FOCAL STABILITY OF RIEMANN METRICS

IVAN KUPKA, MAURICIO PEIXOTO, AND CHARLES PUGH

Abstract. Let M be a complete Riemann manifold with dimension m and metric g. For $p, q \in M$ and $\ell > 0$, let the index $I(g, p, q, \ell)$ be the number of g-geodesics of length ℓ that join p to q. The following generic bounds for this index are the main results we present here. We denote by $\mathcal{R}$ the space of complete Riemann metrics on M.

(a) For each $p \in M$, there is a residual $\mathcal{G}(p) \subset \mathcal{R}$ such that for all $g \in \mathcal{G}(p)$

$$\max_{q,\ell} I(g, p, q, \ell) \leq m + 1.$$

(b) If M is compact, there is a residual $\mathcal{G} \subset \mathcal{R}$ such that for all $g \in \mathcal{G}$

$$\max_{p,q,\ell} I(g, p, q, \ell) \leq 2m + 2.$$

These finiteness results are part of our study of the focal decomposition – i.e., the partition

$$T_pM = \bigsqcup_{i=1}^{\infty} \{v \in T_pM : i = I(g, p, q, \ell),\ q = \exp(v),\ \text{and}\ \ell = |v|\}.$$

Stability of this focal deomposition (as g varies) has a natural meaning, in analogy with structural stability in the theory of dynamical systems, and here we begin an investigation in that direction. Our methods involve the multi-transversality theory of J. Mather and the Bumpy Metric Theorem of R. Abraham, as proved by D. Anosov.

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1. Introduction

A Riemann structure on a manifold M, also called a Riemann metric, is a smooth (i.e., C^∞) choice of an inner product on each fiber of the tangent bundle TM. A Riemann structure generates a smooth exponential map, which we assume is defined on all of TM,

$$\exp : TM \to M.$$

That is, we assume the Riemann structure is complete. If $p \in M$ is a given base point and U is a small neighborhood of the origin in T_pM, then exp sends U diffeomorphically onto a neighborhood of p in M. We consider the totality of geodesics emanating from the base point $p \in M$. Roughly speaking, we are interested in how many of these geodesics will meet again, i.e., focus at some point q, after describing geodesic paths of the same length from p to q.

Definition. The **focal index** of $v \in T_pM$ is the cardinality of the set

$$\{w \in T_pM : |v| = |w| \text{ and } \exp(v) = \exp(w)\}.$$

We denote the focal index as $I(g, v)$. The **focal component of index $\boldsymbol{i}$** at p is

$$\sigma_i = \{v \in T_pM : i = I(g, v)\}.$$

Thus $\sigma_i = \sigma_i(g, p)$. Vectors $v \in \sigma_i$ are equivalent modulo exponentiation to $i - 1$ other vectors in T_pM of equal length.

Definition. The partition of T_pM into its focal components is its **focal decomposition**

$$T_pM = \bigsqcup_i \sigma_i.$$

The tangent bundle has a corresponding focal decomposition

$$TM = \bigsqcup_i \Sigma_i$$

with $\Sigma_i = \bigcup_{p \in M} \sigma_i(g, p)$.

Of course, focal decomposition depends only on the Riemann structure. It is also a global concept: all geodesics passing through p play a role in the construction of the sets σ_i, and all geodesics in the manifold play a role in the construction of the sets Σ_i.

The goal of this paper is to study how the focal decomposition changes as the Riemann structure varies in the space $\mathcal{R}$ of complete, smooth Riemann structures on M. In this direction we rely on the best understood example of focal decomposition, the one on the flat torus in which the connection of our subject with Brillouin zones (a classic topic of solid state physics) is quite clear. See Section 2 below.

When M is compact we equip $\mathcal{R}$ with the uniform C^∞ topology, while in the noncompact case we can use either the C^∞ Whitney topology or the compact open C^∞ topology. (When M is compact, the three topologies on $\mathcal{R}$ coincide.) See [3], page 4, and [13], pages 34-37 and 95. These topologies make $\mathcal{R}$ a **Baire space**, a space in which the countable intersection of open-dense subsets is dense. If $\mathcal{G} \subset \mathcal{R}$ contains a countable intersection of open-dense subsets of $\mathcal{R}$, then $\mathcal{G}$ is called **residual** in $\mathcal{R}$, and its members are called **generic**[1] in $\mathcal{R}$.

Definition. The **pointwise index** of g at $p \in M$ is $I(g,p)$, the largest i for which $\sigma_i(g,p) \neq \emptyset$; the **uniform index** of g is $I(g)$, the largest i for which $\Sigma_i(g) \neq \emptyset$.

Theorem 1.1 (Pointwise Index Theorem). *Given $p \in M^m$, there is a residual set $\mathcal{G}(p) \subset \mathcal{R}$ such that for all $g \in \mathcal{G}(p)$,*

$$I(g,p) \leq m+1.$$

Theorem 1.2 (Uniform Index Theorem). *If M^m is compact then there is a residual set $\mathcal{G} \subset \mathcal{R}$ such that for all $g \in \mathcal{G}$,*

$$I(g) \leq 2m+2.$$

See Sections 6 and 7 for the proofs, and also for a slightly sharper estimate of the uniform index.

Remark. The Pointwise Index Theorem asserts that given p, the generic Riemann structure has no more than $m+1$ geodesics of equal length that join p to some $q \in M$. The Uniform Index Theorem asserts that the generic Riemann structure never has more than $2m+2$ geodesics of equal length that join points of M. Note that, here, a closed geodesic or a geodesic loop counts as two geodesics from a point to itself.

In [16] it is proved that if g is analytic then the focal components are unions of strata of an analytic Whitney stratification, which implies that they are locally finite disjoint unions of boundaryless analytic manifolds. They are locally very regular and, say, none can be a Cantor set. The ideal situation would be to have both analyticity and finiteness of the focal index; we believe this is so.

[1] We follow the standard abuse of language here. By the phrase "the generic g has property P" we mean that the set of g's with property P contains a residual subset of $\mathcal{R}$.

Every Riemann structure for a compact manifold is complete, but it may be of interest to note that even in the noncompact case the *generic* smooth Riemann structure is also complete – a fact that will not be used here.

More precisely, as explained below, there is a topology on the space of analytic Riemann structures that makes it a Baire space, and we conjecture that our index theorems persist.

Analytic Index Conjecture. *The index theorems above remain valid for the generic analytic Riemann structure.*

Two facts support this:

(a) The stepwise transversality technique used to prove the C^∞ Kupka-Smale Theorem has been made to work in the analytic case by Broer and Tangerman [6].
(b) Anosov's proof of the Bumpy Metric Theorem is based on the same stepwise use of transversality.

The natural analytic topology is explicated in [6], and we summarize it here. Start with the simple case $C^\omega = C^\omega([a,b],\mathbb{R})$, the set of real analytic functions $f : [a,b] \to \mathbb{R}$. A neighborhood base at f consists of sets $\mathcal{N}(f,\epsilon,\nu)$ where $f+g \in \mathcal{N}(f,\epsilon,\nu)$ satisfies

(a) $g \in C^\omega$ extends to a complex analytic function G defined on the ν-neighborhood of $[a,b]$ in the complex plane $\mathbb{C}$.
(b) $\sup|G(z)| < \epsilon$.

Since uniform convergence preserves complex analyticity, C^ω is locally complete, and has the Baire property. Except for notation, passage from one to several variables is straightforward. Since uniform convergence of a sequence of bounded complex analytic functions implies the uniform convergence of their derivatives, the analytic topology is finer than the C^∞ topology.

In principle one is led naturally to two types of focal stability, σ-focal stability and Σ-focal stability. In this paper we are concerned mainly with the former. Several stability definitions suggest themselves. The simplest is merely that a perturbation of g has no topological effect on the focal decomposition.

Definition. The Riemann structure g is **absolutely focally stable** at p if it has a neighborhood $\mathcal{N} \subset \mathcal{R}$ such that for each $g' \in \mathcal{N}$, there is a homeomorphism of T_pM to itself that sends $\sigma_i(g,p)$ to $\sigma_i(g',p)$ for all i.

Although the definition of absolute focal stability is concise, it is frequently unverifiable, due to non-compactness of T_pM. In fact, for compact manifolds, we have not found a single example of this type of stability. The next definition is more verifiable, and we adopt it as the primary meaning of focal stability.

Definition. The Riemann structure g is **focally stable** at p if the following condition is satisfied. Given $\epsilon > 0$ and a ball $B_0 \subset T_pM$, there are balls B, B' that contain B_0, and there is a neighborhood $\mathcal{N}$ of g in $\mathcal{R}$ such that: for each $g' \in \mathcal{N}$ there is a homeomorphism $h : B \to B'$ and

(a) $|h(v) - v| < \epsilon$ for all $v \in B$.
(b) h sends $\sigma_i(g, p) \cap B$ to $\sigma_i(g', p) \cap B'$.

The homeomorphism h is called a **focal equivalence**.

It is easy to check that this definition is unaffected by the choice between the compact-open topology and the Whitney topology on $\mathcal{R}$. For it relies only on the geometry of M restricted to a large compact subset of M that contains p.

Definition. Vectors $v, w \in T_pM$ are **focal companions** if they have equal length and equal exponential image. The **companion class** of v consists of v and all its companions.

Clearly, the focal component σ_i splits into a number of companion classes, each containing i vectors, and one could require that the focal equivalence carry companion classes of g to companion classes of g'. It seems likely that the definition of focal stability with this strengthened condition is equivalent to the one without it. For, in the analogous definition of structural stability of a diffeomorphism, there is no difference between requiring that the orbit equivalence preserve orbits as point sets or as $\mathbb{Z}$-parameterized point sets. This was proved by I. Kupka, [15].

Our eventual aim is to verify the following conjecture in general. In Section 4, we do verify it in several specific cases.

Focal Stability Conjecture. *Given $p \in M$, the generic Riemann structure on M is focally stable at p.*

The plan for the rest of the paper is this. After giving some historical discussion in Section 2 and relating the focal decomposition to the cut locus in Section 3, we discuss some examples in Section 4, and verify a simple case of the Focal Stability Conjecture in Section 5. Then we proceed to the proofs of the index theorems in Sections 6 and 7.

2. Focal Decomposition in Other Contexts

To put our current results in perspective, we make some historical comments and discuss how focal decomposition relates naturally to some other areas of mathematics and physics.

The concept of focal decomposition was introduced by M. Peixoto in [19] in the context the 2-point boundary value problem

$$\ddot{x} = f(t, x, \dot{x}), \quad x(t_1) = x_1, \quad x(t_2) = x_2, \tag{1}$$

the simplest and oldest of all boundary value problems. Fixing a base point, say (t_1, x_1), one associates to a second point (t_2, x_2) a non-negative integer or ∞, its index $I(t_2, x_2)$, defined as the number of solutions of the boundary value problem (1). If the second point is the same as the first, the index is defined to be ∞. The sets σ_i of points with index i gives a partition of the plane

$$\mathbb{R}^2 = \sigma_0 \cup \sigma_1 \cup \cdots \cup \sigma_\infty \tag{2}$$

which was the proposed object of study. We nowadays call (2) the focal decomposition of the equation (1) with respect to the base point (t_1, x_1).

Of course this can all be done without specifying a particular base point, and we get the focal decomposition associated to (1) as a partition of $\mathbb{R}^4$ by subsets Σ_i, i.e. the totality of points $(t_1, x_1, t_2, x_2) \in \mathbb{R}^4$ for which the boundary value problem (1) has exactly i solutions. From the work of Peixoto and Thom in [23] the possibility of a general analytic theory became clear, thanks to bringing to the fore a theorem of Hironaka [12], page 42-43, about analytic maps from one manifold to another. We have then an *existence theorem* which says that, under a certain properness condition expressed in terms of the solutions of (1), the sets Σ_i are the unions of strata for an analytic Whitney stratification of (t_1, x_1, t_2, x_2)-space $\mathbb{R}^4$ minus the diagonal $t_1 = t_2$. See [22]. This excludes the possibility of pathological behavior in the sense, say, that no Σ_i can be a Cantor set. Afterwards Kupka and Peixoto [16] put the concept of focal decomposition into the context of geodesics, as explained in Section 1. Then, in contrast with the case of boundary values for differential equations, no extra properness assumption is needed for the corresponding existence theorem. A complete, analytic Riemann structure always produces an analytic Whitney stratification of TM whose strata partition the sets Σ_i. Because of this, it is fair to say that a natural place to study focal decomposition is an analytic Riemann manifold.

When M is a flat torus, the corresponding focal decomposition, surprisingly, reproduces the Brillouin zones of a crystal, to which it bears a close formal relationship. See [5], [7], [8], and [16]. It also leads naturally to relationships with the arithmetic of positive definite quadratic forms. See [20].

A further comment is that the knowledge of the focal decomposition relative to the Euler equation of some action functional may be considered as a prerequisite for its semi-classical quantization via the Feynman path integral method. See [10], page 29, and [21]. As a matter of terminology, we remark that in [16], [20], [23], the expression σ-decomposition is used for what we now call focal decomposition.

3. Focal Decomposition and the Cut Locus

Recall the definition of the cut locus. Fix a base point p in a complete Riemann manifold M. If $v \in T_pM$ and the restriction of exp to the radial segment $[0, v]$ is a minimizing geodesic, but the restriction of exp to any longer segment $[0, (1+\epsilon)v]$ is not minimizing then v belongs to the **tangential cut locus** of p, $\widetilde{C}(p)$. There are just two reasons a vector v belongs to $\widetilde{C}(p)$:

(a) The restriction of exp to the radial segment $[0, v]$ is a minimizing geodesic, but v has at least a focal companion $v' \neq v$. (Necessarily, the companion geodesics are also minimizing so that $v \in \sigma_i$, $i \geq 2$.) The set of such v is $\widetilde{C}_f(p)$.

(b) v has no focal companion other than itself, but it is the first conjugate point along the ray through v. The set of such v is $\widetilde{C}_c(p)$.

Accordingly we write

$$\widetilde{C}(p) = \widetilde{C}_f(p) \sqcup \widetilde{C}_c(p).$$

From [14], Theorem 2.1.14, $\widetilde{C}(p)$ is a closed set in which $\widetilde{C}_f(p)$ is dense.

The exponential image of $\widetilde{C}(p)$ is the **cut locus** $C(p)$. It is a closed nowhere dense subset of M. Its complement, $R(p) = M \backslash C(p)$, is the **range of polar coordinates** at p, an open neighborhood of p whose boundary is $C(p)$. Similarly, the **domain of polar coordinates** at p is the set $\widetilde{R}(p)$ of vectors $v \in T_pM$ such that the restriction of exp to $[0, v]$ is a minimizing geodesic from p to $q = \exp(v) \in R(p)$. The domain of polar coordinates is an open starlike set in T_pM whose boundary is $\widetilde{C}(p)$.

Let $B(p)$ be the connected component of $\mathrm{Int}(\sigma_1)$ which contains the origin in T_pM. In this context it is the **first Brillouin zone** with respect to p. See [21] for a presentation of Brillouin zones on Riemannian manifolds. Since σ_1 contains an open neighborhood of the origin, $B(p) \neq \emptyset$.

Proposition 3.1. *The domain of polar coordinates coincides with the first Brillouin zone,* $\widetilde{R}(p) = B(p)$.

Proof. Take $v \in \widetilde{R}(p)$. The exponential image of $[0, v]$ is the unique minimizing geodesic from p to $q = \exp(v) \in R(p) = M \backslash C(p)$. Hence v has only itself as a companion, so $v \in B(p)$ and $\widetilde{R}(p) \subset B(p)$. Take $v \in \widetilde{C}_f(p)$. The half open segment $[0, v)$ lies in $\widetilde{R}(p)$, and hence is interior to σ_1, while the endpoint v lies in $\bigcup_{i \geq 2} \sigma_i$. Thus $v \in \partial B(p)$, so $\widetilde{C}_f(p) \subset \partial B(p)$. Since $\widetilde{C}_f(p)$ is dense in $\widetilde{C}(p) = \partial \widetilde{R}(p)$, we see that $\partial \widetilde{R}(p) \subset \partial B(p)$. Thus $\widetilde{R}(p)$ is a nonempty, open and closed subset of $B(p)$. By connectedness of $B(p)$, the sets are equal. $\square$

The following proposition shows that there is significance in taking the interior of σ_1 in the definition of $B(p)$.

Proposition 3.2. *The first focal component σ_1 need not be open.*

Proof. Consider an ellipsoid of revolution, M, described by

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1,$$

where $c < a$. It resembles a flying saucer. Take a point p on its equator E, and draw the cut locus $C(p)$. It is an arc on E, opposite to p, whose endpoints are first conjugate points from p along E. As $c/a \to 1$, $C(p)$ shrinks to the point $-p$. For when $c/a = 1$, M is a sphere and $C(p)$ is the point antipodal to p. When c/a becomes small the cut locus arc becomes long and its endpoints converge to p. For then the curvature of M is highly concentrated along the equator.[2] The tangential cut locus $\widetilde{C}(p)$ in T_pM is an ellipse-like curve: the length of its vertical axis is the minor circumference of M, while its horizontal axis is the segment between symmetric first conjugate points v, v' along E from p. See Figure 1. These conjugate points belong to σ_1 but are not interior to σ_1. They exponentiate to the endpoints of the cut locus. Let $B'(p)$ be the connected component of σ_1 that contains the origin. Clearly, in this example $B(p)$ is a proper subset of $B'(p)$, so Proposition 3.1 becomes false if $B'(p)$ is used in place of $B(p)$. $\square$

4. Examples

In this section we present examples that illustrate several variations of the definition of focal stability.

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[2] An ellipsoid like M is used by Klingenberg to show that cut points of type (b) can actually occur. See [14], page 135, example 1, which is ascribed to H. Alkier. It seems to us that some correction is in order: the cut locus is asserted to be a half ellipse normal to the equator, not an arc on the equator, and in particular the length of the cut locus arc is asserted to be independent of the eccentricity of the ellipsoid. In any event, the example shows that cut points of type (b) do occur.

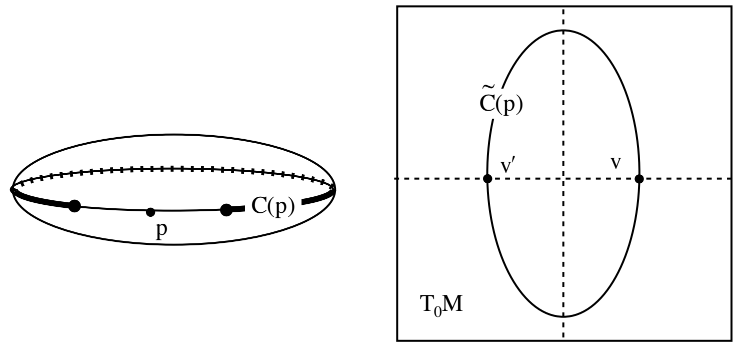

FIGURE 1. The ellipsoid M with the cut locus $C(p)$, and the tangential cut locus $\widetilde{C}(p)$, showing that σ_1 need not be open.

4.1. **Pathology.** We will show that, like the cut locus, the focal decomposition can have complicated local topology. We begin by pointing out that if the Riemann structure happens to be analytic then the focal decomposition is analytically statifiable, and hence not infinitely complicated in a topological sense. Stratifiability means that the focal components σ_i can be expressed as locally finite disjoint unions of strata – boundaryless, relatively open analytic submanifolds of T_pM, the frontiers of which are lower dimensional strata whose tangent bundles are related to those of the higher dimensional strata according to Whitney's conditions (a) and (b). As mentioned in Section 2, analytic stratifiability was obtained as a consequence of Hironaka desingularization. For the map

$$\begin{aligned} E : T_pM &\to M \times [0, \infty) \\ v &\mapsto (\exp(v), |v|) \end{aligned}$$

is proper and analytic. Accordingly there are analytic Whitney stratifications $\mathcal{A} = \{A_j\}$ and $\mathcal{B} = \{B_k\}$ of the domain and target that stratify E. This means that the restriction of E to each A_j is a submersion onto some B_k, the pre-image of each B_k is a finite union of domain strata, and the cardinality of the fiber $E^{-1}(z)$ remains constant as z varies in any B_k. See [12]. It follows that each focal component σ_i is a locally finite union of strata. For let $\mathcal{B}(i)$ denote the collection of those strata B_k for which the cardinality of $E^{-1}(q, r)$ is i as (q, r) varies in B_k. Then σ_i is the union of those A_j in the pre-image of $\bigcup_{B_k \in \mathcal{B}(i)} B_k$. See [16], page 245, and also [24].

Remark. No such stratification is known to exist in the C^∞ case.

Proposition 4.1. *If the Riemann structure is smooth but not analytic, the focal components can be topologically pathological – they need not be stratifiable.*

Proof. Gluck and Singer construct a Riemann structure g on the 2-sphere whose cut locus from the South pole p is an infinite bouquet of arcs at the North pole q,

$$C(p) = \bigcup C_n.$$

See [11] and Figure 2. This Riemann structure g is a perturbation

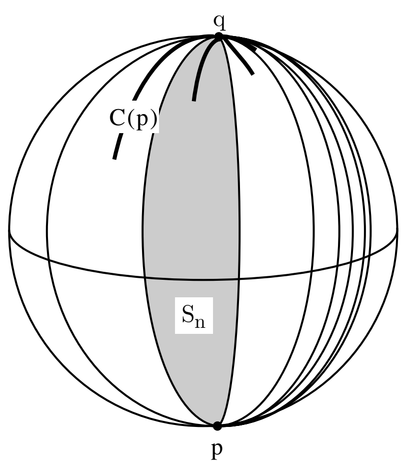

FIGURE 2. The Gluck and Singer cut locus.

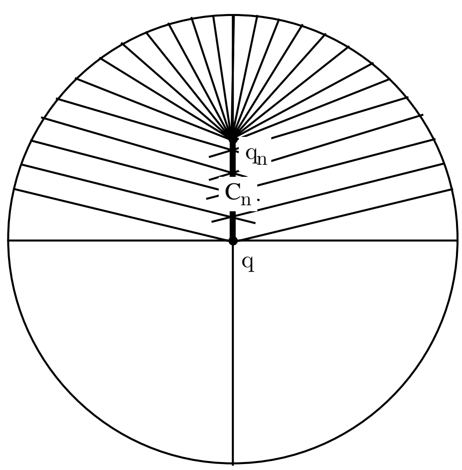

FIGURE 3. The geodesics in the Northern hemisphere of a sector S_n. (The sector's angular aperture as drawn is greatly enlarged.)

of the standard Riemann structure g_0. It is constructed by dividing the sphere into countably many disjoint sectors S_n bounded by pairs of great semi-circles from p to q, and then forcing the g-geodesics in S_n to follow the pattern shown in Figure 3, after Figure 4 of [11].

In the tangent space at p, before g_0 is modified, the focal decomposition is simple: the circles of radius $k\pi$ form σ_∞, $k \in \mathbb{N}$, and the rest of the tangent plane is σ_1. The sector S_n is the exponential image of a wedge W_n bounded by two rays and an arc on the circle of radius π. See Figure 4.

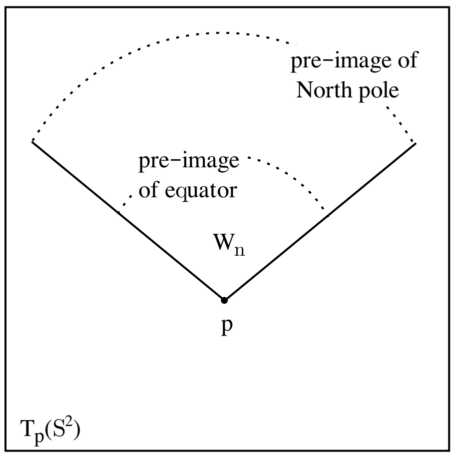

FIGURE 4. The wedge W_n before modification.

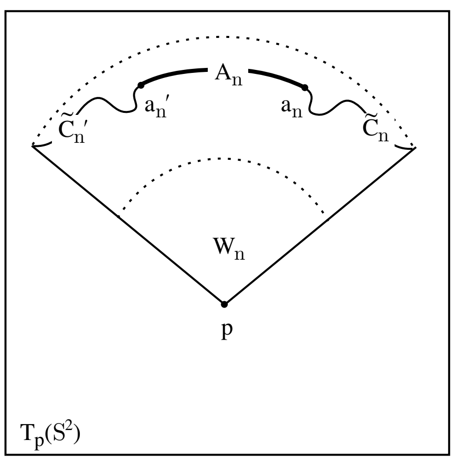

FIGURE 5. The focal decomposition inside the wedge W_n after modification.

After modification, there is present in the wedge an arc A_n contained in σ_∞. It exponentiates to the tip of the arc C_n, q_n. There are also two arcs $\widetilde{C}_n$ and $\widetilde{C}'_n$ contained in σ_2. They join A_n to the corners of W_n symmetrically, and they exponentiate to the arc C_n. See Figure 5. (Actually, by the Angle Lemma in [16], page 246, σ_2 must have empty interior. In fact, all σ_i with $1 < i < \infty$ have empty interior.) The arcs $\widetilde{C}_n$ and $\widetilde{C}'_n$, however, can not be part of such a two dimensional interior of σ_2, because they are in the frontier of σ_1. Hence, if there *is* a stratification of the focal decomposition, these arcs are contained in finitely many one dimensional strata, and their frontier points (the endpoints a_n, a'_n of A_n) are contained in lower dimensional strata, namely points. This requires countably many zero dimensional strata, all contained in the disc of radius π in the tangent space at p, which is inconsistent with the locally finite nature of a stratification. $\square$

4.2. **Bumps.** Here we prove two lemmas used to modify a Riemann structure.

Consider the unit square Σ in the plane $\mathbb{R}^2$, centered at the origin, and draw a right pyramid whose base is Σ. The surface G consisting of the pyramid's upper surface together with $\mathbb{R}^2\setminus\Sigma$ is piecewise linear. Its induced curvature (i.e., its curvature as a subset of $\mathbb{R}^3$) is concentrated at the five vertices: the curvature is negative at the base vertices and positive at the top vertex. Carefully smoothing G yields a surface $\widetilde{G}$ that agrees with G off Σ, and whose induced curvature is zero except in a neighborhood of the five vertices. See Figure 6.

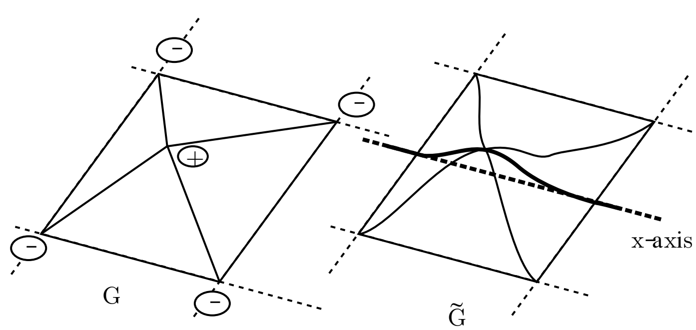

FIGURE 6. Smoothing a pyramid to make a bump.

We refer to $\widetilde{G}$ as a **bump**. The geodesic which projects to the x-axis receives net positive curvature as it crosses the bump. (It is also possible to construct a bump in which a prescribed geodesic receives net negative curvature – a "negative bump".) Let $\beta : \mathbb{R}^2 \to \mathbb{R}$ be the function whose graph is $\widetilde{G}$. Pulling back the induced Riemann structure on $\widetilde{G}$ gives a Riemann structure $g = g(\beta)$ on the plane. Then, by construction we can get:

Lemma 4.2. *The Riemann structure $g = g(\beta)$ satisfies*

(a) *Off the unit square, $g = g_0 =$ the standard, flat Riemann structure on the plane.*
(b) *The involution $(x, y) \mapsto (x, -y)$ is a g-isometry.*
(c) *The x-axis is a reparameterized g-geodesic.*
(d) *$g(\epsilon\beta)$ converges to g_0 in the C^∞ sense as $\epsilon \to 0$.*
(e) *g has non-negative curvature in a neighborhood of the x-axis and positive curvature in a neighborhood of the origin.*

Next we make an estimate like that due to A. Kneser. See [9], page 241.

Lemma 4.3. *A geodesic $\gamma(t), t \geq 0$, contains no point conjugate to $\gamma(0)$ provided that the curvature $K(t)$ along γ satisfies the inequality*

$$\int_0^\infty \|K(t)\|\, t\, dt \leq \frac{1}{2}.$$

Proof. Conjugate points are governed by the Jacobi equation

$$\ddot{J} + K(t)J = 0$$
$$J(0) = 0 \quad \dot{J}(0) = I.$$

If $J(t)$ has no singularity for $t > 0$ then conjugate points do not exist. Write the Jacobi equation as a first order matrix ODE,

$$\dot{X} = Y \quad \dot{Y} = -K(t)X$$
$$X(0) = 0 \quad Y(0) = I.$$

The solution of this ODE exists, and for at least a short time, say for $0 \leq t \leq \delta$,

$$\|Y(t) - I\| < 1.$$

It follows that on a short interval $[0, \delta]$,

$$\big(2t - \|X(t)\|\big)' = 2 - \|X(t)\|' \geq 2 - \|\dot{X}(t)\| = 2 - \|Y(t)\| > 0,$$

and for each unit vector u,

$$\begin{aligned}\langle X(t)u, u\rangle' = \langle Y(t)u, u\rangle &= \langle Iu, u\rangle + \langle (Y(t) - I)u, u\rangle \\ &\geq 1 - \|Y(t) - I\| > 0.\end{aligned}$$

Hence, on a short interval $[0, \delta]$ we have

(a) $\|Y(t) - I\| < 1$.
(b) $(2t\|X(t)\|)' > 0$.
(c) $\inf\{\langle X(t)u, u\rangle' : |u| = 1\} > 0$.

Let Δ be the set of δ such that (a), (b), (c) hold on the interval $[0, \delta]$, and let T be the supremum of Δ. We claim that $T = \infty$. Suppose that $T < \infty$. Since $(2t - \|X(t)\|)' > 0$ on $[0, T)$ and $X(0) = 0$, we know that

$$\|X(t)\| < 2t, \qquad 0 \leq t < T.$$

Thus the solution $(X(t), Y(t))$ does not blow up as $t \to T$ and it extends to an interval larger than $[0, T]$ on which it still solves the ODE. By continuity, $\|X(T)\| \leq 2T$. Also,

$$Y(T) - I = \int_0^T \dot{Y}(t)dt = -\int_0^T K(t)X(t)dt$$

implies that

$$\|Y(T) - I\| \leq \int_0^T \|K(t)\|\|X(t)\|dt < \int_0^T \|K(t)\|2tdt \leq 1.$$

(To get the strict inequality we assumed that K is not identically equal to the zero matrix; but if $K \equiv 0$, then $Y(t) = I$ and the assertion that $\|Y(T) - I\| < 1$ is valid anyway.) Thus (a) is true at $t = T$ and slightly beyond T. Also, at $t = T$ we have

$$\frac{d}{dt}\bigg|_{t=T} (2t\|X(t)\|) \geq 2 - \|Y(T)\| > 0$$

and

$$\frac{d}{dt}\bigg|_{t=T} \langle X(t)u, u\rangle = \langle Y(T)u, u\rangle > 0.$$

These strict inequalities persist for t slightly larger than T, and hence the finite time T could not be the supremum of Δ; i.e., the supremum is ∞, and (a), (b), (c) hold for all time $t \geq 0$. Since $\langle X(0)u, u\rangle = 0$ and $\langle X(t)u, u\rangle' > 0$ for all $t \geq 0$, we see that for all unit vectors u and $t > 0$,

$$\langle X(t)u, u\rangle > 0.$$

Therefore, $J(t) = X(t)$ is non-singular for all $t > 0$ and γ contains no point conjugate to $\gamma(0)$. $\square$

4.3. **Euclidean Examples.**

Proposition 4.4. *The m-sphere equipped with its standard Riemann structure induced from Euclidean space is not focally stable.*

Proofsketch. The focal decomposition for the m-sphere in a tangent plane T_pM consists of $(m-1)$-spheres of radii $k\pi$, $k \in \mathbb{N}$, that form σ_∞, together with the rest of T_pM, which is σ_1. Virtually any perturbation will make other focal components σ_i appear in a neighborhood of the $(m-1)$-sphere of radius π. Therefore the standard Riemann structure on S^m is not focally stable. $\square$

Proposition 4.5. *The induced Riemann structure on the standard two dimensional ellipsoid $E \subset \mathbb{R}^3$ with three unequal axes is not focally stable at the umbillic points.*

Proofsketch. It is well known that E possesses four umbillic points, symmetric with respect to the center of E. Also, every geodesic through one such umbillic, p, necessarily passes through the symmetric umbillic, p', and all these arcs have the same length ρ. Besides, the cut locus of p is p'. See [4], page 412. From this it results that the circles of radius $\rho, 2\rho, \ldots$ in T_pM all belong to σ_∞. From the definition of cut locus, all vectors between these circles belong to σ_1. So we have the previous situation of the sphere and so focal instability. □

If $p \in E$ is not an umbillic, we have yet to analyze the focal decomposition at p.

Proposition 4.6. *The torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ equipped with the flat Euclidean Riemann structure is not focally stable.*

Proofsketch. The focal decomposition of $T_0(\mathbb{T}^2)$ is indicated in Figure 7, where the lines are the perpendicular bisectors of the lattice vectors. The index of $v \in T_0(\mathbb{T}^2$ is 1 plus the number of these lines that pass through v. See [16]. Infinitely many σ_i's are nonempty, a consequence of the fact that the number of solutions of the Diophantine equation

$$x^2 + y^2 = N$$

is unbounded as $N \to \infty$. As will be shown in Section 5, a slight change of the Riemann structure g_0 destroys the symmetry and kills every σ_i, $i \geq 4$. Hence the flat Euclidean Riemann structure on the torus is not focally stable. □

If the focal decomposition of T_pM for a Riemann structure g is absolutely focally stable with respect to the Whitney topology on $\mathcal{R}$, and its focal equivalences $h : T_pM \to T_pM$ can be chosen to approximate the identity map on any compact subset of T_pM, we say for short that g is **Whitney $\boldsymbol{\sigma}$-stable** at p. This means that small perturbations of g in the Whitney topology leave the focal decomposition of T_pM topologically unchanged, and this is expressed by a homeomorphism that moves points in a controlled fashion. The idea applies also to Σ-stability, the simultaneous, coherent stability of the focal decompositions of all the tangent spaces to M, and gives rise to the concept of **Whitney $\boldsymbol{\Sigma}$-stability**.

Proposition 4.7. $\mathbb{R}^m$ *with its Euclidean Riemann structure g_0 has the following properties.*

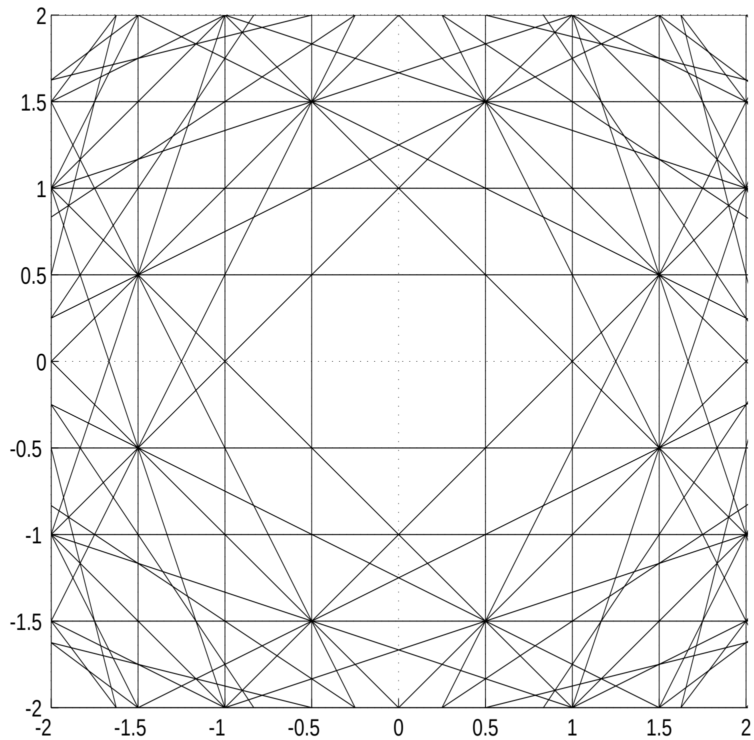

FIGURE 7. The focal decomposition of $T_0(\mathbb{T}^2)$.

(a) *The focal decomposition of each* $T_p\mathbb{R}^m$ *is trivial: no vectors have companions, and all of* T_pM *is* σ_1*.*
(b) g_0 *is Whitney stable at* p*.*
(c) g_0 *is focally stable at* p*.*
(d) g_0 *is not* Σ*-focally stable.*

Proof. (a) This is obvious.

(b), (c). We claim that under Whitney small perturbations of the Riemann structure, all $\sigma_i, i \geq 2$, remain empty. We assume that p is the origin. If g is a Whitney small perturbation of the flat Riemann structure g_0 then

$$\int_0^\infty \|K(t)\|\, t\, dt \ll \frac{1}{2}$$

along any geodesic through the origin. According to Lemma 4.3 there are no conjugate points, so $\exp : T_0\mathbb{R}^m \approx \mathbb{R}^m \to \mathbb{R}^m$ is a local diffeomorphism. It is also proper and therefore, for purely topological reasons, it is a global diffeomorphism. That is, $\sigma_i = \emptyset$ for all $i \geq 2$, and g_0 is absolutely focally stable.

(d) Place an ϵ-bump at the origin and consider a point $-p$ far down the negative x-axis. Geodesics that emanate from $-p$ are nearly parallel straight lines as they approach the origin. They diverge from each other, but very slightly. As they pass through the bump, the lines are focused by a fixed amount, an amount that overcomes their slight divergence, and causes them to cross each other at a point far down the positive x-axis. This shows that although the focal decomposition of $T_0\mathbb{R}^m$, or in fact of $T_p\mathbb{R}^m$ as p ranges over any compact subset of $\mathbb{R}^m$, remains trivial under Whitney small perturbations of g, the global Σ-decomposition can change. □

Remark. In a conversation with one of the authors, R. Tribuzy gave a proof of (b) in Proposition 4.7, different from the one above.

4.4. **The Cylinder and the Silo.** The bump construction applies also to the cylinder and to the singly capped cylinder, or "silo", and demonstrates the difference between absolute focal stability and the kind of relative focal stability in which we control the focal components in a large compact ball in T_pM.

Proposition 4.8. *The cylinder $M = \mathbb{R}^2/\mathbb{Z}$ equipped with the flat Riemann structure g_0 is focally stable but not absolutely focally stable.*

Proof. The focal decomposition of the tangent plane $T_0M = \mathbb{R}^2$ is the following. For $i \geq 3$, $\sigma_i = \emptyset$, while σ_2 consists of lines $(n/2) \times \mathbb{R}$ for $n \in \mathbb{Z}\backslash\{0\}$, and σ_1 is the rest of $\mathbb{R}^2$. If $(n/2, y) \in \sigma_2$ then its focal companion is $(-n/2, y)$.

Ordinary focal stability of g_0 presents no new problems. To verify the lack of absolute focal stability we modify g_0 by pasting on a small positive ϵ-bump B at the point $((1/2), 0)$. See Lemma 4.2 and Figure 8. This produces a new Riemann structure g_ϵ on the cylinder, and clearly g_ϵ converges to g_0 in the C^∞ Whitney sense as $\epsilon \to 0$.

The geodesic circle $S^1 \times \{0\}$ in M remains a geodesic for g_ϵ, say it is $\gamma = \gamma(t)$. Let $\gamma_\pm$ be the restrictions of γ to $\mathbb{R}_\pm$. Every time the forward geodesic γ_+ passes through the bump it experiences a small amount of positive curvature, and nowhere does it experience negative curvature. Eventually, this accumulation of positive curvature causes the birth of a cut point $q_+ = q_+(\epsilon)$ along γ_+,

$$q_+(\epsilon) = \exp_0(v(\epsilon)),$$

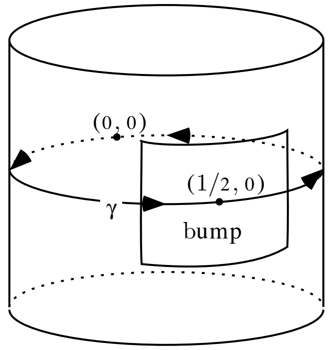

FIGURE 8. The cylinder with a bump.

where $v(\epsilon) = (u(\epsilon), 0) \in \mathbb{R}^2$. As $\epsilon \to 0$, $u(\epsilon) \to +\infty$ continuously. Directly above and below $v(\epsilon)$ on $u(\epsilon) \times \mathbb{R}$ there are focal companions

$$v_\pm(\epsilon) = (u(\epsilon), y_\pm(\epsilon)).$$

The point $b_+(\epsilon) = \exp_0(v_\pm(\epsilon))$ moves continuously along the geodesic circle $S^1 \times \{0\}$, winding infinitely often in the positive sense as $\epsilon \to 0$.

There is of course a symmetric cut point that appears along the reverse geodesic γ_-. It is

$$q_-(\epsilon) = \exp_0(-v(\epsilon)).$$

Note that $-v(\epsilon)$ lies in the left half plane. Corresponding to $-v(\epsilon)$

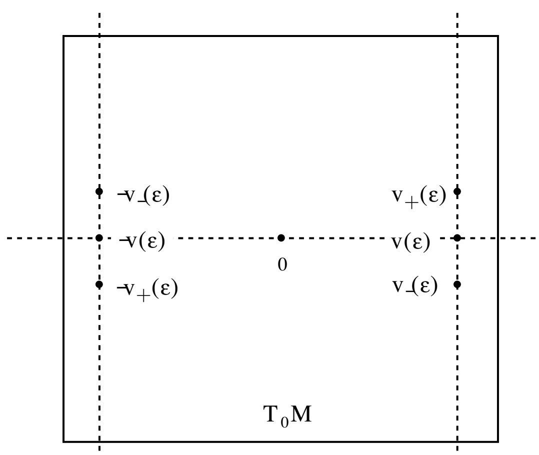

FIGURE 9. Focal companions in T_0M for the cylinder with an ϵ-bump.

there are focal companions

$$-v_\pm(\epsilon) = (-u(\epsilon), y_\pm(\epsilon))$$

directly above and below $-v(\epsilon)$. The point $b_-(\epsilon) = \exp_0(-v_\pm(\epsilon))$ moves along the geodesic circle $S^1 \times \{0\}$, winding infinitely often in the negative sense as $\epsilon \to 0$. See Figure 9. By symmetry and continuity there are infinitely many values $\epsilon = \epsilon_n \to 0$ as $n \to \infty$ such that all four focal companions $\pm v_\pm(\epsilon_n)$ exponentiate to the same point in M. Hence $\sigma_4(\epsilon_n) \neq \emptyset$, and g_0 is not Whitney stable. □

Proposition 4.9. *The plane $\mathbb{R}^2$ equipped with a silo Riemann structure is focally stable but not absolutely focally stable. (The base point p is the top of the silo.)*

Proof. The silo structure on $\mathbb{R}^2$ is gotten by attaching a smoothed hemispherical cap to a cylinder. See Figure 10. The focal decomposition is

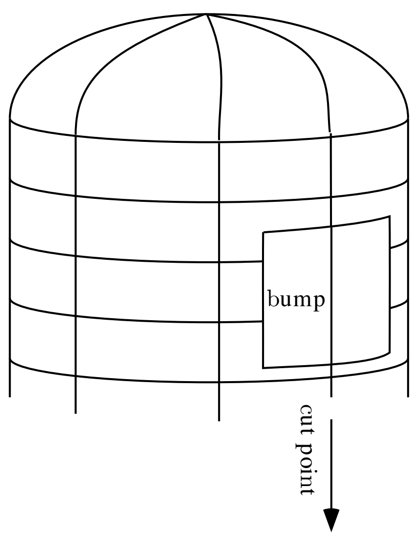

FIGURE 10. A bump on the side of a silo.

the same as for the flat plane: everything is σ_1. By the foregoing analysis, small perturbations of the Riemann structure will not produce conjugate points in any given compact set, and it is easy to see that this precludes $\sigma_i, i \geq 2$, from appearing there. Hence, the silo structure is focally stable.

To check that it is not absolutely focally stable we paste a small bump on the cylinder. See Lemma 4.2. This causes passing geodesics to focus. They will cross far down the cylinder, and their crossing produces points in σ_2. Thus the silo is not absolutely focally stable. □

We mention an interpolation between the silo and the flat plane. It is a "haystack" surface M formed by taking a cone, replacing its vertex with a spherical cap, and smoothing the join. Most of the surface has zero curvature, but some of it has positive curvature. The focal decomposition at the top point p is the same as the flat plane $\sigma_i = \emptyset$ for all $i \geq 2$. We claim that this focal decomposition does not change under Whitney small perturbation. The proof amounts to a version of Lemma 4.3 in which the initial values of the Jacobi equation are

$$J(0) = \sin\alpha\, I \quad \dot{J}(0) = \cos\alpha\, I \qquad 0 \leq \alpha < \pi/2.$$

One checks that if $\int_0^\infty \|K(t)\| t dt$ is small enough then $J(t)$ is defined for all time $t > 0$ and is non-singular.

4.5. **The Hyperbolic Plane.**

Proposition 4.10. *The hyperbolic Riemann structure on $\mathbb{R}^m$ is absolutely Σ-stable. No vectors have companions and this remains true after small perturbations.*

Proof. A small perturbation of the hyperbolic Riemann structure on $\mathbb{R}^m$ still has strictly negative curvature. Hadamard's Theorem ([9], page 149) asserts that for any point p, the exponential map $\exp_p$ is a diffeomorphism from the tangent space at p onto $\mathbb{R}^m$. □

5. Focal Stability

In this section we verify focal stability in a basic two dimensional case, the flat torus. Given a positive definite quadratic form

$$Q = ax^2 + bxy + cy^2$$

we use the standard trivialization of the tangent bundle of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ to define a Riemann structure which is Q in each fiber. Its Gauss curvature is zero, and conversely, if a Riemann structure on the torus has zero Gauss curvature then it arises from such a Q.

Theorem 5.1. *If a, b, c are rationally independent real numbers then the corresponding Riemann structure on the torus is focally stable.*

The focal decomposition is the same on all the tangent spaces, so we fix our basepoint as the origin $O = (0,0)$ and study the focal decomposition in $\mathbb{R}^2 = T_O\mathbb{T}^2$.

A vector $(x, y) \in T_O\mathbb{T}^2$ belongs to σ_k if and only if

$$Q(x + m, y + n) = Q(x, y) \tag{3}$$

for k elements $(m, n) \in \mathbb{Z}^2$.

Definition. For fixed $(m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$, the set of (x, y) that satisfy (3) is the **Brillouin line** $L(m, n, Q)$.

The equation solved by $L(m,n,Q)$ is

$$(2am+bn)x+(bm+2cn)y+Q(m,n)=0,$$

which is checked by direct calculation. Geometrically, $L(m,n,Q)$ is the Q-perpendicular bisector of the segment joining O and $(-m/2,-n/2)$. From the above, it follows that (x,y) belongs to σ_k if and only if through (x,y) there pass exactly $k-1$ Brillouin lines. See [16] and Figure 7.

The following is the key to Theorem 5.1.

Theorem 5.2. *If the coefficients a,b,c of the positive definite quadratic form Q are rationally independent then no three of its Brillouin lines meet at a common point.*

Proof. Suppose not: the Brillouin lines

$$\begin{aligned} L_1:&\quad (2am_1+bn_1)x+(bm_1+2cn_1)y+Q(m_1,n_1)=0\\ L_2:&\quad (2am_2+bn_2)x+(bm_2+2cn_2)y+Q(m_2,n_2)=0\\ L_3:&\quad (2am_3+bn_3)x+(bm_3+2cn_3)y+Q(m_3,n_3)=0 \end{aligned}$$

that meet in a common point. Because the lattice points (m_1,n_1), (m_2,n_2), (m_3,n_3) are distinct and the Brillouin lines are non-parallel, we have

$$\alpha_{12}=\begin{vmatrix} m_1 & n_1\\ m_2 & n_2\end{vmatrix}\neq 0 \quad \alpha_{13}=\begin{vmatrix} m_1 & n_1\\ m_3 & n_3\end{vmatrix}\neq 0 \quad \alpha_{23}=\begin{vmatrix} m_2 & n_2\\ m_3 & n_3\end{vmatrix}\neq 0. \tag{4}$$

The assumption that the three Brillouin lines meet implies that

$$\begin{vmatrix} 2am_1+bn_1 & bm_1+2cn_1 & Q(m_1,n_1)\\ 2am_2+bn_2 & bm_2+2cn_2 & Q(m_2,n_2)\\ 2am_3+bn_3 & bm_3+2cn_3 & Q(m_3,n_3)\end{vmatrix}=0. \tag{5}$$

Expanding (5) gives

$$Aa+Bb+Cc=0$$

where A,B,C are the integers below. Rational independence of a,b,c requires that they be zero:

$$\begin{aligned} A&=\begin{vmatrix} m_1 & n_1 & m_1^2\\ m_2 & n_2 & m_2^2\\ m_3 & n_3 & m_3^2\end{vmatrix}=0\\ B&=\begin{vmatrix} m_1 & n_1 & m_1n_1\\ m_2 & n_2 & m_2n_2\\ m_3 & n_3 & m_3n_3\end{vmatrix}=0\\ C&=\begin{vmatrix} m_1 & n_1 & n_1^2\\ m_2 & n_2 & n_2^2\\ m_3 & n_3 & n_3^2\end{vmatrix}=0. \end{aligned} \tag{6}$$

It remains to show that (4), (6) lead to a contradiction.

Assume that one of the m's or n's is zero; say $m_1 = 0$. From (4), $n_1 m_2 \neq 0$ and $n_1 m_3 \neq 0$. Expanding the determinant $A = 0$ gives

$$m_2(n_1 m_3^2) = m_3(n_1 m_2^2),$$

which implies $m_2 = m_3$. Similarly, $B = 0$ implies that $n_2 = n_3$. But this implies that $\alpha_{23} = 0$, contrary to (4). Thus we may assume that all of the m's and n's are different from zero.

Developing $A = B = C = 0$ according to their third columns gives

$$\begin{aligned}
\alpha_{23} m_1^2 \quad &- \alpha_{13} m_2^2 \quad + \alpha_{12} m_3^2 \quad = 0\\
\alpha_{23} m_1 n_1 &- \alpha_{13} m_2 n_2 + \alpha_{12} m_3 n_3 = 0\\
\alpha_{23} n_1^2 \quad &- \alpha_{13} n_2^2 \quad + \alpha_{12} n_3^2 \quad = 0.
\end{aligned}$$

These homogeneous equations in the α's show that we have

$$\begin{vmatrix} m_1^2 & m_2^2 & m_3^2 \\ n_1^2 & n_2^2 & n_3^2 \\ m_1 n_1 & m_2 n_2 & m_3 n_3 \end{vmatrix} = 0.$$

Expressing the first row of this matrix as a linear combination of the second and third gives

$$\begin{aligned}
m_1^2 &= \lambda n_1^2 + \mu m_1 n_1\\
m_2^2 &= \lambda n_2^2 + \mu m_2 n_2\\
m_3^2 &= \lambda n_3^2 + \mu m_3 n_3.
\end{aligned} \tag{7}$$

Solving the first two equations in (7) for for λ gives

$$\lambda = \frac{\begin{vmatrix} m_1^2 & m_1 n_1 \\ m_2^2 & m_2 n_2 \end{vmatrix}}{\begin{vmatrix} n_1^2 & m_1 n_1 \\ n_2^2 & m_2 n_2 \end{vmatrix}} = -\frac{m_1 m_2}{n_1 n_2}, \tag{8}$$

which is valid since none of the m's and n's are zero. Solving the first and second equation for λ gives

$$\lambda = -\frac{m_1 m_3}{n_1 n_3}. \tag{9}$$

Equations 8 and 9 give

$$\alpha_{23} = \begin{vmatrix} m_2 & n_2 \\ m_3 & n_3 \end{vmatrix} = m_2 n_3 - n_2 m_3 = 0$$

contrary to (4). □

Following Peter Veerman, we refer to an equidistance locus as a *mediatrix*. Thus, the Brillouin line $L(Q,m,n)$ is the mediatrix between O and $(-m,-n)$,

$$L(Q,m,n) = \{z \in \mathbb{R}^2 : d_Q(z,O) = d_Q(z,(-m,-n))\}$$

where d_Q is Q-distance.

Now let g be a Riemann structure on $\mathbb{T}^2$ that smoothly approximates g_0, the flat Riemann structure induced by Q. Lift g to a Riemann structure $\bar{g}$ on $\mathbb{R}^2$. The projection $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$ is a local isometry from $\bar{g}$ to g. The $\bar{g}$-mediatrix

$$L(\bar{g},m,n) = \{z \in \mathbb{R}^2 : d_{\bar{g}}(z,O) = d_{\bar{g}}(z,(-m,-n))\}$$

approximates the corresponding Brillouin line, as explained below. ($d_{\bar{g}}$ is $\bar{g}$-distance.)

Lemma 5.3. *Let B be a disc in $\mathbb{R}^2$. If g C^∞-approximates g_0 then there is a smooth function $\nu : L(Q,m,n) \to \mathbb{R}^2$ such that*

$$L(\bar{g},m,n) \cap B = \operatorname{image}(\nu) \cap B.$$

Furthermore, $\nu(z) - z$ is C^∞-small.

Proof. $L = L(Q,m,n)$ is the zero locus of the function

$$f_Q(z) = d_Q(z,O) - d_Q(z,(-m,-n)).$$

Except at O and $(-m,-n)$, f_Q is smooth. Its gradient at L is nonzero and normal to L. By the Implicit Function Theorem applied to

$$f_{\bar{g}}(z) = d_{\bar{g}}(z,O) - d_{\bar{g}}(z,(-m,-n)),$$

$L(\bar{g},m,n) \cap B$ is the graph of a C^∞-small section of the normal bundle, $\nu : L(Q,m,n) \to L^\perp(Q,m,n)$. □

Proof of Theorem 5.1. We have a positive definite quadratic form $Q = ax^2 + bxy + cy^2$ in which a,b,c are rationally independent. This gives a Riemann structure g_0 on the torus. According to Theorem 5.2 the focal decomposition of g_0 on $T_O\mathbb{T}^2$ consists of a discrete set of points in σ_3, a discrete set of lines that cross pairwise at these points, and the open regions bounded by the polygons they form.

We claim that the focal decomposition is stable under perturbation of the Riemann structure. Let a compact disc B_0 in $\mathbb{R}^2$ and an $\epsilon > 0$ be given. We will find a disc $B \supset B_0$ and a neighborhood $\mathcal{N}$ of g_0 in $\mathcal{R}$ such that for each $g \in \mathcal{N}$ there is a an ϵ-homeomorphism

$$h : B \to B$$

which sends the focal decomposition for g_0 to that for g. This will establish focal stability of g_0.

Choose B so that ∂B is tangent to none of the Brillouin lines. (∂B is "non-critical.") By assumption, there are only a finite number of points in B at which pairs of Brillouin lines meet one another or meet ∂B, and the angles of intersection at these points are bounded away from zero.

By Lemma 5.3 applied to each of the finite number of Brillouin lines L that cross B, there is a neighborhood $\mathcal{N}$ of g_0 in $\mathcal{R}$ such that for each $g \in \mathcal{N}$, the mediatrix $L(\bar{g}, m, n) \cap B$ smoothly approximates the Brillouin line $L(Q, m, n) \cap B$. Thus the web of mediatrices has the same topology as the web of Brillouin lines, and there is an ϵ-diffeomorphism

$$h_0 : B \to B$$

that sends the Brillouin web to the mediatrix web. (A harder construction, but in the same spirit, appears in [18].)

Call $p = \pi(O)$. For each $z \in B$, define $h(z)$ to be the vector $v \in T_p(\mathbb{T}^2)$ such that the $\bar{g}$-geodesic leaving O with initial tangent v arrives at $h_0(z)$ in unit time. That is,

$$\exp^g(h(z)) = \pi(h_0(z)).$$

Note that for g_0, we have $h_0(z) = z$ and $\exp^{g_0} = \pi$. Thus, h is the identity map when $g = g_0$.

Since π is a local isometry it sends $\bar{g}$-geodesics to g-geodesics, and it preserves their length. If a $\bar{g}$-geodesic starts or ends at a lattice point then it projects to a g-geodesic that starts or ends at p. Consider a vector $v \in T_p(\mathbb{T}^2)$ such that its $\bar{g}$-geodesic

$$\bar{\gamma}(t) = \exp^{\bar{g}}(tv)$$

lies in B for $0 \leq t \leq 1$. Call $\ell = |v|$. The projection of $\bar{\gamma}$, $\gamma = \pi \circ \bar{\gamma}$, is a g-geodesic from p to $q = \gamma(1)$ of length ℓ. Let β be a second g-geodesic from p to q of length ℓ. It lifts to a $\bar{g}$-geodesic $\bar{\beta}$ of length ℓ that joins $\bar{q} = \bar{\gamma}(1)$ to a nonzero lattice point. Thus, $\bar{q}$ lies on a mediatrix $L(\bar{g}, m, n)$. The converse is equally clear. If $\bar{q}$ lies on the mediatrix $L(\bar{g}, m, n)$ then it is joined to the lattice point $(-m, -n)$ by a $\bar{g}$-geodesic of length ℓ, and this geodesic projects to a g-geodesic of length ℓ that joins p to q.

Thus, the g-geodesics of length ℓ, other than γ, that join p to q correspond bijectively to $\bar{g}$-geodesics that join nonzero lattice points to $\bar{q}$. This implies that $v \in \sigma_i(g)$ if and only if $\exp^{\bar{g}}(v)$ lies on i mediatrices, and completes the verification that h sends the focal decomposition of g_0 on B to that of g. □

Remark. The previous construction preserves the companion relations because they correspond to Brillouin line intersections.

Remark. From the preceding proof it is clear that the quadratic form Q defines a focally stable Riemann structure if and only if its Brillouin lines meet at most pairwise. We do not know whether this pairwise intersection condition implies that the coefficients of Q are rationally independent, which would give a converse to Theorem 5.1.

Corollary 5.4. *The generic positive definite quadratic form $Q = ax^2 + bxy + cy^2$ defines a Riemann structure on the 2-torus which is focally stable. In particular, focally stable Riemann structures on the 2-torus exist.*

Proof. It is generic that real numbers a, b, c are rationally independent. □

6. The Pointwise Index

The proof of the Theorem 1.1 reduces to a transversality result, namely Theorem 6.1, in the spirit of Mather's multi-transversality theory [17]. For if several geodesics simultaneously emanate from a base point p, it is natural to group them together and consider a multi-exponential map

$$\begin{aligned} E : V_p^k \times \mathcal{R} &\to (M \times \mathbb{R})^k \\ (v_1, \dots, v_k, g) &\mapsto (\exp(v_1), |v_1|, \dots, \exp(v_k), |v_k|), \end{aligned}$$

where exp is the g-exponential,

$$V_p^k = \{(v_1, \dots, v_k) : v_1, \dots, v_k \text{ are distinct nonzero vectors in } T_pM\},$$

and $(M \times \mathbb{R})^k = (M \times \mathbb{R}) \times \cdots \times (M \times \mathbb{R})$. In Mather's notation,

$$V_p^k = (T_pM \setminus \{0\})^{(k)}.$$

We write $E_g = E_g^k = E(\ , g) : V_p^k \to (M \times \mathbb{R})^k$. Also, the diagonal of $(M \times \mathbb{R})^k$ is

$$\Delta = \{(q, \ell, q, \ell, \dots, q, \ell) : q \in M \text{ and } \ell \in \mathbb{R}\}.$$

Obviously, the E_g-pre-image of $(q, \ell)^k \in \Delta$ is a set of k companions and

$$\sigma_k(p) \subset \pi E_g^{-1}(\Delta)$$

where $\pi : V_p^k \to T_pM$ is the projection $\pi(v_1, \dots, v_k) = v_1$.

Theorem 6.1. *Corresponding to the base point p there is a residual $\mathcal{G}(p) \subset \mathcal{R}$ such that if $k \geq 3$ and $g \in \mathcal{G}(p)$ then E_g^k is transverse to Δ.*

Remark. If $k = 1$ the assertion is trivial since $\Delta = M \times \mathbb{R}$, while if $k = 2$ the assertion requires a subtler proof.

Theorem 6.1 requires several lemmas presented below, but first we derive the Pointwise Index Theorem 1.1 from it. The dimension of M is m.

Proof of Theorem 1.1. We claim that if $\mathcal{G}(p)$ is as in Theorem 6.1 and $k > m+1$ then $\sigma_k(p) = \emptyset$. It's a dimension count. We have

$$\dim V_p^k = km \quad \text{and} \quad \text{codimension}\,\Delta = k(m+1) - (m+1).$$

Thus, if $k > m+1$ then for the map $E_g^k : V_p^k \to (M \times \mathbb{R})^k \supset \Delta$, the codimension of Δ exceeds the domain dimension, and transversality implies that the image of E_g^k is disjoint from Δ, which implies that $\sigma_k(p) = \emptyset$. □

The generic properties of the geodesic flow were first investigated by R. Abraham in [1]. He proposed a "Bumpy Metric Theorem" to describe the periodic orbits of the generic geodesic flow, i.e., its closed geodesics, and he outlined a proof. In [3] D. Anosov gives a complete proof of the result. A major step is a perturbation lemma that shows how to move a geodesic by varying the Riemann structure. It is perfectly suited to our purposes as well.

The setting is the second tangent bundle, $T(TM)$. Given a Riemann structure g on M, if v is a nonzero vector in T_pM then its orthogonal complement $g^\perp(v)$ is an $(m-1)$-dimensional subspace of T_pM. It lifts to $(m-1)$-dimensional subspaces in the horizontal/vertical splitting of $T_v(TM)$, and their direct sum, $\widehat{g}^\perp(v)$, is a $(2m-2)$-dimensional subspace of $T_v(TM)$.

If $w \in T_qM$ we refer to q as the **footpoint** of w. It is the image of w under the standard projection $TM \to M$.

A Riemann structure g on M generates a geodesic flow φ^g on TM. In terms of the g-exponential maps it is defined by

$$\varphi_t^g(v) = \frac{d\gamma_v(t)}{dt},$$

where γ_v is the geodesic in M with initial tangent vector v. That is, $\gamma_v(t) = \exp(tv)$. Combining all the arguments gives a map

$$\begin{aligned} \Phi : TM \times \mathbb{R} \times \mathcal{R} &\to TM \\ (v, t, g) &\mapsto \varphi_t^g(v). \end{aligned}$$

It is smooth[3] [3], page 9, and we are interested in knowing to what extent the derivative of Φ with respect to the variable $g \in \mathcal{R}$ is a surjection. Since $\mathcal{R}$ is an open subset of the linear space $\mathcal{S}$ that consists of all smooth symmetric 2-tensors on TM, the derivative (or tangent map) of Φ with respect to g is a linear map from $\mathcal{S}$ to $T(TM)$. We denote it $T_3\Phi$,

$$(T_3\Phi)_{v,t,g} : \mathcal{S} \to T_{\varphi_t^g(v)}(TM).$$

Surjectivity of this derivative is the subject of the following lemma of Anosov [3], page 14.

A parameterized curve $\phi : (a, b) \to X$ has a **simple point** at t_0 if $\phi(t) \neq \phi(t_0)$ for all $t \in (a, b)$, $t \neq t_0$. It is a non-self-intersection point.

Lemma 6.2. *Suppose that* $(v, \ell, g) \in TM \times \mathbb{R} \times \mathcal{R}$*, and the geodesic trajectory in* TM*,* $\varphi_t^g(v)$ *with* $0 < t < \ell$*, has a simple point at some* $t_0 \in (0, \ell)$*. At the point* (v, ℓ, g)*, the range of the derivative of* Φ *with respect to the Riemann structure includes the* $(2m-2)$*-dimensional subspace* $\widehat{g}^\perp(w)$ *where* $w = \varphi_\ell^g(v)$*,*

$$(T_3\Phi)_{v,\ell,g}(\mathcal{S}) \supset \widehat{g}^\perp(w).$$

The simpleness hypothesis refers to the geodesic trajectory in TM. It permits the geodesic curve in M, γ_v, to have some self-intersection. Anosov states, proves, and uses this lemma under the additional assumption that $v = w$, which signifies that v is periodic under the geodesic flow, and its minimum period is ℓ. His proof makes no use of this additional assumption. Besides, he shows that if Z is any neighborhood of the footpoint z_0 of the simple point $\varphi_{t_0}^g(v)$ then we need only deal with Riemann structure perturbations with support in Z. More precisely, if $\mathcal{R}(g, Z)$ denotes those Riemann structures that agree with g off Z, and $\Phi_Z(v, t, g)$ denotes the restriction of Φ to $TM \times \mathbb{R} \times \mathcal{R}(g, Z)$ then

$$(T_3\Phi_Z)_{v,\ell,g}(\mathcal{S}) \supset \widehat{g}^\perp(w). \tag{10}$$

We call z_0 a **control point** because Anosov controls the endpoint w by manipulating the Riemann structure at z_0. In our application of this lemma we only need to control the footpoint of w, i.e., the geodesic's endpoint $\gamma_v(\ell)$. See Figure 11.

[3] Technically, what Anosov shows is that if $2 \leq r < \infty$ and $\mathcal{R}$ is the set of C^r Riemann structures on M then Φ is of class C^r. The case when $r = \infty$ is slightly different because then $\mathcal{R}$ is an open subset of a Fréchet space, not a Banach space. Similarly, in our application of the Abraham Transversality Theorem below, "smooth" should be interpreted to mean "of class C^r for large enough finite r". None of the genericity results for the C^∞ case are affected by this abuse of language, of course.

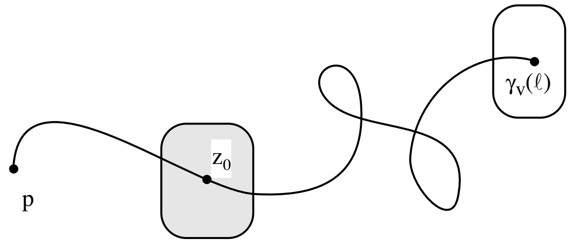

FIGURE 11. Manipulation of the Riemann structure at the control point z_0 causes the geodesic's endpoint to move freely.

Another ingredient that we use is the Abraham Transversality Theorem [2]. It concerns a smooth map

$$F : X \times \mathcal{A} \to Y \supset W$$

where $\mathcal{A}$ is a Banach manifold and X, Y, W are finite dimensional. If, for $a \in \mathcal{A}$, F_a denotes the map $x \mapsto F(x, a)$ then the assertion is

$$F \pitchfork W \quad \Rightarrow \quad F_a \pitchfork W$$

for all a in a residual subset of $\mathcal{A}$. We also make use of the Bumpy Metric Theorem.

A Riemann structure is **bumpy** if

(i) Its closed geodesics are nondegenerate; i.e., the principal eigenvalues of the periodic orbits of its geodesic flow are not roots of unity.
(ii) Given $L > 0$, it has only finitely many closed geodesics of length $\leq L$, and they vary continuously under small perturbations of the Riemann structure.
(iii) Its closed geodesics meet transversally.

(ii) is an easy consequence of (i) and the fact that the geodesic flow on the unit tangent bundle has no equilibria. Similarly, (iii) follows from (i), (ii), and standard transversality reasoning. It follows from (iii) that if M has dimension ≥ 3 then the closed geodesics are disjoint from each other and have no self intersection, while if M has dimension 2 then the intersections and self intersections of the closed geodesics are limited to double points.

Bumpy Metric Theorem. *The generic Riemann structure is bumpy.*

Terminology. A **geodesic loop** is a geodesic curve $\lambda(t) = \exp(tv)$, $0 \leq t \leq 1$, such that $\lambda(1) = \lambda(0)$ and which has only finitely many self intersections, $0 \leq t \leq 1$. See Figure 12. A **geodesic** is a geodesic curve modulo its orientation; thus, a geodesic loop and its reverse are the same geodesic.

Note that a periodic geodesic curve $t \mapsto \exp(tv)$ with prime period $|v|$ is a geodesic loop but $t \mapsto \exp(2tv)$ is not, because all its points are self intersections.

Definition. If $\lambda(t) \neq p$ for $0 < t < 1$ the geodesic loop is **single**. It only meets p at its beginning and end.

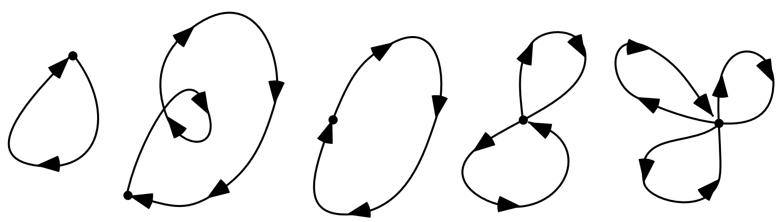

FIGURE 12. Five geodesic loops. The first three are single, the fourth is double, and the fifth is triple.

Lemma 6.3. *Given a base point p and a constant $L > 0$ there is an open dense set $\mathcal{G}(p, L) \subset \mathcal{R}$ such that if $g \in \mathcal{G}(p, L)$ then*

(a) *No closed geodesic of length $\leq L$ passes through p.*
(b) *If λ is a geodesic loop at p of length $\ell \leq L$ then the only other geodesic loop of length ℓ at p is the reverse of λ.*
(c) *Every geodesic loop at p with length $\leq L$ meets p only at its beginning and at its end. None are double.*
(d) *If λ is a geodesic loop at p that is part of a geodesic loop $a{*}\lambda{*}b$ at q of length $\leq L$ then a and b have unequal length. (The notation $a{*}\lambda{*}b$ indicates the concatenation of the three curves.)*

Proof. We claim that the sets

$$\mathcal{G}_a = \{g \in \mathcal{R} : \text{ (a) holds}\} \qquad \mathcal{G}_b = \{g \in \mathcal{R} : \text{ (a), (b) hold}\}$$
$$\mathcal{G}_c = \{g \in \mathcal{R} : \text{ (a) - (c) hold}\} \qquad \mathcal{G}_d = \{g \in \mathcal{R} : \text{ (a) - (d) hold}\}$$

are open dense in $\mathcal{R}$. Continuity of the geodesic flow makes openness easy to check in all cases. Density is the issue.

(a) Let $g \in \mathcal{R}$ be given. By the Bumpy Metric Theorem, we can approximate g by g_1 which has only finitely many closed geodesics of length $\leq L$. Let Γ be their union, and take a diffeomorphism $\psi : M \to M$ that C^∞-approximates the identity map and moves Γ off p. The Riemann structure $g_2 = \psi_* g_1$ approximates g. Because ψ is an isometry from g_1 to g_2, p lies on no closed g_2-geodesic of length $\le L$, which completes the proof of density of $\mathcal{G}_a$.

(b) Let $g \in \mathcal{G}_a$ be given. By Lemma 6.2 the map

$$E : V_p \times \mathcal{G}_a \to M \supset \{p\}$$
$$(v, g) \mapsto \exp_p(v)$$

is transverse to $\{p\}$. (Note. V_p is just $T_p(M) \setminus \{0\}$.) For the geodesic curve $t \mapsto \exp_p(tv)$ is not closed, so it contains many simple points at which to control its endpoint by perturbations of the Riemann structure. By the Abraham Transversality Theorem we get a residual subset $\mathcal{G}_a^* \subset \mathcal{G}_a$ such that if $g \in \mathcal{G}_a^*$ then there are only a finite number of geodesic loops at p having length $\le L$, and these loops are stable under perturbation. They do not split into several loops at p when the Riemann structure is perturbed. A subsequent small perturbation makes these loops (considered as point sets) have different lengths, which verifies density of $\mathcal{G}_b$.

(c) Let $g \in \mathcal{G}_b$ be given. We apply Lemma 6.2 to a geodesic loop λ at p. By induction and the finiteness provided by (b), it is enough to show that if λ is double then a small perturbation of g in $\mathcal{G}_b$ can make it single. Choose simple points q_1, q_2 as shown in Figure 13. Applying Lemma 6.2 at q_1 and the inverse version at q_2 shows that by varying the Riemann structure in neighborhoods of q_1 and q_2 we can move the double point off p without disturbing the geodesic after q_2.

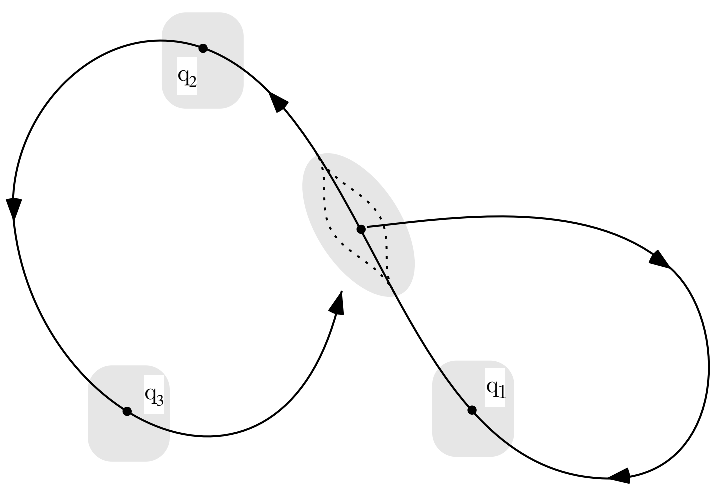

FIGURE 13. Free motion of the midpoint of a double geodesic loop.

(d) Let $g \in \mathcal{G}_c$ be given and let λ be a geodesic loop at p of length $\le L$. Extend λ to a geodesic $\gamma = A{*}\lambda{*}B$ where A and B have length

L. Since $g \in \mathcal{G}_a$, γ is not a closed geodesic, and $A \cap B$ is a finite; say $A \cap B = \{q_1, \dots, q_n\}$. By a small change of g, the lengths of the arcs of A and B between the q_i become rationally independent. (If M has dimension ≥ 3 then g can be perturbed so that $A \cap B = \{p\}$.) If $a{*}\lambda{*}b$ is a geodesic of length $\leq L$ that forms a loop at some q_i then $a \subset A$, $b \subset B$, and a, b have different lengths. This verifies density of $\mathcal{G}_d = \mathcal{G}(p, L)$. □

The next ingredient in our proof of Theorem 6.1 concerns "free spots".

Definition. If $\gamma_1, \dots, \gamma_k$ are geodesics and each γ_j contains a simple point z_j that belongs to no other γ_i then $z_1, \dots, z_k$ are **free spots** on $\gamma_1, \dots, \gamma_k$.

Lemma 6.4. *If $g \in \mathcal{R}(p, L)$ and distinct geodesics $\gamma_1, \dots, \gamma_k$ have equal length and join p to q, $p \neq q$, then there exist free spots $z_1, \dots, z_k$ on $\gamma_1, \dots, \gamma_k$.*

Proof. Geodesics meet one another discretely or in curves. In the former case it is easy to choose a free spot on each γ_j, so suppose that two of the geodesics contain a curve, say $\gamma_1 \cap \gamma_2$ contains a curve μ_0. Extend μ_0 to a maximal curve $\mu \subset \gamma_1 \cap \gamma_2$. Maximality and the fact that γ_1, γ_2 are geodesics imply that

$$\gamma_1 \cap \gamma_2 = \mu \quad \text{and} \quad \mu \text{ joins } p \text{ to } q.$$

Since γ_1, γ_2 are distinct and have equal length, one of them, say γ_1, contains a geodesic loop λ at p, λ meets μ only at p, and

$$\gamma_1 = \lambda{*}\mu.$$

By hypothesis, γ_1 contains no double loop at p, $p \neq q$, and so $\gamma_2 = \mu{*}b_2$ for some loop b_2 at q. Since γ_1 and γ_2 have equal length, so do λ and b_2. See Figure 14.

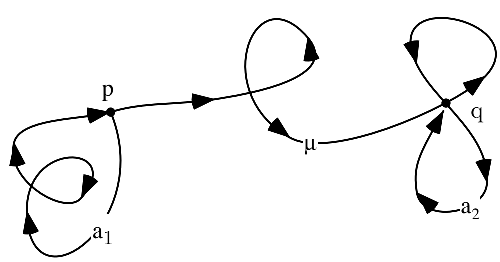

FIGURE 14. γ_1 is the concatenation of a_1 and μ, and γ_2 is the concatenation of μ and a_2. Note that a_1, a_2 have equal length, $p \neq q$, and a_1 must be a single loop at p, while a_2 can be a multiple loop at q.

Now suppose that the geodesic γ_3 also meets γ_1 in a curve, say μ_1. Since $\gamma_2 \neq \gamma_3$, $\mu_1 = \lambda^{-1}$, which denotes λ with its orientation reversed. Thus $\gamma_3 = \lambda^{-1} * b_3$ and

$$b_3^{-1} * \lambda * \mu$$

is a geodesic loop at p in which b_3^{-1} and μ have equal length, a contradiction to property (d) in Lemma 6.3. The upshot is that we can group the geodesics γ_j into ℓ pairs and s singletons,

$$\gamma_1,\ \gamma_2, \ldots, \gamma_{2\ell-1},\ \gamma_{2\ell},\ \gamma_{2\ell+1}, \ldots, \gamma_{2\ell+s}$$

such that

$$\gamma_1 = \lambda_1 * \mu_1,\ \gamma_2 = \mu_1 * b_2, \ldots, \gamma_{2\ell-1} = \lambda_{2\ell-1} * \mu_\ell,\ \gamma_{2\ell} = \mu_\ell * b_{2\ell},$$

the loops b_j are nontrivial, distinct, and all intersections between unpaired geodesics are discrete. Then a satisfactory choice of free spots consists of simple points

$$\begin{aligned} &z_1 \in \lambda_1,\ z_2 \in b_2, \ldots, z_{2\ell-1} \in \lambda_{2\ell-1},\ z_{2\ell} \in b_{2\ell}, \\ &z_{2\ell+1} \in \gamma_{2\ell+1}, \ldots, z_{2\ell+s} \in \gamma_{2\ell+s}. \end{aligned}$$

□

Proof of Theorem 6.1. Consider the open dense set $\mathcal{G}(p, L)$ constructed in Lemma 6.3. The intersection

$$\mathcal{G}(p) = \bigcap_{L \in \mathbb{N}} \mathcal{G}(p, L)$$

is residual in $\mathcal{R}$. We claim that if $k \geq 3$ and $g \in \mathcal{G}$ then

$$E_g^k : V_p^k \to (M \times \mathbb{R})^k$$

is transverse to the diagonal.

Suppose that $g \in \mathcal{G}$ and $E_g(v_1, \ldots, v_k) = (q, \ell)^k \in \Delta$. We claim that $p \neq q$. For if $p = q$ then $k \geq 3$ implies the existence of at least two geodesic loops of equal length at p that are not reverses of each other, contrary to (b) in Lemma 6.3. Lemma 6.4 then gives free spots $z_i \in \gamma_i$ and Lemma 6.2 gives perturbations of g supported in small neighborhoods of the free spots that freely move the endpoint of γ_i. Perturbation of v_i along itself changes the length of v_i. Thus E is transverse to Δ at $(q, \ell)^k$. In fact it is submersive. □

7. The Uniform Index

Given a Riemann structure g on M, given points $p, q \in M$, and given $\ell > 0$, the number of geodesics that join p to q and have length ℓ is denoted $I(g, p, q, \ell)$. (When $p = q$, we distinguish a geodesic loop from the same loop with its orientation reversed.) In the previous section we showed that for each $p \in M$ for all g in a residual set $\mathcal{G}(p) \subset \mathcal{R}$,

$$\max_{q,\ell} I(g,p,q,\ell) \leq m+1.$$

That is, there are most $m+1$ g-geodesics of equal length from p to q. Theorem 1.2 extends the estimate by asserting that for all g in a residual subset $\mathcal{G} \subset \mathcal{R}$,

$$\max_{p,q,\ell} I(g,p,q,\ell) \leq 2m+2$$

That is, there are at most $2m+2$ g-geodesics of equal length from one point to another. In this section, we sharpen this estimate slightly and show also that

$$\max_{p \neq q,\ell} I(g,p,q,\ell) \leq 2m+1.$$

That is, there are at most $2m+1$ g-geodesics of equal length joining distinct points. Throughout we assume that M is compact and has dimension m.

Recall from Section 6 that a geodesic is an unoriented geodesic curve, while a loop is an oriented geodesic curve that starts and ends at the same point.

Definition. An n-tuple of geodesics $(\gamma_1, \ldots, \gamma_n)$ **shingles** if the self intersections of each γ_i are isolated and

$$\gamma_i \setminus (\gamma_{i+1} \cup \cdots \cup \gamma_n) \neq \emptyset, \qquad 1 \leq i \leq n-1.$$

Remark. The geodesics are like shingles on a roof. No γ_i overlaps itself, and the successors of γ_i leave γ_i at least partly exposed. See Figure 15 in which we have unwrapped the a global geodesic Γ containing the shingles and marked the multiple occurrences of a basepoint p. Note that the orientations of the shingles need not be consistent.

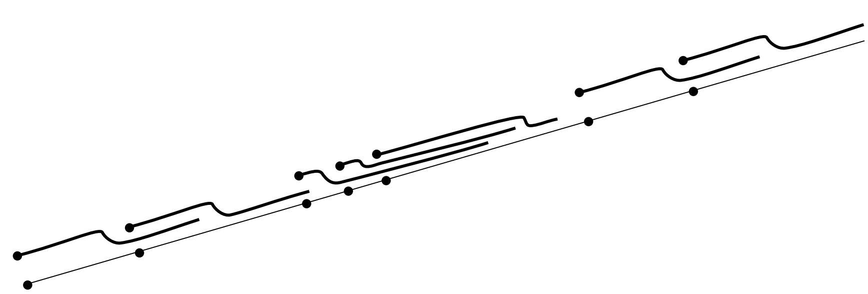

FIGURE 15. The geodesics $\gamma_1, \ldots, \gamma_7$ shingle Γ.

Lemma 7.1. *If a set S of n distinct geodesics of length ℓ lies on a global geodesic Γ, and Γ is non-closed or is closed but has minimal period $> n\ell$ then S can be arranged as an n-tuple that shingles.*

Proof. Fix an orientation on Γ. If Γ is non-closed, let γ_1 be the leftmost of the geodesics in S along Γ, let γ_2 be the second leftmost, etc.

If Γ is a closed geodesic, there is a point $z \in \Gamma$ that lies in none of the geodesics in S because the length of Γ is greater than the total length of the geodesics in S. Starting from z, let γ_1 be the first geodesic in S along Γ, let γ_2 be the second, and so on.

Because Γ does not have period $< \ell$, the geodesics do not self-overlap.

Because the geodesics all have the same length, they shingle. $\square$

Remark. Shingling takes the place of the free spots in the previous section. Lemma 7.1 gives sufficient conditions for shingling, Lemma 7.3 explains how shingling implies submersivity of a multi-exponential map, Corollary 7.5 shows how to get transversality of an augmented multi-exponential map, and these results are used in the proof of Theorem 1.2 to show that for the generic bumpy metric, closed geodesics can be eliminated from sets of sufficiently many companions. Lack of closed geodesics makes the proof of Theorem 1.2 similar to that of Theorem 6.1.

Definition. Recall that non-zero vectors $v, w \in T_pM$ are (focal) companions if they have equal length and equal exponential image. Companions are **friends** if there is a chain $v = v_1, \ldots, v_n = w$ such that the geodesic γ_{v_i} overlaps $\gamma_{v_{i+1}}$, $1 \leq i < n$. Companions whose geodesics are loops are **loop companions**.

Companionship and friendship are equivalence relations, the second refining the first. Also, two geodesics that meet in a nonempty relatively open set lie on a common global geodesic. Thus, all geodesics of vectors in a friendship class lie on a common global geodesic. Since a global geodesic can have many self intersections, including self intersections at p, the geodesics in a friendship class can present a complicated configuration. See Figure 16.

Lemma 7.1 implies that the geodesics of focal companions lying on a sufficiently long global geodesics shingle it.

To manipulate shingling geodesics we recast Anosov's perturbation result, Lemma 6.2, as an implicit function statement.

The time-one map of the geodesic flow gives a smooth mapping

$$\begin{aligned}\Phi : TM \times \mathcal{R} &\to TM \\ (v, g) &\mapsto \varphi_1^g(v).\end{aligned}$$

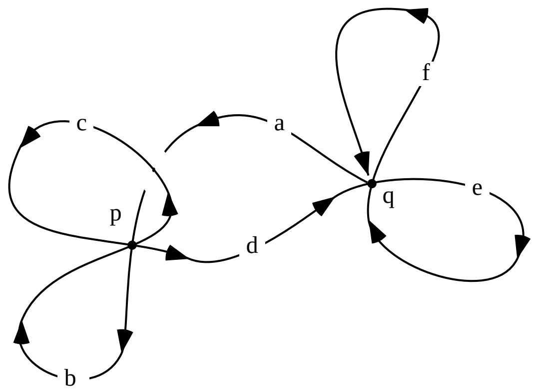

FIGURE 16. Geodesic friends $b{*}c{*}d$, $c{*}d{*}e$, $d{*}e{*}f$, $b^{-1}{*}a^{-1}$.

We want to move the footpoint q of $\Phi(v,g)$ as freely as possible.

We fix p_0, q_0, z_0, t_0, v_0, w_0, g_0, such that: $p_0, z_0, q_0 \in M$, $t_0 \in (0,1)$, $v_0 \in T_{p_0}M$, $w_0 \in T_{q_0}M$, $g_0 \in \mathcal{R}$, and

(a) The geodesic trajectory $\varphi_t^g(v_0)$, $0 \le t \le 1$, has a simple point at $t = t_0$.
(b) $z_0 = \gamma_{v_0}(t_0)$ is the control point.
(c) $w_0 = \Phi(v_0, g_0)$.
(d) $q_0 = \gamma_{v_0}(1)$ is the footpoint of w_0.

See Figure 17.

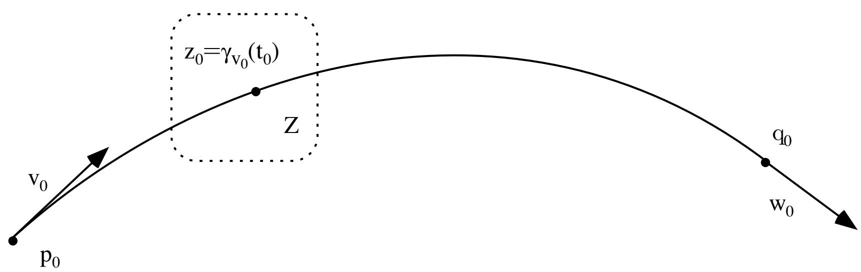

FIGURE 17. The setup of p_0, q_0, z_0, t_0, v_0, w_0.

Lemma 7.2. *Given the previous initial data and $\epsilon > 0$, there are ϵ-small neighborhoods Z, V, W, G of z_0, v_0, w_0, g_0, such that if $v \in V$, $w \in W$, $g \in G$, and*

$$|v|_g = |w|_g,$$

then there is a symmetric 2-tensor field h supported in Z which depends smoothly on v, w, g and reduces to zero when $(v,w,g) = (v_0, w_0, g_0)$ such that $g + h$ is a Riemann structure and

$$\Phi(v, g+h) = w. \tag{11}$$

Proof. Denote the sphere bundle of radius $r_0 = |v_0|$ as $TM(r_0)$. We claim that

$$\text{range}(T_2\Phi)_{v_0,g_0} = T_{w_0}(TM(r_0)). \tag{12}$$

Lemma 6.2 states that the tangent (or derivative) of Φ with respect to the Riemann structure has a range that includes the $(2m-2)$-dimensional plane perpendicular to w_0.

Let β be a bump function supported in an ϵ-neighborhood Z of z_0. The curve $s \mapsto \Phi(v_0, g_0 + s\beta g_0)$ as s passes through 0 gives the rest of the tangent space of $TM(r_0)$ at w_0. For the effect is to slide the footpoint along the geodesic at q_0. We can then choose a $(2m-1)$-dimensional plane Π_0 in the tangent space to $\mathcal{R}$ at g_0 which $T\Phi$ sends isomorphically onto $T_{w_0}(TM(r_0))$. It is tangent to variations of g_0 that are supported in Z. Then Equation 11 is a consequence of the general Implicit Function Theorem, where (v, w, g) is viewed as a parameter that enters the implicit function equation smoothly, and with respect to which the unique solution depends smoothly. □

We use Lemma 7.2 as follows.

Lemma 7.3 (Shingled Perturbation Lemma)**.** *If $v_1, \dots, v_n$ are shingled companions at p then the multi-exponential map*

$$E_p : V_p^n \times \mathcal{R} \to (M \times \mathbb{R})^n$$

is submersive at $(v_1, \dots, v_n, g)$.

Proof. We will show how to move each component of E freely (i.e., surjectively at first order) and independently due to perturbations of the vectors $v_1, \dots, v_n$ (with the base point p held fixed) and the Riemann structure g. To give the submersivity calculation some precision, we introduce a smooth coordinate system on a neighborhood Q of q, say $\psi : Q \to \mathbb{R}^m$, such that $\psi(q) = 0$.

Shingledness implies that we can choose distinct control points $z_i \in \gamma_i \setminus (\gamma_{i+1} \cup \cdots \cup \gamma_n)$, $1 \le i \le n$. Then choose small disjoint neighborhoods Z_i of z_i so that if $i < j$ then $Z_i \cap \gamma_j = \emptyset$.

Fix i, $1 \le i \le n$. By Lemma 7.2 there exists a Riemann structure perturbation h_i in Z_i that freely moves the endpoint q_i. (It does not matter whether the orientation of γ_i agrees with Γ or not.) The perturbation in Z_i has no effect on the endpoints q_j, $j > i$, since $Z_i \cap \gamma_j = \emptyset$.

However, the Riemann structure perturbation in Z_i may cause the endpoints q_k to move, $k < i$. Consider $k = i-1$. If this induced motion of the endpoint q_{i-1} is small, we can apply Lemma 7.2 to *compensate* for the motion by means of a Riemann structure perturbation in Z_{i-1}. This keeps the endpoint q_{i-1} static and does not change the motion of the endpoint q_i. Of course this compensation induces a additional motion of the endpoints q_k, $k \le i-2$. Applying the construction inductively, we see that if we make a very small amount of free motion of the endpoint q_i then it can be compensated by Riemann structure manipulations in $Z_{i-1}, \dots, Z_1$.

The net result is a smooth m-parameter family of Riemann structures $g_i(u) = g_i(u_1, \dots, u_m)$ such that

(a) $g_i(0, \dots, 0) = g$ and $g_i(u) - g$ has support in $Z_1 \cup \dots \cup Z_i$.
(b) If $|u|$ is small then $\psi \circ \exp(v_i, g_i(u)) = u$.
(c) If $i \neq j$ then $\exp(v_j, g_i(u)) = q$.

(b) and (c) can be combined in the equation

$$\psi \circ \exp(v_j, g_i(u)) = \delta_{ij} u.$$

We extend ψ to $\Psi : (M \times \mathbb{R})^n \to \mathbb{R}^{n(m+1)}$ as

$$\Psi(q_1, \ell_1, \dots, q_n, \ell_n) = (\psi(q_1), \ell_1, \dots, \psi(q_n), \ell_n).$$

Now we take a parameter space of dimension $n(m+1)$, and write its general point as

$$(u, b) = (u_{11}, \dots, u_{1m}, b_1, \dots \ \dots, u_{n1}, \dots, u_{nm}, b_n).$$

Then we define

$$\begin{aligned} g(u) &= \sum_{i=1}^{n} g_i(u_{i1}, \dots, u_{im}) \\ F(u, b) &= \Psi \circ E((1+b_1)v_1, \dots, (1+b_n)v_n, g(u)). \end{aligned}$$

Submersivity of E_p at $(v_1, \dots, v_n, g)$ is implied by surjectivity of the derivative of F with respect to (u, b) evaluated at $(u, b) = (0, 0)$. For F is a restriction of E followed by a local diffeomorphism. The i^{th} "component" of F is a vector in $\mathbb{R}^{m+1}$,

$$F_i(u, b) = (\psi \circ \exp((1+b_i)v_i, g(u)), \ \ell + b_i \ell).$$

Note that the perturbation $(1+b_i)v_i$ does not affect the base point p. It merely stretches v_i along itself. Since F_i does not depend on $u_{j1}, \dots, u_{jm}, b_j$ when $i \neq j$, the derivative of F with respect to (u, b) at $(u, b) = (0, 0)$ is block diagonal, with i^{th} block the $(m+1) \times (m+1)$ matrix

$$\begin{bmatrix} 1 & 0 & \dots & 0 & * \\ 0 & 1 & \dots & 0 & \vdots \\ \dots & \dots & \dots & \dots & * \\ 0 & \dots & \dots & 1 & * \\ 0 & \dots & \dots & 0 & \ell \end{bmatrix}.$$

The starred entries are the coordinate expression of the vector tangent to the geodesic $\exp(tv_1)$ at $t = 1$. It is clear that the block diagonal matrix with these blocks is surjective, and thus that E_p is submersive. $\square$

Picture the lemma like this – in a team of n riflemen, each aims his rifle at a target, each rifle rests on a tripod, and the i^{th} rifle's position influences the $(i-1)^{st}$, $(i-2)^{nd}$, etc. (The rifles might be connected by some mechanical linkage.) Initially, each rifle is perfectly on target. Then one (or more) target moves slightly, and perhaps each tripod moves slightly. Can the team adjust, or compensate, so that all the targets will still be hit? The answer is "yes" under the previous conditions.

Next we prove a transversality result.

Proposition 7.4. *Assume that $f : X \times Y \to Z$ is smooth, that $W \subset X \times Z$ is a smooth submanifold, and that*

(a) *The projection $X \times Z \to X$ submerses W to X.*
(b) *The partial derivative $\partial f/\partial y$ is submersive for all $(x, y) \in X$.*

Then the map $F : X \times Y \to X \times Z$ which is a partial graph of f in the sense that

$$F(x, y) = (x, f(x, y))$$

is transverse to W.

Proof. W, X, Y, Z are smooth manifolds. The partial derivative $\partial f/\partial y$ is a linear transformation from a tangent space of Y to a tangent space of Z. The range of $(DF)_{(x,y)}$ includes the result of plugging in vectors $(0, v)$ where v is tangent to Y. Since x does not depend on $y \in Y$,

$$(DF)_{(x,y)}(0, v) = (0, \frac{\partial f}{\partial y}(v)),$$

which implies that the range of DF always includes the Z-factor of the tangent space of $X \times Z$. On the other hand, a tangent space of W always includes a subspace S that projects isomorphically to a tangent space of X. Together, S and the Z-factor of $X \times Z$ span the tangent space of $X \times Z$, which implies transversality. $\square$

Corollary 7.5. *Let E be the multi-exponential map $E : V^n \times \mathcal{R} \to (M \times \mathbb{R})^n$. The map $F : V^n \times \mathcal{R} \to M \times (M \times \mathbb{R})^n$ defined by*

$$F(v, g) = (\pi(v), E(v, g))$$

is transverse to an augmented diagonal

$$\Delta^* = \{(p, p, \ell, \dots, p, \ell) : (p, \ell) \in M \times \mathbb{R}\} \subset M \times (M \times \mathbb{R})^n$$

at all shingled companions (v, g).

Proof. Transversality is a local question, so we may assume that $M = \mathbb{R}^m$ and $TM = \mathbb{R}^m \times \mathbb{R}^m$. Then vectors in TM are written as $(x, \xi) \in \mathbb{R}^m \times \mathbb{R}^m$. The map E is written

$$E(v, g) = E(x, \xi_1, \dots, \xi_n, g) = (e(x, \xi_1, g), |\xi_1|_g, \dots, e(x, \xi_n, g), |\xi_n|_g),$$

where $e(x, \xi, g)$ is the coordinate expression for the g-exponential $\exp_x(\xi)$ and $|\xi|_g$ is g-length. Then F becomes

$$F(x, \xi_1, \dots, \xi_n, g) = (x, E(x, \xi_1, \dots, \xi_n, g)).$$

Take $X = \mathbb{R}^m$, $Y = \mathbb{R}^{nm} \times \mathcal{R}$, $Z = \mathbb{R}^m \times (\mathbb{R}^m \times \mathbb{R})^n$, $W = \Delta^*$, and $f = E$. We claim that Proposition 7.4 is applicable. The submanifold Δ^* submerses to $\mathbb{R}^m$ under the projection of $\mathbb{R}^m \times (\mathbb{R}^m \times \mathbb{R})^n$ to the first $\mathbb{R}^m$-factor, X. According to Lemma 7.3 the partial of f with respect to $y = (\xi_1, \dots, \xi_n, g)$ is submersive at (v, g). Thus Proposition 7.4 applies and F is transverse to Δ^* at (v, g). □

Next we give a globalization result.

Proposition 7.6. *Let X be a σ-compact space, let Y be a Baire space, let D be a countable dense subset of Y, and let P be a property of elements $(x, y) \in X \times Y$. Suppose that each $(x_0, y_0) \in X \times D$ has a neighborhood $X_0 \times Y_0$, and Y_0 has a residual subset S_0 such that property P is true for all $(x, y) \in X_0 \times S_0$. Then there is a residual subset $S \subset Y$ such that property P is true for all $(x, y) \in X \times S$.*

Proof. Let $K \subset X$ be compact and fix $y_0 \in D$, the given dense subset of Y. The hypothsized neighborhoods X_0, Y_0 depend on (x_0, y_0). Accordingly we have an open covering

$$\{X_0(x_0, y_0) : x_0 \in K\}$$

of K. Compactness gives a finite subcover $\{X_0(x_i, y_0) : i = 1, \dots, N\}$. Set

$$X_0(K, y_0) = \bigcup_{i=1}^{N} X_0(x_i, y_0)$$

$$Y_0(K, y_0) = \bigcap_{i=1}^{N} Y_0(x_i, y_0)$$

$$S(K, y_0) = \bigcap_{i=1}^{N} S_0(x_i, y_0).$$

Then $S_0(K, y_0)$ is a residual subset of the neighborhood $Y_0(K, y_0)$ of y_0, and property P is true for all $(x, y) \in X_0(K, y_0) \times S_0(K, y_0)$. In particular, property P is true for all $(x,y) \in K \times S_0(K,y_0)$. Since D is countable dense, the union

$$S(K) = \bigcup_{y_0 \in D} S(K,y_0)$$

is residual in Y, and property P is true for all $(x,y) \in K \times S(K)$. Since X is σ-compact, it is a countable union of compact subsets K_j, and property P is true for all $(x,y) \in X \times S$ where

$$S = \bigcap_{j=1}^{\infty} S(K_j).$$

□

Proposition 7.7. *The following rational independence properties are generic for Riemann structures on M.*

(a) *Closed geodesics have rationally independent lengths.*
(b) *The lengths of the geodesic arcs into which closed geodesics are divided by their intersections with other closed geodesics or by their self intersections are rationally independent.*
(c) *If points p, q lie on a closed geodesic Γ and if three geodesics of length ℓ join p to q and lie off Γ then ℓ and the length of Γ are rationally independent.*

Remark. Rational indepndence is the same as linear independence over the integers. When M has dimension ≥ 3, (b) reduces to (a) since it is generic that closed geodesics are transverse to one another, and hence that closed geodesics are disjoint and have no self intersection.

Lemma 7.8. *Suppose that Γ is a closed geodesic of a bumpy Riemann structure g_0. There is a smooth function*

$$h : \mathbb{R} \times \mathcal{N}_0 \to M$$

such that $\mathcal{N}_0$ is a neighborhood of g_0 in $\mathcal{R}$ and $s \mapsto h(s,g)$ is an arc-length parameterization of $\Gamma(g)$, the unique closed g-geodesic near Γ. The function h is unique up to the choice of $h(0,g)$.

Proof. This follows directly from the Implicit Function Theorem and the fact that a bumpy Riemann structure has elementary closed geodesics. □

Lemma 7.9. *It is a generic property of a Riemann structure on a surface M^2 that if p, q are distinct points of a closed geodesic Γ then*

(a) *if Γ has a self intersection at p then at most two geodesics of equal length join p to q off Γ.*

(b) *if* Γ_1 *is a second closed geodesic that passes through* p *then at most two geodesics of equal length join* p *to* q *off* $\Gamma \cup \Gamma_1$.

Proof. Assume that g_0 is bumpy, Γ_0 is a closed geodesic, $p_0, q_0 \in \Gamma_0$ are distinct, and three geodesics of length ℓ join p_0 to q_0. Label them $\gamma_i(t) = \exp_{p_0}(tv_{0i})$ for $i = 1, 2, 3$ and $0 \le t \le 1$.

(a) Assume also that Γ_0 has a self intersection at p_0. Referring to Lemma 7.8, if g approximates g_0, choose the parameterization h of $\Gamma(g)$ so that $h(0, g_0) = p_0$, and $p = h(0, g)$ is the unique self intersection point of $\Gamma(g)$ near p_0. Introduce smooth coordinates at p_0, q_0, and consider the map

$$\begin{aligned} H : \mathbb{R} \times \mathbb{R}^6 \times \mathcal{R} &\to \mathbb{R}^8 \\ (s, v_1, v_2, v_3, g) &\mapsto (\exp_p(v_1) - q, \exp_p(v_2) - q, \exp_p(v_3) - q, \\ &\qquad |v_2| - |v_1|, |v_3| - |v_1|) \end{aligned}$$

where $q = h(s, g)$. The geodesics $\gamma_1, \gamma_2, \gamma_3$ do not lie on closed geodesics because p_0 is a self intersection point and g_0 is generic. Thus they shingle. By Lemma 7.3 the map H is submersive at the point $(s_0, v_{01}, v_{02}, v_{03}, g_0)$, where $q_0 = h(s_0, g_0)$. By the Abraham Transversality Theorem, there is a neighborhood of g_0 on which the map H_g is transverse to the origin in $\mathbb{R}^8$. The origin has codimension 8 while the domain space has dimension 7, so transversality implies empty intersection. That is, for the generic g near g_0 and all (s, v_1, v_2, v_3) near $(s_0, v_{01}, v_{02}, v_{03})$, the corresponding geodesics of equal length miss Γ. Proposition 7.6 makes this local fact global: for the generic $g \in \mathcal{R}$, no three geodesics of equal length connect a self intersection point of a closed geodesic Γ to another point of Γ.

(b) If Γ_1 is a second closed geodesic through p_0, then we repeat the calculation where $p = h(0, g)$ is the unique point near p_0 at which the closed geodesics $\Gamma(g)$, $\Gamma_1(g)$ intersect. □

Proof of Proposition 7.7. (a) Consider two closed geodesics Γ_1, Γ_2 of the bumpy Riemann structure g_0. They meet at worst in isolated points, so there exists a control point $z \in \Gamma_1 \setminus \Gamma_2$. Changing the Riemann structure in a neighborhood of z changes the length of Γ_1 and leaves Γ_2 alone. For any constant k this gives an open-dense subset of a neighborhood of g_0 whose elements satisfy the condition: for all integers $k_1, k_2 \in [-k, k]$,

$$k_1 \operatorname{length} \Gamma_1 + k_2 \operatorname{length} \Gamma_2 \neq 0.$$

Proposition 7.6 completes the proof of (a).

(b) The proof is similar to that of (a), except now we independently alter the lengths of the arcs into which a closed geodesic is divided by its intersection with another closed geodesic or with itself.

(c) Let Γ_0 be a closed geodesic of a generic bumpy Riemann structure g_0, and let $\gamma_1, \gamma_2, \gamma_3$ be three geodesics of length ℓ_0 that lie off Γ_0 and join points $p_0, q_0 \in \Gamma_0$.

Case 1. $p_0 \neq q_0$. Referring to Lemma 7.8, for g near g_0 let $h(s, g)$ parameterize the nearby closed geodesic $\Gamma(g)$. Set $p = h(t, g)$ and $q = h(s, g)$ where $p_0 = h(t_0, g_0)$, $q_0 = h(s_0, g_0)$, and $t - t_0$, $s - s_0$ are small. Fix integers k_1, k_2. Introduce local coordinates at p_0 and q_0, and consider the map

$$\begin{aligned} H : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{3m} \times \mathcal{R} \to& \mathbb{R}^{3m} \times \mathbb{R}^3 \\ (t, s, v_1, v_2, v_3, g) \mapsto& (\exp_p(v_1) - q, \exp_p(v_2) - q, \exp_p(v_3) - q, \\ & |v_2| - |v_1|, |v_3| - |v_1|, k_1|v_1| + k_2 \operatorname{length} \Gamma(g)). \end{aligned}$$

Since $p_0 \neq q_0$, Lemma 7.9 implies that the geodesics γ_i do not lie on a closed geodesic. By Lemma 7.1 they shingle. By Lemma 7.3, H is submersive at $(t_0, s_0, v_1, v_2, v_3, g_0)$. By the Abrham Transversality Theorem, for the generic g near g_0, H_g is transverse to the origin in $\mathbb{R}^{3m+3}$. Since the domain dimension is $3m + 2 < 3m + 3$, transversality implies empty intersection, so it is locally generic that $k_1\ell + k_2 \operatorname{length} \Gamma \neq 0$. Proposition 7.6 converts this local genericity to global genericity.

Case 2. $p_0 = q_0$. By Lemma 7.9, Γ_0 does not self intersect at p_0, nor is there a second closed geodesic through p_0. The geodesics $\gamma_1, \gamma_2, \gamma_3$ shingle. Consider the map as in Case 1, except now $t = s$. That is,

$$\begin{aligned} H : \mathbb{R} \times \mathbb{R}^{3m} \times \mathcal{R} \to& \mathbb{R}^{3m} \times \mathbb{R}^3 \\ (t, v_1, v_2, v_3, g) \mapsto& (\exp_p(v_1) - p, \exp_p(v_2) - p, \exp_p(v_3) - p, \\ & |v_2| - |v_1|, |v_3| - |v_1|, k_1|v_1| + k_2 \operatorname{length} \Gamma(g)). \end{aligned}$$

By Lemma 7.3, H is submersive at $(t_0, v_1, v_2, v_3, g_0)$. By the Abraham Transversality Theorem, for the generic g near g_0, H_g is transverse to the origin in $\mathbb{R}^{2m+3}$. Since the domain dimension is less than the codimension of the origin in $\mathbb{R}^{2m+3}$, transversality means empty intersection. Thus, it is locally generic that $k_1\ell + k_2 \operatorname{length} \Gamma \neq 0$. Again Proposition 7.6 promotes the local genericity to global genericity. □

Lemma 7.10. *It is a generic property of a Riemann structure on M^m that if p, q are distinct points of a closed geodesic Γ then at most three geodesics of equal length join p to q off Γ.*

Proof. Case 1. $m = 2$. Suppose there are four geodesics $\gamma_1, \dots, \gamma_4$ of length ℓ that join p to q and lie off Γ. Since it is a generic property that closed geodesics on a surface have at worst isolated double points, and isolated pairwise intersections, there is at most one of the γ_i that lies on a closed geodesic, say it is γ_1. (If Γ self intersects at p, γ_1 could equal Γ. If Γ has a simple point at p, γ_1 would lie on a distinct closed geodesic Γ_1. Note that γ_1 may overlap itself many times. For ℓ can be much greater than the length of Γ_1. This would make γ_1 problematic in terms of shingling.) The other three geodesics $\gamma_2, \gamma_3, \gamma_4$ join p to q off $\Gamma \cup \gamma_1$, and by Lemma 7.9, this is one too many.

Case 2. $m \geq 3$. We repeat the calculation in the proof of Lemma 7.9, but this time both the points p, q vary and there are four geodesics. We have a map

$$\begin{aligned} H : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{4m} \times \mathcal{R} \to& \mathbb{R}^{4m+3} \\ (t, s, v_1, \dots, v_4, g) \mapsto& (\exp_p(v_1) - q, \dots, \exp_p(v_4) - q, \\ & |v_2| - |v_1|, |v_3| - |v_1|, |v_4| - |v_1|) \end{aligned}$$

where $p = h(t, g)$ and $q = h(t + s, g)$. Again we have submersivity and transversality to the origin. This time the dimensions are

$$\text{dod} = 2 + 4m \qquad \text{cod} = 4m + 3.$$

Again we get empty intersection generically. $\square$

Proof of Theorem 1.2. We assume that the Riemann structure is generic in the sense that it obeys the conclusions of Proposition 7.7, Lemma 7.9, and Lemma 7.10. We assert that

$$\begin{aligned} \max_{p \neq q, \ell} I(g, p, q, \ell) &\leq 2m + 1 \\ \max_{p, q, \ell} I(g, p, q, \ell) &\leq 2m + 2. \end{aligned}$$

Case 1. $p \neq q$ and $m \geq 3$. We start with $2m + 2$ geodesics on M^m that join p and q and derive a contradiction to genericity of the Riemann structure. Because $m \geq 3$, at most two are tangent to a closed geodesic at p. This leaves $2m$. Since $2m > 3$, this contradicts Lemma 7.10. Thus, for the generic bumpy Riemann structure, none of our $2m + 2$ geodesics is tangent to a closed geodesic. Therefore they shingle and the multiexponential map

$$E : V^{2m+2} \times \mathcal{R} \to (M \times \mathbb{R})^{2m+2}$$

is transverse to the diagonal. The Abraham Transversality Theorem gives a residual subset such that E_g is transverse to the diagonal. The dimensions are

$$\mathrm{dod} = m + m(2m+2) = 2m^2 + 3m$$
$$\mathrm{cod} = (2m+2)(m+1) - (m+1) = 2m^2 + 3m + 1$$

so transversality implies empty intersection.

Case 2. $p \neq q$ and $m = 2$. We start with 6 geodesics of length ℓ that join p to q, and derive a contradiction to genericity of the Riemann structure.

At most four of the geodesics are tangent to closed geodesics at p, because closed geodesics only intersect pairwise or have double self intersections.

If none of the six geodesics is tangent to a closed geodesic at p, the proof is the same as when $m \geq 3$.

If only one or two of the six geodesics are tangent to a closed geodesic Γ at p, this leaves at least four which have equal length and join distinct points of Γ. By Lemma 7.10, this is one too many and the Riemann structure is not generic.

If three or four of the six geodesics are tangent to a closed geodesic Γ at p then Γ has a self intersection at p, and since $p \neq q$, p and q are antipodal along Γ. Moreover, q is another double point of Γ. Since the geodesics all have the same length ℓ, this contradicts Proposition 7.7, and the Riemann structure is not generic.

Finally if two of the six geodesics are tangent to a closed geodesic Γ which is simple at p, and if at least one of the remaining four geodesics is tangent to a second closed geodesic Γ_1 at p, then p is antipodal to q along Γ, and Proposition 7.7 is contradicted.

Case 3. $p = q$. We start with $2m + 3$ (oriented) geodesic loops at p and derive a contradiction to genericity of the Riemann structure. At least $m + 2$ of the loops are distinct as point sets. Suppose that one of them lies on a closed geodesic Γ. Then ℓ is an integer multiple of the length of Γ. This leaves $m + 1 \geq 3$ loops at p of length ℓ at p. Proposition 7.7 states that for the generic g, ℓ and the length of Γ are rationally independent, contrary to the fact that ℓ is an integer multiple of the length of Γ. Thus it is ungeneric that any of our $m+2$ physically distinct loops lie on a closed geodesic.

All the $m + 2$ loops at p lie on non-closed geodesics. They shingle. By Corollary 7.5 the multiexponential map

$$F : (v, g) \mapsto (\pi(v), E(v, g))$$

is transverse to the augmented diagonal. The Abraham Transversality Theorem gives a locally residual set of bumpy Riemann structures g, such that F_g is transverse to the augmented diagonal. The dimensions are

$$\mathrm{dod} = m + m(m+2) = m^2 + 3m$$
$$\mathrm{cod} = m + (m+1)(m+2) - (m+1) = m^2 + 3m + 1$$

Thus, transversality implies empty intersection, and it is generic that there are at most $m+1$ unoriented geodesic loops of equal length, i.e., there are at most $2m+2$ oriented geodesic loops at any point. □

Remark. The estimate of 6 for the maximum number of companions for the generic Riemann structure on a surface is sharp, which can be seen as follows. Let S be a surface formed by smoothing three thin vertical cylinders pasted at the vertices of an equilateral triangle T in the xy-plane. See Figure 18. Encircle the cylinders with three geodesic

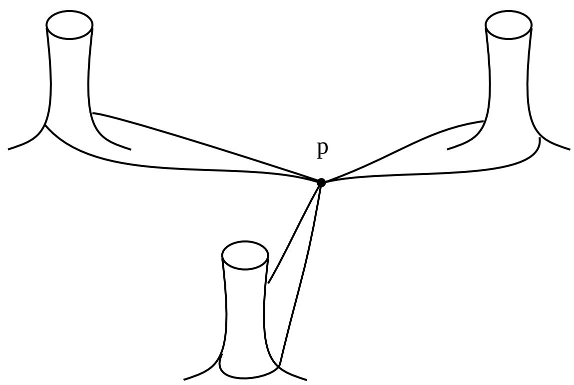

Figure 18. The surface S.

loops emanating from the center p of T. This gives six geodesic loops of equal length at p because we distinguish a loop from its reverse. It is easy to check that for each perturbation of g there is a new point p' near p and six new geodesic loops of equal length at p'.

Remark. If g has negative sectional curvature the local-to-global proof of Theorem 1.2 given above can be made more immediately global by using the following observations.

(a) Geodesics joining points p, q are unique in their homotopy class.
(b) The geodesics depend smoothly on p, q.
(c) The geodesics lift to the universal covering space $\overline{M}$ of M, which is homeomorphic to $\mathbb{R}^m$, and the lifted geodesics have no self intersections.

Remark. We believe our generic index estimates are sharp in all dimensions, but it seems probable that better estimates hold if we make extra assumptions on the Riemann structure. For example, as we showed in Section 5, it is generic that a *flat* Riemann structure on the 2-torus has uniform index 3, whereas Theorem 1.2 predicts only that the index is ≤ 6.

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Department de Mathematiques, Université de Paris, Paris, France
E-mail address: kupka@math.jussieu.fr

IMPA, Estrada Dona Castorina, 110, Rio de Janeiro, Brazil
E-mail address: peixoto@impa.br

Mathematics Department, University of California, Berkeley California, 94720, U.S.A.
E-mail address: pugh@math.berkeley.edu