], we have

$$\begin{aligned}
\langle\mu^R, (\sigma')^{1/4}P_N - (\sigma')^{-1/4}Q_N\rangle &= (\sigma')^{1/4}P_N(\mathbf{w}^T) - (\sigma')^{-1/4}Q_N(\mathbf{w}^T) \\
&= -2[1 + O(\varepsilon)]\int_{w_2^T-2\varepsilon}^{w_2^T} a(w_1^T - w)(w - (w_2^T - 2\varepsilon))\,dw.
\end{aligned} \tag{4.24}$$

Since the integrand in (4.24) above is nonzero and continuous on the interval $(w_2^T - 2\varepsilon, w_2^T)$, we can rescale g^N_ε by dividing it by $\int_{w_2-2\varepsilon}^{w_2^T} a(w_1^T - w)(w - (w_2 - 2\varepsilon))\,dw$. Then the first term in (4.23) converges to the nonzero constant -2. A similar argument shows that the second term in (4.23), after a similar rescaling, also converges to -2, which is a contradiction. Then the support of ν is a point.

□

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