Immersions with fractal set of points of zero Gauss-Kronecker curvature

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Abstract

We construct, for any "good" Cantor set F of S^{n-1}, an immersion of the sphere S^n with set of points of zero Gauss-Kronecker curvature equal to $F \times D^1$, where D^1 is the 1-dimensional disk. In particular these examples show that the theorem of Matheus-Oliveira strictly extends two results by do Carmo-Elbert and Barbosa-Fukuoka-Mercuri.

1 Introduction

Let $x : M^n \to \mathbb{R}^{n+1}$ be a codimension one Euclidean immersion of an orientable manifold M. Let $N : M^n \to S^n$ be the associated Gauss map in the given orientation. Recall that $A = dN$ is self-adjoint and its eigenvalues are the principal curvatures. We denote $H_n = det(dN)$ the Gauss-Kronecker curvature and $rank(x) := rank(N) := \min_{p \in M} rank(d_p N)$.

A compact set $F \subset S^n$ is called a *good* Cantor set if $S^n - F = \bigcup_{i \in \mathbb{N}} U_i$, is the disjoint union of open balls U_i in S^n (with the standard metric) of radius bounded by a small constant $\delta_0 = \delta_0(n)$ (to be choosen later).

Our main result is :

Theorem A. *For any $F \subset S^{n-1}$ a good Cantor set, there are immersions $x : S^n \to \mathbb{R}^{n+1}$ such that $rank(x) = n-1$, the Gauss-Kronecker curvature is non-negative and $\{p \in S^n : H_n(p) = 0\} = F \times D^1$, where D^1 is the 1-dimensional disk.*

Before starting some comments, we briefly recall the definition :

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Definition 1.1. A complete orientable hypersurface M has finite geometrical type (in [BFM] sense) if M is equal to a compact manifold $\overline{M}$ minus a finite number of points (called *ends*), the Gauss map $N : M \to S^n$ extends continuously to each "end" and the set of points of zero Gauss-Kronecker curvature is contained in a finite union of submanifolds of dimension $\leq n-2$; M has finite total curvature if $\int_M |A|^n < \infty$, where $|A| = (\sum_{i=1}^{n} k_i^2)^{1/2}$, k_i are the principal curvatures. We remark that, as showed in [dCE], if M has finite total curvature then M has finite geometrical type.

The motivation of our theorem A are the following results. In a recent work, do Carmo and Elbert [dCE] show that :

Theorem 1.2 (do Carmo, Elbert). *If M is a hypersurface of finite total curvature with Gauss-Kronecker curvature $H_n \neq 0$ everywhere then M is topologically a sphere minus a finite number of points.*

In fact, this result can be improved, as showed in [BFM] :

Theorem 1.3 (Barbosa, Fukuoka, Mercuri). *If M is a $2n$ dimensional hypersurface of finite geometrical type such that $\{p \in M : H_n(p) = 0\}$ is a subset of a finite union of submanifolds with dimension less than $n-1$ then the hypersurface is a sphere minus* two *points. If M is minimal, M is the $2n$-catenoid.*

In general, it is not easy to obtain the hypothesis of the result by Barbosa, Fukuoka and Mercuri for arbitrary immersions, since the classical theorems (Sard, Moreira [M]) treat only the critical values (in our case, the spherical image of points of zero curvature). With this difficult in mind, Matheus and Oliveira [MO], using the concept of Hausdorff dimension, generalize the previous results :

Theorem 1.4 (Matheus, Oliveira). *If $x : M \to \mathbb{R}^{n+1}$ is an immersion of finite geometrical type such that $rank(x) \geq k$ and the $(k - [\frac{n}{2}])$- dimensional Hausdorff measure of $N(\{p \in M : H_n(p) = 0\})$ is zero then M is topologically a sphere minus a finite number of points ($[r]$ = integer part of r). If M is minimal, M is the catenoid.*

However, it is not obvious the existence of immersions satisfying Matheus-Oliveira's hypothesis which does not satisfies the assumptions of Barbosa-Fukuoka-Mercuri. The question about existence of "fractal immersions" was posed to second author by Walcy Santos during the Differential Geometry seminar at IMPA. The main goal of this paper is to show how one can construct immersions with "large" fractal set of points of zero curvature. Clearly, the existence of such immersions follows from our theorem A. The outline of proof of theorem A is :

- First, we construct immersions φ of S^n with rank equal to k, $\{H_n = 0\} := \{p : H_n(p) = 0\} = S^k \times D^{n-k}$ (D^{n-k} is a $(n-k)$-dimensional disk and S^k is a round sphere) and $N(\{H_n = 0\}) = S^k$;

- Second, if U is a ball in S^{n-1}, we produce a modification $\varphi|_U$ of φ such that $\varphi_U = \varphi$ in $(S^{n-1} - U) \times D^1$ and the Gauss-Kronecker curvature of $\varphi_U(U \times D^1)$ does not vanishes;

- Finally, we consider $F = S^{n-1} - \bigcup_{i=1}^{\infty} U_i$ and we produce the desired immersion by induction, using the two steps above.

In next three sections, we are going to make precise the steps described above. In other words, we describe explicitly the immersions with the properties commented in the previous outline.

To finish this introduction, we observe that in the special case of "good" Cantor sets, even if it has positive measure, the manifold can be the sphere (this occurs because the "singular" set has "good" geometry). In particular, the theorems of Matheus-Oliveira are not sharp. Moreover, the proof of our thereom A shows that the round balls can be replaced by sets with a well-defined "distance function". However, the proof only works in codimension 1 (i.e., for good Cantor sets $F \subset S^{n-1}$), by technical reasons. This is showed in last section of this paper.

2 Some immersions with cylindrical pieces

The main result of this section is the following lemma :

Lemma 2.1. *There are immersions $\varphi : S^n \to \mathbb{R}^{n+1}$ such that rank$(\varphi) = k$ and $\{p \in S^n : H_n(p) = 0\} = S^k \times D^{n-k}$, where S^k is a round sphere in $\mathbb{R}^{k+1}$ with radius $0 < \sqrt{\gamma} < 1$, D^{n-k} is the $(n-k)$-dimensional disk of radius $\sqrt{\alpha}$, $\alpha < \frac{1}{2}$ and $\alpha < \gamma < 1 - \alpha$.*

The ideia of the proof of this lemma is flatten the boundary of a hemisphere such that the curvature is positive everywhere except at the boundary. Now take a cylinder and glue isometrically the boundaries of the cylinder and the hemisphere (see figure 1 after the proof of lemma 2.1). Gromov [G] uses this ideia (in other context) to construct some examples of manifolds of nonpositive curvature with special properties. Because the authors does not know any reference in literature where these examples are constructed in details (the known references deals only with higher codimension surgeries [GL]), we present the proof of the lemma 2.1.

Proof of lemma 2.1. Fix some $\alpha < \beta < \gamma$. We write $\mathbb{R}^{n+1} \ni p = (x, y)$, where $x \in \mathbb{R}^{k+1}$ and $y \in \mathbb{R}^{n-k}$. We denote $||.||$ the Euclidean metric. Consider some real function $\nu \in C^\infty$ s.t. $\nu(r) \equiv \gamma$ if $0 \le r \le \alpha$, $\nu(r) \equiv 1$ if $\beta \le r \le 1$ and ν is strictly increasing in $[\alpha, \beta]$. The immersion $\varphi : S^n \to \mathbb{R}^{n+1}$ is $\varphi(x, y) = (\theta(||y||^2) \cdot x, y) = (z, w)$, where $\theta(r) = \sqrt{\frac{\nu(r) - r \cdot \mu(r)}{1-r}}$, μ is a convenient real function (it will be defined later). We take μ such that $\mu(r) \equiv 0$ if $0 \le r \le \alpha$ and $\mu(r) \equiv 1$ if $\beta \le r \le 1$. These implies that $\varphi(x, y) = (x, y)$ if $||y||^2 \ge \beta$, $\varphi(x, y) = (\frac{\sqrt{\gamma}}{\sqrt{1-||y||^2}} \cdot x, y)$ if $||y||^2 \le \alpha$. In other words, φ has a spherical piece and a cylindrical piece. Now, it is sufficient to define φ (i.e., μ) in such way that the Gauss-Kronecker curvature is positive except at the cylindrical piece. By definition, using $y = w$, $(x, y) \in S^n \Rightarrow ||x||^2 + ||y||^2 = 1$, we have $||z||^2 + \mu(||w||^2) \cdot ||w||^2 = \nu(||w||^2)$. I.e., if

$$f(z, w) = ||z||^2 + \mu(||w||^2) \cdot ||w||^2 - \nu(||w||^2),$$

then $f|_{\varphi(S^n)} \equiv 0$. Thus, the Gauss map is $N(z, w) = \frac{\text{grad } f}{||\text{grad } f||}$. But $\frac{\partial f}{\partial z_i} = 2 \cdot z_i$, $\frac{\partial f}{\partial w_j} = 2 \cdot w_j \cdot \{\mu + ||w||^2 \cdot \mu' - \nu'\}$. To simplify, we denote $c_1(r) = \mu + r \cdot \mu' - \nu'$. So, $||\text{grad } f|| = 2 \cdot \sqrt{\nu + ||w||^2 \cdot (c_1^2 - \mu)}$. We denote $c_2(r) = \sqrt{\nu + r \cdot (c_1^2 - \mu)}$. Then,

$$N(z, w) = \frac{1}{c_2(||w||^2)} \cdot (z,\ c_1(||w||^2) \cdot w).$$

Clearly, $\frac{\partial N_l}{\partial z_i} = 0$ if $l \neq i$, $\frac{\partial N_l}{\partial z_i} = \frac{1}{c_2}$ if $l = i$. Analogously, $\frac{\partial N_l}{\partial w_j} = 2 \cdot (\frac{1}{c_2})' \cdot z_l \cdot w_j$ if $l \le k+1$ and, if $l \ge k+2$,

$$\frac{\partial N_l}{\partial w_j} = \begin{cases} 2 \cdot (\frac{c_1}{c_2})' \cdot w_j \cdot w_l & \text{if } l \neq j \\ 2 \cdot (\frac{c_1}{c_2})' \cdot w_j \cdot w_l + \frac{c_1}{c_2} & \text{if } l = j \end{cases}$$

These computations implies that :

$$\mathbf{dN} = \begin{pmatrix} \frac{1}{c_2} \cdot I_{k+1} & \bigstar \\ 0 & A + \frac{c_1}{c_2} \cdot I_{n-k} \end{pmatrix},$$

where I_m is the $(m \times m)$-identity matrix and $A = 2 \cdot (\frac{c_1}{c_2})' \cdot [w_j \cdot w_l]_{jl}$ (w_j is the j-th component of w). We observe that A has eigenvalues 0 with multiplicity $(n - k - 1)$ and $2 \cdot (\frac{c_1}{c_2})' \cdot ||w||^2$ with multiplicity 1. In particular, it is easy that the principal curvatures of φ are : $\frac{1}{c_2}$ with multiplicity k, $\frac{c_1}{c_2}$ with multiplicity $n - k - 1$ and $2 \cdot (\frac{c_1}{c_2})' \cdot ||w||^2 + \frac{c_1}{c_2}$ with multiplicity 1 (since $\det(dN - \lambda \cdot Id) = \det(\frac{1}{c_2} \cdot Id) \cdot \det(A + \frac{c_1}{c_2} \cdot Id - \lambda \cdot Id)$). We need $H_n > 0$ if $||y||^2 = ||w||^2 > \alpha$, $H_n = 0$ if $||w||^2 \le \alpha$. To solve this, we consider $\psi \in C^\infty$, $\psi|_{[0,\alpha]} \equiv 0$ and $\psi|_{[\beta,1]} \equiv 1$. The admissible μ satisfies $\frac{c_1}{c_2} = \rho$, $2r\rho' + \rho = \psi$, where $\rho|_{[0,\alpha]} \equiv 0$, $\rho|_{[\beta,1]} \equiv 1$ also. As we will see later, this property is sufficient to conclude the proof. In order to make the following rigorous, fix any $0 < \widetilde{\alpha} < \alpha$ and $\beta < \widetilde{\beta} < 1$. The second ODE

$$(*)\ 2r\rho' + \rho = \psi$$

can be explicitly solved : $(*) \iff \rho' + \frac{\rho}{2r} = \frac{\psi}{2r} \iff \left(e^{\int_\alpha^r \frac{1}{2s}\, ds} \cdot \rho\right)' = \frac{\psi}{2r}$.

Integrating (and using $\rho(\alpha) = 0$), we get $(*) \Leftrightarrow \rho(r) = \frac{1}{\sqrt{2r}} \cdot \int\limits_\alpha^r \frac{\psi}{\sqrt{2s}}\, ds$, for $r \in [\widetilde{\alpha}, \widetilde{\beta}]$. Now, we solve the (implicit) ODE : $(**)\ \frac{c_1}{c_2} = \rho$. Observe that $(**) \Leftrightarrow c_1^2 = \rho^2\nu + c_1^2 r\rho - \mu r\rho^2 \Leftrightarrow \frac{c_1^2}{\nu - r\mu} = \frac{\rho^2}{1 - r\rho^2} := \Delta^2$. Integrating and making the change of variables $u = \nu - r\mu$, we obtain, by definition of c_1, $-\int_\gamma^{\nu - r\mu} \frac{du}{\sqrt{u}} = \int_\alpha^r \Delta(s)\, ds \Leftrightarrow 2\sqrt{\gamma} - 2\sqrt{\nu - r\mu} = \int_\alpha^r \Delta$. I.e.,

$$\mu = \frac{1}{r}\Big[\nu - \big(\sqrt{\gamma} - \frac{1}{2}\int_\alpha^r \Delta\big)^2\Big]$$

(in the interval $[\widetilde{\alpha}, \widetilde{\beta}]$, where $\frac{1}{r}$ makes sense). Because ψ satisfies $\psi|_{[0,\alpha]} \equiv 0$, $\mu|_{[0,\alpha]} \equiv 0$ holds. It remains only $\mu|_{[\beta,1]} \equiv 1$, for some ψ carefully choosen. However, $\mu = 1 \iff (***)\ r = \nu - \left(\sqrt{\gamma} - \frac{1}{2}\int_\alpha^r \Delta\right)^2$, for all $r \geq \beta$. By definition, $\Delta = \frac{1}{\sqrt{1-r}}$, for $r \geq \beta$. In particular, if $r \geq \beta$, $\int_\alpha^r \Delta = \int_\alpha^\beta \Delta + \int_\beta^r \Delta = \int_\alpha^\beta \Delta + 2\sqrt{1-\beta} - 2\sqrt{1-r}$. Then $(***) \iff \frac{1}{2}\int_\alpha^\beta \Delta = \sqrt{\gamma} - \sqrt{1-\beta}$.

Fix $\epsilon > 0$ small and consider ψ_1 s.t. $\psi_1|_{[\alpha+\epsilon,\beta]} \geq 1 - \epsilon$, ψ_1 strictly increasing in $[\alpha, \beta]$ and $\psi_1|_{[0,\alpha]} \equiv 0$, $\psi_1|_{[\beta,1]} \equiv 1$. For a sufficiently small $\epsilon > 0$, if ρ_1 and Δ_1 denotes the functions associated to ψ_1, then $\rho_1|_{[\alpha+\epsilon,\beta]} \geq 1 - 2\epsilon$. In particular, $\Delta_1 \geq (1 - 3\epsilon)\frac{1}{\sqrt{1-r}} \Rightarrow \frac{1}{2}\int_\alpha^\beta \Delta_1 \geq (1 - 4\epsilon) \cdot \{\sqrt{1-\alpha} - \sqrt{1-\beta}\}$. By hypothesis, $\gamma < 1 - \alpha$ that imply $\frac{1}{2}\int_\alpha^\beta \Delta_1 \geq \sqrt{\gamma} - \sqrt{1-\beta}$. With a similar argument, we can take ψ_0 s.t. $\psi_0|_{[0,\alpha]} \equiv 0$, $\psi_0|_{[\beta,1]} \equiv 1$ and $\psi_0|_{[\alpha,\beta-\epsilon]} \leq \epsilon$. If $\epsilon > 0$ is small, we get $\frac{1}{2}\int_\alpha^\beta \Delta_0 \leq \sqrt{\gamma} - \sqrt{1-\beta}$, where Δ_0 is the associated function to ψ_0. Now, consider the linear combination $\psi_t = (1-t)\psi_0 + t\psi_1$. An easy verification is that ψ_t has the desired values in the intervals $[0, \alpha]$ and $[\beta, 1]$, for any $t \in [0, 1]$. By a continuity argument, it is not difficult that there is some t_0 such that $\frac{1}{2}\int_\alpha^\beta \Delta_{t_0} = \sqrt{\gamma} - \sqrt{1-\beta}$. Finally, we consider $\psi = \psi_{t_0}$. The Gauss-Kronecker curvature has the properties of the lemma for the previous ψ. To see this, we proceed as below :

By the definitions, $rank(\varphi) = k$, since H_n is positive everywhere except at the cylindrical piece $S^k \times D^{n-k}$ and $S^k \times D^{n-k}$ has exactly k positive principal curvatures. Moreover, $\psi > 0$ if $r > \alpha$ implies that $\{H_n = 0\} = S^k \times D^{n-k}$. At this point, the informations about the radii of S^k and D^{n-k} are clear. This concludes the proof. q.e.d.

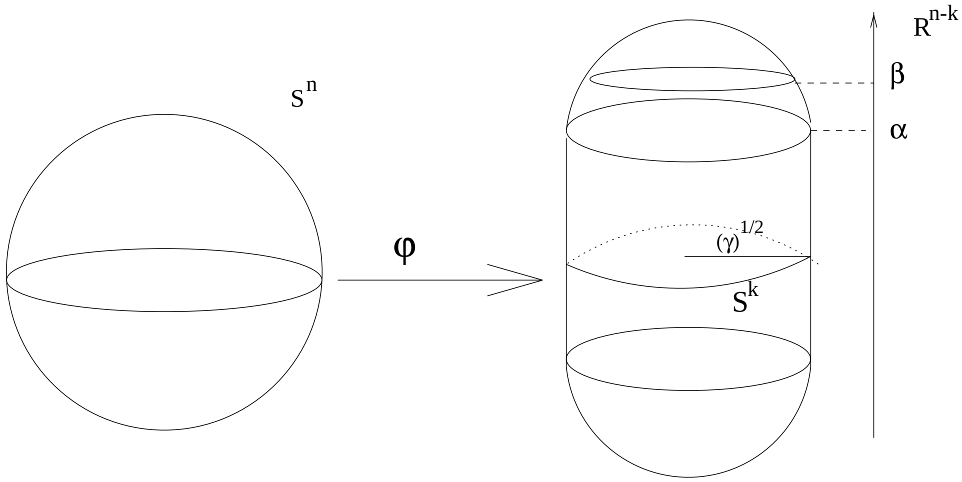

Figure 1: Immersion with large region of zero curvature

The proof of lemma 2.1 is illustrated in figure 1. The conditions on the numbers α, β, γ are necessary. In other case, our immersions creates some undesired regions of negative Gauss-Kronecker. The role of these numbers in the construction of φ is explained in figure 1.

In next section, we show how one can perturbate φ to destroy regions of zero curvature. With this is easy to obtain fractal immersions, by sucessively destroying "large" regions of zero curvature in such way that the set of zero curvature is fractal at the end of this process.

3 Removing zero Gauss-Kronecker curvature

Before giving the statement and the proof of the main result of this paragraph, we briefly describe the ideia of the lemma. Take the immersion φ and an open set U of S^{n-1}. We want to perturbe φ such that the region correspondent to $U \times D^1$ has positive curvature (i.e., we want to "inflate" the set), and this perturbation should glue smoothly with the other region $(S^{n-1} - U) \times D^1$ of the cylinder $S^{n-1} \times D^1$. This can be done because the region of positive curvature $\{(z, w) : |w| > \alpha\}$ permits the change of curvature in the "goal" region. Moreover, this method does not have points of negative curvature since the curvature of S^{n-1} is positive. The ideia is more clear contained in figure 2 below.

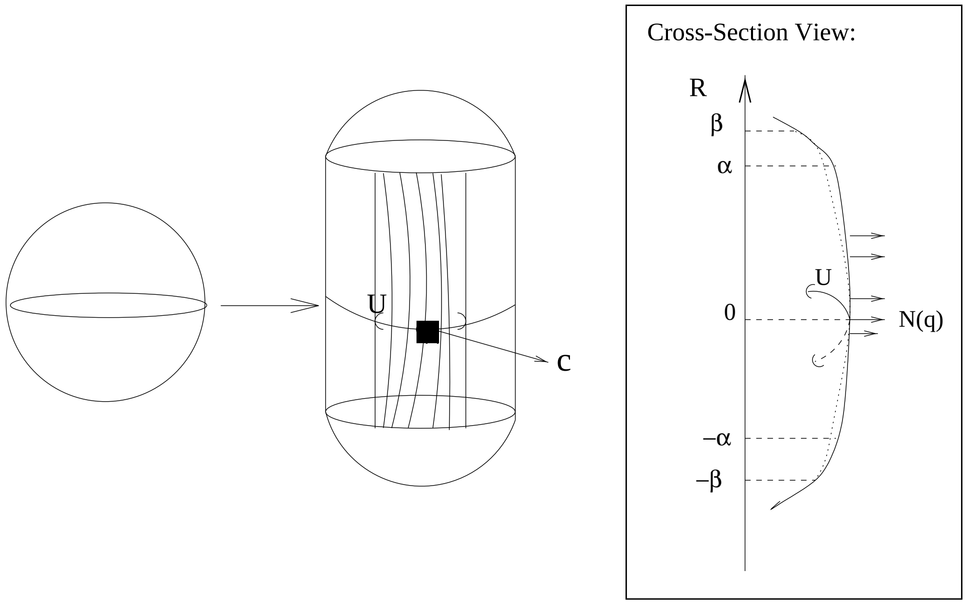

Figure 2: Removing regions with zero curvature

With this figure in mind, the main lemma of this section is :

Lemma 3.1. *If U is an open ball of S^{n-1} with radius bounded by $\delta_0 = \delta_0(n-1)$, and $\varphi : S^n \to \mathbb{R}^{n+1}$ is the immersion constructed above, then there exists a perturbation φ_U of φ such that $\varphi_U = \varphi$ in $(S^{n-1} - U) \times D^1 \bigcup \{(z, w) : |w| \geq \beta\}$ and the Gauss-Kronecker curvature of $\varphi_U(U \times D^1)$ is positive.*

Proof of lemma 3.1. Let $U \subset S^{n-1}$ be a round ball of radius δ and center c. Denote by $d(z) = \frac{1}{2}\mathrm{dist}(z, c)^2$, where dist is the distance function in S^{n-1}. We will consider a perturbation of φ by normal variation, i.e., $F_t(q) = q + t \cdot f(q) \cdot N(q)$, for $q \in \varphi(S^n)$ (here $N(q)$ denotes the normal vector to $M := \varphi(S^n)$ at q). To prove the result, we need to show that a suitable $f : M \to \mathbb{R}$, $t > 0$ have the desired properties. In order to verify these properties, we first calculate the normal vector $N_t(q_t)$ to $M_t := F_t(M)$ at $q_t := F_t(q)$. By definition,

$$dF_t(q)v = v + t.f(q).dN(q) \cdot v + t.(df(q).v) \cdot N(q).$$

In particular,

$$\begin{aligned} 0 &= <N_t(q_t), dF_t(q).v> \\ &= <N_t(q_t), v> +t.(df(q).v).\ <N_t(q_t), N(q)> + \\ &\quad +t.f(q).<N_t(q_t), dN(q).v> \end{aligned}$$

$\forall\ v \in T_qM$.

So, if $\delta_0 = \delta_0(n-1)$ is small, we can define on U a global ortonormal frame $\{e_i(q)\}_{i=1}^n$ which diagonalizes $dN(q)$. If $k_i(q)$ are the respectives principal curvatures then the previous equation implies:

$$< N_t(q_t), e_i(q) >= -\frac{t \cdot df(q) \cdot e_i(q)}{1+t.f(q).k_i(q)} \cdot < N_t(q_t), N(q) > .$$

Moreover, $||N_t(q_t)||^2 = 1$ says that:

$$< N_t(q_t), N(q) >= \left[\sqrt{1 + t^2 \sum_{i=1}^{n} \frac{(df(q)e_i(q))^2}{(1+tf(q)k_i(q))^2}} \right]^{-1/2} := \zeta_t(q).$$

In particular, we can write:

$$N_t(q_t) = \zeta_t(q). \left[\sum_{i=1}^{n} -\frac{tdf(q)e_i(q)}{1+tf(q)k_i(q)} \cdot e_i(q) + N(q) \right].$$

Differentiating the last expression:

$$\begin{aligned} dN_t(q_t) \cdot v &= d\zeta_t(q) \cdot v \Big[\sum_{i=1}^{n} -\frac{tdf(q)e_i(q)}{1+tf(q)k_i(q)} \cdot e_i(q) + N(q) \Big] + \\ &\quad +\zeta_t(q) \cdot \Big[\sum_{i=1}^{n} -\frac{tdf(q)e_i(q)}{1+tf(q)k_i(q)} \cdot de_i(q) \cdot v + dN(q) \cdot v \Big] + \\ &\quad +\zeta_t(q) \cdot \left[\sum_{i=1}^{n} -t \cdot \frac{e_i(q)}{(1+tf(q)k_i(q))} \cdot \big[< d^2 f(q) \cdot v, e_i(q) > \right. \\ &\quad \left. + < \mathrm{grad} f(q), de_i(q) \cdot v > \big] \right] - \\ &\quad -\zeta_t(q) \cdot \left[\sum_{i=1}^{n} -t \cdot e_i(q) \Big[\frac{-tdf(q)e_i(q)}{(1+tf(q)k_i(q))^2} \cdot \big(df(q)v \cdot k_i(q) + \right. \\ &\quad \left. +f(q)dk_i(q).v \big) \Big] \right] \end{aligned}$$

Recall that we want to know the value of $\det dN_t(q_t)$. But it is not difficult that $\frac{\partial}{\partial t}\det dN_t(q_t)|_{t=0} = (\frac{1}{c_2})^{n-1}\cdot <\frac{\partial}{\partial t}dN_t(q_t)|_{t=0}\cdot e_n(q), e_n(q)> + \sum_{i=1}^{n-1}(\frac{1}{c_2})^{n-2}\cdot \psi\cdot <\frac{\partial}{\partial t}dN_t(q_t)|_{t=0}\cdot e_i(q), e_i(q)>$. In fact this follows from the fact that the determinant $\det A$ is the multilinear alternating n-form of the columns vectors $\det(A.e_1, \ldots, A.e_n)$. Thus,

$$\frac{d}{dt}\det A(t)|_{t=0} = \sum_{i=1}^{n}\det(A(t).e_1, \ldots, A'(t).e_i, \ldots, A(t).e_n)|_{t=0}.$$

Since $e_i(q)$, for $i = 1, \ldots, n-1$, are eigenvectors of $dN(q)$ with eigenvalue $\frac{1}{c_2}$ and e_n is an eigenvector of $dN(q)$ with eigenvalue ψ (see proof of lemma 2.1), if we define $A(t) = dN_t(q_t)$, $e_i = e_i(q)$, the formula above gives the claim.

We observe that the formula above uses (implicitly) that the codimension $(n-k)$ of the sphere S^k is 1. See remark 5.2 below.

However,

$$\begin{aligned}\frac{\partial}{\partial t}dN_t(q_t)|_{t=0}\cdot v = {} & \frac{\partial}{\partial t}\zeta_t(q)|_{t=0}\cdot dN(q)\cdot v + \\ & + \lim_{t\to 0}\zeta_t(q)\cdot\Big[-\sum_{i=1}^{n} df(q)e_i(q)de_i(q)\cdot v \\ & -\sum_{i=1}^{n} e_i(q)\cdot(<d^2f(q)v, e_i(q)> + <\text{grad} f(q), de_i(q)v>)\Big] + \\ & +\frac{\partial}{\partial t}d\zeta_t(q)|_{t=0}\cdot v\cdot N(q).\end{aligned}$$

Taking $v = e_i(q), i = 1, \ldots, n$, we have :

$$<\frac{\partial}{\partial t}dN_t(q_t)|_{t=0}.e_i(q), e_i(q)> = - < d^2f(q)e_i(q), e_i(q) > .$$

By the geometry of our immersion, we choose $f(q) = f(z, w) = l_0.\lambda(d(z))\cdot\sigma(w)$ (see figure 2). Here σ is a convex function ($\sigma'' < 0$) s.t. $\sigma(0) = 0$, e.g., $\sigma(w) = -w^2/2$ (at least in $[0, \alpha_0]$, $\alpha < \alpha_0 < \beta$ close to α) and λ is a bump function s.t. $\lambda \equiv 1$ if $t \leq 0$, $\lambda \equiv 0$ if $t \geq \delta$, $\lambda = e^{-1/\delta - t}$ if t is close to δ and λ is strictly decreasing in $(0, \delta)$. An easy calculation shows that $< d^2f(q)e_n(q), e_n(q) > = l_0.\lambda(d(z))\cdot\sigma''(w)\cdot < \frac{\partial}{\partial w}, e_n(q) >^2$. If α_0 is sufficiently close to α, $< \frac{\partial}{\partial w}, e_n(q) >^2 \geq 1/2$; $< d^2f(q)e_i(q), e_i(q) > = l_0.\sigma(w)\cdot\left[\lambda''||\text{grad}(d)||^2 + \lambda'\Delta d\right]$.

Then,

$$\frac{\partial}{\partial t} \det dN_t|_{t=0} = l_0.(\frac{1}{c_2})^{n-2}.\big[(\frac{1}{c_2}) <\frac{\partial}{\partial w}, e_n(q)>^2 .\lambda.\sigma'' + \psi\sigma\{\lambda''||\mathrm{grad}(d)||^2 + \lambda'\Delta d\}\big].$$

To complete the proof, we show that the last expression is positive in $\{(z,w) : d(z) < \delta, |w| \leq \alpha_0\}$ and it is small in $\{(z,w) : |w| \geq \beta\}$. This is sufficient beacuse the derivative is positive imply that the curvature increases in the "goal" region, and the derivative is small (possible negative) does not creates regions of negative curvature since the curvature starts positive in the construction.

Since $\frac{1}{c_2} \geq \sqrt{\gamma}$, $<\frac{\partial}{\partial w}, e_n(q)>^2 \geq \frac{1}{2}$, l_0 is arbitrarily small and the term $\big[(\frac{1}{c_2}) \cdot \lambda.\sigma''. <\frac{\partial}{\partial w}, e_n(q)>^2 + \psi\sigma\{\lambda''||\mathrm{grad}(d)||^2 + \lambda'\Delta d\}$ is uniformly bounded, the region $\{(z,w) : |w| \geq \beta\}$ remains with positive curvature. At the critcal region $\{(z,w) : d(z) < \delta, |w| \leq \alpha_0\}$, we consider two cases : $d(z)$ close to δ and $d(z)$ far from δ.

If $d(z)$ is close to δ, $\lambda'' = \big(\frac{1}{(\delta-t)^4} - 2 \cdot \frac{1}{(\delta-t)^3}\big) \cdot \lambda$ and $\lambda' = -\frac{1}{(\delta-t)^2} \cdot \lambda$. So, if $d(z)$ is close to δ, i.e., $\delta - d(z)$ is close to zero, the term $\lambda''||\mathrm{grad}(d)||^2 + \lambda'\Delta d$ is positive. Since, σ, σ'' is negative and ψ is positive, this concludes the first case.

If $d(z)$ is far from δ, the term is $\lambda''||\mathrm{grad}(d)||^2 + \lambda'\Delta d$ is bounded and λ is positive and far away from zero. So, if α_0 is sufficiently close to α, ψ is small (and positive). This concludes the second case.

This completes the proof. q.e.d.

4 Proof of Theorem A

Proof of Theorem A. Consider the good Cantor set $F = S^{n-1} - \bigcup_{i=1}^{\infty} U_i$, where U_i are round balls. We fix an immersion $\varphi = \varphi_0$ given by lemma 2.1. For each i, let $\varphi_i = \varphi_{U_i}$ a perturbation of φ with support $\{(z,w) : d(z) < \delta_i, |w| \leq \alpha_0^i\}$, where δ_i is the radius of U_i, $\alpha_0^i \geq \alpha$. Althought the existence of φ_i with the previous properties is not explicitly stated in lemma 3.1, this is contained in the proof. Finally, define $x = \lim_n \varphi_n \circ \cdots \circ \varphi_1 \circ \varphi_0$. Observe that U_i are pairwise disjoint implies that the support of φ_i are disjoint. So the limit immersion x above exists and satisfies the desired properties. q.e.d.

5 Final Remarks

We finish the paper with two remarks. The first remark is a possible generalization of theorem A for open sets more general than round balls. The second remark is an explanation about the restrictive hypothesis on the codimension.

Remark 5.1. We can replace in theorem A the round balls by sets with "distance functions" (i.e., bump functions with support *equal* to the open set) whose gradient is positive and bounded Laplacian. This follows from a carefull read of lemma 3.1, the only place where properties of distance functions were used.

Remark 5.2. The proof of lemma 3.1 works for codimension 1 since for higher codimensions, the determinat formula has a critical point of order $(n-k-1)$ (the determinant is morally $t^{n-k-1} \cdot \det A_t$, where A_t are positive matrices close to A_0). So, our trick of calculate the first derivative of this family and shows that the determinant increases does not work (the critical point is "flat").

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