Measuring the degree of pointedness of a closed convex cone: a metric approach

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We introduce the concept of radius of pointedness for a closed convex cone in a finite dimensional Hilbert space. Such radius measures the degree of pointedness of the cone: the bigger the radius, the higher its degree of pointedness. We also discuss the question of measuring the degree of solidity of a closed convex cone. Pointedness and solidity radiuses are related to each other through a simple duality formula. Explicit computations are carried out for several classical cones appearing in the literature.

1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. Throughout this work we assume that $2 \leq \dim H < \infty$. Finite dimensionality of H is a crucial hypothesis that will always be in force.

There is a great variety of interesting subsets of

$$\mathcal{C}(H) = \{K \subset H : \ K \ \text{ is a nonempty closed convex cone}\},$$

two of which are

$$\mathcal{P}(H) = \{K \in \mathcal{C}(H) : \ K \ \text{ is pointed}\},$$

$$\mathcal{S}(H) = \{K \in \mathcal{C}(H) : \ K \ \text{ is solid}\}.$$

The terminology that we are using is standard: pointed cones are those which don't contain a line, whereas solid cones are those having nonempty interior. Pointed cones and solid cones abound in the literature, and play a significant role in many areas of applied mathematics.

We mention that only few authors have addressed the question of measuring the degree of pointedness and the degree of solidity of a cone. As far as pointedness is concerned, reference [10] uses a dimensional criterion: the degree of "nonpointedness" of a closed convex cone K is the dimension of its linearity space

$$\text{lin}K = K \cap (-K).$$

A pointed cone is one whose lineality space is zero-dimensional. If $\text{lin}K$ has the same dimension as the underlying space H, then K achieves the highest possible degree of nonpointedness. This classification scheme does not discriminate within the class of pointed cones itself, and this is precisely what we would like to do.

The notation used in this work is mostly standard; however, a partial list is provided for the reader's convenience:

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$$
\begin{array}{ll}
B_H = \{x \in H : \ \|x\| \le 1\} & \text{(closed unit ball in } H) \\
S_H = \{x \in H : \ \|x\| = 1\} & \text{(unit sphere in } H) \\
K^- = \{y \in H : \ \langle y, x\rangle \le 0 \ \ \forall x \in K\} & \text{(polar cone of } K) \\
K^+ = \{y \in H : \ \langle y, x\rangle \ge 0 \ \ \forall x \in K\} & \text{(dual cone of } K) \\
L^\perp = \{y \in H : \ \langle y, x\rangle = 0 \ \ \forall x \in L\} & \text{(orthogonal space of } L)
\end{array}
$$

2 Radius of pointedness: basic properties

Here we propose a “metrical" way of measuring the degree of pointedness of a closed convex cone. Indeed, our approach relies heavily on the use of the metric

$$\delta(K_1, K_2) = \sup_{\|x\| \le 1} |\operatorname{dist}[x, K_1] - \operatorname{dist}[x, K_2]|,$$

where the notation $\operatorname{dist}[x, K]$ refers to the Euclidean distance from x to K. Since this metric is going to be at the core of our discussion, a few preliminary words are useful at this early stage. First of all, it must be mentioned that $\delta(K_1, K_2)$ coincides with the classical Pompeiu-Hausdorff distance between the truncated sets $K_1 \cap B_H$ and $K_2 \cap B_H$ (cf. [1, 9]). The metric δ admits also the formulation

$$\delta(K_1, K_2) = \max\{e[K_1 \cap B_H, K_2]\ , e[K_2 \cap B_H, K_1]\}, \tag{2.1}$$

where

$$e[C, D] = \sup_{z \in C} \operatorname{dist}[z, D]$$

stands for the excess of C over D. The formulation (2.1) is sometimes more convenient when it comes to practical computations. From a theoretical viewpoint, what must be recalled is that

$$\text{the metric space } (\mathcal{C}(H), \delta) \text{ is compact.} \qquad ([5], \text{ Prop. } 2.1)$$

Also of relevance is the fact that

$$\mathcal{P}(H) \text{ is an open set in } (\mathcal{C}(H), \delta). \qquad ([5], \text{ Prop. } 2.2)$$

The later condition amounts to saying that each $K \in \mathcal{P}(H)$ is the center of a ball

$$U_r(K) = \{Q \in \mathcal{C}(H) : \ \delta(K, Q) < r\}$$

contained in $\mathcal{P}(H)$. It is then natural to ask how large can be the radius r of such a ball. This question leads to the estimation of the least upper bound

$$f(K) = \sup\{r \in [0, 1] : \ U_r(K) \subset \mathcal{P}(H)\}. \tag{2.2}$$

This number is what we call the radius of pointedness of K. Some comments on the expression (2.2) are in order. First of all, restricting the variable r to the interval $[0, 1]$ is not accidental. There is no need to go beyond $r = 1$ because $U_r(K) = \mathcal{C}(H)$ for any $r > 1$. When $r = 0$, the ball $U_r(K)$ is empty, and therefore the constraint $U_r(K) \subset \mathcal{P}(H)$ is trivially satisfied, regardless of whether K is in $\mathcal{P}(H)$ or not. In short, the number (2.2) is actually well defined for any element of $\mathcal{C}(H)$, and

$$f(K) = 0 \quad \Longleftrightarrow \quad K \text{ is not pointed.}$$

The radius of pointedness of K admits the obvious characterization

$$f(K) = \inf_{Q \in \mathcal{M}(H)} \delta(K, Q), \tag{2.3}$$

with $\mathcal{M}(H) = \mathcal{C}(H) \backslash \mathcal{P}(H)$ denoting the complement of $\mathcal{P}(H)$.

The equality (2.3) confirms that $f : \mathcal{C}(H) \to [0, 1]$ attains its lowest value 0 at any non-pointed cone. The attainability of the highest value 1 is an issue addressed in the next proposition. In what follows, the term “zero cone" refers to the element $O_H = \{0\} \in \mathcal{C}(H)$, and by a “ray" we mean a set of the form $\mathbb{R}_+ a = \{ta : t \in \mathbb{R}_+\}$, with $a \neq 0$.

PROPOSITION 2.1. Let $K \in \mathcal{C}(H)$. Then,

$$f(K) = 1 \quad \Longleftrightarrow \quad K \text{ is either a ray or the zero cone.}$$

Proof. We use the characterization (2.3), together with the expression (2.1). Since clearly

$$\delta(O_H, K) = 1 \qquad \forall K \neq O_H,$$

it follows that $f(O_H) = 1$. Consider now an arbitrary ray $\mathbb{R}_+ a$ with $\|a\| = 1$. Take any $Q \in \mathcal{M}(H)$. Select a unit vector $b \in H$ so that the line $\mathbb{R}b$ is contained in Q. By using (2.1), one obtains

$$\delta(\mathbb{R}_+ a, Q) \geq \sup_{z \in Q \cap B_H} \mathrm{dist}[z, \mathbb{R}_+ a] \geq \sup_{z \in \mathbb{R}b \cap B_H} \mathrm{dist}[z, \mathbb{R}_+ a]. \tag{2.4}$$

The maximum on the right-hand side of (2.4) is attained at $z = b$ if $\langle a, b\rangle \leq 0$, and at $z = -b$ if $\langle a, b\rangle \geq 0$. In either case, the maximum is equal to 1. Thus, $\delta(\mathbb{R}_+ a, Q) = 1$, proving in this way that $f(\mathbb{R}_+ a) = 1$. Suppose now that $K \in \mathcal{C}(H)$ is neither a ray nor the zero cone. We must prove that $f(K) < 1$. Without loss of generality, assume that K is not a linear subspace. For the sake of clarity, we distinguish between two cases.

Case 1: K is solid. This assumption ensures the existence of an hyperplane separating strictly K from its polar cone K^-. In other words, there is a nonzero vector $v \in H$ such that

$$\langle v, x\rangle \geq 0 \ \ \forall x \in K \quad \text{and} \quad \langle v, x\rangle < 0 \ \ \forall x \in K^- \backslash \{0\}. \tag{2.5}$$

Since the half-space $Q = \{x \in H : \langle v, x\rangle \geq 0\}$ is not pointed and contains K, it follows that

$$f(K) \leq \delta(K, Q) = \sup_{z \in Q \cap B_H} \mathrm{dist}[z, K].$$

The above supremum is attained at some vector $z_0 \in Q$ with $\|z_0\| = 1$. The separation property (2.5) implies that $z_0 \notin K^-$, and therefore

$$f(K) \leq \mathrm{dist}[z_0, K] = (\|z_0\|^2 - \mathrm{dist}^2[z_0, K^-])^{1/2} < 1,$$

the equality appearing in the above line being a consequence of Moreau's decomposition theorem [7].

Case 2: K is not solid. This case can be brought to the previous one by working in the linear space spanned by K, i.e.

$$\mathrm{span}K = K - K.$$

The cone K can be strictly separated from its relative polar cone

$$K^{\ominus} = \{y \in \mathrm{span}K : \langle y, x\rangle \leq 0 \ \ \forall x \in K\},$$

meaning that (2.5) should be written with $K^{\ominus}$ instead of K^-, and with a nonzero vector v living in $\mathrm{span}K$. Observe that $K^{\ominus} \neq \{0\}$ because $K \neq \mathrm{span}K$. Since K is not a ray, the relative half-space

$$Q = \{x \in \mathrm{span}K : \langle v, x\rangle \geq 0\}$$

contains not only K, but also a linear subspace of dimension ≥ 1. The remainder of the argument is as before. $\square$

Remark: Proposition 2.1 says, in particular, that all rays have the same radius of pointedness, namely, the highest possible. This information is somehow reassuring because we see rays as extremely pointed objects. The orientation of the ray is, of course, totally irrelevant.

The next theorem collects some basic facts concerning the function f. The notation $\mathrm{Isom}(H)$ stands for the set of linear isometries on H (i.e., linear operators $U : H \to H$ such that $\|Ux\| = \|x\| \ \forall x \in H$).

THEOREM 2.2. The function $f : \mathcal{C}(H) \to [0,1]$ enjoys the following properties:

(a) Lipschitz continuity: $|f(K_1) - f(K_2)| \leq \delta(K_1, K_2) \quad \forall K_1, K_2 \in \mathcal{C}(H)$;

(b) invariance property: $f(U(K)) = f(K) \quad \forall K \in \mathcal{C}(H), U \in \mathrm{Isom}(H)$;

(c) evenness: $f(-K) = f(K) \quad \forall K \in \mathcal{C}(H)$;

(d) surjectivity: $\{f(K) : \ K \in \mathcal{C}(H)\} = [0,1]$.

Proof. Lipschitz continuity of f is a direct consequence of the representation (2.3). In order to prove item (b), observe that a linear isometry $U : H \to H$ leaves invariant the set $\mathcal{M}(H)$, i.e.

$$U[\mathcal{M}(H)] = \{U(P) : \ P \in \mathcal{M}(H)\} \ = \mathcal{M}(H),$$

and also preserves distances between cones, i.e.

$$\delta(U(K_1), U(K_2)) = \delta(K_1, K_2) \qquad \forall K_1, K_2 \in \mathcal{C}(H).$$

Hence,

$$f(U(K)) = \inf_{Q \in \mathcal{M}(H)} \delta(U(K), Q) = \inf_{P \in \mathcal{M}(H)} \delta(U(K), U(P)) = \inf_{P \in \mathcal{M}(H)} \delta(K, P) = f(K).$$

Condition (c) is a particular case of (b). Surjectivity of f can be proven in a very elegant manner by invoking the deformation map of Iusem and Seeger [5]. Choose a cone $K \in \mathcal{C}(H)$ which is neither pointed, nor a linear subspace. Select a unit vector $a \in K$ such that

$$\langle a, x \rangle \geq 0 \quad \forall x \in K.$$

The existence of such a vector a is guaranteed by Gaddum's theorem ([2], Thm. 2.1). Consider now the function $\gamma : [0,1] \to \mathcal{C}(H)$ defined by

$$\gamma(t) = \{(1-t)v + t\|v\|a : \ v \in K\}.$$

As indicated in [5], γ is a continuous path joining $\gamma(0) = K$ and $\gamma(1) = \mathbb{R}_+ a$. As a consequence, the composition $f \circ \gamma : [0,1] \to \mathbb{R}$ attains the values

$$(f \circ \gamma)(0) = f(K) = 0 \quad \text{and} \quad (f \circ \gamma)(1) = f(\mathbb{R}_+ a) = 1.$$

By continuity, it attains also all the intermediate values. $\square$

Remark: Surjectivity of $f : \mathcal{C}(H) \to [0,1]$ can also be proven by using a more elementary argument. In Section 5, we shall compute the radius of pointedness of an arbitrary revolution cone. With this computation at hand, it will be clear that

$$\forall r \in [0,1], \ \text{ there is a revolution cone } K \in \mathcal{C}(H) \ \text{ such that } \ f(K) = r. \tag{2.6}$$

The revolution cone in (2.6) is not unique because we can place the axis of revolution pointing toward a different direction and this change will not modify the radius of pointedness. By contrast, the angle of revolution is unique, and it is given by $\theta = \arccos(r)$. To see this, we will have to wait until Section 5.

3 Radius of pointedness: intrinsic character

Up to now, we have considered K as an object lying in the linear space H. However, one can also see K as an object lying in the linear space $\mathrm{span}K$. It is therefore important to compare the number $f(K)$ with the intrinsic radius of pointedness

$$f_{\mathrm{intrinsic}}(K) = \inf\{\delta(K, Q) : \ Q \in \mathcal{M}(H) \ \text{ such that } Q \subset \mathrm{span}K\}$$

of the cone $K \neq \{0\}$. As we shall see in the next proposition, both numbers are equal. To prove this result, one needs first to state a technical lemma.

LEMMA 3.1. Let $K \in \mathcal{C}(H)$ be contained in some linear subspace $L \subset H$. Then,

$$\delta(\overline{P_L(Q)}, K) \leq \delta(Q, K) \qquad \forall Q \in \mathcal{C}(H), \tag{3.1}$$

where $P_L : H \to H$ denotes the orthogonal projection onto L, and $\overline{P_L(Q)}$ stands for the closure of $P_L(Q) = \{P_L(x) : x \in Q\}$.

Proof. To avoid trivialities, assume that $K \neq \{0\}$ and $Q \neq \{0\}$. Projecting an element of $\mathcal{C}(H)$ over the linear space L produces a convex cone which may not be closed. The closure operation in (3.1) has been added just to make sure that we remain in $\mathcal{C}(H)$. The orthogonal projection $P_L : H \to H$ is a linear mapping satisfying

$$\|P_L(u) - P_L(v)\| \leq \|u - v\| \qquad \forall u, v \in H.$$

In particular, for each $x \in K \subset L$, one has

$$\|x - v\| \geq \|x - P_L(v)\| \quad \forall v \in H.$$

By taking the infimum with respect to $v \in Q$, one gets

$$\mathrm{dist}[x, Q] \geq \inf_{v \in Q} \|x - P_L(v)\| = \mathrm{dist}[x, \overline{P_L(Q)}],$$

which, in turns, implies that $e[K \cap B_H, Q] \geq e[K \cap B_H, \overline{P_L(Q)}]$. We now claim that

$$e[Q \cap B_H, K] \geq e[\overline{P_L(Q)} \cap B_H, K]. \tag{3.2}$$

To prove this claim, we pick up a vector $z \in \overline{P_L(Q)} \cap B_H$ such that

$$\mathrm{dist}[z, K] = e[\overline{P_L(Q)} \cap B_H, K].$$

In other words, z is a maximum of the function $\mathrm{dist}[\,\cdot\,, K]$ over the set $\overline{P_L(Q)} \cap B_H$. We may assume without loss of generality that $z \neq 0$, because otherwise the right-hand side of (3.2) vanishes, in which case the inequality holds trivially. By positive homogeneity, we can suppose that $\|z\| = 1$. We do not know whether z belongs to $P_L(Q)$, but we can write

$$z = \lim_{n\to\infty} P_L(x_n)$$

for some sequence $\{x_n\}_{n\in\mathbb{N}}$ of nonzero vectors lying in Q. Observe that

$$\mathrm{dist}[P_L(x_n), K] \;\leq\; \mathrm{dist}[z, K] + \varepsilon_n \;\leq\; 1 + \varepsilon_n,$$

with $\varepsilon_n = \|z - P_L(x_n)\| \to 0$ as $n \to \infty$. Hence,

$$\begin{aligned}(1 + \varepsilon_n)^2 \|P_{L^\perp}(x_n)\|^2 &\geq \mathrm{dist}[P_L(x_n), K]^2 \|P_{L^\perp}(x_n)\|^2 \\ &= \mathrm{dist}[P_L(x_n), K]^2 (\|x_n\|^2 - \|P_L(x_n)\|^2),\end{aligned}$$

with $P_{L^\perp} = I - P_L$ being the orthogonal projection onto $L^\perp$. It follows that

$$\mathrm{dist}[P_L(x_n), K]^2 \;\leq\; \mathrm{dist}[P_L(\hat{x}_n), K]^2 \|P_L(x_n)\|^2 + (1 + \varepsilon_n)^2 \|P_{L^\perp}(\hat{x}_n)\|^2,$$

where $\{\hat{x}_n\}_{n\in\mathbb{N}}$ is the sequence obtained by normalizing each x_n. By taking a subsequence if necessary, one may assume that $\{\hat{x}_n\}_{n\in\mathbb{N}}$ converges to some unit vector $\hat{x}$. A simple rearrangement shows that

$$\mathrm{dist}[P_L(x_n), K]^2 \;\leq\; \mathrm{dist}[P_L(\hat{x}_n), K]^2 + \|P_{L^\perp}(\hat{x}_n)\|^2 + \gamma_n, \tag{3.3}$$

with

$$\gamma_n = \mathrm{dist}[P_L(\hat{x}_n), K]^2 (\|P_L(x_n)\|^2 - 1) + [(1 + \varepsilon_n)^2 - 1] \|P_{L^\perp}(\hat{x}_n)\|^2$$

going to 0 as $n \to \infty$. On the other hand, one can prove that

$$\mathrm{dist}[y, K]^2 \;\geq\; \mathrm{dist}[P_L(y), K]^2 + \|P_{L^\perp}(y)\|^2 \quad \forall y \in H. \tag{3.4}$$

To see this, write

$$\begin{aligned}\operatorname{dist}[y,K]^2 &= \|y-\pi_K(y)\|^2 = \|P_L(y)+P_{L^\perp}(y)-\pi_K(y)\|^2\\ &= \|P_L(y)-\pi_K(y)\|^2+\|P_{L^\perp}(y)\|^2 \geq \operatorname{dist}[P_L(y),K]^2+\|P_{L^\perp}(y)\|^2,\end{aligned}$$

where $\pi_K(y)$ is the point in K at shortest distance from y. The combination of (3.3) and (3.4) yields the inequality

$$\operatorname{dist}[\hat{x}_n,K]^2 \geq \operatorname{dist}[P_L(x_n),K]^2-\gamma_n.$$

By passing to the limit, one gets $\operatorname{dist}[\hat{x},K] \geq \operatorname{dist}[z,K]$. Since $\hat{x}\in Q\cap B_H$, it follows that

$$e[Q\cap B_H,K] \geq \operatorname{dist}[\hat{x},K],$$

completing the proof of (3.2), and of the lemma. □

PROPOSITION 3.2. Suppose that $K\in\mathcal{C}(H)$ is not the zero cone. Then,

(a) there exists $D\in\mathcal{M}(H)$ such that $D\subset \operatorname{span}K$ and $\delta(K,D)=f(K)$;

(b) $f(K)=f_{\text{intrinsic}}(K)$.

Proof. It is enough to check (a), item (b) being an immediate consequence. Let L be the linear space spanned by K. We know that there exists $Q\in\mathcal{M}(H)$ such that $\delta(K,Q)=f(K)$. Two cases are possible:

Case 1: $L^\perp\cap \operatorname{lin}Q\neq\{0\}$. Pick up a unit vector v in this intersection. Since $v\in L^\perp$ and $K\subset L$, we have that $\|v-x\|\geq 1$ for all $x\in K$. This implies that $d(v,K)=1$, and therefore $1=\delta(K,Q)=f(K)$. From the very definition of f, we get that $\delta(R,K)=1$ for all $R\in\mathcal{M}(H)$. Condition (a) is fulfilled because one can take D as any nonpointed cone contained in L, for instance, L itself.

Case 2: $L^\perp\cap \operatorname{lin}Q=\{0\}$. It follows that the projection onto L of any line contained in Q is also a line. Therefore, $P_L(Q)$ contains a line, and $D=\overline{P_L(Q)}$ belongs to $\mathcal{M}(H)$. By Lemma 3.1, we conclude that

$$f(K)\leq\delta(D,K)\leq\delta(Q,K)=f(K),$$

proving again what has been stated in (a). □

4 Radius of solidity

Knowing that $\mathcal{S}(H)$ is an open set in the metric space $(\mathcal{C}(H),\delta)$, we simply write

$$g(K)=\sup\{r\in[0,1]:\ U_r(K)\subset\mathcal{S}(H)\},$$

and refer to this number as the radius of solidity of K. It is clear that

$$g(K)=\inf_{Q\in\mathcal{N}(H)}\delta(K,Q), \tag{4.1}$$

with $\mathcal{N}(H)=\mathcal{C}(H)\backslash\mathcal{S}(H)$ denoting the complement of $\mathcal{S}(H)$.

The next theorem shows that the radius of solidity of a cone K is just the same as the radius of pointedness of its dual cone K^+. Such a result is obtained by exploiting the well known properties of the duality mapping $[\,\cdot\,]^+:\mathcal{C}(H)\to\mathcal{C}(H)$. Since these properties will be extensively used later on, this is a good opportunity to write them down in a detailed manner. What need to be recalled is that

(a) $(K^+)^+=K \quad \forall K\in\mathcal{C}(H)$; (biduality theorem [8])

(b) $\delta(K_1^+,K_2^+)=\delta(K_1,K_2) \quad \forall K_1,K_2\in\mathcal{C}(H)$; (isometry theorem [11])

(c) $\mathcal{S}(H)=\{K^+:\ K\in\mathcal{P}(H)\}$.

THEOREM 4.1. Let $K\in\mathcal{C}(H)$. Then, $g(K)=f(K^+)$ and $f(K)=g(K^+)$.

Proof. Is is enough to write

$$g(K)=\inf_{Q\in\mathcal{N}(H)}\delta(K,Q)=\inf_{Q\in\mathcal{N}(H)}\delta(K^+,Q^+)=\inf_{P\in\mathcal{M}(H)}\delta(K^+,P)=f(K^+).$$

The second formula is obtained by biduality. □

COROLLARY 4.2. The function $g:\mathcal{C}(H)\to[0,1]$ enjoys the following properties:

(a) Lipschitz continuity: $|g(K_1)-g(K_2)|\leq\delta(K_1,K_2)$ $\forall K_1,K_2\in\mathcal{C}(H)$;

(b) invariance property: $g(U(K))=g(K)$ $\forall K\in\mathcal{C}(H),U\in Isom(H)$;

(c) evenness: $g(-K)=g(K)$ $\forall K\in\mathcal{C}(H)$;

(d) surjectivity: $\{g(K):\ K\in\mathcal{C}(H)\}=[0,1]$.

Proof. It is a matter of combining Theorem 4.1 with Theorem 2.2. □

COROLLARY 4.3. Let $K\in\mathcal{C}(H)$. Then,

$$g(K)=1\quad\Longleftrightarrow\quad K\text{ is either a half-space or the whole space }H.$$

Proof. This time one combines Theorem 4.1 with Proposition 2.1. □

Remark: Corollary 4.3 says, in particular, that all half-spaces have the same radius of solidity, namely, the highest possible. If a closed convex cone is strictly contained in a half-space, then its radius of solidity is strictly smaller than 1.

In what follows, the notation

$$\mathrm{Proj}[K,\mathcal{M}(H)]=\{Q\in\mathcal{M}(H):\ f(K)=\delta(K,Q)\}$$

refers to the set of cones in $\mathcal{M}(H)$ achieving the minimal distance (2.3). According to standard terminology, this set corresponds to the metric projection of K into $\mathcal{M}(H)$. Similarly,

$$\mathrm{Proj}[K,\mathcal{N}(H)]=\{Q\in\mathcal{N}(H):\ g(K)=\delta(K,Q)\}$$

is set of cones in $\mathcal{N}(H)$ achieving the minimal distance (4.1), that is to say, it is the metric projection of K into $\mathcal{N}(H)$. As complement to Theorem 4.1, one has:

THEOREM 4.4. Let $K\in\mathcal{C}(H)$. Then,

$$Q\in\mathrm{Proj}[K,\mathcal{N}(H)]\quad\Longleftrightarrow\quad Q^+\in\mathrm{Proj}[K^+,\mathcal{M}(H)].$$

Proof. See the proof of Theorem 4.1. □

Remark: A completely analogous duality theory can be developed in terms of the polarity mapping $[\,\cdot\,]^-:\mathcal{C}(H)\to\mathcal{C}(H)$. The relationship $K^-=-K^+$ allows us to switch from one framework to the other one.

We end this section with a remark concerning the "intrinsic" radius of solidity of a cone. What could be such a concept? A simple duality argument leads us to introduce

$$g_{\mathrm{intrinsic}}(K)=\inf\{\delta(K,Q):\ Q\in\mathcal{N}(H)\ \text{ such that } Q\supset\mathrm{lin}K\}$$

for any $K\in\mathcal{C}(H)$ such that $K\neq H$.

PROPOSITION 4.5. Take any $K\in\mathcal{C}(H)$ different from the whole space H. Then,

(a) there exists $Q \in \mathcal{N}(H)$ such that $Q \supset \mathrm{lin}K$ and $\delta(K,Q) = g(K)$;

(b) $g(K) = g_{\mathrm{intrinsic}}(K)$.

Proof. A simple computation shows that $g_{\mathrm{intrinsic}}(K) = f_{\mathrm{intrinsic}}(K^+)$, so everything boils down to combining Proposition 3.2 and Theorem 4.1. □

5 Revolution cones

In this section we focuss our attention in a very particular class of closed convex cones. The term "revolution cone" refers to a set of the form

$$\mathrm{rev}(a,\theta) = \{x \in H : \|x\| \cos\theta \le \langle a, x\rangle\}, \tag{5.1}$$

with $a \in H$ being a unit vector, and $\theta \in [0, \pi/2]$. The representation (5.1) is unique in the sense that

$$\mathrm{rev}(a_1,\theta_1) = \mathrm{rev}(a_2,\theta_2) \quad \Longrightarrow \quad (a_1,\theta_1) = (a_2,\theta_2).$$

According to standard practice, $\mathbb{R}_+ a$ is called the "axis of revolution" of the cone (5.1), while the number θ is referred to as the "angle of revolution" of (5.1).

The dual cone of a revolution cone is again a revolution cone, more precisely

$$[\mathrm{rev}(a,\theta)]^+ = \mathrm{rev}(a, \pi/2 - \theta). \qquad \text{([3], p.120)}$$

PROPOSITION 5.1. For any unit vector $a \in H$ and any $\theta \in [0, \pi/2]$, one has

$$f(\mathrm{rev}(a,\theta)) = \cos\theta \quad \text{and} \quad g(\mathrm{rev}(a,\theta)) = \sin\theta. \tag{5.2}$$

Proof. To compute the radius of solidity of $\mathrm{rev}(a,\theta)$, we start by examining the distance

$$\begin{aligned}\delta(\mathrm{rev}(a,\theta), \mathbb{R}_+ a) &= \sup\{\mathrm{dist}[x, \mathbb{R}_+ a] : \|x\| = 1,\ x \in \mathrm{rev}(a,\theta)\}\\ &= \sup\{[1 - \langle a, x\rangle^2]^{1/2} : \|x\| = 1,\ \cos\theta \le \langle a, x\rangle\}.\end{aligned}$$

From the last line, one sees that

$$\delta(\mathrm{rev}(a,\theta), \mathbb{R}_+ a) = [1 - (\cos\theta)^2]^{1/2} = \sin\theta.$$

Since $\mathbb{R}_+ a$ is not solid, we got the upper bound $g(\mathrm{rev}(a,\theta)) \le \sin\theta$. We now prove that this upper bound cannot be sharpened. Let Q be an element of $\mathcal{N}(H)$ at minimal distance from $\mathrm{rev}(a,\theta)$. First of all, one can show that

$$\mathbb{R}_+ a \subset Q. \tag{5.3}$$

If this was not the case, then one could apply a separation argument and exhibit a point $\tilde{x} \in \mathrm{rev}(a,\theta) \cap B_H$ such that $\mathrm{dist}[\tilde{x}, Q] > \sin\theta$, contradicting in this way the optimality of Q. Secondly, since the cone Q is not solid, it must be contained in some hyperplane

$$b^\perp = \{y \in H : \langle b, y\rangle = 0\}, \quad \text{with } \|b\| = 1.$$

Due to the inclusion (5.3), this hyperplane contains the ray $\mathbb{R}_+ a$. In short, $\langle a, b\rangle = 0$. One can easily check that

$$x_0 = (\cos\theta)\, a + (\sin\theta)\, b$$

is a unit vector lying in $\mathrm{rev}(a,\theta)$, and

$$\sin\theta = \mathrm{dist}[x_0, b^\perp] \le \mathrm{dist}[x_0, Q].$$

Hence, $\sin\theta \le \delta(\mathrm{rev}(a,\theta), Q)$, proving in this way our claim. The first equality in (5.2) follows from a duality argument: it suffices to apply Theorem 4.1. □

Example: The Lorentz (or ice-cream) cone is the archetypal example of a revolution cone. Its n-dimensional version corresponds to

$$\Lambda_n = \{x \in \mathbb{R}^n : \ [x_1^2 + \cdots + x_{n-1}^2]^{1/2} \ \leq x_n \ \}.$$

One supposes, of course, that $n \geq 3$. By writing

$$\Lambda_n = \{x \in \mathbb{R}^n : \|x\| \leq \sqrt{2}\ \langle a, x\rangle\} \quad \text{with} \ \ a = (0, \cdots, 0, 1),$$

one sees that Λ_n is a revolution cone with angle of revolution $\theta = \pi/4$. Hence, $f(\Lambda_n) = g(\Lambda_n) = \sqrt{2}/2$.

6 Majorization and minorization techniques

Optimal solutions of problems defined in a normed space can be identified with the help of necessary optimality conditions. Computing the radius of pointedness of a given cone $K \in \mathcal{C}(H)$ is, in general, a very hard task, and this is because the minimization process (2.3) is taking place in a metric space without linear structure. One way of coping with the lack of optimality conditions is by using a traditional technique of majorization and minorization. As shown in the next proposition, the radius of pointedness of K admits as lower bound the expression

$$f_\sharp(K) = \inf_{\|z\|=1} \ \max \{\text{dist}[z, K], \ \text{dist}[-z, K]\}. \tag{6.1}$$

The auxiliary function $f_\sharp : \mathcal{C}(H) \to [0, 1]$ deserves to be studied on its own right. A wealth of information on this function is provided in our work [6], but here we go straight to the main point:

LEMMA 6.1. For any $K \in \mathcal{C}(H)$, one has $f_\sharp(K) \leq f(K)$.

Proof. Take $Q \in \mathcal{M}(H)$ such that $f(K) = \delta(K, Q)$. Pick up a unit vector $z \in H$ such that $\mathbb{R}z \subset Q$. Since both z and $-z$ belong to $Q \cap B_H$, we have

$$f(K) = \delta(K, Q) \geq \sup_{x \in Q \cap B_H} \text{dist}[x, K] \ \geq \ \max\{\text{dist}[z, K], \text{dist}[-z, K]\} \ \geq f_\sharp(K),$$

completing the proof in this way. □

Evaluating $f_\sharp$ is, of course, much simpler than evaluating f. Lemma 6.1 is used next to derive lower bounds for the radiuses of pointedness of cones with special structure. Recall that

$$K \in \mathcal{C}(H) \ \text{ is said to be } \begin{cases} \text{infra-dual if } K \subset K^+, \\ \text{supra-dual if } K \supset K^+, \\ \text{self-dual if } K = K^+. \end{cases}$$

Infra-dual cones are also called acute cones because they satisfy the acute angles property:

$$K \ \text{ is infra-dual} \quad \Longleftrightarrow \quad \langle x, y\rangle \geq 0 \quad \forall x, y \in K.$$

One can easily prove that infra-dual cones are pointed and that supra-dual cones are solid. In fact, these statements admit a more precise formulation:

PROPOSITION 6.2. For any $K \in \mathcal{C}(H)$, one has:

(a) if K is infra-dual, then $f(K) \geq \sqrt{2}/2$;

(b) if K is supra-dual, then $g(K) \geq \sqrt{2}/2$.

Proof. Before starting with the proof, just a small remark on the geometry of Hilbert spaces. The fact that $\|\cdot\|$ derives from the inner product $\langle\cdot, \cdot\rangle$, allows us to write

$$\|z - x\|^2 + \|z + y\|^2 = \|z\|^2 + \|z - x + y\|^2 + 2\langle x, y\rangle \qquad \forall x, y, z \in H,$$

and, in particular,

$$\langle x,y\rangle \geq 0 \text{ and } \|z\|=1 \quad\Longrightarrow\quad \|z-x\|^2+\|z+y\|^2\geq 1. \tag{6.2}$$

Having said this, take now a unit vector $z\in H$ such that

$$f_\sharp(K)=\max\,\{\mathrm{dist}[z,K],\ \mathrm{dist}[-z,K]\}.$$

Hence,

$$f_\sharp(K)=\max\{\|z-\pi_K(z)\|,\|-z-\pi_K(-z)\|\}=\max\{\|z-\pi_K(z)\|,\|z+\pi_K(-z)\|\}.$$

Since the vectors $x=\pi_K(z)$ and $y=\pi_K(-z)$ belong to the infra-dual cone K, their inner product cannot be negative. We can apply (6.2) and obtain

$$\|z-\pi_K(z)\|^2+\|z+\pi_K(-z)\|^2\geq 1. \tag{6.3}$$

The maximum of the two terms in the left-hand side of (6.3) must be at least $1/2$, so one gets

$$\max\{\|z-\pi_K(z)\|,\|z+\pi_K(-z)\|\}\geq\sqrt{2}/2.$$

The conclusion is that $f_\sharp(K)\geq\sqrt{2}/2$. It suffices now to recall Lemma 6.1. As far as (b) is concerned, just apply item (a) to the cone K^+ and invoke Theorem 4.1. □

By obvious reasons, we refer to $f_\sharp$ as being a lower auxiliary function. Finding a nice upper auxiliary function is not a trivial matter. One must look for a function $f^\sharp$ enjoying two properties: on the one hand side, $f^\sharp$ should be a “sharp" upper estimate for f, and, on the other hand, the computation of $f^\sharp$ should not be too difficult. As a compromise between these two conflicting objectives, we suggest considering the expression

$$f^\sharp(K)=\inf_{\|z\|=1} m_K(z),$$

with

$$m_K(z)=\sup\,\{\mathrm{dist}[x+\alpha z,K]:\ (x,\alpha)\in K\times\mathbb{R},\ \|x+\alpha z\|=1\}. \tag{6.4}$$

To better understand the reason leading to this choice, we must enter into the proof of the next lemma.

LEMMA 6.3. For any $K\in\mathcal{C}(H)$, one has

$$f^\sharp(K)=\inf\{\delta(K,Q):\ Q\in\mathcal{M}(H)\text{ such that }Q\supset K\}. \tag{6.5}$$

In particular, $f(K)\leq f^\sharp(K)$.

Proof. Denote by $\tilde f(K)$ the term on the right-hand side of (6.5). Consider an arbitrary unit vector $z\in H$. The expression (6.4) can be written in the form

$$m_K(z)=\sup_{c\in S_H\cap(K+\mathbb{R}z)}\mathrm{dist}[c,K].$$

By positive homogeneity, one can also write

$$m_K(z)=\sup_{c\in B_H\cap(K+\mathbb{R}z)}\mathrm{dist}[c,K]=\sup_{c\in B_H\cap\overline{K+\mathbb{R}z}}\mathrm{dist}[c,K],$$

with the upper bar denoting the closure operation. Now, observe that $\overline{K+\mathbb{R}z}\in\mathcal{C}(H)$ contains the cone K, and therefore

$$m_K(z)=\delta(K,\overline{K+\mathbb{R}z})\ \geq\tilde f(K),$$

showing that $f^\sharp(K)\ \geq\tilde f(K)$. For the reverse inequality, we rely on a general property of the metric δ, to wit

$$K_1\subset K_2\subset K_3\quad\Longrightarrow\quad\delta(K_1,K_2)\leq\delta(K_1,K_3). \tag{6.6}$$

Take $Q \in \mathcal{M}(H)$ such that $Q \supset K$ and $\delta(K,Q) = \tilde{f}(K)$. The existence of such a Q follows from a compactness argument. Since Q contains a line, there is a unit vector $z \in H$ such that

$$K \subset \overline{K + \mathbb{R}z} \subset Q.$$

By applying (6.6), one gets

$$\delta(K, \overline{K + \mathbb{R}z}) \leq \delta(K,Q).$$

This yields $f^\sharp(K) \leq \delta(K,Q) = \tilde{f}(K)$, and completes the proof. $\square$

What assumptions on K ensure there is no gap between $f_\sharp(K)$ and $f^\sharp(K)$? Taking care of this question is of great importance because the answer would provide a practical rule for computing the radius of pointedness of K. As a preliminary step in this direction, we exhibit below an alternative characterization of $f_\sharp$.

LEMMA 6.4. If $K \in \mathcal{C}(H)$ is not the zero cone, then

$$f_\sharp(K) = \sqrt{1 - \frac{[\mathrm{diam}(K \cap S_H)]^2}{4}} = \sqrt{\frac{1 + \cos\theta_{\max}(K)}{2}}, \tag{6.7}$$

with $\mathrm{diam}(K \cap S_H)$ denoting the diameter of the set $K \cap S_H$, and $\theta_{\max}(K)$ being the largest angle that can be formed by picking up two unit vectors in the cone K.

Proof. The proof is quite involved and takes several pages. It is given in our work [6]. $\square$

THEOREM 6.5. Suppose that $K \in \mathcal{C}(H)$ is neither a ray, nor the zero cone. Let $L = \mathbb{R}(u - v)$, with $u, v \in K \cap S_H$ such that $\|u - v\| = \mathrm{diam}(K \cap S_H)$. Assume the following hypothesis:

$$\begin{cases} \text{each } c \in K + L \text{ can be decomposed in the form} \\ c = a + b, \text{ with } a \in K,\ b \in L, \text{ and } \langle a, b\rangle \geq 0. \end{cases} \tag{6.8}$$

In such case,

(a) $f_\sharp(K) = f(K) = f^\sharp(K)$;

(b) $K + L$ is a member of $\mathcal{M}(H)$ lying at minimal distance from K.

Proof. Since K is not a ray, one has $u \neq v$, and one can define $z = \|u-v\|^{-1}(u-v)$. Notice, incidentally, that u, v are two unit vectors in K achieving the maximal angle $\theta_{\max}(K)$. This fact is easy to check. What is more difficult to prove is that

$$f_\sharp(K) = \mathrm{dist}[z, K] = \mathrm{dist}[-z, K], \tag{6.9}$$

but this has been done in [6]. Another point that must be mentioned from the very beginning is that

$$\begin{cases} L \cap K = \{0\} & \text{if } K \text{ is pointed} \\ L \subset K & \text{if } K \text{ is not pointed,} \end{cases} \tag{6.10}$$

so, in either case, the cone $K + L$ is closed. The proof of (6.10) is not difficult. Let us prove that

$$K \text{ is pointed} \quad \Longrightarrow \quad u - v \notin K \text{ and } v - u \notin K.$$

Suppose that K is pointed, so that $\langle u, v\rangle > -1$. If $u - v$ were in K, then z would be unit vector in K such that

$$\langle z, v\rangle \ = \ \frac{\langle u, v\rangle - 1}{\|u - v\|} \ = \ -\frac{1 - \langle u, v\rangle}{\sqrt{2(1 - \langle u, v\rangle)}} \ = \ -\sqrt{\frac{1 - \langle u, v\rangle}{2}} \ < \ \langle u, v\rangle,$$

contradicting the fact that u, v achieve the maximal angle of K. In a similar way one proves that $v - u \notin K$. So, in the pointed case, the line L touches K only at the origin. On the contrary, if K is not pointed, then $u = -v$ and the line L is wholly contained in K. This concludes the proof of (6.10). Now, by Lemmas 6.1 and 6.3, we have

$$f_\sharp(K) \le f(K) \le f^\sharp(K) \le \delta(K, K+L) \ = \sup_{c \in (K+L)\cap S_H} \|c - \pi_K(c)\|.$$

To obtain (a) and (b), it is enough to show that

$$\|c - \pi_K(c)\| \le f_\sharp(K) \qquad \forall c \in (K+L) \cap S_H. \tag{6.11}$$

Take any $c \in (K+L)\cap S_H$ and decompose it in the form (6.8). First of all, observe that $a + \pi_K(b)$ belongs to K. By definition of π_K,

$$\|c - \pi_K(c)\| \le \|c - [a + \pi_K(b)]\| = \|a + b - a - \pi_K(b)\| = \|b - \pi_K(b)\|.$$

To proceed further, write $b = \eta z$ with $\eta \in \mathbb{R}$. By positive homogeneity of π_K, one gets

$$\|b - \pi_K(b)\| = \|\eta z - \pi_K(\eta z)\| \le \eta \|z - \pi_K(z)\| \quad \text{if} \quad \eta \ge 0,$$

$$\|b - \pi_K(b)\| = \|\eta z - \pi_K(\eta z)\| \le -\eta \| - z - \pi_K(-z)\| \quad \text{if} \quad \eta \le 0.$$

In short,

$$\|b - \pi_K(b)\| \le |\eta| \max\{\mathrm{dist}[z, K], \mathrm{dist}[-z, K]\}.$$

But,

$$1 = \|c\|^2 = \|a\|^2 + \eta^2 \|z\|^2 + 2\langle a, \eta z\rangle \ge \eta^2.$$

Hence, $|\eta| \le 1$, and

$$\|c - \pi_K(c)\| \le \max\{\mathrm{dist}[z, K], \mathrm{dist}[-z, K]\} = f_\sharp(K),$$

the last equality being due to (6.9). □

Remark: We are using the decomposability assumption (6.8) because it has a clear geometric meaning and doesn't need a further explanation. The conclusion of Theorem 6.5 remains true, however, if one uses the weaker assumption

$$\begin{cases} \text{each } c \in (K+L)\cap S_H \text{ can be decomposed in the} \\ \text{form } \ c = a + b, \text{ with } \ a \in K, \ b \in L, \ \text{ and } \ \|b\| \le 1. \end{cases} \tag{6.12}$$

The decomposability requirement (6.12) looks perhaps a little bit more technical, but it can rephrased in terms of a simple set-inclusion, namely $(K+L)\cap S_H \subset K + (L \cap B_H)$.

7 Pareto cone and Loewner cone

In this section we evaluate the radiuses of pointedness of two important self-dual cones. Theorem 6.5 plays a key role in both cases.

7.A. Pareto cone. The Pareto cone (or positive orthant) in $\mathbb{R}^n$ is simply

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : \ x_1 \ge 0, \ \cdots, \ x_n \ge 0\}.$$

The largest angle that can be formed in this cone is $\theta_{\max}(\mathbb{R}^n_+) = \pi/2$. This maximal angle is attained by choosing an arbitrary pair among the canonical vectors

$$e_1 = (1, 0, \cdots, 0), \ \ \cdots, e_n = (0, 0, \cdots, 1).$$

According to Lemma 6.4, one has $f_\sharp(\mathbb{R}^n_+) = \sqrt{2}/2$. The same estimate holds for the radius of pointedness:

COROLLARY 7.1. The radius of pointedness of the Pareto cone is $f(\mathbb{R}^n_+) = \sqrt{2}/2$. A member of $\mathcal{M}(\mathbb{R}^n)$ at minimal distance from $\mathbb{R}^n_+$ is, for instance,

$$\mathbb{R}^n_+ + \mathbb{R}(e_1 - e_2) = \{x \in \mathbb{R}^n : \ x_1 + x_2 \ge 0, x_3 \ge 0, \cdots, x_n \ge 0\}.$$

Proof. We consider the first two canonical vectors, but the same argument applies to any other choice. If $L = \mathbb{R}(e_1 - e_2) = \{(\eta, -\eta, 0, \cdots, 0) : \eta \in \mathbb{R}\}$, then

$$\begin{aligned} \mathbb{R}^n_+ + L &= \{(x_1 + \eta, x_2 - \eta, x_3, \cdots, x_n) : x \in \mathbb{R}^n_+, \ \eta \in \mathbb{R}\} \\ &= \{x \in \mathbb{R}^n : x_1 + x_2 \geq 0, x_3 \geq 0, \cdots, x_n \geq 0\}. \end{aligned}$$

Any $c \in \mathbb{R}^n_+ + L$ can be decomposed as sum

$$c = (\frac{c_1 + c_2}{2}, \frac{c_1 + c_2}{2}, c_3, \cdots, c_n) + (\frac{c_1 - c_2}{2}, \frac{c_2 - c_1}{2}, 0, \cdots, 0)$$

of two orthogonal vectors, the first in $\mathbb{R}^n_+$ and the second in L. This takes care of the decomposability assumption (6.8). Theorem 6.5 yields then the desired conclusion. □

Three final comments on the Pareto cone are in order. First, the half-space

$$Q = \{x \in \mathbb{R}^n : x_1 + \cdots + x_n \geq 0\},$$

which could be thought as a good candidate for achieving the minimal distance to $\mathbb{R}^n_+$, turns out to be a bad choice. Indeed,

$$\delta(Q, \mathbb{R}^n_+) = \sqrt{1 - 1/n} \ > \sqrt{2}/2 \qquad \forall n > 2.$$

Second, the cone which realizes the minimum in the definition of $f(\mathbb{R}^n_+)$ is not unique: our construction exhibits at least $n(n-1)/2$ of them, corresponding to all possible ways of selecting a pair among the n canonical vectors. And, third, a duality argument shows that

$$[\mathbb{R}^n_+ + \mathbb{R}(e_1 - e_2)]^+ = \mathbb{R}^n_+ \cap [\mathbb{R}(e_1 - e_2)]^\perp = \{x \in \mathbb{R}^n_+ : \ x_1 = x_2\}$$

is a member in $\mathcal{N}(\mathbb{R}^n)$ at minimal distance from $\mathbb{R}^n_+$. The radius of solidity of $\mathbb{R}^n_+$ is, of course, $g(\mathbb{R}^n_+) = \sqrt{2}/2$.

7.B. Loewner cone. The linear space $\mathrm{Sym}(n)$ of symmetric matrices of order $n \times n$ is equipped with the usual inner product $\langle A, B \rangle = \mathrm{trace}(AB)$. The Loewner cone

$$\mathrm{Sym}_+(n) = \{A \in \mathrm{Sym}(n) : \ A \text{ is positive semidefinite } \}$$

is yet another example of self-dual cone.

We shall compute the radius of pointedness of $\mathrm{Sym}_+(n)$ by using Theorem 6.5. Instead of the decomposability requirement (6.8), we will check the weaker condition (6.11). As can be seen from the proof of Theorem 6.5, the condition (6.11) is all what is needed to arrive at the same conclusion. To check (6.11) in this particular setting, we rely on a technical lemma concerning diagonal perturbations of positive semidefinite matrices. As usual, the notation $\mathrm{diag}(d_1, , \cdots, d_n)$ refers to the diagonal matrix having the elements $d_1, , \cdots, d_n$ on the diagonal.

LEMMA 7.2 (diagonal perturbation lemma). Let $X \in \mathrm{Sym}(n)$ be a positive semidefinite matrix, and $D = \pm\mathrm{diag}(1, -1, 0, \cdots, 0)$. Then, all the eigenvalues of $X + D$ are nonnegative, except possibly the smallest one. For this special eigenvalue, one has the lower estimate

$$-\frac{\sqrt{2}}{2} \, \|X + D\| \leq \lambda_{\min}(X + D).$$

Proof. It is a long exercise in matrix analysis. To avoid interrupting the flow of the exposition, we leave the details for the appendix added at the end of the paper. □

One could formulate a more general diagonal perturbation lemma, but this is not our aim. Without further ado, we state:

COROLLARY 7.3. The radius of pointedness of the Loewner cone is $f(\mathrm{Sym}_+(n)) = \sqrt{2}/2$. A member of $\mathcal{M}(\mathrm{Sym}(n))$ at minimal distance from $\mathrm{Sym}_+(n)$ is, for instance,

$$\mathcal{Q} = \mathrm{Sym}_+(n) + \mathbb{R}(E_1 - E_2),$$

with $E_1 = \mathrm{diag}(1, 0, 0, \cdots, 0)$ and $E_2 = \mathrm{diag}(0, 1, 0, \cdots, 0)$.

Proof. Observe that $E_1 \in \mathrm{Sym}_+(n)$ and $E_2 \in \mathrm{Sym}_+(n)$ achieve the maximal angle

$$\theta_{\max}(\mathrm{Sym}_+(n)) = \pi/2.$$

By Lemma 6.4, we know that $f_\sharp(\mathrm{Sym}_+(n)) = \sqrt{2}/2$. To apply Theorem 6.5, we need to check that

$$\mathrm{dist}[C, \mathrm{Sym}_+(n)] \leq \sqrt{2}/2 \tag{7.1}$$

for any matrix $C \in \mathcal{Q}$ of unit length. Before checking (7.1), it is useful to recall the general formula

$$\mathrm{dist}[B, \mathrm{Sym}_+(n)] = \Big[\sum_{i=1}^{n} [\min\{0, \lambda_i(B)\}]^2\Big]^{1/2} \qquad \forall B \in \mathrm{Sym}(n), \tag{7.2}$$

where the $\lambda_i(B)$'s are the eigenvalues of B. So, take

$$C = A + \eta(E_1 - E_2) \quad \text{with } \ A \in \mathrm{Sym}_+(n) \quad \text{and } \ \eta \in \mathbb{R},$$

and suppose that $\|C\| = 1$. If $\eta = 0$, then $\mathrm{dist}[C, \mathrm{Sym}_+(n)] = 0$ and we are done. Otherwise, we can write C in the form

$$C = \|X + D\|^{-1}(X + D),$$

with X and D as in Lemma 7.2. The case $D = \mathrm{diag}(1, -1, 0, \cdots, 0)$ occurs if $\eta > 0$, and the case $D = -\mathrm{diag}(1, -1, 0, \cdots, 0)$ occurs if $\eta < 0$. There are two possibilities regarding the sign of $\lambda_{\min}(X+D)$. If this eigenvalue is nonnegative, then $C \in \mathrm{Sym}_+(n)$ and (7.1) holds trivially. When the smallest eigenvalue of $X + D$ is negative, the situation is not problematic either. Indeed, Lemma 7.2 yields

$$|\lambda_{\min}(C)| \leq \sqrt{2}/2,$$

and formula (7.2) allows us to obtain $\mathrm{dist}[C, \mathrm{Sym}_+(n)] \leq \sqrt{2}/2$. □

8 Elliptic cones

To see Theorem 6.5 in action in a more involved setting, we discuss now the important class of elliptic cones. For notational convenience, we work in the space $H = \mathbb{R}^n \times \mathbb{R}$ equipped with the usual inner product

$$\langle (y, r), (x, t) \rangle = \langle y, x \rangle + rt.$$

We use the term "elliptic cone" to refer to a set of the form

$$\mathcal{E}(A) = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : \sqrt{\langle x, Ax \rangle} \leq t\},$$

where the matrix $A \in \mathrm{Sym}(n)$ is assumed to be positive semidefinite. The term "nondegenerate elliptic cone" is used when $A \in \mathrm{Sym}(n)$ is positive definite. This is actually the most interesting case, because a singular matrix produces necessarily a nonpointed elliptic cone. In fact,

$$f(\mathcal{E}(A)) = 0 \quad \Longleftrightarrow \quad A \ \text{ is singular.}$$

The next two lemmas set up the stage for the coming discussion.

LEMMA 8.1. The polar of a nondegenerate elliptic cone is a nondegenerate elliptic cone. More precisely, for any positive definite $A \in \mathrm{Sym}(n)$, one has

$$[\mathcal{E}(A)]^+ = \mathcal{E}(A^{-1}).$$

Proof. Take $(x,t)\in\mathcal{E}(A)$ and $(y,r)\in\mathcal{E}(A^{-1})$. Then,

$$\langle (y,r),(x,t)\rangle = rt+\langle y,x\rangle = rt+\langle A^{-1/2}y, A^{1/2}x\rangle \ \geq\ rt-\|A^{-1/2}y\|\,\|A^{1/2}x\| \ \geq 0,$$

where the last inequality follows from the fact that $r\geq \langle y,A^{-1}y\rangle^{1/2}$ and $t\geq\langle x,Ax\rangle^{1/2}$. It follows that $\mathcal{E}(A^{-1})\subset[\mathcal{E}(A)]^+$. For the reverse inclusion, take $(y,r)\in[\mathcal{E}(A)]^+$ and suppose that (y,r) does not belong to $\mathcal{E}(A^{-1})$, i.e.

$$r<\langle y,A^{-1}y\rangle^{1/2}.$$

Consider the vector $z\in\mathbb{R}^n$ defined by

$$z=-\langle y,A^{-1}y\rangle^{-1/2}\ A^{-1}y.$$

Since $\langle z,Az\rangle\ =1$, it follows that $(z,1)\in\mathcal{E}(A)$. On the other hand,

$$\langle (y,r),(z,1)\rangle = r-\frac{\langle y,A^{-1}y\rangle}{\langle y,A^{-1}y\rangle^{1/2}} = r-\langle y,A^{-1}y\rangle^{1/2}\ <0,$$

contradicting the fact that $(y,r)\in[\mathcal{E}(A)]^+$. □

LEMMA 8.2. If $A\in\mathrm{Sym}(n)$ is a nonnegative multiple of the identity matrix, say $A=aI$, then $\mathcal{E}(A)$ is a revolution cone with angle of revolution θ given by $\cos\theta=\sqrt{a/(1+a)}$.

Proof. It is elementary. □

Proposition 5.1 tells us how to compute the radius of pointedness of a revolution cone. With such an estimate at hand, it is possible to address now the more general case of an elliptic cone.

THEOREM 8.3. For any positive semidefinite matrix $A\in\mathrm{Sym}(n)$, the radius of pointedness of $\mathcal{E}(A)$ is given by

$$f(\mathcal{E}(A))=\sqrt{\frac{\lambda_{\min}(A)}{1+\lambda_{\min}(A)}}\ , \tag{8.1}$$

with $\lambda_{\min}(A)$ denoting the smallest eigenvalue of A.

Proof. Decompose $A\in\mathrm{Sym}(n)$ in the usual form $A=UDU^T$, with $D=\mathrm{diag}(\lambda_1,\cdots,\lambda_n)$ containing the eigenvalues $\lambda_1\leq\ \cdots\ \leq\lambda_n$ of A arranged in a nondecreasing order. The orthonormal matrix U is formed with the corresponding eigenvectors. A simple exercise in linear algebra shows that

$$\mathcal{E}(A)=(U\oplus 1)[\mathcal{E}(D)],$$

with $U\oplus 1:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}$ being the linear isometry defined by $(U\oplus 1)(z,t)=(Uz,t)$. The invariance property stated in Theorem 2.2 tells us that

$$f(\mathcal{E}(A))=f(\mathcal{E}(D)),$$

so everything boils down to the analysis of the diagonal case. Our first task will be evaluating the lower estimate $f_\sharp(\mathcal{E}(D))$. Consider the points $u,v\in\mathbb{R}^n\times\mathbb{R}$ given by

$$u=(1+\lambda_1^{-1})^{-1/2}\ (\lambda_1^{-1/2},0,\ldots,0,1),$$
$$v=(1+\lambda_1^{-1})^{-1/2}\ (-\lambda_1^{-1/2},0,\ldots,0,1).$$

Since u,v are unit vectors lying in $\mathcal{E}(D)$, one has

$$[\mathrm{diam}(S_{\mathbb{R}^n\times\mathbb{R}}\cap\mathcal{E}(D))]^2\geq\|u-v\|^2=4/(1+\lambda_1),$$

and therefore

$$[f_\sharp(\mathcal{E}(D))]^2=1-\frac{[\mathrm{diam}(S_{\mathbb{R}^n\times\mathbb{R}}\cap\mathcal{E}(D))]^2}{4}\ \leq\frac{\lambda_1}{1+\lambda_1}.$$

On the other hand, the inequality

$$\lambda_1 \|z\|^2 \leq \langle z, Dz \rangle \quad \forall z \in \mathbb{R}^n,$$

yields the inclusion $\mathcal{E}(D) \subset \mathcal{E}(\lambda_1 I)$, the latter set being a revolution cone with angle of revolution θ given by

$$\cos^2\theta = \lambda_1/(1+\lambda_1).$$

Since $f_\sharp$ is a reverse monotone function, it follows that

$$[f_\sharp(\mathcal{E}(D))]^2 \ \geq [f_\sharp(\mathcal{E}(\lambda_1 I))]^2 \ = \cos^2\theta = \lambda_1/(1+\lambda_1).$$

Summarizing, the vectors u, v achieve the maximal angle that can be formed in $\mathcal{E}(D)$, and

$$f_\sharp(\mathcal{E}(D)) = \sqrt{\lambda_1/(1+\lambda_1)}.$$

Our second task consists in showing that $f_\sharp(\mathcal{E}(D)) = f(\mathcal{E}(D))$. Here is where Theorem 6.5 enters into action. The crucial point, of course, is checking the decomposability assumption (6.8). Since

$$u - v = (1+\lambda_1^{-1})^{-1/2}(2\lambda_1^{-1/2}, 0, \cdots, 0),$$

it follows that the line L through $u - v$ is given by $L = \{(\eta, 0, \ldots, 0) : \eta \in \mathbb{R}\}$. In view of the definition of $\mathcal{E}(D)$, we conclude that

$$\mathcal{E}(D) + L = \{(z,t) \in \mathbb{R}^n \times \mathbb{R} : \ [\sum_{j=2}^{n} \lambda_j z_j^2]^{1/2} \leq t\}.$$

Notice that any $(z,t) \in \mathcal{E}(D) + L$ can be written as sum

$$(z,t) = (0, z_2, \cdots, z_n, t) + (z_1, 0, \cdots, 0, 0)$$

of two orthogonal vectors, the first in $\mathcal{E}(D)$ and the second in L. This means that we are authorized to apply Theorem 6.5, and we are able to arrive at the desired conclusion. □

COROLLARY 8.4. The function $A \in \mathrm{Sym}_+(n) \mapsto f(\mathcal{E}(A))$ is continuous and concave.

Proof. It follows from (8.1). □

The proof of Theorem 8.3 contains a wealth of information concerning the radius of pointedness of an elliptic cone $\mathcal{E}(A)$. One gets not only the exact estimate of $f(\mathcal{E}(A))$, but also one see how to construct a nonpointed cone achieving the minimal distance to $\mathcal{E}(A)$. Eigenvectors associated to the smallest eigenvalue of A serve for this purpose. Without further ado, we state:

THEOREM 8.5. Let $A \in \mathrm{Sym}(n)$ be positive definite. Let $x_{\min}(A)$ be an eigenvector of A corresponding to the eigenvalue $\lambda_{\min}(A)$. Then,

$$Q = \mathcal{E}(A) + [\mathbb{R}\, x_{\min}(A)] \times \{0\}$$

is a member of $\mathcal{M}(\mathbb{R}^n \times \mathbb{R})$ at minimal distance from $\mathcal{E}(A)$.

Proof. From the proof of Theorem 8.3, we know already that $\mathcal{E}(D) + L$ is a member of $\mathcal{M}(\mathbb{R}^n \times \mathbb{R})$ at minimal distance from $\mathcal{E}(D)$. Since $U \oplus 1$ is a linear isometry, it follows that

$$Q = (U \oplus 1)[\mathcal{E}(D) + L] \tag{8.2}$$

is a member of $\mathcal{M}(\mathbb{R}^n \times \mathbb{R})$ at minimal distance from $(U \oplus 1)[\mathcal{E}(D)] = \mathcal{E}(A)$. We just need to work out a bit more the term on the right-hand side of (8.2). As a matter of computation, one gets

$$(U \oplus 1)[\mathcal{E}(D) + L] = (U \oplus 1)[\mathcal{E}(D)] + (U \oplus 1)[L] = \mathcal{E}(A) + \{(\eta u_1, 0) : \ \eta \in \mathbb{R}\},$$

where $u_1 = x_{\min}(A)$ is the first column of U. □

As brought into light by Theorems 8.3 and 8.5, the spectral nature of the matrix A gives us an important clue regarding the pointedness degree of the elliptic cone $\mathcal{E}(A)$. This theme could be developed further by examining, for instance, the geometric multiplicity of the smallest eigenvalue $\lambda_{\min}(A)$. For reasons of space limitation, we will not discuss here all these subtleties. We mention, however, a few words concerning the solidity radius of an elliptic cone.

THEOREM 8.6. For any positive definite matrix $A \in \mathrm{Sym}(n)$, one has:

(a) the radius of solidity of $\mathcal{E}(A)$ is given by

$$g(\mathcal{E}(A)) = 1/\sqrt{1+\lambda_{\max}(A)}\,, \tag{8.3}$$

with $\lambda_{\max}(A)$ denoting the largest eigenvalue of A.

(b) a member of $\mathcal{N}(\mathbb{R}^n \times \mathbb{R})$ at minimal distance from $\mathcal{E}(A)$ is

$$P = \mathcal{E}(A) \cap [x_{\max}(A)]^{\perp} \times \mathbb{R},$$

with $[x_{\max}(A)]^{\perp} \subset \mathbb{R}^n$ denoting the space of vectors that are orthogonal to $x_{\max}(A)$.

Proof. The proof of item (a) is a matter of exploiting the duality results established in Theorem 4.1 and Lemma 8.1. Indeed,

$$g(\mathcal{E}(A)) = f([\mathcal{E}(A)]^+) = f(\mathcal{E}(A^{-1})) = \sqrt{\frac{\lambda_{\min}(A^{-1})}{1+\lambda_{\min}(A^{-1})}}.$$

Plugging $\lambda_{\min}(A^{-1}) = [\lambda_{\max}(A)]^{-1}$ in the last expression, one gets the announced formula. For proving item (b), we combine Theorems 4.4 and 8.5. □

COROLLARY 8.7. For any positive semidefinite matrix $A \in \mathrm{Sym}(n)$, one has

$$[f(\mathcal{E}(A))]^2 + [g(\mathcal{E}(A))]^2 \leq 1. \tag{8.4}$$

Equality in (8.4) occurs if and only if A is a nonnegative multiple of the identity matrix.

Proof. If A is singular, then $f(\mathcal{E}(A)) = 0$ and (8.4) holds trivially. If A is nonsingular, then

$$[f(\mathcal{E}(A))]^2 + [g(\mathcal{E}(A))]^2 = \frac{\lambda_{\min}(A)}{1+\lambda_{\min}(A)} + \frac{1}{1+\lambda_{\max}(A)} \leq \frac{\lambda_{\min}(A)}{1+\lambda_{\min}(A)} + \frac{1}{1+\lambda_{\min}(A)} = 1.$$

Equality in (8.4) occurs if and only if $\lambda_{\min}(A) = \lambda_{\max}(A)$. □

9 The region of radial configurations

Pointedness and solidity radiuses describe somehow the geometry of a closed convex cone. Since these numbers are related to each other, the region

$$\Omega = \{(f(K), g(K)) : \ K \in \mathcal{C}(H)\}$$

of radial configurations is strictly contained in the square $[0,1] \times [0,1]$. For instance, the pair $(p,s) = (1,1)$ is not a radial configuration because f and g cannot achieve the value 1 at the same time. What about pairs like $(p,s) = (1/3, 2/3)$ or $(p,s) = (\sqrt{2}/2, 3/4)$? We state below two results concerning the set Ω.

PROPOSITION 9.1 . The region of radial configurations is closed and path-connected. Also, it is symmetric in the sense that $(p,s) \in \Omega$ if and only if $(s,p) \in \Omega$.

Proof. Ω is the image of the compact set $\mathcal{C}(H)$ under the continuous function $K \mapsto (f(K), g(K))$. This proves that Ω is closed. Throwing away from $\mathcal{C}(H)$ all the cones that are linear subspaces will not change the set Ω. More precisely,

$$\Omega = \{(f(K), g(K)) : \ K \in \mathcal{C}_1(H)\}, \quad \text{with} \quad \mathcal{C}_1(H) = \{K \in \mathcal{C}(H) : \ K \neq -K\}.$$

As shown in [5], Prop. 7.3, the set $\mathcal{C}_1(H)$ is path-connected. This takes care of the path-connectedness of Ω. For the last statement of the proposition, just apply Theorem 4.1. □

PROPOSITION 9.2. Suppose that $\dim H \geq 3$. For any $(p, s) \in [0,1] \times [0,1]$ such that $p^2 + s^2 \leq 1$, there is a cone $K \in \mathcal{C}(H)$ for which $(f(K), g(K)) = (p, s)$.

Proof. We consider $H = \mathbb{R}^3$, but similar examples can be constructed in higher dimensional spaces.

Case $p > 0, s > 0$. The positive numbers $a_1 = p^2/(1-p^2)$ and $a_2 = (1-s^2)/s^2$ are well defined. The condition $p^2 + s^2 \leq 1$ implies that $a_1 \leq a_2$. Take $A = \mathrm{diag}(a_1, a_2)$ and look at the elliptic cone $\mathcal{E}(A) \subset \mathbb{R}^3$. Theorems 8.3 and 8.6 yield

$$f(\mathcal{E}(A)) = \sqrt{a_1/(1+a_1)} = p \ \text{ and } \ g(\mathcal{E}(A)) = 1/\sqrt{1+a_2} = s.$$

Case $p > 0, s = 0$. The idea is intersecting a revolution cone with an hyperplane. Take, for instance, $K = \{x \in \mathbb{R}^3 : \ x_1 = 0, \ p[x_2^2 + x_3^2]^{1/2} \leq x_3\}$. By combining Propositions 3.2 and 5.1, one gets $f(K) = p$. Since K is not solid, one has, of course, $g(K) = 0$.

Case $p = 0, s > 0$. This case is derived from the previous one by using a duality argument.

Case $p = 0, s = 0$. Take, for instance, $K = \{x \in \mathbb{R}^3 : x_1 = 0, x_3 \geq 0\}$, which is neither pointed nor solid. □

Proposition 9.2 provides a lower estimate for the region of radial configurations:

$$\{(p, s) \in [0,1] \times [0,1] : \ p^2 + s^2 \leq 1\} \ \subset \Omega. \tag{9.1}$$

There are reasons to suspect that (9.1) is actually an equality, but this is something that we cannot ascertain at the time being. We leave as open the following (difficult) question:

$$\text{is there a cone } \ K \in \mathcal{C}(H) \ \text{ such that } \ [f(K)]^2 + [g(K)]^2 > 1? \tag{9.2}$$

Notice that if such K exists, it cannot be an elliptic cone (cf. Corollary 8.7).

10 Appendix

This appendix is devoted to the proof of the diagonal perturbation lemma. We start by recalling a celebrated interlacing property between the eigenvalues $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$ of a matrix $A \in \mathrm{Sym}(n)$ and the eigenvalues $\lambda_1(B) \leq \cdots \leq \lambda_n(B)$ of a perturbed version $B = A \pm qq^T$.

THEOREM (interlacing property). Let $A \in \mathrm{Sym}(n)$ and $q \in \mathbb{R}^n$ be given. Then,

$$\lambda_k(A) \leq \lambda_{k+1}(A \pm qq^T) \leq \lambda_{k+2}(A) \quad \forall k \in \{1, \cdots, n-2\}.$$

Proof. This result can be found, for instance, in the book by Horn and Johnson [4], Thm. 4.3.4. □

The next result provides a more precise information on the localization of the eigenvalues of $A \pm qq^T$. We shall consider only the case $A - qq^T$, the case $A + qq^T$ can be treated in a similar way. Recall that

$$\rho(A) = \{\mu \in \mathbb{R} : \ A - \mu I \text{ is nonsingular}\} \quad \text{and} \quad \mu \in \rho(A) \mapsto (A - \mu I)^{-1}$$

are called, respectively, the resolvant set and the resolvant mapping of $A \in \mathrm{Sym}(n)$.

THEOREM (eigenvalue localization). Let $A \in \mathrm{Sym}(n)$ and $q \in \mathbb{R}^n$ be a nonzero vector. If μ is an eigenvalue of $A - qq^T$, then exactly one of the following two alternatives occurs:

(a) μ is an eigenvalue of A;

(b) $\mu \in \rho(A)$ and $\langle q, (A - \mu I)^{-1} q \rangle = 1$.

Proof. This result is probably known. The proof is simple and runs as follows. Suppose μ is an eigenvalue of $A - qq^T$, but not an eigenvalue of A. Let $x \in \mathbb{R}^n$ be an eigenvector of $A - qq^T$ associated to the eigenvalue μ. Then, $Ax - \langle q, x\rangle q = \mu x$. Notice that $\langle q, x \rangle \neq 0$. Since

$$\hat{x} = \langle q, x\rangle^{-1} x = (A - \mu I)^{-1} q \tag{A.1}$$

is an eigenvector of $A - qq^T$ associated to the eigenvalue μ, one has the relation

$$\mu \, \langle \hat{x}, \hat{x} \rangle = \langle \hat{x}, (A - qq^T)\hat{x} \rangle. \tag{A.2}$$

By plugging (A.1) into (A.2), one arrives finally at the relation (b). □

We now are ready to take care of the lemma on diagonal perturbations:

Proof of LEMMA 7.2. Consider, for instance, the case $D = E_2 - E_1 = e_2 e_2^T - e_1 e_1^T$. The matrix $B = X + E_2$ is positive semidefinite, and therefore $0 \leq \lambda_1(B) \leq \lambda_2(B - E_1)$. This shows that $X + D = B - E_1$ admits at most one negative eigenvalue. Suppose $\mu = \lambda_{\min}(X + D)$ is negative, otherwise there is nothing more to prove. Decompose $X = U \Lambda U^T$, with Λ being the diagonal matrix formed with the eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ of X, and U being an orthonormal matrix whose columns are corresponding eigenvectors. Observe that $X + D$ has the same norm as

$$U^T(X + D)U = U^T(X + E_2 - E_1) = \Lambda + U^T E_2 U - U^T E_1 U = \Lambda + vv^T - qq^T,$$

where q, v are the first and second row of the matrix U. Since the unit vectors q and v are orthogonal, one has $\|vv^T - qq^T\| = 2$ and

$$\|X + D\| = \|\Lambda + vv^T - qq^T\| = \Big[2 + \sum_{j=1}^{n} \lambda_j^2 + 2 \sum_{j=1}^{n} \lambda_j (v_j^2 - q_j^2)\Big]^{1/2}. \tag{A.3}$$

Let us keep (A.3) waiting for a while, and have a look at the eigenvalue $\hat{\mu} = \lambda_{\min}(X - E_1)$. One can easily check that $0 > \mu \geq \hat{\mu} \geq -1$, so that $|\mu| \leq |\hat{\mu}| \leq 1$. Observe now that $U^T(X - E_1)U = \Lambda - qq^T$. Due to the previous theorem, we know that $\hat{\mu}$ satisfies the resolvent equality $\langle q, (\Lambda - \hat{\mu} I)^{-1} q\rangle = 1$, which in this case reduces to

$$\sum_{j=1}^{n} (\lambda_j - \hat{\mu})^{-1} q_j^2 \; = 1.$$

A simple computation shows that

$$1 = \sum_{j=1}^{n} \frac{q_j^2}{\lambda_j + |\hat{\mu}|} \leq \sum_{j=1}^{n} \frac{q_j^2}{|\hat{\mu}|\lambda_j + |\hat{\mu}|} = |\hat{\mu}|^{-1} \sum_{j=1}^{n} \frac{q_j^2}{\lambda_j + 1}.$$

But the convexity of the square function yields

$$\Big(\sum_{j=1}^{n} \frac{q_j^2}{\lambda_j + 1}\Big)^2 \; \leq \sum_{j=1}^{n} q_j^2 \Big(\frac{1}{\lambda_j + 1}\Big)^2.$$

This and the general inequality

$$\Big(\frac{1}{t+1}\Big)^2 \leq 1 + \frac{t^2}{2} - t \qquad \forall t \geq 0,$$

lead us to

$$\Big(\sum_{j=1}^{n} \frac{q_j^2}{\lambda_j + 1}\Big)^2 \; \leq 1 + \frac{1}{2} \sum_{j=1}^{n} \lambda_j^2 - \sum_{j=1}^{n} \lambda_j q_j^2.$$

Hence,

$$|\mu| \leq |\hat{\mu}| \leq \sum_{j=1}^{n} \frac{q_j^2}{\lambda_j + 1} \leq \Big[1 + \frac{1}{2}\sum_{j=1}^{n} \lambda_j^2 - \sum_{j=1}^{n} \lambda_j q_j^2\Big]^{1/2} \leq \frac{\sqrt{2}}{2}\|X + D\|,$$

completing the proof in this way. □

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