Non-wandering sets with non-empty interior

F. Abdenur, C. Bonatti and L.J. Díaz *

March 19, 2003

Abstract

We study diffeomorphisms of a closed connected manifold whose non-wandering set has non-empty interior and conjecture that C^1-generic diffeomorphisms whose non-wandering set has non-empty interior are transitive. We first prove this conjecture in three cases: hyperbolic, partially hyperbolic with two hyperbolic bundles, and tame diffeomorphisms (in the first case, the conjecture is a folklore result and in the second one it follows adapting the proof in [B]).

We study this conjecture without global assumptions and prove that, generically, a homoclinic class with non-empty interior is either the whole manifold or else accumulated by infinitely many different homoclinic classes. Finally, we prove that, generically, homoclinic classes and non-wandering sets with non-empty interior are weakly hyperbolic (existence of a dominated or a volume hyperbolic splitting).

1 Introduction

When the dynamics of a system (here a diffeomorphism f) is complicated, one tries to study the global dynamics of the system by looking at those regions of the ambient space which concentrate the recurrences and the intricate parts of the dynamics. The non-wandering set is one of the most common sets used by dynamicists for localizing the complexity (other possibilities are, for instance, the limit set, the closure of the periodic points or the chain recurrent set). Examples of diffeomorphisms on compact manifolds having very simple dynamics are those whose non-wandering set $\Omega(f)$ is finite (for instance, the Morse-Smale systems). On the contrary, diffeomorphisms whose non-wandering set is the whole manifold exhibit very complex dynamics and have played an important role in our understanding of global dynamics. The best-known examples of these diffeomorphisms are certainly the irrational rotations of the circle or of the torus T^n, and the Anosov diffeomorphisms (and also flows). The first ones are perhaps more perfect: all the orbits have the same behaviour (f is minimal and uniquely ergodic). Unfortunately, they are very fragile and can be destroyed by small perturbations. In contrast, the second class of examples presents a very rich variety of behaviours and are robust. In the seventies [Sh] and [M_1], and more recently [BD_1] and [BV], presented examples of non-hyperbolic robustly transitive diffeomorphisms (in particular, the non-wandering set is robustly the whole manifold).

We study here an intermediate situation between the two extremes possibilities described above: we consider diffeomorphisms with a big non-wandering set (non-empty interior) which is

*This work was partially supported by the agreement Brasil-France in mathematics, Faperj and CNPq. F.A. is supported by a Prodoc/CAPES fellowship and would like to thank UFRJ where he spent some of the period during which this paper was written. The authors acknowledge the very kind hospitality of Departamento de Matemática (PUC-Rio), IMPA, and Laboratoire de Topologie (Université de Dijon) during their visits while preparing this paper.

not the whole manifold. The following two questions arise now in a natural way: do such diffeomorphisms exist generically? Can one derive some global consequences for the dynamics of the diffeomorphism from this local property? In fact, the robustly transitive diffeomorphisms cited above are the unique known examples of locally-generic diffeomorphisms whose non-wandering set has non-empty interior. Actually, we conjecture that *C^1-generic diffeomorphisms whose non-wandering set has non-empty interior are transitive* (see Section 1.1). For instance, it is well-known that this conjecture is true if f is an Axiom A diffeomorphism. Indeed, by the Spectral Decomposition Theorem, see [Sm], an Axiom A diffeomorphism f whose non-wandering set has non-empty interior must have a basic set Λ with non-empty interior (the local product structure implies that the boundary of such a basic set is empty). We also present two other context where the conjecture holds.

1.1 The conjecture

Throughout this paper M denotes a compact connected boundaryless d-dimensional manifold, $d \geq 2$, and $\mathrm{Diff}^1(M)$ the space of C^1 diffeomorphisms on M, endowed with the usual C^1 topology. We begin by stating some basic definitions.

Given an open subset $\mathcal{U}$ of $\mathrm{Diff}^1(M)$, a subset $\mathcal{R}$ of $\mathcal{U}$ is *residual in* $\mathcal{U}$ if $\mathcal{R}$ contains the intersection of a countable family of open and dense subsets of $\mathcal{U}$; in this case the set $\mathcal{R}$ is dense in $\mathcal{U}$. A set $\mathcal{R} \subset \mathrm{Diff}^1(M)$ is said to be *residual* if it is residual in all of $\mathrm{Diff}^1(M)$. A property (P) is *residual* or *generic* in $\mathcal{U}$ if it holds for all diffeomorphisms which belong to some residual subset of $\mathcal{U}$. We say that the property (P) is *locally generic* if there is a non-empty open set of $\mathrm{Diff}^1(M)$ in which (P) is generic; the property (P) is said to be *residual* or *generic* if it is residual in all of $\mathrm{Diff}^1(M)$.

The *non-wandering set* of a diffeomorphism f, denoted by $\Omega(f)$, is the set of points x such that, for every neighbourhood U of x, there is some natural number n with $f^n(U) \cap U \neq \emptyset$. This set is closed and f-invariant.

A diffeomorphism $f \in \mathrm{Diff}^1(M)$ is *transitive* if there is some $x \in M$ such that the ω-limit set $\omega_f(x)$ of x coincides with M (i.e., the forward orbit of x is dense in the ambient manifold). Observe that if a diffeomorphism is transitive then $\Omega(f)$ is the whole manifold.

Conjecture 1. *There is a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ of diffeomorphisms f such that if $\Omega(f)$ has non-empty interior then the diffeomorphism is transitive.*

In this paper we collect some partial results in the direction of this conjecture:

- In Section 2 we study the conjecture assuming some global hypotheses. We first give a simple proof of the conjecture in the hyperbolic case (Theorem 1); this is a well known folklore result. We also prove it in the strongly partially hyperbolic case (there is a partially hyperbolic splitting with two hyperbolic bundles defined on the whole manifold). In this case the result (Theorem 2) follows by Brin's arguments in [B]. Finally, we prove the conjecture for tame diffeomorphisms (diffeomorphisms whose nonwandering set consists, locally generically, of finitely many homoclinic classes), see Theorem 3.

- In Section 3 we prove that the property of a homoclinic class or the nonwandering set containing a fixed open set is a locally generic property (Theorem 4). We also relate this property to Lyapunov stability. Finally, in Theorem 5 we prove that, generically, the dynamics at a homoclinic class with non-empty interior which is not the whole manifold is wild: such a homoclinic class is accumulated by infinitely many pairwise disjoint homoclinic classes. In positive terms: an isolated homoclinic class of a generic diffeomorphisms with non-empty interior is the whole manifold, thus the conjecture holds for isolated homoclinic classes.

- We give local versions of the global results above in Section 4. We prove in Theorem 6 that, generically, any homoclinic class which is strongly partially hyperbolic and has non-empty interior is the whole manifold.
- Finally, in Section 5 we prove that any homoclinic class with non-empty interior satisfies a weak form of hyperbolicity, namely existence of a dominated splitting or volume hyperbolicity, depending on the context, see Theorems 7 and 8.

2 Cases where the conjecture holds: hyperbolic, strongly partially hyperbolic, and tame diffeomorphisms

In this section we present three situations where the conjecture is settled: the hyperbolic case, the *strongly partially hyperbolic* case (there is a partially hyperbolic splitting defined on the whole manifold with two hyperbolic directions), and the case of *tame diffeomorphisms* (generic diffeomorphisms having finitely many homoclinic classes). We will give the precise definitions later.

2.1 The hyperbolic case

In the hyperbolic case the conjecture is well known. For completeness we give the proof here.

Theorem 1. *Any hyperbolic set Λ of a diffeomorphism $f \in \mathrm{Diff}^1(M)$, contained in $\Omega(f)$ and with non-empty interior, is the whole manifold. Thus the diffeomorphism f is Anosov.*

Proof. The theorem of existence of stable and unstable manifolds asserts that they contain transverse differentiable disks $W^s_{loc}(x)$, $W^u_{loc}(x)$ of radius $\delta > 0$ through each point $x \in \Lambda$ and that these disks vary continuously with x.

Let x belong to the interior of Λ and fix $\eta > 0$ such that the ball of radius η centered at x is contained in Λ. As $\Lambda \subset \Omega(f)$ there is a sequence $x_n \in \Lambda$ converging to x and a sequence $i_n \to +\infty$ such that $f^{i_n}(x_n)$ converges to x. For large n, the disk of radius $\frac{\eta}{2}$ centered at x_n in the local unstable manifold of x_n is contained in Λ. Using the uniform expansion of f in the unstable direction and the continuity of the unstable manifolds, one gets that the whole unstable manifold of x is contained in the closure $\overline{\Lambda} \subset \Lambda$ of the interior of Λ. By continuity of the unstable manifolds and the compacity of Λ, the same holds for any point $y \in \overline{\Lambda}$, in fact, for any $y \in \overline{\Lambda}$, the closure $\overline{W^u(y)}$ is contained in $\overline{\Lambda}$. Arguing in the same way with f^{-1} one gets the same property for the stable manifolds.

Assume by contradiction that $\overline{\Lambda}$ is not the whole manifold M and consider now a point z in the boundary $\partial\overline{\Lambda}$. The union of the stable manifolds of the points of $W^u(z)$ is a neighborhood of z contained in Λ, which is a contradiction finishing the proof. $\square$

Remark 2.1. *The result of Theorem 1 is false if the hyperbolic set Λ is not assumed to be contained in $\Omega(f)$. For instance, consider an Axiom A diffeomorphism on a compact surface having a Plykin attractor A and a Plykin repellor R whose basins have non-empty intersection. Consider now a disk $D \subset W^s(A) \cap W^u(R)$ where the stable foliation of $W^s(A)$ is transverse to the unstable foliation of $W^u(R)$. Now the closure of the orbit of D is a hyperbolic set with non-empty interior but which is different from the whole manifold.*

2.2 The strongly partially hyperbolic case

Before discussing the proof in the strongly partially hyperbolic case let us define precisely the different types of hyperbolicity we will consider in this paper.

2.2.1 Forms of hyperbolicity

Definition 2.2. (Forms of hyperbolicity).

- *Consider a diffeomorphism f and an f-invariant set Λ. A Df-invariant splitting $E \oplus F$ of TM over Λ is* dominated *if the fibers of the bundles have constant dimension and there are a metric $||\cdot||$ and a natural number $n \in \mathbb{N}$ such that*

$$||Df^n(x)_E|| \cdot ||Df^{-n}(x)_F|| < \frac{1}{2} \quad \textit{for all } x \in \Lambda.$$

  In this case we say that the splitting is n-dominated.

- *On some occasions one needs to consider splittings having more than two bundles, say $E_1 \oplus E_2 \oplus \cdots \oplus E_k$; this splitting is* dominated *if, for all $i = 1, \cdots, (k-1)$, the splitting $(\oplus_{j=1}^{i} E_j) \oplus (\oplus_{j=i+1}^{k} E_j)$ is dominated.*

- *We say that the splitting $E_1 \oplus E_2 \oplus \cdots \oplus E_k$ is the* finest dominated splitting *if any bundle of the splitting is* undecomposable, *that is no E_i admits a dominated splitting.*

- *The dominated splitting $E_1 \oplus E_2 \oplus \cdots \oplus E_k$ is* volume hyperbolic *if there is n such that the derivative of f^n uniformly contracts the (induced) volume in E_1 and uniformly expands the volume in E_k.*

- *The splitting $E_1 \oplus E_2 \oplus \cdots \oplus E_k$ is* partially hyperbolic *if at least one of the extremal bundles E_1 or E_k is hyperbolic. If both bundles E_1 and E_k are hyperbolic (contracting and expanding, respectively) the splitting is* strongly partially hyperbolic. *In this case, the bundles E_1 and E_k are called the* stable *and* unstable bundles, *respectively, and denoted by E^s and E^u.*

Let $\mathcal{PH}(M) \subset \mathrm{Diff}^1(M)$ denote the set of *partially hyperbolic* diffeomorphisms: that is, $\mathcal{PH}(M)$ is the (open) space of C^1 diffeomorphisms f having a partially hyperbolic splitting defined on the whole M. If the splitting is strongly partially hyperbolic we say that the diffeomorphism is *strongly partially hyperbolic.* We denote by $\mathcal{SPH}(M) \subset \mathrm{Diff}^1(M)$ this set of diffeomorphisms.

The definitions of partially hyperbolic and strongly partially hyperbolic sets (for compact invariant sets) are given in the natural way.

Recall that any $f \in \mathcal{SPH}(M)$ has two invariant foliations $\mathcal{F}^u_f$ and $\mathcal{F}^s_f$ tangent to E^u and to E^s (the unstable and stable bundles of f) respectively, called the *strong stable and unstable foliations*, see [BP, HPS].

Finally, a set Σ is called *saturated by a foliation $\mathcal{F}$* if it contains any leaf passing through its points.

2.2.2 Strongly partially hyperbolic dynamics

We need a definition that will play a key role in this paper:

Definition 2.3. *For any $f \in \mathcal{SPH}(M)$ and any point $p \in M$ the* accessibility class *of p is the smallest saturated set for both foliations $\mathcal{F}^u_f$ and $\mathcal{F}^s_f$ containing p. In other words, it is the set of points q which can be joined to p by a path consisting of finitely many segments contained in leaves of $\mathcal{F}^s_f$ or $\mathcal{F}^u_f$.*

A diffeomorphism $f \in \mathcal{SPH}(M)$ is accessible *if there is a unique accessibility class (necessarily equal to M).*

There is the following result on the accessibility classes of partially hyperbolic diffeomorphisms:

Theorem ([DW]). *There is a C^1-open and dense subset $\mathcal{A}$ of $\mathcal{SPH}(M)$ such that every $f \in \mathcal{A}$ is accessible.*

We have now the following result (which implies the conjecture in this setting):

Theorem 2. *Let $\mathcal{A}$ be a C^1-open set of $\mathcal{SPH}(M)$ consisting of accessible diffeomorphisms. Then any diffeomorphism $f \in \mathcal{A}$ whose non-wandering set has non-empty interior is transitive.*

This result is a generalization of the result in [B] asserting that *every accessible diffeomorphism $f \in \mathcal{SPH}(M)$ with $\Omega(f) = M$ is transitive.* In fact, Theorem 2 follows immediately using the nice argument of Brin in [B].

Proof. The proof follows noting that the saturation by $\mathcal{F}^u$ (or by $\mathcal{F}^s$) of any non-empty open subset U of $\Omega(f)$ also is contained in $\Omega(f)$. Since by hypothesis f is an accessible diffeomorphism, the saturation of U by $\mathcal{F}^s$ and $\mathcal{F}^u$ is the whole manifold, thus $\Omega(f) = M$. Finally, [B] implies the transitivity of f.

We prove, for instance, that the saturation of U by $\mathcal{F}^u$ is contained in $\Omega(f)$. Take any point $x \in U \subset \Omega(f)$. Arbitrarily close to x there are sequences of points $y_n \in U$ and of positive numbers k_n such that the sequences y_n and $f^{k_n}(y_n)$ converge to x and the k_n go to $+\infty$. For each y_n we can take a disk D_n^u centered at y_n, contained in U and in the leaf of $\mathcal{F}^u$ containing y_n, and having uniform radius. Then, by the continuous dependence of the leaves of $\mathcal{F}^u$ and since the disks are expanded in the unstable direction, the sequence of disks $f^{k_n}(D_n)$ accumulates to the leaf of x. By construction, these disks are contained in $\Omega(f)$, thus, since $\Omega(f)$ is closed, the leaf of x is also contained in $\Omega(f)$. The proof of the theorem is now complete. $\square$

2.3 Tame diffeomorphisms case

To prove the conjecture in the tame case let us define the *homoclinic class* $H(p, f)$ of a hyperbolic periodic point p of a diffeomorphism f as the closure of the transverse intersections of the stable and unstable manifolds of the orbit of p [1]. We say that two saddles p and q of f are *homoclinically related* if the unstable manifold of p and the stable manifold of q have non-empty transverse intersection, and the same happens with the unstable manifold of p and the stable one of q. We observe that two homoclinically related saddles are in the same homoclinic class. The converse is false in general, for instance, a homoclinic class may contain points having different indices (dimension of the stable bundle) and which therefore cannot be homoclinically related. Homoclinic classes are compact f-invariant transitive sets, thus contained in $\Omega(f)$. Moreover, in the case of Axiom A diffeomorphisms they are precisely the basic sets in the Spectral Decomposition Theorem.

[Ab$_1$] introduces two types of generic diffeomorphisms: there is a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such $\mathcal{T} \cup \mathcal{W} = \mathcal{R}$ where:

- $\mathcal{T}$ is the set of *tame diffeomorphisms,* that is, diffeomorphisms f having a C^1-neighbourhood $\mathcal{U}_f$ such that every $g \in \mathcal{U}_f \cap \mathcal{R}$ has finitely many homoclinic classes (which are pairwise disjoint) and such that the number of the classes is constant in $\mathcal{U}_f$. Moreover, the union of these homoclinic classes is the whole non-wandering set;

- $\mathcal{W}$ is the set of *wild diffeomorphisms,* that is, diffeomorphisms f having a C^1-neighbourhood $\mathcal{U}_f$ such that every $g \in \mathcal{U}_f \cap \mathcal{R}$ has infinitely many homoclinic classes (which are pairwise disjoint).

[1] For simplicity, we use the notation $W^{s,u}(p, f)$ to denote the orbit of the stable/unstable manifold of the orbit a point p.

Theorem 3. *Any homoclinic class of a tame diffeomorphism $f \in \mathrm{Diff}^1(M)$ with non-empty interior is the whole manifold. Thus, in such a case, the diffeomorphism f is transitive.*

Proof. To prove the theorem we use the following result in [CM]: *the residual set $\mathcal{R}$ above can be taken such that every $f \in \mathcal{T}$ has finitely many attractors whose basins cover an open and dense subset of M and a finitely many repellors whose basins cover an open and dense subset of M.* The notion of attractor Λ in [CM] means that there is a neighbourhood U of Λ such that $f(\overline{U}) \subset U$ and $\cap_{n\in\mathbb{N}} f^n(U) = \Lambda$. The notion of repellor is analogous.

Observe that if f is tame and $\Omega(f)$ has non-empty interior then there is a homoclinic class $H(p, f)$ containing an open set. The theorem in [CM] implies that the class $H(p, f)$ is simultaneously an attractor and a repellor. We claim that $H(p, f) = M$, thus f is transitive. Otherwise, there are open neighbourhoods U and V of $H(p, f)$ contained in the basins of the attraction and repulsion of f (meaning, $f(\overline{U}) \subset U$ and $\cap_{n\in\mathbb{N}} f^n(U) = \Lambda$, and $f^{-1}(\overline{V}) \subset V$ and $\cap_{n\in\mathbb{N}} f^{-n}(V) = \Lambda$) such that the closure of V contained in U. Take now $x \in U \setminus V$, since x belongs to the basin of attraction of $H(p, f)$ there is $n > 0$ such that $f^n(x) \in V$, contradicting that the backward orbit of V is contained in V. This completes the proof of the theorem. □

3 The interior of non-wandering sets and homoclinic classes. General arguments

In this section we derive some consequences for the topological dynamics of diffeomorphisms whose non-wandering set has non-empty interior. We also consider the case of homoclinic classes with non-empty interior. We point out that in this section we will not make use of any hyperbolic-like hypothesis.

The first result in this section says that, generically, if the non-wandering set (or a homoclinic class) of a diffeomorphism f has nonempty interior then there is some fixed open set $O \subset M$ that persistently belongs to the interior of the non-wandering set (or to the continuation of the homoclinic class). To be precise:

Theorem 4. *There is a residual set $\mathcal{R} \subset \mathrm{Diff}^1(M)$ such that if $f \in \mathcal{R}$ then, for every open set $U \subset M$ whose closure is contained in $\Omega(f)$, there is a neighborhood $\mathcal{U} \subset \mathrm{Diff}^1(M)$ of f such that if $g \in \mathcal{R} \cap \mathcal{U}$ then $U \subset H(p^g, g) \subset \Omega(g)$, where p^g is the continuation for g of some saddle $p \in U$ of the diffeomorphism f.*

One can also consider homoclinic classes which are Lyapunov stable for both f and f^{-1}.

Definition 3.1. *Given a diffeomorphism $f \in \mathrm{Diff}^1(M)$ and Λ an f-invariant compact subset of M, the set Λ is* Lyapunov stable *for f if for any open neighborhood U of Λ there is some smaller neighborhood $V \subset U$ of Λ such that $f^k(V) \subset U$ for all $k \in \mathbb{N}$.*

We will see that that homoclinic classes which are Lyapunov stable for f and f^{-1} are a generalization of the classes with non-empty interior. In fact, it is easy to see that, generically, a homoclinic class of a diffeomorphism f with non-empty interior is Lyapunov stable for both f and f^{-1}, see Lemma 3.3. This is the reason that leads us to consider this a bit more general case

Theorem 5. *There is a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that if $f \in \mathcal{R}$ and $H(p, f)$ is a homoclinic class of f such that:*

- *$H(p, f)$ is Lyapunov stable for both f and f^{-1},*
- *the homoclinic class $H(p, f)$ is not the whole manifold.*

Then, there are infinitely many (pairwise disjoint) homoclinic classes that accumulate to $H(p,f)$. Moreover, this property holds locally generically: for every g close to f in a residual set the homoclinic class $H(p^g,g)$ is accumulated by infinitely many different homoclinic classes.

This theorem can be reformulated as follows:

Theorem. *Generically, every isolated class with non-empty interior is the whole manifold.*

In the statement above the expression "isolated" can mean either *locally maximal* (i.e., there is a neighborhood of the homoclinic class whose maximal invariant set is precisely the homoclinic class) or Ω-*isolated* (i.e., there is a neighborhood of the homoclinic class which does not intersect any nonwandering point other than the points of the homoclinic class). Indeed, [CMP, Theorem A] shows that generically a homoclinic class is Ω-isolated if and only if it is locally maximal.

Remark 3.2. *The arguments in Section 2.3 proving that tame diffeomorphisms having a homoclinic class with non-empty interior are transitive (Theorem 3) also show that tame diffeomorphisms having a homoclinic class which is Lyapunov stable for f and f^{-1} are transitive.*

The proof of Conjecture 1 for tame diffeomorphisms implies that (when the the interior of $H(p,f)$ is non-empty) then the dynamics of f is *wild*, but this result does not give any information about the *local* dynamics at $H(p,f)$. Theorem 5 says that this dynamics is also locally wild at $H(p,f)$.

3.1 C^1-generic machinery and general results

We start by stating a result describing the generic C^1-dynamics obtained putting together the General Density Theorem in [P] and the results on homoclinic classes in [BD$_2$, CMP, BC]:

Theorem ([P, BD$_2$, CMP, BC]). (Generic diffeomorphisms). *There is a residual set $\mathcal{R}_0$ of $\mathrm{Diff}^1(M)$ of diffeomorphisms f such that:*

- **(G1)** *the set of periodic points of f is dense in $\Omega(f)$;*
- **(G2)** *the diffeomorphism f is Kupka-Smale, in particular, every periodic point of f is hyperbolic;*
- **(G3)** *two saddles are in the same transitive set if and only if their homoclinic classes are equal;*
- **(G4)** *any two homoclinic classes of f are either disjoint or equal;*
- **(G5)** *the homoclinic classes depend continuously on $g \in \mathcal{R}_0$;*
- **(G6)** *the closures of the stable and unstable manifolds of periodic points depend continuously on $g \in \mathcal{R}_0$;*
- **(G7)** *the closure of the stable and of the unstable manifolds of any periodic p point of f are Lyapunov stable for f^{-1} and f, respectively;*
- **(G8)** *the homoclinic class of a periodic point of f is the intersection of the closures of its invariant manifolds;*
- **(G9)** *any connected component of $\Omega(f)$ with non-empty interior is periodic and its orbit is a homoclinic class;*
- **(G10)** *any set Λ which is simultaneously Lyapunov stable for f and f^{-1} is a* weak attractor *and a* weak repellor, *where* weak attractor *means that there is a family of neighbourhhoods $(U_n)_{n\in\mathbb{N}}$ such that $U_{n+1} \subset U_n$, $f(\overline{U_n}) \subset U_n$ and $\cap_n U_n = \Lambda$; a* weak repellor *is a weak attractor for f^{-1};*

(G11) *the set $\Omega(f)$ depends continuously on $f \in \mathcal{R}_0$.*

Items (G6) and (G11) follow from items (G1) and (G2) and the lower semi-continuity (on $\mathrm{Diff}^1(M)$) of the closure of the set of hyperbolic periodic points. Items (G1) and (G2) are obtained in [P], item (G3) in [BD_2], items (G4–G5) and (G7–G8) in [CMP], and items (G9–G10) in [BC].

Putting together the facts in the theorem above one has the following:

Lemma 3.3. *Consider the residual subset $\mathcal{R}_0$ of $\mathrm{Diff}^1(M)$ defined above verifying (G1–G11). Then, for every $f \in \mathcal{R}_0$, any homoclinic class $H(p,f)$ with non-empty interior coincides with the closures of the stable and unstable manifolds of p, so that it is Lyapunov stable for f and f^{-1}.*

Proof. By the Generic Diffeomorphisms Theorem, the interior of $H(p,f)$ contains a saddle q, which is homoclinically related to p, hence the closures of $W^s(p,f)$ and $W^s(q,f)$ are equal. Similarly, the closures of $W^u(p,f)$ and $W^u(q,f)$ coincide. Observing that both invariant manifolds of q are contained in the interior of $H(p,f)$, one has that their closures are contained in $H(p,f)$. On the other hand, as $f \in \mathcal{R}_0$,

$$H(p,f) = H(q,f) \subset \overline{W^s(q,f)} \cap \overline{W^u(q,f)}.$$

Putting together these three facts one gets that $H(p,f)$ and the closures of $W^s(p,f)$ and $W^u(p,f)$ are equal. Now the lemma follows from item (G7) in the theorem. □

We also have the following easy lemma describing a Lyapunov stable (for f and f^{-1}) homoclinic class.

Lemma 3.4. *Let f be any diffeomorphism in $\mathrm{Diff}^1(M)$ and p a saddle of f.*

- *If the homoclinic class $H(p,f)$ is Lyapunov stable for f then $H(p,f) = \overline{W^u(p)}$.*
- *If the homoclinic class $H(p,f)$ is Lyapunov stable for f^{-1} then $H(p,f) = \overline{W^s(p)}$.*

Proof. We first see that if the homoclinic class $H(p,f)$ is Lyapunov stable for f then the closure of the unstable manifold of p is contained in $H(p,f)$. As $H(p,f)$ is contained in the closure of $W^u(p)$, this implies the first part of the lemma (the second follows analogously by considering f^{-1}).

Since a homoclinic class is closed and f-invariant, it is enough to see that $W^u(p,f) \subset H(p,f)$. For that, it suffices to see that $W^u(p,f)$ is contained in any neighbourhood of $H(p,f)$. But this follows immediately from the Lyapunov stability: given any neighbourhood U of $H(p,f)$, consider a neighbourhood V of $H(p,f)$ whose forward orbit is contained in U and a local unstable manifold $W^u_{loc}(p)$ of p contained in V. Now, by construction, the whole forward orbit of $W^u_{loc}(p)$ (which is $W^u(p)$) is contained in U. □

From Lemma 3.4 and condition (G7) we immediately obtain:

Lemma 3.5. *Consider the residual subset $\mathcal{R}_0$ of $\mathrm{Diff}^1(M)$ defined above. Then, for every $f \in \mathcal{R}_0$ and every saddle p of f,*

- *the homoclinic class $H(p,f)$ is Lyapunov stable for f if and only if $H(p,f) = \overline{W^u(p)}$;*
- *the homoclinic class $H(p,f)$ is Lyapunov stable for f^{-1} if and only if $H(p,f) = \overline{W^s(p)}$.*

Lemma 3.6. *There is a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$, such that, for any $f \in \mathcal{R}$ and any homoclinic class $H(p,f)$ of f, there is a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$ such that the continuation p^g of p is defined for all $g \in \mathcal{U}$ and:*

- *either $H(p^g, g)$ is Lyapunov stable for any $g \in \mathcal{R} \cap \mathcal{U}$,*
- *or $H(p^g, g)$ is not Lyapunov stable for any $g \in \mathcal{R} \cap \mathcal{U}$.*

Proof. First recall that, by Lemma 3.5, for every diffeomorphism $f \in \mathcal{R}_0$ and any saddle p of f, the homoclinic class $H(p, f)$ is Lyapunov stable if and only if $H(p, f) = \overline{W^u(p)}$.

Let $f \in \mathcal{R}_0$ and p a periodic saddle of f. Consider a neighborhood $\mathcal{U}_0$ of f where the continuation of p is well defined. Notice that, by (G5) and (G6), $H(p^g, g)$ and $\overline{W^u(p^g, g)}$ vary continuously on $g \in \mathcal{R}_0 \cap \mathcal{U}_0$.

Let A be the set of diffeomorphisms $g \in \mathcal{R}_0 \cap \mathcal{U}_0$ such that $H(p^g, g) \neq \overline{W^u(p^g, g)}$. As these two sets are compact and depend continuously on $g \in \mathcal{R}_0 \cap \mathcal{U}_0$, the set A is an open subset of $\mathcal{R}_0 \cap \mathcal{U}_0$. This shows that the property of $H(p, f)$ not being Lyapunov stable is open in $\mathcal{R}_0$.

Let $\tilde{A}$ be an open set of $\mathcal{U}_0$ such that $A = \mathcal{R}_0 \cap \tilde{A}$. Denote by $\tilde{B}$ the complement of the closure of $\tilde{A}$ in $\mathcal{U}_0$ and write $B = \mathcal{R}_0 \cap \tilde{B}$. This set is open in $\mathcal{R}_0$ and, for any $g \in B$, the homoclinic class $H(p^g, g)$ is equal to the closure of the unstable manifold of p^g. Thus it is Lyapunov stable. This shows that the property of $H(p, f)$ being Lyapunov stable is open in $\mathcal{R}_0$.

The announced residual subset $\mathcal{R}$ is obtained by considering a countable collection of pairs $(\mathcal{U}, p)$, where $\mathcal{U}$ is an open set of a countable basis of the topology of $\mathrm{Diff}^1(M)$ and p is a hyperbolic periodic point varying continuously on $g \in \mathcal{U}$. □

3.2 Persistent localization of the interior of the non-wandering set

3.2.1 Proof of Theorem 4

Let $O \subset M$ be a non-empty open set, and $\mathcal{R}_0 \subset \mathrm{Diff}^1(M)$ be the residual subset in the Generic Diffeomorphisms Theorem. Denote by $A(O) \subset \mathcal{R}_0$ the set of diffeomorphisms $f \in \mathcal{R}_0$ such that the closure of O is not contained in $\Omega(f)$. As $\Omega(f)$ depends continuously on $f \in \mathcal{R}_0$, the set $A(O)$ is open in $\mathcal{R}_0$. Then there is an open subset $\tilde{A}(O)$ of $\mathrm{Diff}^1(M)$ such that $A(O) = \tilde{A}(O) \cap \mathcal{R}_0$.

Denote $\tilde{B}(O)$ the complement in $\mathrm{Diff}^1(M)$ of the closure of $\tilde{A}(O)$ (that is, in formula: $\tilde{B}(O) = \mathrm{Diff}^1(M) \setminus \overline{\tilde{A}(O)}$). Consider now $B(O) = \mathcal{R}_0 \cap \tilde{B}(O)$, it is an open subset of $\mathcal{R}_0$. By construction, $\tilde{A}(O) \cap \tilde{B}(O)$ is an open and dense subset of $\mathrm{Diff}^1(M)$ so that the set $C(O)$ defined as $C(O) = A(O) \cup B(O)$ is an open and dense subset of $\mathcal{R}_0$.

Consider now a countable basis of open sets O_n of M and, for each n, the sets $A(O_n)$, $B(O_n)$ and $C(O_n)$ defined as above. Let $\mathcal{R}$ be the intersection $\mathcal{R} = \bigcap_n C(O_n)$. This is a residual subset of $\mathrm{Diff}^1(M)$ (being a countable intersection of residual subsets).

Consider a diffeomorphism $f \in \mathcal{R}$ and let U be some open set such that $\overline{U} \subset \Omega(f)$. Consider a finite covering $O_{i_1}, \ldots, O_{i_k}$ of the compact set $\overline{U}$ by open sets O_{i_j} whose closure is contained in $\Omega(f)$. By construction, for every i_j, f does not belong to $A(O_{i_j})$ so that f belong to the open set $\tilde{B}(O_{i_j})$. Moreover, as f belongs to $\mathcal{R}_0$, there is a hyperbolic periodic point p of f in U. Choose an open neighborhood $\mathcal{U}$ of f contained in the finite intersection of the open sets $\tilde{B}(O_{i_j})$, and small enough such that the continuation p^g of p is defined for all $g \in \mathcal{U}$ and belongs to U.

Let $g \in \mathcal{R} \cap \mathcal{U}$. As $g \in (\mathcal{R} \cap \tilde{B}(O_{i_j})) \subset B(O_{i_j})$, one has that $\overline{O_{i_j}} \subset \Omega(g)$, thus $\overline{U}$ is contained in $\Omega(g)$. Moreover, as $g \in \mathcal{R}_0$, properties (G3–G4) and (G9) guarantee that U is contained in the homoclinic class $H(p^g, g)$. This ends the proof of the theorem. □

In fact, the same proof gives the following (probably well known) result of general topology:

Scholium 3.7. *Let X be a Baire space, $\mathcal{R} \subset X$ a residual subset, and Y a compact metric space. Assume that, for every $x \in \mathcal{R}$, $\phi(x)$ is a compact subset of Y depending continuously on x for the Hausdorff topology. Then there is a residual subset $\mathcal{S} \subset \mathcal{R}$ of X such that the closure of the interior of $\phi(x)$ varies continuously on $x \in \mathcal{S}$.*

3.2.2 Proof of Theorem 5

Consider $f \in \mathcal{R}_0$ and a Lyapunov stable (for f and f^{-1}) homoclinic class $H(p,f)$. Property (G10) ensures that $H(p,f)$ is simultaneously a weak attractor and a weak repellor. This allows us to construct two families $\{U_n\}_{n\in\mathbb{N}}$ and $\{V_n\}_{n\in\mathbb{N}}$ of open neighborhoods of $H(p,f)$ such that:

- $\overline{V_{n+1}} \subset U_{n+1} \subset \overline{U_{n+1}} \subset V_n \subset \overline{V_n} \subset U_n$, for all n,
- $f(\overline{U_n}) \subset U_n$ and $f(\overline{V_n}) \supset V_n$, for all n.

By construction, the sets $D_n = \overline{U_n} \setminus V_n$ have non-empty interior and are compact, disjoint and strictly f-invariant ($f(D_n) \subset \mathrm{int}(D_n)$), thus Properties (G1) and (G2) imply that there is a hyperbolic periodic point p_n in D_n. The strict invariance of the D_n implies that the homoclinic classes of the p_n are disjoint, ending the proof of the first part of the theorem.

The final part of the theorem follows from Lemma 3.6. □

4 Local strong partial hyperbolicity

In this section we state a local version of Theorem 2, without the assumption of a global partially hyperbolic splitting.

Theorem 6. *There is a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that if $f \in \mathcal{R}$ and $H(p,f)$ is a homoclinic class of f such that:*

- *there are a C^1-neighborhood $\mathcal{U}$ of f and a residual subset $\mathcal{S}$ of $\mathcal{U}$ such that, for any $g \in \mathcal{S}$, the homoclinic class $H(p,g)$ is Lyapunov stable for both g and g^{-1}, and*
- *$H(p,f)$ is strongly partially hyperbolic,*

then the homoclinic class $H(p,f)$ is the whole manifold. In particular, the diffeomorphism f is transitive.

This result follows using a local version of the accessibility results in [DW] (see Proposition 4.1).

As an immediate consequence of Lemma 3.3 we now have the following.

Corollary 1. *There is a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that if $f \in \mathcal{R}$ and $H(p,f)$ is a homoclinic class of f such that:*

- *$H(p,f)$ has non-empty interior, and*
- *$H(p,f)$ is strongly partially hyperbolic,*

then the homoclinic class $H(p,f)$ is the whole manifold. In particular, the diffeomorphism f is transitive.

4.1 Lyapunov stable Homoclinic classes

In this section we study homoclinic classes which are Lyapunov stable for both f and f^{-1}. By Lemma 3.3, this case includes the classes with non-empty interior.

To prove Theorem 6 we need a local version of the accessibility theorem in [DW] which we explain in the next section.

4.1.1 Local accessibility

Consider an invariant open set V which admits (in fact, whose tangent space admits) a strongly partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$. Then at each point of V we can define local leaves tangent to E^s and to E^u, called *s- and u-paths.* An *su-path from p to q* is a path from p to q consisting of finitely many s and u-paths.

Proposition 4.1. (Stable local accessibility). *Let $f \in \mathcal{R}_0$ be a generic diffeomorphism and L a compact f^{-1} and f-invariant set which is Lyapunov stable for both f^{-1} and f. Let U be a compact f-invariant neighbourhood of L which admits a strongly partially hyperbolic splitting $T_U M = E^s \oplus E^c \oplus E^u$. Then there are:*

- *an f^{-1} and f-invariant compact K neighbourhood of L contained in the interior of U,*
- *a neighbourhood $\mathcal{U}_f$ of f in $\mathrm{Diff}^1(M)$, and*
- *an open and dense subset $\mathcal{A}$ of $\mathcal{U}_f$*

such that, for every g in $\mathcal{A}$ and any pair of points x and y in the same connected component of K, there is a su-path (for the foliations associated to g) joining x and y contained in U

Observe that since, in general, f-invariant sets are not connected, we need to consider points in the same connected component of the set (points in different components can not be joined by su-arcs).

The obstruction to apply the arguments in [DW] to our setting is:

Invariance problem: The main difficulty to translate the result in [DW] to our setting is that, a priori, the su-path can leave, after iterations by diffeomorphisms g close to f, the initial neighbourhood U where the splitting is defined. To avoid this problem we make a strong use of the Lyapunov stability, which allows us to find g-invariant neighbourhoods of the initial L contained in the region where the splitting is defined. For that we need some preparatory lemmas.

Lemma 4.2. *Let L, $L \neq M$, be an f-invariant set which is Lyapunov stable for f and f^{-1}. Then for each neighbourhood V of L there is an f-invariant compact neighbourhood Λ of L contained in the interior of V.*

Proof. Take a compact neighbourhood W of L contained in the interior of V. By the Lyapunov stability of L for f, there is a new neighbourhood W_1 of L contained in W such that its whole forward orbit is contained in W. Let W_2 be the union of the sets $f^i(W_1)$, $i \geq 0$. Applying the Lyapunov stability of L for f^{-1} to the neighbourhhod W_2 of L, we get a new neighbourhood W_3 of L such that its backward orbit is contained in W_2. Note that, by construction, the forward orbit of W_3 is contained in W_2. Thus the complete orbit of W_3 is contained in W_2. The set Λ is now the closure of the orbit of W_3, which is contained in the closure of W □

Lemma 4.3. *Let L be a compact f^{-1} and f-invariant set such that*

- *it is Lyapunov stable for both f^{-1} and f,*
- *there is a strongly partially hyperbolic splitting $T_L M = E^s \oplus E^c \oplus E^u$ defined over L.*

Then there is a compact f^{-1} and f-invariant neighbourhood U of L where the extension of the strongly partially hyperbolic bundle is defined. Moreover, the neighbourhood U of L can be taken arbitrarily small.

From now on, in this section, f-invariant will mean invariant by f and f^{-1}.

Proof. First observe that, by a standard cone-field construction, we can extend the splitting $E^s \oplus E^c \oplus E^u$ defined on L to a small neighbourhoood W of the set L. A priori, such a neighbourhood is not f-invariant. To get an f-invariant compact neighbourhood of L we apply Lemma 4.2 taking W a neighbourhood where the extension of the splitting is defined and $U = \Lambda$. $\square$

Remark 4.4. *Let U be an f-invariant compact neighbourhhod of L as in Lemma 4.3. Then, for every g close enough to f, any g-invariant set K_g contained in U is strongly partially hyperbolic.*

Lemma 4.5. *Under the hypotheses of Lemma 4.3, there are*

- *a neighbourhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$ and*
- *arbitrarily small f-invariant compact neighbourhoods U_1 and U_2 of L, U_2 contained in the interior of U_1, having a strong partially hyperbolic splitting*

such that, for every $g \in \mathcal{U}$, there are g-invariant compact sets K_1^g and K_2^g which are strongly partially hyperbolic and satisfy $U_3 \subset K_2^g \subset \mathrm{int}(U_2) \subset K_1^g \subset \mathrm{int}(U_1)$ for some neighbourhood U_3 of L.

Proof. Since the set L is Lyapunov stable and $f \in \mathcal{R}_0$, by property (G10) there is a compact neighbourhood U_1 of L (which we can take arbitrarily small) such that $g(U_1) \subset \mathrm{int}(U_1)$ for all g close to f. Similarly, and again by (G10), there is a compact neighbourhood V_1 of L, $V_1 \subset \mathrm{int}(U_1)$, such that $g^{-1}(V_1) \subset \mathrm{int}(V_1)$ for all g close to f. This implies that the complete g-orbit of V_1 is contained in U_1. Taking the closure of the previous orbit we get a g-invariant set K_g^1. By construction, the set K_1^g is contained in U_1 and contains V_1. Moreover, by Remark 4.4, the set K_1^g is strongly partially hyperbolic. This completes the construction of the sets U_1 and K_g^1 for g close to f.

We now construct the sets U_2 and K_2^g. Arguing as above, by the Lyapunov stability of L for f and f^{-1}, there is a compact neighbourhood U_2 of L, $U_2 \subset \mathrm{int}(V_1)$ with $g(U_2) \subset \mathrm{int}(U_2)$ for all g close to f. The definitions of the set V_2 and the compact g-invariant set K_2^g follow as above. Moreover, by Remark 4.4, the set K_2^g is strongly partially hyperbolic. Finally, by construction, $V_2 \subset K_2^g \subset U_2 \subset V_1 \subset K_1^g \subset U_1$. Taking $U_3 = V_2$, this ends the proof of the lemma. $\square$

We are now ready to explain how the accessibility proof in [DW] is adapted to our context.

Using the Lyapunov stability of L for f and f^{-1} and applying Lemma 4.5, we get compact f-invariant neighbourhoods U_1, U_2 and U_3 of L, $U_{i+1} \subset \mathrm{int}(U_1)$, and the g-invariant sets K_1^g and K_2^g, g close to f. Applying Lemma 4.2 to U_3, we get an f-invariant set K contained in the interior of U_3, thus contained in K_2^g for all g close to f.

Consider now the sets U_2 and K. Then the orbits by g of these sets are contained in K_1^g, thus in U_1. Take now $U_1 = U$.

The idea now is to consider for points x of K small segments centered at x and tangent or quasi-tangent to the directions of the splitting. If the segments are small, they are contained in U_2 and their g-orbits are contained in K_1^g, so they do not escape from U. This solves the invariance problem mentioned above.

The proof of [DW] only involves local arguments and is done in two steps:

- The first step consists in proving accessibility modulo small central disks (one can join two points by an su-path and some small paths contained in small fixed central disks).
- The second step shows that, after local perturbations, two points in the same central path can be joined by an su-path.

We now give the main steps (explaining how they hold or are adapted in our case). Quoting and adapting [DW], the key of the proof is to find a family of (small) *central disks* (i.e., approximately tangent to the central direction E^c) centered at points of K (thus contained in U) which is large enough so that the diffeomorphism is *su-accessible modulo these central disks*: for every pair of points x and y in the same connected component of K there are finitely many su-paths of f, the first going from x to some central disk, the last going from some central disk to y, and the intermediate paths beginning and ending in central disks, each path beginning where the previous one ends. This property is robust (for any diffeomorphism close to f) for the (same) family of central disks.

The construction in [DW] also makes use of the following generic assumption (verified in our setting): periodic points of any period k are isolated. Moreover the disks are taken small enough to guarantee that they are highly non-recurrent (this is guaranteed by performing small translations in the u-direction).

Given any family of central disks $\mathcal{D}$ with small diameters and highly non-recurrent for f (this corresponds to the notion of *admissibility* in [DW]) then there is g arbitrarily close to f which is stably accessible modulo a family of central disks (uniformly) contained in the interior of the disks of $\mathcal{D}$. These results correspond to [DW, Lemmas 1.1-3]. Having this property, the result follows considering only local arguments (this is done in [DW, Section 2]). This completes our sketch of the proof of the proposition.

4.2 End of the proof of Theorem 6

Consider now a diffeomorphism $f \in \mathcal{R}_0$ (the residual subset of $\mathrm{Diff}^1(M)$ in the Generic Diffeomorphisms Theorem). Then the homoclinic class of any saddle of f depends continuously on the diffeomorphisms of $\mathcal{R}_0$. Take a saddle p of f such that the homoclinic class $H(p, f)$ verifies the hypotheses of Theorem 6. We argue by contradiction, assuming that $H(p, f)$ is not the whole manifold, and then obtaining an explosion of this homoclinic class for diffeomorphisms in $\mathcal{R}_0$ close to f, violating the continuous dependence of the homoclinic classes for diffeomorphisms in $\mathcal{R}_0$, recall condition (G5).

Associated to $H(p, f)$ there are f-invariant compact neighbourhoods K and U of $H(p, f)$, K contained in the interior of U, a neighbourhood $\mathcal{U}$ of f, and the open and dense subset $\mathcal{A}$ of $\mathcal{U}$ of locally accessible diffeomorphisms (modulus K and U) given by Proposition 4.1.

By shrinking the neighbourhood $\mathcal{U}$ of f, we can assume that, for every $g \in \mathcal{U} \cap \mathcal{R}_0$, the homoclinic class of p^g is contained in the interior of K. To get the contradiction it suffices to see that, for all $g \in \mathcal{A} \cap \mathcal{R}_0$, the whole K is contained in $H(p^g, g)$.

Consider now a diffeomorphism $g \in (\mathcal{R}_0 \cap \mathcal{U} \cap \mathcal{A})$ verifying Lemma 3.5. Then there is a saddle q of g such that $H(q, g)$ coincides with the closures of both $W^s(q)$ and $W^u(q)$. Thus these closures are contained in K.

Observe now that, for any saddle $t \in K$ of g, the closures of $W^s(t)$ and $W^u(t)$ are saturated sets by the strong stable foliation $\mathcal{F}^s_g$ and the strong unstable foliation $\mathcal{F}^u_g$ of g, respectively, (these foliations are defined in the set U). This implies that the accessibility class of q in U is contained in $H(q, g)$. Since $g \in \mathcal{A}$ the whole K is contained in $H(q, g)$. This gives the contradiction. The proof of the theorem is now complete. □

5 Hyperbolic-like properties of the interior of the non-wandering set

In this section, we consider diffeomorphisms whose non-wandering set has non-empty interior, paying a special attention to the particular case when the non-wandering set is the whole manifold. When this property is persistent (either robust or locally generic) or when reducing the study to some residual subset of $\mathrm{Diff}^1(M)$, we prove the existence of a dominated splitting. Moreover, when the non-wandering set is the whole manifold the dominated splitting is volume hyperbolic (recall Definition 2.2).

We begin by recalling the following result:

Theorem ([BDP]). (Generic dichotomy). *There is a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that, for every diffeomorphism $f \in \mathcal{R}$ and any homoclinic class $H(p, f)$ of a saddle p of f,*

- *either there is a dominated splitting defined on $H(p, f)$,*
- *or $H(p, f)$ is contained in the closure of the (infinite) set of sinks and sources of f.*

Mañé's Ergodic Closing Lemma [M_2] and the previous theorem almost immediately give the following generalization:

Corollary 2. *There is a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that, for every diffeomorphism $f \in \mathcal{R}$ and any nontrivial homoclinic class $H(p, f)$ of a saddle p of f,*

- *either $H(p, f)$ is volume hyperbolic*
- *or $H(p, f)$ intersects the closure of the (infinite) set of sinks and sources of f.*

This result generalizes [Ab_2, Corollary A].

Proof. Observe that the set $S(f)$ defined as the closure of the set of sinks and sources of f depends lower semi-continuously on f. Thus there is a residual subset $\mathcal{G}$ of $\mathrm{Diff}^1(M)$ where $S(f)$ depends continuously. Intersecting the residual sets $\mathcal{G}$ and $\mathcal{R}_0$ (the set in the Generic Diffeomorphisms Theorem) one gets a residual subset $\mathcal{R}_1$ where the sets $S(f)$ and $H(p, f)$ depend continuously.

If $S(f) \cap H(p, f) \neq \emptyset$ we are done. Otherwise, there is a neighbourhood V of $H(p, f)$ such that, for every g in $\mathcal{R}_1$ close to f, $S(g) \cap V = \emptyset$. Finally, if $H(p, f)$ is not volume hyperbolic, after applying the Ergodic Closing Lemma (arguing exactly as in [BDP, Section 6.2]) one gets a diffeomorphism arbitrarily close to f having a sink (if we are considering the first bundle of the splitting) or a source (if we are considering the last bundle) close to $H(p, f)$, thus in V. Since to have a sink/source is a robust property, we can assume that $g \in \mathcal{R}_1$, contradicting that $S(g)$ is disjoint from V. $\square$

Theorem 7. *Let M be a compact boundaryless manifold. Assume that there are a C^1-open subset $\mathcal{U} \subset \mathrm{Diff}^1(M)$ and a residual subset $\mathcal{R}$ of $\mathcal{U}$ such that, for every $f \in \mathcal{R}$, one has $\Omega(f) = M$. Then every $g \in \mathcal{U}$ is volume hyperbolic (there is a volume hyperbolic splitting defined on the whole manifold).*

Observe that, if the dimension of M is 3 then at least one of the extremal bundles of the splitting is 1-dimensional and thus it is either hyperbolic (and the diffeomorphism is Anosov) or partially hyperbolic. Finally, if the dimension is 2, the splitting is hyperbolic (and the diffeomorphism is Anosov). This result extends the results in [M_2, DPU, BDP] for robustly transitive diffeomorphisms (in dimensions 2, 3 and any dimension, respectively).

Corollary 3. *There is a residual subset $\mathcal{R} \subset \mathrm{Diff}^1(M)$ such that every $f \in \mathcal{R}$ with $\Omega(f) = M$ is volume hyperbolic.*

The corollary follows from Theorem 7 after observing that, for generic diffeomorphisms, the property $\Omega(f) = M$ is locally generic, see Theorem 4.

We stress that Theorem 7 *cannot* be obtained *only* using the techniques in [BDP] (see the Generic Dichotomy Theorem above). On the one hand, by (G1-G2), for generic diffeomorphisms

f, the set $\Omega(f)$ is the closure of the set of periodic points, and so the closure of the union of the homoclinic classes. On the other hand, by the Generic Dichotomy Theorem, if $\Omega(f) = M$ persistently then each homoclinic class admits a dominated splitting (sinks and sources cannot be obtained by perturbations). Now one is tempted to use the naive idea of "gluing" together the dominated splittings of the different homoclinic classes in order to obtain a dominated splitting on the whole non-wandering set (that is, on M). This does not work, for the splittings over the various homoclinic classes may be of different types (for instance, in dimension 3, of types $E^s \oplus E^c$, $E^c \oplus E^u$ or $E^s \oplus E^c \oplus E^u$ according to the different homoclinic classes). Now since by (G9) there is a unique homoclinic class, we can bypass this problem. Actually, this remark is also a sketch of the proof of Theorem 7 (see next section for the proof).

Observe that in Theorem 7 the type of the splitting can change when g varies along $\mathcal{U}$.

Finally, we give a local version of Corollary 3.

Theorem 8. *There is a residual set $\mathcal{R} \subset \mathrm{Diff}^1(M)$, such that for any diffeomorphism $f \in \mathcal{R}$ and any (non-empty) connected open set U whose closure $\overline{U}$ is contained in $\Omega(f)$, one has the following: there is a periodic point p of f such that the homoclinic class $H(p, f)$ contains $\bar{U}$ and admits a dominated splitting.*

We point out that we do not know if the splitting in the theorem is volume hyperbolic. The proof of the volume hyperbolicity in Theorem 7 involves Mañé's Ergodic Closing Lemma to create a sink or a source (this yields a contradiction when the non-wandering set is persistently the whole manifold). However, we did not succeed here in placing the new sink/source inside the set U, and thus we cannot derive a contradiction. So let us pose two questions:

Problem 5.1. *Let $H(p, f)$ be a class which is locally generically Lyapunov stable for f and f^{-1}.*

- *Does $H(p, f)$ contain an open set?*
- *Does $H(p, f)$ admit a dominated splitting?*

By the comments above, a positive answer to the first question also answers positively the second one.

5.1 Proof of Theorem 7

Let $\mathcal{U}$ and $\mathcal{R}$ be an open set and a residual subset of it as in the theorem (for all $f \in \mathcal{R}$, $\Omega(f) = M$). By property (G9), if $f \in \mathcal{R}_0$ and $\Omega(f) = M$ then $\Omega(f) = H(p, f)$ for any saddle p of f (the last assertion follows from (G4)). Consider now the residual subset $\mathcal{R}' = \mathcal{R} \cap \mathcal{R}_0$ of $\mathcal{U}$, $\mathcal{R}_0$ is the residual set in Generic Diffeomorphisms Theorem.

Remark 5.2. *Any diffeomorphism g in $\mathcal{U}$ has neither sinks nor sources: since sinks and sources are robust, the existence of a sink/source for g implies that the non-wandering set of any h close to g is not the whole manifold.*

This remark and the Dichotomy Theorem above imply that, for every $g \in \mathcal{R}'$, the homoclinic class $H(p^g, g) = M$ has a dominated splitting $E \oplus F$. Thus $TM = E \oplus F$. Since to have a dominated splitting is a stable property (this is a folklore result, for instance, see [BDP, Lemma 1.4]), any diffeomorphism g which is C^1-close to f has a dominated splitting $TM = E^g \oplus F^g$, which is the continuation of $E \oplus F$. This completes the proof of the domination.

Consider now the finest dominated splitting (see Definition 2.2) of TM for g, say $TM = E_1^g \oplus E_2^g \oplus \cdots \oplus E_k^g$. We need to prove that the first bundle is uniformly contracting and the last one uniformly expanding. This follows from the Ergodic Closing Lemma (arguing exactly as in [BDP, Section 6.2]: if the bundle E_1^g is not uniformly contracting one obtains h close to g (thus in $\mathcal{U}$) with a source, contradicting Remark 5.2.

5.2 Proof of Theorem 8

We can assume that f is a generic diffeomorphism verifying the properties given by the Generic Diffeomorphisms and Generic Dichotomy theorems. Hence, by property (G9), there is a saddle $p \in U$ of f such that $U \subset H(p, f)$. By Theorem 4, there is a neighbourhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$ and a residual subset $\mathcal{G}$ of $\mathcal{U}$ such that, for all $g \in \mathcal{G}$, the open set U is contained in $H(p^g, g)$. Thus we can assume that the generic diffeomorphisms close to f verify this condition.

We claim that $H(p, f)$ has a dominated splitting. Otherwise (by the dichotomy result) there is g close to f with a sink or a source in U. Thus there is $h \in \mathcal{G}$ having a sink or a source in U. So U is not contained in $H(p^h, h)$, which is a contradiction.

Finally, if U is contained in $H(p, f)$, the whole orbit of $\overline{U}$ is also contained in $H(p, f)$. Thus the closure K_f of such a orbit is a compact f-invariant set contained in $H(p, f)$, so it has a dominated splitting.

References

[Ab$_1$] F. Abdenur, *Generic robustness of spectral decompositions,* to appear in Ann. Scient. Éc. Norm. Sup..

[Ab$_2$] F. Abdenur, *Attractors of generic diffeomorphisms are persistent,* Nonlinearity, **16**(1), 301-312, (2003).

[Ar] M.-C. Arnaud, *Creation de connexions en topologie C^1,* Ergod. Th. and Dynam. Syst., **21**, 339-381, (2001).

[BC] Ch. Bonatti and S. Crovisier, *Récurrence et généricité,* short version preprint Dijon (2003) and complete version in preparation.

[BD$_1$] Ch. Bonatti and L.J. Díaz, *Persistent nonhyperbolic transitive diffeomorphims,* Ann. of Math., **143**, 357–396, (1996).

[BD$_2$] Ch. Bonatti and L.J. Díaz, *Connexions hétéroclines et généricité d'une infinité de puits ou de sources,* Ann. Scient. Éc. Norm. Sup., 4^e série, t32, 135–150, (1999).

[BDP] Ch. Bonatti, L.J. Díaz, and E.R. Pujals, *A C^1-generic dichotomy for diffeomorphisms: Weak forms of hyperbolicicity or infinitely many sinks or sources,* to appear in Ann. Math.

[BDU] Ch. Bonatti, L.J. Díaz, and R. Ures, *Minimality of strong stable and unstable foliations for partially hyperbolic diffeomorphisms,* Journal of the Inst. of Math. Jussieu, **1** (4), 513–541, (2002).

[BV] Ch. Bonatti and M. Viana, *SRB measures for partially hyperbolic attractors: the mostly contracting case,* Israel J. of Math., **115**, 157–193, (2000).

[B] M. Brin, *Topological transitivity of a certain class of dynamical systems, and flows of frames on manifolds of negative curvature,* Funkcional. Anal. i Priložen., **9**(1), 9–19, (1975).

[BP] M. Brin and Ya. Pesin, *Partially hyperbolic dynamical systems,* Math. ISSR Izv., **8**, 177–208, (1974).

[CM] C. Carballo and C. Morales, *Homoclinic classes and finitude of attractors for vector fields in n-manifolds,* to appear in Bull. of London Math. Soc..

[CMP] C. Carballo, C. Morales, and M.J. Pacífico, *Homoclinic classes for $\mathcal{C}^1$-generic vector fields,* to appear in Ergodic Th. and Dynam. Syst..

[DPU] L.J. Díaz, E.R. Pujals, and R. Ures, *Partial hyperbolicity and robust transitivity,* Acta Mathematica, **183**, 1-43, (1999).

[DR] L.J. Díaz and J. Rocha, *Partially hyperbolic and transitive dynamics generated by heteroclinic cycles,* Ergodic Th. and Dyn. Syst., **25**, 25-76, (2001).

[DW] D. Dolgopyat and A. Wilkinson, *Stable accessibility is C^1-dense,* to appear in Asterisque.

[H] S. Hayashi, *Connecting invariant manifolds and the solution of the C^1-stability and Ω-stability conjectures for flows,* Ann. of Math., **145**, 81-137, (1997).

[HPS] M. Hirsch, C. Pugh, and M. Shub, *Invariant manifolds,* Lecture Notes in Math., **583**, Springer Verlag, (1977).

[M_1] R. Mañé, *Contributions to the stability conjecture,* Topology, **17**, 386-396, (1978).

[M_2] R. Mañé, *An ergodic closing lemma,* Ann. of Math., **116**, 503-540, (1982).

[P] C. Pugh, *The closing lemma,* Amer. Jour. of Math., **89**, 956-1009, (1967).

[Sh] M. Shub,*Topological transitive diffeomorphism on T^4*, Lect. Notes in Math., **206**, 39 (1971).

[Sm] S. Smale, *Differentiable dynamical systems,* Bull. A.M.S., **73**, 747-817, (1967).
21, 249-302, (2001).

Flavio Abdenur (flavio@impa.br)
Instituto Nacional de Matemática Pura e Aplicada, IMPA
Estrada Da. Castorina 110, 22460-320 Rio de Janeiro RJ, Brazil

Christian Bonatti (bonatti@u-bourgogne.fr)
Université de Bourgogne, Laboratoire de Topologie, UMR 5584 du CNRS,
BP 47 870, 21078 Dijon Cedex, France

Lorenzo J. Díaz (lodiaz@mat.puc-rio.br)
Dep. Matemática PUC-Rio
Marquês de S. Vicente 225, 22453-900 Rio de Janeiro RJ, Brazil