SPATIALLY PERIODIC SOLUTIONS IN RELATIVISTIC ISENTROPIC GAS DYNAMICS

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ABSTRACT. We consider the initial value problem, with periodic initial data, for the Euler equations in relativistic isentropic gas dynamics, for ideal polytropic gases which obey a constitutive equation, relating pressure p and density ρ, $p = \kappa^2\rho^\gamma$, with $\gamma \geq 1$, $0 < \kappa < c$, where c is the speed of light. Global existence of periodic entropy solutions for initial data sufficiently close to a constant state follows from a celebrated result of Glimm and Lax (1970). We prove that given any periodic initial data of locally bounded total variation, satisfying the physical restrictions $0 < \inf_{x\in\mathbb{R}} \rho_0(x) < \sup_{x\in\mathbb{R}} \rho_0(x) < +\infty$, $\|v_0\|_\infty < c$, where v is the gas velocity, there exists a globally defined spatially periodic entropy solution for the Cauchy problem, if $1 \leq \gamma < \gamma_0$, for some $\gamma_0 > 1$, depending on the initial bounds. The solution decays in L^1_{loc} to its mean value as $t \to \infty$.

1. Introduction

We consider the nonlinear hyperbolic system of conservation laws which describes the motion of one dimensional isentropic relativistic gas in Euler coordinates,

$$\begin{cases} \frac{\partial}{\partial t}\big(\frac{p+\rho c^2}{c^2}\frac{v^2}{c^2-v^2} + \rho\big) + \frac{\partial}{\partial x}\big((p+\rho c^2)\frac{v}{c^2-v^2}\big) = 0, \\ \frac{\partial}{\partial t}\big((p+\rho c^2)\frac{v}{c^2-v^2}\big) + \frac{\partial}{\partial x}\big((p+\rho c^2)\frac{v^2}{c^2-v^2} + p\big) = 0, \end{cases} \tag{1.1}$$

with given periodic initial data, with, say, period 1, and locally bounded total variation,

$$(\rho(x,0), v(x,0)) = (\rho_0(x), v_0(x)). \tag{1.2}$$

Here ρ is the density, v is the velocity, $p = \kappa^2\rho^\gamma$, $1 < \gamma < 2$, is the pressure, $\kappa < c$, and c is the speed of light. The initial data are subject to the physical bounds

$$0 < \inf_{x\in\mathbb{R}} \rho_0(x) \leq \sup_{x\in\mathbb{R}} \rho_0(x) < +\infty, \quad \sup_{x\in\mathbb{R}} p'(\rho_0(x)) < c^2, \quad \|v_0\|_\infty < c. \tag{1.3}$$

Let $U = (U_1, U_2)$, with

$$U_1 = \frac{p+\rho c^2}{c^2}\frac{v^2}{c^2-v^2} + \rho, \qquad U_2 = (p+\rho c^2)\frac{v}{c^2-v^2}, \tag{1.4}$$

and denote $U_0(x) = (U_{1\,0}(x), U_{2\,0}(x))$ the vector function corresponding to $(\rho_0(x), v_0(x))$. Set

$$U_\Pi = \int_0^1 U_0(x)\,dx. \tag{1.5}$$

The main purpose of this paper is to prove the following result.

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Theorem 1.1. *Suppose the periodic initial data ρ_0, v_0 satisfy* (1.3)*, and*

$$\operatorname{Var}_\Pi[\rho_0] + \operatorname{Var}_\Pi[\log \frac{c+v_0}{c-v_0}] < +\infty, \tag{1.6}$$

where by Var_Π *we mean the total variation over one period. Then, there exists* $\gamma_0 > 1$ *such that, for all* $1 \le \gamma < \gamma_0$*, there exists a globally defined spatially periodic entropy solution* $U(x,t)$ *of problem* (1.1),(1.2)*, assuming values in a compact subset of* $\{\rho > 0,\, p' < c^2,\, v^2 < c^2\}$*, with locally bounded total variation, defined through the Glimm difference scheme. The solution* $U(x,t)$ *satisfies*

$$\lim_{t\to\infty} \int_0^1 |U(x,t) - U_\Pi|\, dx = 0. \tag{1.7}$$

We recall that global existence of entropy solutions for initial data in L^∞ sufficiently close to a constant state follows from the celebrated result of Glimm and Lax [9], which, in the periodic case, also ensures decay to the mean value in the L^∞ norm at a rate $O(t^{-1})$. So, here we will be concerned with periodic initial data subjected only to the physical restrictions (1.3), but we also impose the regularity condition (1.6). To this, we need to restrict γ to be sufficiently close to 1, depending on the bounds for the initial data. As we explain below, the decay given by (1.7) will be a direct consequence of the fact that $U(x,t)$ assumes values in a compact subset of $\{\rho > 0,\, p' < c^2,\, v^2 < c^2\}$, as an application of a general decay result in [2] combined with a well known compactness result of DiPerna [6].

Discontinuous solutions of the relativistic Euler equations were first considered in the pioneering paper of Smoller and Temple [15], where it was shown the global existence of BV entropy solutions of the Cauchy problem for (1.1), with $\gamma = 1$, for initial data in $BV(\mathbb{R})$, satisfying the physical restrictions (1.3). Their result is based on the striking observation that the shock curves in the relativistic case with $\gamma = 1$, in the plane of the natural Riemann invariants, (z, w), possess the same geometrical property observed by Nishida [11] in the non-relativistic case with $\gamma = 1$, namely, the shock curves of both families starting from any arbitrary point in the (z, w)-plane may be obtained by translation of the corresponding curves starting from a fixed point, say, $(0, 0)$. Also for initial data in $BV(\mathbb{R})$, satisfying the physical restrictions (1.3), it was shown in [3] the global existence of BV entropy solutions of the Cauchy problem (1.1),(1.2) for $1 < \gamma < \gamma_0$, for some γ_0 depending on the bounds for the initial data. We remark that neither of these results can apply to periodic initial data. We refer to [15] for an account of the physical derivation of (1.1) and for references in the physics literature. We also refer to [12] for a better understanding on the physical ground concerning (1.1).

Returning to system (1.1), one easily sees that in the limit, as $c \to \infty$, the Euler equations for isentropic gas dynamics are recovered:

$$\begin{cases} \frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho v) = 0, \\ \frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho v^2 + \kappa^2 \rho^\gamma) = 0. \end{cases} \tag{1.8}$$

Recall that in Lagrangian coordinates system (1.8) reads

$$\begin{cases} \frac{\partial}{\partial t}\tau - \frac{\partial}{\partial y} v = 0, \\ \frac{\partial}{\partial t} v + \frac{\partial}{\partial y}(\kappa^2 \tau^{-\gamma}) = 0, \end{cases} \tag{1.9}$$

where $\tau = \rho^{-1}$ represents the specific volume of the gas.

Next, before closing this section, we give an exposition of the main points involved in the proof of Theorem 1.1. We start considering the non-relativistic case because it is simpler and so allows a neater outline of all steps.

1.1. **The Non-Relativistic Case.** We now explain our method for constructing global periodic solutions of the isentropic gas dynamics equations, with large total variation and oscillation, as long as the adiabatic exponent γ is close to 1, by considering the simpler non-relativistic case. So, we set $m = \rho v$, $U = (\rho, m)$, consider the equations (1.8) with a periodic initial data

(1.10)

$$U(x,0) = U_0(x), \quad x \in \mathbb{R}, \quad U_0(x+1) = U_0(x), \quad U_0 \in BV_{loc}(\mathbb{R})^2, \quad \rho_0(x) > 0.$$

Recalling the well known equivalence between BV entropy solutions in Eulerian and in Lagrangian coordinates (see [16]), setting $V = (\tau, v)$, we may consider, alternatively, system (1.9) with initial data

$$V(y,0) = V_0(y) := \Big(\frac{1}{\rho_0(X_0(y))}, \frac{m_0(X_0(y))}{\rho_0(X_0(y))}\Big), \tag{1.11}$$

where $X_0(y)$ is implicitly defined by the equation

$$y = \int_0^{X_0(y)} \rho_0(x)\,dx. \tag{1.12}$$

Observe that $V_0(y + \rho_\Pi) = V_0(y)$, for all $y \in \mathbb{R}$, where

$$\rho_\Pi = \int_0^1 \rho_0(x)\,dx,$$

and also that $V_0 \in BV_{loc}(\mathbb{R})$ with

$$\mathrm{Var}_\Pi(V_0(y)) \le C\,\mathrm{Var}_\Pi(U_0(x)),$$

where $C > 0$ depends on the lower bound for ρ_0 and the upper bound for m_0. To simplify the notation we write x and U instead of y and V, and assume, with no loss of generality, that $\rho_\Pi = 1$.

System (1.9) has eigenvalues

$$\lambda_1(U) = -\kappa\sqrt{\gamma}\,\tau^{-\frac{\gamma-1}{2}}, \qquad \lambda_2(U) = \kappa\sqrt{\gamma}\,\tau^{-\frac{\gamma-1}{2}}.$$

We want to construct a periodic solution for (1.9),(1.11) using a periodic version of the Glimm scheme. We recall that, by this method, approximate solutions $U^h(x,t)$ are constructed as follows. Set $l = \Delta x$, $h = \Delta t$, with

$$\frac{l}{2h} \ge \sup_{U \in \mathcal{V}, i=1,2} |\lambda_i(U)|,$$

where $\mathcal{V}$ is a set where U^h takes its values and for which the right-hand member of the inequality is finite. Set

$$U^h(x,0) = U_0(jl+0) \qquad x \in ((j-1/2)l, (j+1/2)l)$$

and, for $0 < t < h$, we define $U^h(x,t)$ by solving the Riemann problems for the discontinuities at $x = (j+1/2)l$, $j \in \mathbb{Z}$. Then, assuming that $U^h(x,t)$ has been defined for $0 \le t < nh$, we define

$$U^h(x,nh) = U((j+\alpha_n)l, nh-0), \qquad \text{for } (j-1/2)l < x < (j+1/2)l,\ j \in \mathbb{Z},$$

where $\alpha_n \in (-1/2, 1/2)$ is a number randomly chosen. Then, we define $U^h(x,t)$, for $nh < t < (n+1)h$, by solving the Riemann problems for the discontinuities at $((j+1/2)l, nh)$, $j \in \mathbb{Z}$. This procedure can be reiterated as long as U^h takes values in $\mathcal{V}$ and the Riemann problems can be solved. To guarantee these conditions, in general, the main point is to control the growth of the spatial total variation of $U^h(x,t)$ as t increases. In the usual non-periodic case we assume that the initial data is in $BV(\mathbb{R})$, and, in particular, the limit $\lim_{x\to+\infty} U_0(x) = U_\infty$ exist, hence, control of the total variation implies control of $\|U^h - U_\infty\|_\infty$. Therefore, controlling the spatial total variation of U^h, we may guarantee that it takes its values in a suitable neighborhood of U_∞.

In the periodic case the situation is more complicated since we miss a state like U_∞ which is assumed by $U^h(x,t)$, for all $t > 0$, as long as it can be defined. The state that approximates best this role, is the mean value

$$U_\Pi = (\tau_\Pi, v_\Pi) := \int_0^1 U_0(x)\,dx.$$

Each of its coordinates (not the vector itself) may be viewed as a value assumed by the corresponding coordinate of $U_0(x)$. Although

$$(U^h(t))_\Pi := \int_0^1 U^h(x,t)\,dx$$

is only piecewise constant with jumps at $t = nh$, if $U^h(x,t)$ converges almost everywhere to a weak solution of (1.9),(1.11), $U(x,t)$, then we must have $(U(t))_\Pi = U_\Pi$. So, the idea here is: (i) show first that there is a time $T > 0$, independent of h, so that all the U^h can be defined up to $t = T$, by controlling $\mathrm{Var}_\Pi(U^h(t))$ and $\|U^h(t) - U_\Pi\|_\infty$; (ii) show that if $|(U^h(T))_\Pi - U_\Pi|$ is small enough, which may be achieved for very small h, the initial situation is recovered, roughly speaking, so that we can construct our approximate solutions through the time interval $[T, 2T]$, and so on. Finally, one uses a diagonal argument to show the convergence of a subsequence of U^h in the whole $\mathbb{R}^2_+$.

The control of $\mathrm{Var}_\Pi(U^h(t))$ is possible because the system (1.9) (as well as (1.8)) belongs, in a disguised way, to the, so called, Bakhvalov class, characterized by four conditions, $A_1, \dots, A_4$, described in Section 3. This remarkable fact was proved by DiPerna [5]. We recall that (1.9) is endowed with a pair of (natural) Riemann invariants, namely,

$$z = v - \frac{2\kappa\sqrt{\gamma}}{\gamma-1}\tau^{-\frac{\gamma-1}{2}}, \qquad w = v + \frac{2\kappa\sqrt{\gamma}}{\gamma-1}\tau^{-\frac{\gamma-1}{2}}, \tag{1.13}$$

that is, z, w satisfy $\nabla z(U)\nabla F(U) = \lambda_1(U)\nabla z(U)$, $\nabla w(U)\nabla F(U) = \lambda_2(U)\nabla w(U)$, where $F(U) = (-v, \kappa^2\tau^{-\gamma})$. Clearly, any function of a Riemann invariant is also a Riemann invariant. We refer to [4, 13, 14] for the basic facts about the theory of conservation laws. For an exposition of Bakhvalov conditions we refer the reader to Section 3.

Let

$$\sigma = w + z, \qquad \eta = w - z.$$

It is sometimes more convenient to work with the coordinates (σ, η) than with the Riemann invariants (z, w). Denote

$$W(a,k) = \{(\sigma,\eta) \,:\, |\sigma - a| \le k\eta\}.$$

In [5], DiPerna proved the following theorem.

Theorem 1.2 (DiPerna [5]). *There exists a 2-parameter family of transformations $T(a,\theta) : (z,w) \to (z'(z), w'(w))$, $a \in \mathbb{R}$, $\theta > 0$, and positive constants $k, c_1, c_2(k)$ which have the following property. For sufficiently small k, $T(a,\theta)$ maps the shock curves of* (1.9) *in*

$$\tilde{W}(a,k) = W(a,k) \cap \{c_1/\theta < \eta < c_2(k)/\theta\}$$

onto shock curves which satisfy A_i, $i = 1,2,3,4$ in the $z' - w'$ variables. Furthermore,

$$\lim_{k\to 0} c_2(k) = \infty. \tag{1.14}$$

Actually, DiPerna's analysis in [5] misses a clear determination of the way in which the bounds to be imposed on the initial data can grow when γ decreases to 1. To make this study precise, besides (1.14), it is necessary to use the fact, demonstrated in our analysis, that k and $c_2(k)$ can be chosen such that

$$\frac{k}{\gamma - 1} \to \infty, \qquad c_2(k)k \to 0, \qquad \text{as } \gamma \to 1+. \tag{1.15}$$

Let

$$\varpi(\tau) = -\log\tau, \qquad \varpi_0(y) = \varpi(\tau_0(y)),$$
$$\sigma_\Pi = 2v_\Pi, \quad \eta_\Pi = \eta(\tau_\Pi), \quad \varpi_\Pi = \varpi(\tau_\Pi).$$

We assume $(\tau_0(y), v_0(y)) \in \mathcal{R}[V(\gamma)]$, for all $y \in \mathbb{R}$, where

$$\mathcal{R}[V(\gamma)] = \{\, \varpi_\Pi - V(\gamma) \le \varpi \le \varpi_\Pi + V(\gamma)\,\} \cap \{\, |\sigma - \sigma_\Pi| \le V(\gamma)\,\},$$

and $V(\gamma)$ is a positive decreasing function defined for $\gamma > 1$ satisfying

$$V(\gamma) \to \infty, \quad \frac{(\gamma-1)V(\gamma)}{k} \to 0, \quad \frac{V(\gamma)}{c_2(k)} \to 0, \qquad \text{as } \gamma \to 1+. \tag{1.16}$$

We also denote

$$\tilde{\mathcal{R}}[V(\gamma)] = \{\, \eta_\Pi - V(\gamma) \le \eta \le \eta_\Pi + V(\gamma)\,\} \cap \{\, |\sigma - \sigma_\Pi| \le V(\gamma)\,\}.$$

An important observation is that there is an absolute constant c_0, independent of γ, such that

$$\frac{c_0}{2}(\varpi(\tau_1) - \varpi(\tau_2)) \le \eta(\tau_1) - \eta(\tau_2) \le 2c_0(\varpi(\tau_1) - \varpi(\tau_2)), \tag{1.17}$$
$$\text{if } \tau_1 \le \tau_2 \text{ and } \varpi_\Pi - N_0V(\gamma) \le \varpi(\tau_i) \le \varpi_\Pi + N_0V(\gamma),\ i = 1,2,$$

and, in particular,

$$\mathcal{R}[N_0V(\gamma)] \subset \tilde{\mathcal{R}}[2c_0N_0V(\gamma)],$$

for any given positive integer N_0, provided γ is sufficiently close to 1, due to (1.16). The Riemann invariants z', w' to which Theorem 1.2 refers are defined by

$$z' = 1 - \exp(2\theta(a/2 - z)), \qquad w' = -1 + \exp(2\theta(w - a/2)). \tag{1.18}$$

We choose

$$a = \sigma_\Pi, \qquad \theta = \frac{c_2(k)}{\eta_\Pi + 6c_0V(\gamma)}, \tag{1.19}$$

and define

$$z'' = \frac{\exp(-\theta\eta_\Pi)}{\theta} z', \qquad w'' = \frac{\exp(-\theta\eta_\Pi)}{\theta} w'. \tag{1.20}$$

Using (1.13), (1.15) and (1.17), it is not difficult to see that there exist absolute constants c_1, c_2 such that

$$\begin{aligned} c_1(|\sigma(U_1)-\sigma(U_2)|+|\varpi(U_1)-\varpi(U_2)|) &\le |z''(U_1)-z''(U_2)|+|w''(U_1)-w''(U_2)| \\ &\le c_2(|\sigma(U_1)-\sigma(U_2)|+|\varpi(U_1)-\varpi(U_2)|), \end{aligned} \tag{1.21}$$

if $(\sigma(U_i),\varpi(U_i))\in\mathcal{R}[N_0V(\gamma)]$, for any given positive integer N_0, provided that γ is sufficiently close to 1. Now, if the approximate solution $U^h(x,t)$ assumes values in a region for which Bakhvalov's conditions A_1-A_4, recalled in Section 3, are satisfied relatively to the Riemann invariants z', w' (and, hence, also relatively to z'', w'') then a periodic version of the main result in [1] (cf. [7]) implies that there exists an absolute constant c_3 such that

$$\operatorname{Var}_\Pi[(z'',w'')(U^h(t))]\le c_3\operatorname{Var}_\Pi[(z'',w'')(U^h(0))], \tag{1.22}$$

and so there is an absolute constant c_4 such that

$$\operatorname{Var}_\Pi[(\sigma,\varpi)(U^h(t))]\le c_4\operatorname{Var}_\Pi[(\sigma,\varpi)(U_0)]. \tag{1.23}$$

We then assume that

$$c_4\operatorname{Var}_\Pi[(\sigma,\varpi)(U_0)]<V(\gamma). \tag{1.24}$$

Concerning (1.22), the key point in Bakhvalov's proof of an inequality like this one, in [1], is the introduction of a functional which, restricted to solutions of Riemann problems with a left state $\mathbf{U}_l$ and a right state $\mathbf{U}_r$, denoted by $(\mathbf{U}_l\mathbf{U}_r)$, is defined by $F[(\mathbf{U}_l\mathbf{U}_r)]=([z''(\delta_1)])_-+([w''(\delta_2)])_-$, where the first term is the absolute value of the increment in z'' across the first wave, if it is a shock, and 0 otherwise, and the second is the absolute value of increment of w'' across the second wave, if it is a shock, and 0 otherwise. Thanks to A_1–A_4 this functional is then proven to satisfy

$$F[(\mathbf{U}_l\mathbf{U}_r)]\le F[(\mathbf{U}_l\mathbf{U}_m)]+F[(\mathbf{U}_m\mathbf{U}_r)],$$

with equality holding if $\mathbf{U}_m$ is a value assumed by $(\mathbf{U}_l\mathbf{U}_r)$. An essential feature of the above relation is that it does not involve quadratic terms, as opposed to the original interaction estimate in [8], which is a crucial point for a periodic version of Glimm's method. Extending, in a natural way, the above functional to the periodic approximate solutions and using periodicity, (1.22) follows.

Now, we choose $N_0=3$ and show that, for γ sufficiently close to 1, Bakhvalov's conditions are satisfied in $\mathcal{R}[3V(\gamma)]$, relatively to the Riemann invariants z', w'; this implies that the approximate solution can be defined for the first two steps so that it assumes values in a region where Bakhavalov's conditions are satisfied relatively to the Riemann invariants z', w'. To this, it suffices to show that

$$\tilde{\mathcal{R}}[6c_0V(\gamma)]\subset\tilde W(\sigma_\Pi,k), \tag{1.25}$$

if γ is sufficiently close to 1. Now, clearly, due to (1.16), we have

$$\frac{c_1}{c_2(k)}(\eta_\Pi+6c_0V(\gamma))<\eta_\Pi-6c_0V(\gamma),$$

and

$$k(\eta_\Pi-6c_0V(\gamma))\ge 6c_0V(\gamma),$$

if γ is sufficiently close to 1, which proves (1.25).

As in the classical case, we easily prove that there is a constant $C(\gamma)$, depending on γ, such that

$$\int_0^1 \left|(\sigma,\varpi)(U^h(x,t_1)) - (\sigma,\varpi)(U^h(x,t_2))\right| dx \le C(\gamma)(|t_1-t_2|+h), \tag{1.26}$$

for $t_1, t_2 \in [0,T]$, as long as $U^h(t)$ is defined and satisfies (1.23) in the interval $[0,T]$.

Now, let us investigate the growth of the oscillation of the approximate solutions with time. Denoting by $C(\gamma)$ a positive constant depending on γ, inpendent of h, T, not necessarily the same as in (1.26), we have

$$\begin{aligned}
&|(\sigma,\varpi)(U^h(x,t)) - (\sigma_\Pi,\varpi_\Pi)| \le |(\sigma,\varpi)(U^h(x,t)) - (\sigma,\varpi)((U^h(t))_\Pi)| \\
&+ |(\sigma,\varpi)((U^h(t))_\Pi) - (\sigma,\varpi)(U_\Pi)| \le |\sigma(x,t) - (\sigma^h(t))_\Pi| + |\varpi(\tau^h(x,t)) - \varpi((\tau^h(t))_\Pi)| \\
&\quad + C(\gamma)(t+h) \le \mathrm{Var}_\Pi[(\sigma,\varpi)(U^h(t))] + C(\gamma)(t+h) \le c_4\,\mathrm{Var}_\Pi[(\sigma,\varpi)(U_0)] + C(\gamma)(t+h),
\end{aligned}$$

so that, if $T > h$ is such that $2C(\gamma)T < V(\gamma) - c_4\,\mathrm{Var}_\Pi[U_0]$, we have that the approximate solution may be defined up to a time $T' > T$, independent of h, satisfying

$$|(\sigma,\varpi)(U^h(x,t)) - (\sigma_\Pi,\varpi_\Pi)| < V(\gamma), \qquad 0 \le t \le T,$$

in particular, $U^h(t) \in \mathcal{R}[V(\gamma)]$, for $0 \le t \le T$. We may assume $T = m_0 h_0$, for some $m_0 \in \mathbb{N}$, and $h_0 > 0$ such that all h to be used are of the form $h = 2^{-p}h_0$, for some $p \in \mathbb{N}$. In particular, $T \in \mathbb{N}h$, for all time-steps h considered. Now, assume an approximate solution has been defined up to $t = NT$, for some $N \in \mathbb{N}$, satisfying

$$|(\sigma,\varpi)(U^h(x,t)) - (\sigma_\Pi,\varpi_\Pi)| < V(\gamma), \qquad 0 \le t \le NT,$$

and suppose

$$|(\sigma,\varpi)((U^h(NT))_\Pi) - (\sigma,\varpi)(U_\Pi)| < V(\gamma) - c_4\,\mathrm{Var}_\Pi[(\sigma,\varpi)(U_0)] - 2C(\gamma)T. \tag{1.27}$$

Hence, as above, for $t > NT$ such that $U^h(t)$ is defined and assume values in $\mathcal{R}[3V(\gamma)]$, we have

$$\begin{aligned}
&|(\sigma,\varpi)(U^h(x,t)) - (\sigma_\Pi,\varpi_\Pi)| \le |(\sigma,\varpi)(U^h(x,t)) - (\sigma,\varpi)((U^h(t))_\Pi)| \\
&+ |(\sigma,\varpi)((U^h(t))_\Pi) - (\sigma,\varpi)((U^h(NT))_\Pi)| + |(\sigma,\varpi)((U^h(NT))_\Pi) - (\sigma,\varpi)(U_\Pi)| \\
&\quad \le c_4\,\mathrm{Var}_\Pi[(\sigma,\varpi)(U_0)] + C(\gamma)(t - NT + h) + |(\sigma,\varpi)((U^h(NT))_\Pi) - (\sigma,\varpi)(U_\Pi)|,
\end{aligned}$$

so that $U^h(t)$ may be defined up to a time $NT + T'$ with $T' > T$, independent of h, such that

$$|(\sigma,\varpi)(U^h(x,t)) - (\sigma_\Pi,\varpi_\Pi)| < V(\gamma), \qquad 0 \le t < NT + T'. \tag{1.28}$$

The above argument provides the reiteration procedure introduced in [7]. That is, assuming the reiteration has been carried out until the N-th step, we see, from (1.28), that the approximate solutions can then be defined until a time $NT + T'$, with $T' > T$. Then, using Glimm's argument in [8], for the consistence of his scheme, we can obtain a subsequence of h's and a set $\Theta_N \subset \Pi_{n=1}^\infty(-1/2, 1/2)$, of measure 1, such that the U^h are defined, for h less than certain h_N, and converge in $C([0, NT+T'), L^1_{loc}(\mathbb{R}))$ to a weak solution of (1.9). In particular, (1.27), with N replaced by $N+1$, holds and we can advance one more step, continuing this way indefinitely.

1.2. **An overview of the relativistic case.** The situation in the relativistic case becomes more complicated because, first, the proof of the analogous of Theorem 1.2 (see Theorem 4.1), including the new estimates (1.15), requires yet more technical calculations, second, we now miss completely a reference value, such as U_Π above, since the variables that are conserved, namely, $U=(U_1,U_2)$ given in (1.4), are not nicely related with the (natural) relativistic Riemann invariants

$$w=\frac{1}{2}\log\frac{c+v}{c-v}+c\int_0^\rho\frac{\sqrt{p'}}{p+sc^2}ds,\quad z=\frac{1}{2}\log\frac{c+v}{c-v}-c\int_0^\rho\frac{\sqrt{p'}}{p+sc^2}ds,$$

or the corresponding $\sigma=w+z$, $\eta=w-z$, and transforming to Lagrangian coordinates in this case would not change this situation. The latter forces us to change the argument a bit, as follows.

Let be given an initial data $(\rho_0(x),v_0(x))$, periodic with, say, period 1, with bounded total variation over one period and satisfying $0<\underline{\rho}<\inf_{x\in\mathbb{R}}\rho_0(x)<\sup_{x\in\mathbb{R}}\rho_0(x)<\bar\rho$, $\|v_0\|<c$, with $p'(\bar\rho)<c^2$. We do not impose any size restriction neither on $\mathrm{Var}_\Pi(\rho_0,v_0)$, nor on $\underline{\rho}$ and $\bar\rho$. We will need to use the fact that the region

$$\Omega=\left\{(z,w)\,:\,z\ge\inf_x z(x,0),\; w\le\sup_x w(x,0)\right\}$$

is invariant for the solution of Riemann problems. Let $(\rho^h(x,t),v^h(x,t))$ denote the approximate solution in (ρ,v)-coordinates, constructed by Glimm's method, as above. The invariance of Ω implies that, while $\eta^h(x,t)=\eta(\rho^h(x,t))$ assumes values in an interval $[\eta(\underline{\rho})-V(\gamma),\eta(\bar\rho)+V(\gamma)]$, for $V(\gamma)$ as in the non-relativistic case, we must have that $\sigma^h(x,t)=\sigma(v^h(x,t))$ assumes values in an interval $|\sigma|\le\bar{\bar\sigma}(\gamma)$, with $\bar{\bar\sigma}(\gamma)$ determined by $V(\gamma)$ using the bounds for the region Ω. We prove that

$$\{|\sigma|\le\bar{\bar\sigma}(\gamma)\}\cap\{\eta(\underline{\rho})-V(\gamma)<\eta<\eta(\bar\rho)+V(\gamma)\}\subset\tilde W(a,k),\tag{1.29}$$

for γ sufficiently close to 1, where $\tilde W(a,k)$ is as in the non-relativistic case and, by the analogous of Theorem 1.2 (see Theorem 4.1 below), has the property that (1.1) satisfies Bakhvalov's conditions in its image by the map $T(\theta,a):(z,w)\mapsto(z'(z),w'(w))$. Similarly to the non-relativistic case, we choose θ,a conveniently so that (1.29) holds true.

Using the inequality corresponding to (1.26), for the approximate solutions in the conservative variables $(U_1^h(x,t),U_2^h(x,t))$, we then obtain rough estimates from above and from below for the mean value $(\rho^h(t))_\Pi$, in the form

$$\check\rho(\gamma)<(\rho^h(t))_\Pi<\hat\rho,\qquad\text{for } 0\le t<T',\tag{1.30}$$

as long as

$$2\check\rho(\gamma)<(\rho^h(0))_\Pi<\frac{\hat\rho}{2},\tag{1.31}$$

where $\hat\rho$ does not depend on γ, and $\check\rho(\gamma)$ is a suitable function of γ satisfying

$$\eta(\check\rho(\gamma))>\eta(\underline{\rho})-\frac{V(\gamma)}{2},\tag{1.32}$$

for γ close to 1. The estimate (1.30), then, tell us that $\rho^h(x,t)$ assumes a value in the interval $(\check\rho(\gamma),\hat\rho)$, for $0\le t<T'$, where we agree that we can redefine $\rho^h(x,t)$ in any discontinuity point (x_0,t) such that $\rho^h(x_0,t)$ is any suitable value in the interval between $\rho^h(x_0-0,t)$ and $\rho^h(x_0+0,t)$, observing that this redefinition does

not changes the $\mathrm{Var}_\Pi(\rho^h(\cdot,t))$. In terms of the variables (σ,η), this tells us that $\eta^h(x,t)$ assumes a value from the interval $(\eta(\check\rho(\gamma)),\eta(\hat\rho))$, for each $0\le t<T'$.

The point is that an estimate as (1.31) *follows directly from estimates for* $(U_{1,0})_\Pi$ *and from the estimate* $|\sigma|<\overline{\overline{\sigma}}$.

Now, arguing as in the non-relativistic case, we may find constants $c_0,\dots,c_4$ playing analogous roles, and we have, for $1<\gamma<\gamma_0$, for a certain $\gamma_0>1$, (cf. (1.24))

$$c_0^2c_4\,\mathrm{Var}_\Pi[(\sigma,\eta)(\rho_0,v_0)]<\frac{V(\gamma)}{2}. \tag{1.33}$$

Since, Bakhvalov's conditions ensure that

$$\mathrm{Var}_\Pi[(\sigma^h(t),\eta^h(t))]\le c_0^2c_4\,\mathrm{Var}_\Pi[(\sigma,\eta)(\rho_0,v_0)]$$

as long as the values of $(\sigma^h(s),\eta^h(s))$ lie in $\tilde W(a,k)$, for $0\le s\le t$, (1.29), (1.30), (1.32) and (1.33) imply that $(\sigma^h(x,t),\eta^h(x,t))$ assumes values in the rectangle $\{|\sigma|<\overline{\overline{\sigma}}(\gamma)\}\cap\{\eta(\underline{\rho})-V(\gamma)<\eta<\eta(\bar\rho)+V(\gamma)\}$, for $0\le t<T'$. Now, since (1.31) depends only on an estimate for $(U_{1,0})_\Pi$ and the estimate $|\sigma|<\overline{\overline{\sigma}}$, we may reiterate the argument in intervals $[NT,NT+T')$, $N\in\mathbb{N}$, provided $(U_1^h(NT))_\Pi$ is sufficiently close to $(U_{1,0})_\Pi$, as in the general approach of [7] used in subsection 1.1.

1.3. **Decay of the periodic solutions.** Here we briefly recall how the decay property of the periodic solutions obtained in the first part of Theorem 1.1 can be achieved. In sum, it is a consequence of the fact that they take values on compact subsets of $\{\rho>0,\ p'<c^2,\ v^2<c^2\}$ and the fact that (1.1) is strictly hyperbolic and genuinely nonlinear over this compact set, by applying the followings results in [2] and [6]. Let us consider the Cauchy problem for a general $n\times n$ system of conservation laws

$$U_t+F(U)_x=0, \tag{1.34}$$

$$U(x,0)+U_0(x). \tag{1.35}$$

Theorem 1.3 (Chen & Frid [2]). *Assume that* (1.34) *is endowed with a strictly convex entropy, and let* $U\in L^\infty(\mathbb{R}\times\mathbb{R}_+)$ *be a periodic entropy solution of* (1.34)-(1.35), *with, say, period 1. Denote* $U^T(x,t)=U(Tx,Tt)$. *If the sequence* U^T, $T\to\infty$, *is pre-compact in* $L^1_{loc}(\mathbb{R}\times\mathbb{R}_+)$ *then one has*

$$ess\lim_{t\to\infty}\int_0^1|U(x,t)-U_\Pi|\,dx=0,$$

where $U_\Pi=\int_0^1U_0(x)\,dx$.

Theorem 1.4 (DiPerna [6]). *Assume* (1.34) *is a strictly hyperbolic genuinely nonlinear* 2×2 *system. Let* U^T, $T\in I$, *for some index set* I, *be a family of entropy solutions of initial value problems for* (1.34), *which is uniformly bounded in* $L^\infty(\mathbb{R}\times\mathbb{R}_+)$. *Then* U^T *is pre-compact in* $L^1_{loc}(\mathbb{R}\times\mathbb{R}_+)$.

We remark that the existence of a strictly convex entropy for (1.1), defined on a compact set where the periodic solution assumes its values, is a consequence of a well known result of Lax (see [10]), by using the results in Section 2 below. The fact that solutions constructed by Glimm's method are entropy solutions is also proved in [10].

The remaining of this paper is organized as follows. In Section 2, we recall several properties of system (1.1). In Section 3, we recall Bakhavalov's and DiPerna's conditions. In Section 4, we state Theorem 4.1, which is our extension of Theorem 1.2 to the relativistic case, including the new asymptotic information (1.15), mentioned above. In Section 5, we prove the existence part of Theorem 1.1. Finally, in Section 6, we give the rather technical proof of Theorem 4.1.

2. Properties of the system (1.1)

In this section we collect some properties of the system (1.1). The proofs can be found in [15] and [3].

Lemma 2.1 (cf. [15], p.79). *(i) The mapping* $(U_1, U_2) \to (\rho, v)$*, as given by (1.4), is one-to-one and the determinant of its Jacobian is non zero when* $\rho > 0$, $|v| < c$.
(ii) The system (1.1) is strictly hyperbolic and genuinely nonlinear when

$$|v| < c, \quad 0 < \sqrt{p'(\rho)} < c,$$

and has two real eigenvalues

$$\lambda_1 = \frac{v - \sqrt{p'}}{1 - \frac{v\sqrt{p'}}{c^2}}, \quad \lambda_2 = \frac{v + \sqrt{p'}}{1 + \frac{v\sqrt{p'}}{c^2}}. \tag{2.1}$$

(iii) There is the pair of "classical" Riemann invariants

$$w = \frac{1}{2}\log\frac{c+v}{c-v} + c\int_0^\rho \frac{\sqrt{p'(s)}}{p(s)+sc^2}ds, \quad z = \frac{1}{2}\log\frac{c+v}{c-v} - c\int_0^\rho \frac{\sqrt{p'(s)}}{p(s)+sc^2}ds, \tag{2.2}$$

the mapping $(w, z) \to (\rho, v)$ *is one-to-one and the determinant of the corresponding Jacobian is non zero when* $\rho > 0$, $|v| < c$.

Remark 2.1. In the view of the above lemma we will often refer to a given state of the system (1.1) in different state spaces by marking coordinates with the same label, for example, U_R, (z_R, w_R) and (ρ_R, v_R) are assumed to be connected by (1.4) and (2.2).

From the results in [3] it follows that it is possible to choose

$$z = R_1(w; z_L, w_L), \qquad z = R_2(w; z_L, w_L), \quad w < w_L; \tag{2.3}$$

as the parametrization of shock curves of the first and the second family with the given state on the left (z_L, w_L), and

$$z = L_1(w; z_R, w_R), \qquad z = L_2(w; z_R, w_R), \quad w > w_R. \tag{2.4}$$

as the parametrization of shock curves of the first and the second family with the given state on the right (z_R, w_R). Moreover the shock curves have the following properties.

Lemma 2.2 (cf. [3], p.1623). *Let* (ρ_L, v_L) *and* (ρ_R, v_R) *be two states (on the left and right) connected by the shock of the first family then* $v_R < v_L$, $\rho_R > \rho_L$, $1 < \frac{\partial R_1}{\partial w}(w_R; z_L, w_R) < +\infty$, *and* $1 < \frac{\partial L_1}{\partial w}(w_L; z_R, w_R) < +\infty$. *If* (ρ_L, v_L) *and* (ρ_R, v_R) *connected by the shock of the second family then* $v_R < v_L$, $\rho_R < \rho_L$, $0 < \frac{\partial R_2}{\partial w}(w_R; z_L, w_L) < 1$, *and* $0 < \frac{\partial L_2}{\partial w}(w_R; z_L, w_R) < 1$.

The shocks are admissible in the sense of Lax, when $\rho > 0$. It follows from the next lemma, since we assume that $p = \kappa\rho^\gamma$ with $\gamma > 1$.

Lemma 2.3 (cf. [3], p.1613). *If $p(\rho)$ satisfies $p'(\rho) > 0$, $p''(\rho) \geq 0$, then Lax shock conditions hold, i.e.,*

$$\lambda_1(U_R) < s < \lambda_1(U_L),\ \ s < \lambda_2(U_R),$$

for 1-shocks, and

$$s > \lambda_1(U_L),\ \ \lambda_2(U_R) < s < \lambda_2(U_L),$$

for 2-shocks.

In addition, the shock curves are monotone in the way described by the next lemma.

Lemma 2.4 (cf. [3], p.1619). *Given the left state (ρ_L, v_L) the shock curves are star-like in $(\rho,\, v)$ plane, when $1 \leq \gamma \leq 2$.*

As was noted in [15] the system (1.1) is invariant under a Lorenz transformation, meaning that if $(\bar{t},\, \bar{x})$ is a reference frame that moves with velocity μ, as measured in $(t,\, x)$ frame, then the system (1.1) does not change when rewritten in $(\bar{t},\, \bar{x})$ coordinates, provided that velocities of particles are calculated in the barred reference frame. The correspondence between velocities of a particle in unbarred frame, v, and barred frame, $\bar{v}$, is given by the rule

$$v = \frac{\mu + \bar{v}}{1 + \mu\bar{v}/c^2}.$$

The density $\rho(t, x)$ is invariant under a Lorenz transformation. Moreover, there is an invariant functional of velocities .

Lemma 2.5 (cf. [15], p.74). *Let velocities of two fluid particles be given by v_R and v_L, as measured in a coordinates $(t,\, x)$, and $\bar{v}_R$ and $\bar{v}_L$, as measured in coordinates $(\bar{t},\, \bar{x})$, obtained form $(t,\, x)$ by a Lorenz transformation, then*

$$\log\frac{c + v_R}{c - v_R} - \log\frac{c + v_L}{c - v_L} = \log\frac{c + \bar{v}_R}{c - \bar{v}_R} - \log\frac{c + \bar{v}_L}{c - \bar{v}_L}.$$

The above lemma and (2.2) motivate the introduction of functions σ and η by

$$\sigma = w + z = \log\left(\frac{c + v}{c - v}\right), \tag{2.5}$$

$$\eta = w - z = 2c\int_0^\rho \frac{\sqrt{p'(s)}}{p(s) + sc^2}ds = \frac{2\sqrt{\gamma}}{\gamma - 1}\arctan\frac{\kappa\rho^{\frac{\gamma-1}{2}}}{c}. \tag{2.6}$$

Let us consider the shock curves in $(\sigma,\, \eta)$ plane, which is obtained from $(z,\, w)$ plane through rotation by the angle $-\frac{\pi}{4}$.

Remark 2.2. Throughout the paper we will use the same notation for the shock curves in $(z,\, w)$ and $(\sigma,\, \eta)$ planes.

The use of $(\sigma,\, \eta)$ coordinates as a state space will prove to be useful due to the following fact.

Lemma 2.6. *For any two states (σ_0, η_0) and (σ_1, η_0) and for each $i = 1, 2$, two shock curves $\sigma = R_i(\eta; \sigma_0, \eta_0)$ and $\sigma = R_i(\eta; \sigma_1, \eta_0)$ are identical up to a translation along σ-axes.*

Proof. Let v_0 be such that $\sigma_0 = \sigma(v_0)$, where $\sigma(v)$ is given by (2.5). Let the new reference frame $(\bar{t},\, \bar{x})$ to be chosen in such a way that σ_0 computed in the new reference frame is 0. This is the case if $(\bar{t}, \bar{x})$ moves relative to the given reference frame with velocity v_0. Then, by Lemma 2.5, $\bar{\sigma}_1 = \sigma_1 - \sigma_0$, and similarly

$$R_i(\eta; \sigma_1, \eta_0) - \sigma_1 = R_i(\eta; \sigma_1 - \sigma_0, \eta_0) - (\sigma_1 - \sigma_0).$$

As a result we get

$$R_i(\eta; \sigma_1, \eta_0) = R_i(\eta; \sigma_1 - \sigma_0, \eta_0) + \sigma_0.$$

Choose $\sigma_0 = \sigma_1$. We conclude that

$$R_i(\eta; \sigma_1, \eta_0) = \sigma_1 + R_i(\eta; 0, \eta_0).$$

□

Remark 2.3. The above lemma also holds for $\sigma = L_i(\eta; \sigma_0, \eta_0)$, $i = 1, 2$.

For the purpose of the paper it will be sufficient to use the Rankine-Hugoniot condition in the following form.

Lemma 2.7. *Let $(\rho_L, v_L) = (\rho, v)$ and $(\rho_R, v_R) = (\rho_R, 0)$ be two states (on the left and right) connected by a shock, then*

$$v = c^2 \sqrt{\frac{(p - p_R)(\rho - \rho_R)}{(p + \rho_R c^2)(p_R + \rho c^2)}}, \tag{2.7}$$

where $p_R = p(\rho_R)$.

3. Bakhvalov and DiPerna's conditions

Generalizing Nishida's method of proof of the existence of solutions of isothermal gas dynamics with initial data from the class $L^\infty \cap BV_{loc}(\mathbb{R})$, Bakhvalov, in [1], introduced a class of 2x2 strictly hyperbolic and genuinely non-linear systems, characterized by the particular geometry of the shock curves in the plane of Riemann invariants, for which existence result for the same class of initial data can be proven. We follow [5] in the exposition of Bakhvalov conditions. Consider a strictly hyperbolic, genuinely nonlinear system

$$\partial_t \mathbf{U} + \partial_x F(\mathbf{U}) = 0, \tag{3.1}$$

where $\mathbf{U} = (U_1,\, U_2)$ and $F(\mathbf{U}) = (f_1(U),\, f_2(U))$. Let $\lambda_1 < \lambda_2$ be the characteristic speeds of (3.1). Let z, w be a pair of Riemann invariants for (3.1) such that in its domain of definition the map $(U_1, U_2) \mapsto (z, w)$ is one-to-one. Let the shock curves of the first and second family be parameterized by

$$\begin{array}{ll} z = R_1(w; z_0, w_0),\ w \le w_0; & z = L_1(w; z_0, w_0),\ w \ge w_0 \\ z = R_2(w; z_0, w_0),\ w \le w_0; & z = L_2(w; z_0, w_0),\ w \ge w_0 \end{array}. \tag{3.2}$$

In the above, state (z, w) is a state which can be connected on the left (L_i) and on the right (R_i) to (z_0, w_0) by a shock of the i^{th} family. Finally, let

$$\Omega = \left\{ (z, w)\ :\ z \ge \inf_x z(x, 0),\ w \le \sup_x w(x, 0) \right\}. \tag{3.3}$$

The next hypotheses impose conditions on the shock curves under which the solvability of Cauchy problem with locally bounded variation is obtained.

A_1 : $\sup_{i,\Omega} |\lambda_i(z,w)| < \infty.$

A_2 : $\forall (z,w) \in \Omega,\ 1 < \dfrac{\partial R_1}{\partial w},\ \dfrac{\partial L_1}{\partial w} < +\infty,\ 0 < \dfrac{\partial R_2}{\partial w},\ \dfrac{\partial L_2}{\partial w} < 1,\ w \neq w_0$

A_3 : If $z_r = R_i(w_r; z_l, w_l)$, $i = 1, 2$, then shock curves $z = R_i(w; z_l, w_l)$, $w \le w_l$ and $z = L_i(w; z_r, w_r)$, $w \ge w_r$ intersect only in points (z_l, w_l), (z_r, w_r).

A_4 : If four points (z_l, w_l), (z_r, w_r), (z_m, w_m) and $(\hat{z}_m, \hat{w}_m)$ satisfy $z_m = R_2(w_m; z_l, w_l)$, $z_r = R_1(w_r; z_m, w_m)$, $\hat{z}_m = R_1(\hat{w}_m; z_l, w_l)$ and $z_r = R_2(w_r; \hat{z}_m, \hat{w}_m)$, then $(z_l - \hat{z}_m) + (\hat{w}_m - w_r) \le (w_l - w_m) + (z_m - z_r)$.

We say that system (3.1) belongs to Bakhvalov's class over Ω if it satisfies $A_1 - A_4$.

Theorem 3.1 (Bakhvalov, [1]). *If a strictly hyperbolic, genuinely nonlinear system* (3.1) *satisfies conditions* $A_1 - A_4$*, over the domain* Ω *given by* (3.3)*, then the Cauchy problem has a solution for arbitrary initial data* $U_0(x)$ *in* BV_{loc}.

Remark 3.1. The region considered by Bakhvalov is a subset of our Ω, so his theorem is a little stronger than the above statement.

The proof of Theorem 3.1 (see [1]) involves the construction of a non-increasing in time functional of approximate solutions, obtained by the Glimm scheme. This functional is a constant on solutions of Riemann problems and defined as follows. Let $\mathbf{U}_l$ and $\mathbf{U}_r$ be the left and right constant states in the Riemann problem and $(\mathbf{U}_l\mathbf{U}_r)(t,x)$ denote the corresponding solution. It consists of a shock (or rarefaction) wave of first kind followed be a shock (or rarefaction) wave of second kind.

Definition 3.1. Define $F[(\mathbf{U}_l\mathbf{U}_r)] = ([z(\delta_1)])_- + ([w(\delta_2)])_-$, where the first term is the absolute value of the increment in z across the first wave, if it is a shock, and 0 otherwise, and the second is the absolute value of increment of w across the second wave, if it is a shock, and 0 otherwise.

Lemma 3.1 (Bakhavalov, [1]). *If a strictly hyperbolic, genuinely nonlinear system* (3.1) *satisfies conditions* $A_1 - A_4$*, over the domain* Ω *given by* (3.3)*, then for any three states* $\mathbf{U}_l$*,* $\mathbf{U}_m$ *and* $\mathbf{U}_r$ *in* Ω *we have*

$$F[(\mathbf{U}_l\mathbf{U}_r)] \le F[(\mathbf{U}_l\mathbf{U}_m)] + F[(\mathbf{U}_m\mathbf{U}_r)], \tag{3.4}$$

and equality holds in (3.4) *if* $\mathbf{U}_m$ *is a value assumed by* $(\mathbf{U}_l\mathbf{U}_r)(x,t)$.

We will need the local version of above lemma. For any set B in (z, w) plane define $R[B]$ to be the set of all values of any Riemann problem with initial data in B.

Lemma 3.2. *Let* B_0 *and* B_1 *be rectangles in* (z, w) *plane with the property that* $R[R[B_0]] \subset B_1$ *and system* (3.1) *verifies Bakhvalov conditions* A_i*,* $i = 1, .., 4$*, when restricted to* B_1*. Then for any three states* $\mathbf{U}_l$*,* $\mathbf{U}_m$ *and* $\mathbf{U}_r$ *in* B_0 *we have*

$$F[(\mathbf{U}_l\mathbf{U}_r)] \le F[(\mathbf{U}_l\mathbf{U}_r)] + F[(\mathbf{U}_l\mathbf{U}_r)], \tag{3.5}$$

and equality holds in (3.4) *if* $\mathbf{U}_m$ *is a value assumed by* $(\mathbf{U}_l\mathbf{U}_r)(x,t)$.

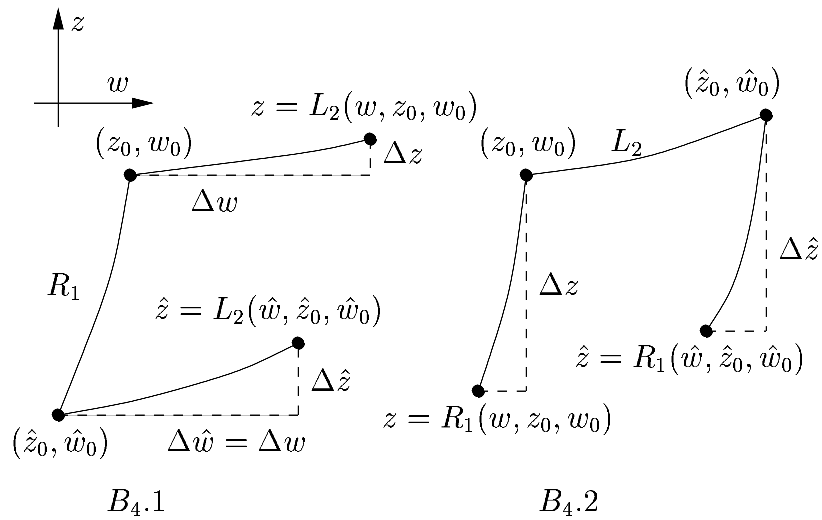

FIGURE 1. Hypothesis B_4.

It is convenient for our subsequent analysis to substitute condition A_4 by the following stronger condition introduced by DiPerna in [5]. Define

$$\begin{aligned} R_1(z_0, w_0) &= \{(z,w) : z = R_1(w; z_0, w_0),\ w \le w_0\}, \\ L_2(z_0, w_0) &= \{(z,w) : z = L_2(w; z_0, w_0),\ w \ge w_0\}, \end{aligned}$$

and

$$\Delta w = w - w_0,\ \Delta \hat{w} = \hat{w} - \hat{w}_0,\ \Delta z = z - z_0,\ \Delta \hat{z} = \hat{z} - \hat{z}_0.$$

Condition B_4 consists of the following.

$B_4.1$: Let $(\hat{z}_0, \hat{w}_0) \in R_1(z_0, w_0)$. If $z = L_2(w; z_0, w_0)$, $\hat{z} = L_2(\hat{w}; \hat{z}_0, \hat{w}_0)$ and $\Delta \hat{w} = \Delta w$ then $\Delta \hat{z} \ge \Delta z$.

$B_4.2$: Let $(\hat{z}_0, \hat{w}_0) \in L_2(z_0, w_0)$. If $z = R_1(w; z_0, w_0)$, $\hat{z} = R_1(\hat{w}; \hat{z}_0, \hat{w}_0)$ and $\Delta \hat{z} = \Delta z$ then $\Delta \hat{w} \ge \Delta w$.

The above conditions depend on the choice of the pair of Riemann invariants. For the classical ones, the system (1.1) satisfies conditions $A_1 - A_3$. This follows from the lemmas in section 2. The principal difficulty is the fact that neither B_4 nor A_4 holds for the classical Riemann invariants of system (1.1). The situation is parallel to that of the system of non-relativistic isentropic gas dynamics, in which case it was shown by DiPerna in [5] that it is still possible to find a pair of Riemann invariants z', w' for which the system satisfies $A_1 - A_3$, B_4, at least locally.

4. THE SHOCK CURVES

First, we state the result concerning the geometry of the shock curves in the plane of Riemann invariants. Let $\overline{\overline{\rho}} > 0$, with

$$\sqrt{p'(\overline{\overline{\rho}})} < c(1 - \delta_0), \tag{4.1}$$

for some $\delta_0 > 0$, independent of γ, and $\bar{\bar{\sigma}} < +\infty$ be given. Let $W(a,k) = \{(\sigma,\eta) \, : \, |\sigma - a| < k\eta\}$, where $a > 0$ is a constant such that $|a| < \bar{\bar{\sigma}}$. Let

$$\tilde{W}(a,k) = W(a,k) \cap \{(\sigma,\eta) \, : \, 0 < \eta < \eta(\bar{\bar{\rho}}),\, |\sigma| < \bar{\bar{\sigma}}\}.$$

Define a map $T(a,\theta) : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$\begin{aligned} T(a,\theta) \quad &: \quad (z,w) \to (z',w'), \\ z' \quad &= \quad 1 - \exp 2\theta(a/2 - z), \\ w' \quad &= \quad -1 + \exp 2\theta(w - a/2), \\ &\quad\;\; \theta \in \mathbb{R}. \end{aligned} \tag{4.2}$$

This map written in $\sigma = z + w$, $\eta = w - z$ variables has the following form

$$\begin{aligned} \sigma' &= \exp\theta(\eta + \sigma - a) - \exp\theta(\eta - \sigma + a), \\ \eta' &= \exp\theta(\eta + \sigma - a) + \exp\theta(\eta - \sigma + a) - 2. \end{aligned}$$

The next theorem and its proof is analogous to Theorem 3.3 in [5], except for the second part, in which we investigate the dependence of system (1.1) on γ.

Theorem 4.1. *There are positive constants c_1, c_2, depending on $(\gamma, k, \bar{\bar{\sigma}}, \delta_0)$, such that shock curves in $\tilde{W}(a,k) \cap \{\frac{c_1}{\theta} < \eta < \frac{c_2}{\theta}\}$ are mapped by $T(a,\theta)$ to shock curves satisfying hypothesis A_1, A_2, A_3, B_4. Moreover, we can choose $\bar{\bar{\sigma}} = \bar{\bar{\sigma}}(\gamma) \to +\infty$ and $k = k(\gamma) \to 0$, when $\gamma \to 1$, such that the following limits hold.*

$$\frac{k}{\gamma - 1} \to +\infty,\; c_2 \to +\infty,\; c_2 k \to 0,\; \frac{c_1}{c_2} \to 0, \quad \text{as } \gamma \to 1+. \tag{4.3}$$

We postpone the proof of Theorem 4.1 to Section 6.

5. Cauchy Problem

In this section we prove the existence part of Theorem 1.1; the decay (1.7) was already explained in the subsection 1.3. We carry out the construction of periodic, weak solution by the adaptation of Glimm's scheme, as was introduced by H.Frid in [7].

Proof of Theorem 1.1. In contrast with non-relativistic case, for the physically meaningful values of ρ the sound speed must be smaller than speed of light, that is, $p'(\rho) < c^2$. In the equation of state we assume that $\kappa < c$. Then, for all $\gamma_0 > 1$ there is $\hat{\rho}(\gamma_0) > 0$ such that $p'(\hat{\rho}) = c^2$. It is easy to see that $\hat{\rho}$ increases to $+\infty$ as $\gamma \to 1+$. We restrict γ to be so small that

$$\gamma \in (1, \gamma_0), \text{ where } \gamma_0 \text{ is such that } \bar{\rho} < \hat{\rho}(\gamma_0). \tag{5.1}$$

We will impose further restrictions on γ_0 in the course of the proof. Bounds (1.3) imply that there is $\bar{\sigma} > 0$ such that

$$\sup_{x \in \mathbb{R}} |\sigma(v_0(x))| < \bar{\sigma}. \tag{5.2}$$

Consider the set

$$\Omega = \left\{ (z,w) \, : \, w \le \sup_x w_0(x),\, z \ge \inf_x z_0(x) \right\}.$$

Ω is an invariant set for a Riemann problem formed with any two states in it. Next, we define the set which serves as an "admissible" region for approximate solutions defined through Glimm scheme.

Let $\overline{\overline{\sigma}} = \overline{\overline{\sigma}}(\gamma)$ be the function from Theorem 4.1; its explicit form, $\overline{\overline{\sigma}} = \bar\sigma + \beta\log(\gamma-1)^{-1}$, $\beta>0$, is given by Lemma 6.9 of Section 6.

Lemma 5.1. *There exist absolute constants δ_0, $\alpha>\beta$ and γ_0 and functions $\underline{\underline{\rho}} < \bar\rho$, $\overline{\overline{\rho}} > \bar\rho$ such that for any (η,σ) in $\Omega\cap\left\{\eta(\underline{\underline{\rho}})<\eta<\eta(\overline{\overline{\rho}})\right\}$ and $1<\gamma<\gamma_0$ we have*

$$\underline{\underline{\rho}} \le \rho \le \overline{\overline{\rho}}, \quad |\sigma| \le \overline{\overline{\sigma}}, \quad \sqrt{p'(\overline{\overline{\rho}})} < c(1-\delta_0). \tag{5.3}$$

Moreover,

$$\underline{\underline{\rho}} = \bar\rho(\gamma-1)^{\alpha}, \tag{5.4}$$

$$\eta(\overline{\overline{\rho}}) = \eta(\bar\rho) - \epsilon\log(\gamma-1), \tag{5.5}$$

for some $\epsilon>0$.

Remark 5.1. Conditions $\rho<\overline{\overline{\rho}}$ and $|\sigma|<\overline{\overline{\sigma}}$ are needed to apply the results of the previous section, whereas condition $\rho>\underline{\underline{\rho}}$ is needed to obtain the lower bound for the average of $\rho(t,\cdot)$.

Proof. Let us choose $\overline{\overline{\rho}}>\bar\rho$ from the equation

$$\eta(\overline{\overline{\rho}}) - \eta(\bar\rho) = \epsilon\log(\gamma-1)^{-1}, \tag{5.6}$$

with $\epsilon>0$ so small that

$$\frac{\kappa}{c}\overline{\overline{\rho}}^{(\gamma-1)/2} < 1-\delta_0, \tag{5.7}$$

for some $\delta_0>0$. Such $\overline{\overline{\rho}}$ exists. Indeed, from (5.6), upon use of (2.6), we have

$$\frac{\kappa}{c}\overline{\overline{\rho}}^{(\gamma-1)/2} = \tan\left[\arctan\frac{\kappa}{c}\bar\rho^{(\gamma-1)/2} + \epsilon\frac{\gamma-1}{2\sqrt{\gamma}}\log(\gamma-1)^{-1}\right]. \tag{5.8}$$

By assumptions on initial data (1.3), $\frac{\kappa}{c}\bar\rho^{(\gamma-1)/2} < 1-\hat\delta_0$, for some $\hat\delta_0>0$. Also, the function $\frac{\gamma-1}{2\sqrt{\gamma}}\log(\gamma-1)^{-1}$ is bounded. We now can choose ϵ such that the argument of tan in (5.8) is less then $\pi/4$, verifying by this (5.7). By the choice of $\overline{\overline{\rho}}$, (5.6), we trivially have $\overline{\overline{\rho}}>\bar\rho$.

Let $\underline{\underline{\rho}} = \bar\rho(\gamma-1)^{\alpha}$, $\alpha>0$ Then, for some $\psi\in(\bar\rho(\gamma-1)^{\alpha},\ \bar\rho)$,

$$\begin{aligned}
\eta(\bar\rho)-\eta(\underline{\underline{\rho}}) &= \frac{2\sqrt{\gamma}}{\gamma-1}\left(\arctan\frac{\kappa\bar\rho^{(\gamma-1)/2}}{c} - \arctan\frac{\kappa\underline{\underline{\rho}}^{(\gamma-1)/2}}{c}\right)\\
&= \frac{\kappa 2\sqrt{\gamma}}{c(\gamma-1)}\frac{1}{1+k^2c^{-2}\psi^{\gamma-1}}\bar\rho^{(\gamma-1)/2}\left(1-(\gamma-1)^{\alpha(\gamma-1)/2}\right)\\
&\le \frac{2\sqrt{\gamma}\kappa}{c}\bar\rho^{(\gamma-1)/2}\frac{1-(\gamma-1)^{\alpha(\gamma-1)/2}}{\gamma-1}\\
&= \frac{\sqrt{\gamma}\kappa}{c}\bar\rho^{(\gamma-1)/2}\alpha\log(\gamma-1)^{-1}\left(1+o(1)\right)\\
&= \frac{\sqrt{p'(\bar\rho)}}{c}\left(1+o(1)\right)\alpha\log(\gamma-1)^{-1}.
\end{aligned} \tag{5.9}$$

Note that by assumptions on initial data, $\frac{\kappa}{c}\bar\rho^{(\gamma-1)/2} < 1-\hat\delta_0$, for some $\hat\delta_0>0$. We thus obtain that $\eta(\bar\rho)-\eta(\underline{\underline{\rho}}) \le \beta\log(\gamma-1)^{-1}$ for all γ close to 1, if we choose $\alpha>\beta>0$ such that $(1-\delta_0)\alpha<\beta$.

Since we restrict (σ,η) to the set Ω we have

$$\begin{aligned}\sigma+\eta &\le \sup_x\left\{\frac{1}{2}\log\frac{c+v_0(x)}{c-v_0(x)}+\frac{2\sqrt{\gamma}}{\gamma-1}\arctan\frac{\kappa\rho_0(x)^{(\gamma-1)/2}}{c}\right\}\\ &\le \bar\sigma+\eta(\bar\rho),\end{aligned}$$

from where we derive $\sigma\le\bar\sigma+\eta(\bar\rho)-\eta(\underline{\underline{\rho}})\le\bar\sigma+\beta\log(\gamma-1)^{-1}=\bar{\bar\sigma}$. Analogously, $\sigma\ge-\bar\sigma-\eta(\bar\rho)+\eta(\underline{\underline{\rho}})=-\bar{\bar\sigma}$. □

Define

$$\Omega_a=\{|\sigma|<\bar{\bar\sigma}\}\cap\left\{\eta(\bar{\bar\rho})<\eta<\eta(\underline{\underline{\rho}})\right\}.$$

Let $U=(U_1(x),\,U_2(x))$ be a vector function in $BV_{loc}(\mathbb{R})$, periodic with period 1, such that $(\sigma\circ U(x),\,\eta\circ U(x))$, obtained through (1.4), (2.5) and (2.6), assumes values in Ω_a, for all $x\in\mathbb{R}$, and such that

$$\left|\int_0^1(U(x)-U_0(x))\,dx\right|<\frac{1}{2}\underline{\rho}. \tag{5.10}$$

We can obtain the bounds on the average of $\rho\circ U(x)$. From the definition of function U_1 in (1.4) we get

$$U_1\ge\rho. \tag{5.11}$$

Also, since values of $(\sigma,\,\eta)$ belong to Ω_a, it follows that $v^2<c^2$, $c^2-v^2=c^2\frac{4e^\sigma}{(1+e^\sigma)^2}\ge c^2e^{-\bar{\bar\sigma}}$, $\frac{p}{\rho c^2}<1$ and thus

$$U_1=\rho\left(\frac{p}{\rho c^2}+1\right)\frac{v^2}{c^2-v^2}+\rho\le 3e^{\bar{\bar\sigma}}\rho. \tag{5.12}$$

Similarly, for initial data $(\rho_0(x),\,\sigma_0(x))$ satisfying bounds (1.3) and (5.2) we have

$$\underline{\rho}<U_{0,1}(\cdot)<3e^{\bar\sigma}\bar\rho, \tag{5.13}$$

which implies the following estimate for the space average over the period Π of $U_{0,1}$.

$$\underline{\rho}<(U_{0,1})_\Pi<3e^{\bar\sigma}\bar\rho. \tag{5.14}$$

Using (5.10), (5.11) and (5.12) we arrive at the following inequalities.

$$\frac{e^{-\bar{\bar\sigma}}}{6}\underline{\rho}<(\rho\circ U)_\Pi<4e^{\bar\sigma}\bar\rho. \tag{5.15}$$

Remark 5.2. The inequalities in (5.15) imply, in particular, that $\rho\circ\mathbf{U}(x)$ takes a value in the interval $(\check\rho,\,\hat\rho)$, where

$$\check\rho=e^{-\bar{\bar\sigma}}\underline{\rho}/6,\ \hat\rho=4e^{\bar\sigma}\bar\rho, \tag{5.16}$$

and we agree that we can redefine $\rho\circ U(x)$ in any discontinuity point x_0 such that $\rho\circ U(x_0)$ is any suitable value in the interval between $\rho\circ U(x_0-0)$ and $\rho\circ U(x_0+0)$, observing that this redefinition does not changes the $\mathrm{Var}_\Pi(\rho\circ U)$.

Remark 5.3. A computation similar to (5.9) shows that there is a positive constant $\varepsilon<\beta$ such that for any $(\eta,\,\sigma)\in\Omega\cap\{\eta(\check\rho)<\eta<\eta(\hat\rho)\}$ and γ is close to 1 holds that

$$|\sigma|<\hat\sigma=\bar\sigma-\varepsilon\log(\gamma-1). \tag{5.17}$$

Define

$$\Omega_c = \{|\sigma| < \hat{\sigma}\} \cap \{\eta(\check{\rho}) < \eta < \eta(\hat{\rho})\}\,. \tag{5.18}$$

Let us apply Theorem 4.1 with $\overline{\overline{\rho}}$ and $\overline{\overline{\sigma}}$ chosen as above and

$$a = 0,\, \theta = \frac{c_2}{2\eta(\hat{\rho})}. \tag{5.19}$$

Map $T(0,\theta)$ takes shock curves in

$$\tilde{W} = \{|\sigma| < k\eta\} \cap \left\{|\sigma| < \overline{\overline{\sigma}}\right\} \cap \left\{\max\{2\frac{c_1}{c_2}\eta(\hat{\rho}),\, \eta(\underline{\underline{\rho}})\} < \eta < \min\{2\eta(\hat{\rho}),\, \eta(\overline{\overline{\rho}})\}\right\} \tag{5.20}$$

to shock curves satisfying hypotheses $A_1 - A_4$. Let

$$\tilde{\mathcal{R}}[V(\gamma)] \equiv \{\eta(\check{\rho}) - V(\gamma) < \eta < \eta(\hat{\rho}) + V(\gamma)\} \cap \{|\sigma| < \hat{\sigma} + V(\gamma)\}\,. \tag{5.21}$$

Lemma 5.2. *There exist a positive, monotone increasing function $V(\gamma)$, $1 < \gamma < 2$, with $V(\gamma) = o(-\log(\gamma-1))$ as $\gamma \to 1+$, and $\gamma_0 > 1$ such that for all $1 < \gamma < \gamma_0$ we have*

$$\Omega_c \subset \tilde{\mathcal{R}}[V(\gamma)] \subset \Omega_a \subset \tilde{W}. \tag{5.22}$$

Proof. The first inclusion in (5.22) holds trivially for any $V(\gamma) > 0$. To find $V(\gamma)$ with properties stated in the lemma and to prove the second inclusion it is enough to show that

$$\eta(\overline{\overline{\rho}}) - \eta(\hat{\rho}) \geq -\epsilon_1 \log(\gamma - 1), \tag{5.23}$$

$$\eta(\check{\rho}) - \eta(\underline{\underline{\rho}}) \geq -\epsilon_2 \log(\gamma - 1), \tag{5.24}$$

for some $\epsilon_i > 0$, $i = 1,2$ and γ close to 1 and then, use the fact that $\overline{\overline{\sigma}} - \hat{\sigma} = (\varepsilon - \beta)\log(\gamma - 1)$, $\varepsilon < \beta$. Consider

$$\begin{aligned} \eta(\overline{\overline{\rho}}) - \eta(\hat{\rho}) &= \eta(\overline{\overline{\rho}}) - \eta(\bar{\rho}) + \eta(\bar{\rho}) - \eta(\hat{\rho}) \\ &\geq \frac{\epsilon}{2}\log(\gamma - 1)^{-1}, \end{aligned}$$

by (5.6) and the fact that $\eta(\hat{\rho}) - \eta(\bar{\rho}) = \eta(4e^{\bar{\sigma}}\bar{\rho}) - \eta(\bar{\rho})$ is bounded independently of γ close to 1 we establish (5.24) with $\epsilon_1 = \frac{\epsilon}{2}$. Recalling the definitions $\check{\rho} = \underline{\rho}e^{-\overline{\overline{\sigma}}}/6 = \underline{\rho}e^{\bar{\sigma}}(\gamma - 1)^{\beta}/6$, $\underline{\underline{\rho}} = \bar{\rho}(\gamma - 1)^{\alpha}$, $\alpha > \beta > 0$, we have the following inequality

$$\begin{aligned} \eta(\check{\rho}) - \eta(\underline{\underline{\rho}}) &= \eta\left(\frac{\underline{\rho}e^{-\bar{\sigma}}}{6}(\gamma - 1)^{\beta}\right) - \eta(\bar{\rho}(\gamma - 1)^{\alpha}) \\ &\geq \frac{2\kappa\sqrt{\gamma}}{c}\left[\frac{\underline{\rho}e^{-\bar{\sigma}}}{6}(\gamma - 1)^{\beta}\right]^{(\gamma-1)/2} \frac{1 - \left[\frac{6\bar{\rho}e^{\bar{\sigma}}}{\underline{\rho}}(\gamma - 1)^{(\alpha-\beta)}\right]^{(\gamma-1)/2}}{\gamma - 1} \\ &= -\epsilon_2 \log(\gamma - 1), \end{aligned} \tag{5.25}$$

for some $\epsilon_2 > 0$. This verifies (5.24). To prove that $\Omega_a \subset \tilde{W}$ we show that for γ sufficiently close to 1 next conditions hold.

$$\min\{2\eta(\hat{\rho}),\, \eta(\overline{\overline{\rho}})\} = \eta(\overline{\overline{\rho}}), \tag{5.26}$$

$$\max\{2\frac{c_1}{c_2}\eta(\hat{\rho}),\, \eta(\underline{\underline{\rho}})\} = \eta(\underline{\underline{\rho}}), \tag{5.27}$$

$$\eta(\overline{\overline{\rho}})k - \overline{\overline{\sigma}} > 0. \tag{5.28}$$

Since $(\gamma-1)\eta(\hat\rho)>0$ uniformly in γ close to 1 and

$$\begin{aligned}\eta(\overline{\overline{\rho}})-\eta(\hat\rho) &= \eta(\overline{\overline{\rho}})-\eta(\bar\rho)+\eta(\bar\rho)-\eta(\hat\rho)\\ &\le 2\epsilon\log(\gamma-1)^{-1},\end{aligned}$$

by (5.6) and the fact that $\eta(\hat\rho)-\eta(\bar\rho)=\eta(4e^{\bar\sigma}\bar\rho)-\eta(\bar\rho)$ is bounded independently of γ close to 1 we conclude (5.26). On the other hand, since $c_1/c_2\to 0$ as $\gamma\to 1+$ by Theorem 4.1, (5.27) holds. By Lemma 6.9, $\overline{\overline{\sigma}}=\bar\sigma-\beta\log(\gamma-1)$ and $k=(\gamma-1)^{-\iota(\gamma-1)}-1=\iota(\gamma-1)\log(\gamma-1)^{-1}(1+o(1))$ and $\beta/\iota<2\arctan(\kappa/c)$. Thus

$$\begin{aligned}\eta(\overline{\overline{\rho}})k-\overline{\overline{\sigma}} &> \iota 2\sqrt{\gamma}\arctan\frac{\kappa(\bar\rho)^{(\gamma-1)/2}}{c}(1+o(1))\log(\gamma-1)^{-1}\\ &- \bar\sigma-\beta\log(\gamma-1)^{-1}=-\epsilon_3\log(\gamma-1),\end{aligned}$$

with $\epsilon_3=2\iota\arctan\frac{\kappa}{c}-\beta$. This verifies the last condition (5.28). □

Remark 5.4. For a function $V(\gamma)$ as in Lemma 5.2 and a positive constant N, (5.22) holds also for $NV(\gamma)$, provided that γ is sufficiently close to 1.

Lemma 5.3. *There exists a constant $C_0>0$, independent of γ, such that*

$$|\eta(\rho_1)-\eta(\rho_2)|\le C_0|\rho_1-\rho_2|, \tag{5.29}$$

if $\underline{\rho}\le\rho_i\le\bar\rho$, $i=1,2$ and provided that γ is sufficiently close to 1.

Proof. This follows immediately from the definition of $\eta(\rho)$ in (2.6). □

Let $(z',\, w')$ be the pair of Riemann invariants given by Theorem 4.1, when $a=0$, $\theta=\frac{c_2}{2\eta(\hat\rho)}$. It is convenient to use a normalized version of $(z',\, w')$. Define $(z'',\, w'')$ by

$$z''=\frac{\exp(-\theta\eta(\check\rho))}{\theta}z',\qquad w''=\frac{\exp(-\theta\eta(\check\rho))}{\theta}w'. \tag{5.30}$$

The next lemma establishes that increments in $(z'',\, w'')$ are comparable with corresponding increments in $(\sigma,\, \eta)$.

Lemma 5.4. *There exists an absolute constant $C>0$ such that*

$$C^{-1}(|\sigma_1-\sigma_2|+|\eta_1-\eta_2|)\le|z_1''-z_2''|+|z_1''-w_2''|\le C(|\sigma_1-\sigma_2|+|\eta_1-\eta_2|), \tag{5.31}$$

if $(\sigma_i,\, \eta_i)\in\tilde{\mathcal{R}}[KV(\gamma)]$, $z_i''=z''(\sigma_i,\eta_i)$, and $w_i''=w''(\sigma_i,\eta_i)$, for any $K>0$, provided that γ is close to 1.

Proof. Let us consider dependence $z''=z''(\sigma,\, \eta)$, and $w''=w''(\sigma,\eta)$. The determinant of Jacobian matrix, $J=\frac{\partial(z'',w'')}{\partial(\sigma,\eta)}$, equals $-2\exp(\theta(\eta-\eta(\check\rho)))$ and each element $|J_{ij}|$, $i,j=1,2$, is smaller than $exp(\theta|\sigma|+\theta|\eta-\eta(\check\rho)|)$. The set $\tilde{\mathcal{R}}[KV(\gamma)]$ is convex in $(\sigma,\, \eta)$ plane and, consequently, using notation

$$||J||=\sup_{(\sigma,\eta)\in\tilde{\mathcal{R}}[KV(\gamma)]} e^{\theta|\sigma|+\theta|\eta-\eta(\check\rho)|}, \tag{5.32}$$

we can write the following estimate

$$\begin{aligned}|z_1''-z_2''|+|z_1''-w_2''| &\ge \inf_{(\sigma,\eta)\in\tilde{\mathcal{R}}[KV(\gamma)]} -2e^{(\theta(\eta-\eta(\check\rho))}||J||(|\sigma_1-\sigma_2|+|\eta_1-\eta_2|),\\ |z_1''-z_2''|+|w_1''-w_2''| &\le \sup_{(\sigma,\eta)\in\tilde{\mathcal{R}}[V(\gamma)]} -2e^{(\theta(\eta-\eta(\check\rho))}||J||(|\sigma_1-\sigma_2|+|\eta_1-\eta_2|)\end{aligned}$$

Let us show that for the choice of θ, (5.19), it holds that $\theta(\eta(\hat\rho)-\eta(\check\rho)+2KV(\gamma)+\hat\sigma)\to 0$, with $\gamma\to 1+$. This will suffice to conclude the lemma since, for $(\sigma,\eta)\in\tilde{\mathcal{R}}[KV(\gamma)]$, η ranges over the interval $[\eta(\check\rho)-KV(\gamma),\,\eta(\hat\rho)+KV(\gamma)]$ and $|\sigma|<\hat\sigma+KV(\gamma)$. By Lemma 5.2, $V(\gamma)=o\left(-\log(\gamma-1)\right)\to\infty$ as $\gamma\to 1+$. The estimate similar to (5.25) shows that $\eta(\hat\rho)-\eta(\check\rho)=O\left(-\log(\gamma-1)\right)$, and, by (5.17), $\hat\sigma=\bar\sigma-\varepsilon\log(\gamma-1)$. The definition of $\eta=\eta(\rho)$ and $\hat\rho=4e^{\bar\sigma}\bar\rho$ imply that

$$\eta(\hat\rho)=2\frac{\sqrt{\gamma}}{(\gamma-1)}\arctan\frac{\kappa}{c}(4e^{\bar\sigma}\bar\rho)^{(\gamma-1)/2}=2\frac{\sqrt{\gamma}}{(\gamma-1)}(\arctan\frac{\kappa}{c}+o(1)). \tag{5.33}$$

By Lemma 6.9, $k=\iota(\gamma-1)\log(\gamma-1)^{-1}(1+o(1))$, and so $\eta(\hat\rho)k=\iota 2\sqrt{\gamma}\arctan(\frac{\kappa}{c})\log(\gamma-1)^{-1}(1+o(1))$. Since, by Theorem 4.1, $c_2k\to 0$, we have for $\gamma\to 1+$ that

$$\begin{aligned}\theta(\eta(\hat\rho)-\eta(\check\rho)+2KV(\gamma)+\hat\sigma) &= \frac{c_2k}{2\eta(\hat\rho)k}(\eta(\hat\rho)-\eta(\check\rho)+V(\gamma)+\bar\sigma-\varepsilon\log(\gamma-1))\\ &= \frac{o(1)O\left(-\log(\gamma-1)\right)}{2\iota\gamma\arctan(\frac{\kappa}{c})(1+o(1))\log(\gamma-1)^{-1}}\to 0.\end{aligned}$$

□

Now, we construct a sequence of approximate solutions, $U^h(x,t)$, using Glimm's method, as recalled in subsection 1.1. We consider in detail the solutions of Riemann problems that make up U^h. Denote $(\sigma^h,\eta^h)=(\sigma\circ U^h,\eta\circ U^h)$, $(z^h,w^h)=(z\circ U^h,w\circ U^h)$. Assume that $U^h(x,t)$ can be constructed in the interval $[0,nh]$, satisfying (5.10). Then $\eta^h(\cdot,(n-1)h)$ takes a value in $[\eta(\check\rho),\eta(\hat\rho)]$. Furthermore, since $(z^h,w^h)(x,t)$ assumes values in Ω for all $(x,t)\in\mathbb{R}\times[0,(n-1)h]$, it follows from Remark 5 that $(\sigma^h(\cdot,(n-1)h),\eta^h(\cdot,(n-1)h))$ takes a value in Ω_c. Denote this value by (σ_c,η_c). If

$$\mathrm{Var}_\Pi[(\sigma^h(\cdot,t),\eta^h(\cdot,t))]<V(\gamma),\quad t\in[0,(n-1)h], \tag{5.34}$$

then, upon using the fact that Bakhvalov's condition A_2 holds in (z,w) plane, we obtain

$$(\sigma^h(\cdot,t),\eta^h(\cdot,t))\in\left\{(\sigma,\eta)\,:\,|\sigma-\sigma_c|+|\eta-\eta_c|<2V(\gamma)\right\},\quad t\in[(n-1)h,nh], \tag{5.35}$$

$$(\sigma^h(\cdot,t),\eta^h(\cdot,t))\in B_0=\left\{(\sigma,\eta)\,:\,|\sigma-\sigma_c|+|\eta-\eta_c|<4V(\gamma)\right\},\quad t\in[nh,(n+1)h]. \tag{5.36}$$

By the same argument, using notation in Section 3, we derive the inclusion

$$R[R[B_0]]\subset B_1=\left\{(\sigma,\eta)\,:\,|\sigma-\sigma_c|+|\eta-\eta_c|<16V(\gamma)\right\}. \tag{5.37}$$

We have $B_1\subset\tilde{\mathcal{R}}[16V(\gamma)]$ and so $B_1\subset\tilde W$, provided that γ is close to 1, by Remark 5.4. Theorem 4.1 ensures that map $T(0,\theta)$ takes shock curves in B_1 onto shock curves satisfying Bakhvalov's conditions A_1-A_4. It is straightforward to see that $B_i'=T(0,\theta)[B_i]$, $i=0,1$ are rectangles in (z',w') plane and moreover, by (5.37), $R[R[B_0']]\subset B_1'$.

Let $F[(\mathbf{U}_l\mathbf{U}_r)]$ be the functional given by Definition 3.1, using (z',w') instead of (z,w) as the pair of Riemann invariants. Approximate solutions $U^h(\cdot,\nu h+0)$ are piecewise constant when $\nu=0,\dots,n$. Assume, as we may, that $\Delta x=l=2^{-k_l}$, for some $k_l\in\mathbb{N}$. define

$$\mathbf{F}[U^h](t):=\sum_{1\le j\le 2^{k_l}}F[(U^h((j-1)l,\nu h+0)U^h(jl,\nu h+0))],\ t\in[\nu h,(\nu+1)h). \tag{5.38}$$

We have the following corollary of Lemma 3.2.

Corollary 5.1.

$$\mathbf{F}[U^h](nh) \leq \mathbf{F}[U^h]((n-1)h) \leq \cdots \leq \mathbf{F}[U^h](0). \tag{5.39}$$

Moreover, for $t \in [0,(n+1)h)$*, we have*

$$\frac{1}{2} DV_{\Pi}[({z'}^h, {w'}^h)(\cdot,t)] \leq \mathbf{F}[U^h](t) \leq \mathrm{Var}_{\Pi}[({z'}^h, {w'}^h)(t,\cdot)], \tag{5.40}$$

where DV_{Π} *stands for the decreasing variation over one period.*

Proof. The first inequality in (5.1) follows from Lemma 3.2, recalling the construction of Glimm's approximate solution, using, in case $-1/2 < \alpha_n \leq 0$,

$$\begin{aligned} F[(U^h((j-1)l,(n-1)h+0)U^h(jl,(n-1)h+0))] &\\ = F[(U^h((j-1)l, nh-0)U^h((j+\alpha_n)l, nh-0))] &\\ + F[(U^h((j+\alpha_n)l, nh-0)U^h(jl, nh-0))], & \end{aligned}$$

$$\begin{aligned} F[(U^h((j-1+\alpha_n)l, nh-0)U^h((j-1)l, nh-0))] &\\ + F[(U^h((j-1)l, nh-0)U^h((j+\alpha_n)l, nh-0))] &\\ \geq F[(U^h((j-1+\alpha_n)l, nh-0)U^h((j+\alpha_n)l, nh-0)] & \end{aligned}$$

and periodicity. In case $0 < \alpha_n < 1/2$, we proceed similarly, only replacing $j-1$ by j in both relations above. The subsequent inequalities in (5.39) are reiterations of the first one with time steps $(n-k)h$ and $(n-k-1)h$ instead of nh and $(n-1)h$, for $k = 1, \ldots, n-1$. As for inequalities (5.40), the first one holds because z' and w' decrease in shocks and increase in rarefaction waves, and also because, by property A_2, we have $|[w(\delta_1)]| \leq |[z(\delta_1)]|$ and $|[z(\delta_2)]| \leq |[w(\delta_2)]|$, where $\delta_1(\delta_2)$ is a shock of the first(second) family. The last inequality in (5.40) is then obvious. □

We can use the non-increasing property of the functional $\mathbf{F}$ to bound the total variation per period of the approximate solutions.

Lemma 5.5. *For any* t*, such that* $0 < t < (n+1)h$

$$\mathrm{Var}_{\Pi}\left[({z'}^h, {w'}^h)\right](t) \leq 4\,\mathrm{Var}_{\Pi}\left[({z'}^h, {w'}^h)\right](0). \tag{5.41}$$

Proof. Let $0 < t < (n+1)h$. Since the approximate solutions are periodic with period 1, we have

$$\mathrm{Var}_{\Pi}\left[({z'}^h, {w'}^h)(\cdot,t)\right] \leq 2DV_{\Pi}\left[({z'}^h, {w'}^h)(\cdot,t)\right]. \tag{5.42}$$

Using (5.39), (5.40) and inequality $\mathrm{Var}_{\Pi}[({z'}^h, {w'}^h)](0) \leq \mathrm{Var}_{\Pi}[(z'_0, w'_0)]$ we get

$$\begin{aligned} \mathrm{Var}_{\Pi}\left[({z'}^h, {w'}^h)\right](t) \leq 4\mathbf{F}\left[U^h\right](t) \quad &\leq \quad 4\mathbf{F}\left[U^h\right](0) \\ &\leq \quad 4\,\mathrm{Var}_{\Pi}\left[({z'}^h, {w'}^h)\right](0) \leq 4\,\mathrm{Var}_{\Pi}\left[(z'_0, w'_0)\right]. \end{aligned}$$

□

Since (z'', w'') are constant multiples of (z', w') it also holds that for t specified in the Lemma

$$\mathrm{Var}_\Pi\left[({z''}^h, {w''}^h)\right](t) \le 4\,\mathrm{Var}_\Pi\left[(z''_0, w''_0)\right].$$

The values of U^h lie in $\tilde{\mathcal{R}}[16V(\gamma)]$. Lemma 5.4 can be used, with $K = 16$, to write that for all γ sufficiently small,

$$\mathrm{Var}_\Pi\left[(\sigma_h, \eta_h)\right](t) \le 4C^2\,\mathrm{Var}_\Pi\left[(\sigma_0, \eta_0)\right].$$

Finally, by Lemma 5.3 we obtain that

$$\mathrm{Var}_\Pi\left[(\sigma_h, \eta_h)\right](t) \le 4C^2 C_0\,\mathrm{Var}_\Pi\left[(\sigma_0, \rho_0)\right].$$

We thus impose the following restriction on $\mathrm{Var}_\Pi\left[(\sigma_0, \rho_0)\right]$:

$$\mathrm{Var}_\Pi\left[(\sigma_0, \rho_0)\right] < (4C^2C_0)^{-1}V(\gamma). \tag{5.43}$$

We have then obtained that, under the above restriction on total variation of initial data,

$$\mathrm{Var}_\Pi\left[(\sigma^h, \eta^h)\right](t) < V(\gamma), \quad t \in [0,\,(n+1)h). \tag{5.44}$$

Remark 5.5. As a consequence of (5.44) we obtain that $\mathrm{Var}_\Pi\left[U^h\right](t) < \tilde{C}(\gamma, \Omega_a)$, for $t \in [0,\,(n+1)h)$ and some $\tilde{C} > 0$.

The following lemma is proved following the same procedures as in [8] but using also periodicity (cf. [7]).

Lemma 5.6. *There exists $\tilde{C}_1 > 0$ depending on γ and Ω_a such that for all $0 \le s < t < (n+1)h$ we have*

$$\int_0^1 |U^h(x,t) - U^h(x,s)|\,dx < \tilde{C}_1(|t-s| + h). \tag{5.45}$$

In particular, there is a $T' > 0$ and $h_0 > 0$, both independent of h, n, such that $U^h(\cdot, t)$ verifies condition (5.10)*, i.e.,*

$$\left|\int_0^1 (U^h(x,t) - U_0(x))\,dx\right| < \frac{\rho}{2}, \tag{5.46}$$

for $0 < h < h_0$ and $0 \le t < \min(T', (n+1)h)$.

Now, let us take $T > 0$ satisfying $h_0 < T < T'$. Iterating the above argument we obtain a sequence of approximate solutions $U^h(x,t)$ with $h < h_0$ and $t \in [0, T')$ such that $U^h(x,t) \in \Omega_a$ and (5.44) holds. Applying results of [8] we conclude that there is a sequence $h_k \to 0$ and a vector function U such that $U^{h_k} \to U$ in $C([0,T]; L^1_{loc}(\mathbb{R}))^2$ and $a.e.\,x$ for each t in $[0,T]$. Moreover there is a set $\Theta_0 \subset \prod_1^\infty(-1/2, 1/2)$, with measure 1, such that, for any sampling sequence $\{\alpha_n\}$ from Θ_0, U is a weak entropy solution of (1.1) in $\mathbb{R} \times [0, T')$. Suppose that for some integer N there is a sequence $\tilde{h}_k$, a set $\Theta_N \subset \prod_1^\infty(-1/2, 1/2)$ of measure 1, such that $U^{\tilde{h}_k}$ is defined on time interval $[0, NT]$, satisfies (5.44) and (5.46) for $t \in [0, NT]$ and $U^{\tilde{h}_k} \to U^N$ as $k \to +\infty$ in $C([0,NT]; L^1_{loc}(\mathbb{R}))^2$ and $a.e.\,x$ for each t in $[0, NT]$, which is a weak entropy solution of (1.1) in $\mathbb{R} \times [0, (N-1)T + T')$, for all sampling sequences $\{\alpha_n\}$ in

Θ_N. Since U^N is a weak solution we have $\int_0^1 U^N(x,NT)\,dx = \int_0^1 U_0(x)\,dx$. Thus, for $NT < t < (N-1)T + T'$,

$$\begin{aligned} \left|\int_0^1 U^{\tilde{h}_n}(x,t) - U_0(x)\,dx\right| &\le \left|\int_0^1 U^{\tilde{h}_n}(x,t) - U^{\tilde{h}_n}(x,NT)\,dx\right| \\ &+ \left|\int_0^1 U^{\tilde{h}_n}(x,NT) - U_0(x)\,dx\right| \\ &< \tilde{C}_1(t - NT + \tilde{h}_n) + ||U^{\tilde{h}_n}(\cdot,NT) - U^N(\cdot,NT)||_{L^1(0,1)}. \end{aligned}$$

The right hand side is smaller than $\frac{\rho}{2}$ when $\tilde{h}_k$ is sufficiently small because of the way we have chosen T and fact that $||U^{\tilde{h}_k}(\cdot,NT) - U^N(\cdot,NT)||_{L^1(0,1)} \to 0$, as $k \to \infty$. For these $\tilde{h}_k$, the sequence $U^{\tilde{h}_k}(x,t)$ can be defined on the interval $[0,(N+1)T]$ while keeping estimates (5.44) and (5.46). Extracting a convergent subsequence we obtain a vector function U^{N+1} which coincide with U^N when $t \in [0,NT]$ and which is a weak entropy solution of (1.1) for all sampling sequences $\{\alpha_n\}$ in $\Theta_{(N+1)} \subset \Theta_N$, of measure 1. This process can be reiterated and, consequently, a weak entropy solution defined by $U(x,t) = U^N(x,t)$, for $t \in [0,NT]$, exists for all times $t > 0$. Finally, U satisfies bounds (5.44) and (5.46) and takes values in the bonded set Ω_a. □

6. Proof of Theorem 4.1

We give the proof in the sequence of lemmas. We start by investigating the conditions under which property A_2 is preserved by the map $T(a,\theta)$.

Lemma 6.1 (DiPerna, [5], p.249). *A_2 is equivalent to the requirement that*

$$-\infty < \frac{\partial L_1}{\partial \eta}, \frac{\partial R_1}{\partial \eta} < -1, \qquad 1 < \frac{\partial L_2}{\partial \eta}, \frac{\partial R_2}{\partial \eta} < \infty \qquad \text{for } \eta \ne \eta_0.$$

Lemma 6.2 (DiPerna, [5], p.249). *The image of the shock curve $\sigma = R_i(\eta;\sigma_0,\eta_0)$ or $\sigma = L_i(\eta;\sigma_0,\eta_0)$, $i = 1,2$ under $T(a,\theta)$ satisfies A_2 if*

$$|\sigma - a| \le \frac{1}{2\theta}\left|\log\left\{\frac{\sigma_\eta + (-1)^i}{\sigma_\eta - (-1)^i}\right\}\right|. \tag{6.1}$$

By the following lemma we estimate σ_η for the shocks in $\tilde{W}(a,k)$.

Lemma 6.3. *There exists an absolute constant $\hat{C} > 0$, such that*

$$\left|\frac{\partial R_i}{\partial \eta}\right|, \left|\frac{\partial L_i}{\partial \eta}\right| \le \hat{C} e^{\overline{\overline{\sigma}}} \left(\frac{k+1}{k-1}\right)^{2\frac{\gamma+1}{\gamma-1}}, \tag{6.2}$$

if $\sigma = R_i(\eta;\sigma_0,\eta_0)$ and $\sigma = L_i(\eta;\sigma_0,\eta_0)$ lie in $\tilde{W}(a,k)$.

Proof. The estimate is straightforward. We consider only the curve $\sigma = L_2(\eta;\sigma_0,\eta_0)$, for the proof is similar for all other cases. From the discussion of section 2 it follows that it is possible to choose the reference frame $(\bar{t},\,\bar{x})$ in such a way that $v_0 = 0$, where $\sigma_0 = \sigma(v_0)$. In the computation to follow we are going to use some shorthand notations;

$$\delta\rho = \rho - \rho_0 > 0,\ \delta p = p(\rho) - p(\rho_0),\ p' = p'(\rho),\ p'_0 = p'(\rho_0). \tag{6.3}$$

Then, by the parametrization of the Hugoniot curve (2.7), and formula for σ, (2.5), we have:

(6.4)

$$
\begin{aligned}
\frac{d\sigma}{d\eta} &= \frac{2c}{c^2-v^2}\frac{dv}{d\eta} = \frac{c^3}{c^2-v^2}\sqrt{\frac{(p+\rho_0c^2)(p_0+\rho c^2)}{\delta p\delta\rho}} \\
&\quad\times\left\{\frac{(p'\delta\rho+\delta p)(p+\rho_0c^2)(p_0+\rho c^2)-\delta p\delta\rho(p'(p_0+\rho c^2)+c^2(p+\rho_0c^2))}{(p+\rho_0c^2)^2(p_0+\rho c^2)^2}\right\}\rho_\eta \\
&= \frac{c^3}{c^2-v^2}\sqrt{\frac{(p+\rho_0c^2)(p_0+\rho c^2)}{\delta p\delta\rho}} \\
&\quad\times\left\{\frac{(p'+\frac{\delta p}{\delta\rho})(p+\rho_0c^2)(p_0+\rho c^2)-\delta p(p'(p_0+\rho c^2)+c^2(p+\rho_0c^2))}{(p+\rho_0c^2)^2(p_0+\rho c^2)^2}\right\}\rho_\eta\delta\rho \\
&= \frac{c^3}{c^2-v^2}\sqrt{\frac{\delta\rho}{\delta p}(p+\rho_0c^2)(p_0+\rho c^2)} \\
&\quad\times\left\{p'(p_0+\rho c^2)+\frac{\delta p}{\delta\rho}(p+\rho_0c^2)\right\}\frac{\rho_\eta(p_0+\rho_0c^2)}{(p+\rho_0c^2)^2(p_0+\rho c^2)^2} \\
&\le \frac{c^3}{c^2-v^2}\sqrt{\frac{\delta\rho}{\delta p}(p+\rho_0c^2)(p_0+\rho c^2)}\frac{(p+\rho c^2)}{2c\sqrt{p'}}p' \\
&\quad\times\frac{(p+\rho c^2+p_0+\rho_0c^2)(p_0+\rho_0c^2)}{(p+\rho_0c^2)^2(p_0+\rho c^2)^2} \\
&\le \frac{1}{2}\frac{c^2}{c^2-v^2}\sqrt{\frac{p'}{p_0'}}\frac{(p+\rho c^2+p_0+\rho_0c^2)(p_0+\rho_0c^2)(p+\rho c^2)}{(p+\rho_0c^2)^{3/2}(p_0+\rho c^2)^{3/2}} \\
&\le \frac{1}{2}\frac{1}{1-\frac{v^2}{c^2}}\sqrt{\frac{p'}{p_0'}}\frac{2\rho^2\rho_0(\frac{p}{\rho c^2}+1)^2(\frac{p_0}{\rho_0c^2}+1)}{\rho^{3/2}\rho_0^{3/2}},
\end{aligned}
$$

where in the first inequality we substituted $\rho_\eta = \frac{p+\rho c^2}{2c\sqrt{p'}}$ and used the fact $0 < \frac{\delta p}{\delta\rho} < p'$. In the second inequality we used $p_0' < \frac{\delta p}{\delta\rho}$, and in the last the fact that $p+\rho c^2 \ge p_0+\rho_0c^2$, all true since $\rho > \rho_0$. We remind that ρ_0 refers to the state on the right and $\sigma = L_2(\eta,\sigma_0,\eta_0)$ is the shock curve of the second family. Now, for all states (ρ,v) in $\tilde W(a,k)$ we have

$$
\begin{gathered}
\sqrt{p'(\rho)} < c, \text{ or } \tfrac{p}{\rho c^2} < \gamma, \\
\left|\tfrac{v}{c}\right| \le \tfrac{1-e^{-\overline{\overline{\sigma}}}}{1+e^{-\overline{\overline{\sigma}}}}, \text{ or } 1-\left(\tfrac{v}{c}\right)^2 \ge e^{-\overline{\overline{\sigma}}} \quad .
\end{gathered}
$$

Hence, from (6.4), after some calculations we arrive at

$$
\sigma_\eta \le C_4 e^{\overline{\overline{\sigma}}}\sqrt{\frac{p'\rho^2}{p_0'\rho_0^2}} = C_4 e^{\overline{\overline{\sigma}}}\left[\frac{\rho}{\rho_0}\right]^{(\gamma+1)/2}.
$$

Given η_0, any η, such that $\sigma = L_2(\eta, \sigma_0, \eta_0)$ lies in $W(a,k)$, can be written as $\eta = \epsilon\eta_0$, where $1 \le \epsilon \le \frac{k+1}{1-k}$. Also, from the formula (2.6) and the bound (4.1) we find

$$\rho = \left[\frac{c}{\kappa}\tan\frac{\gamma-1}{2\sqrt{\gamma}}\eta\right]^{2/(\gamma-1)}, \quad \frac{\gamma-1}{2\sqrt{\gamma}}\eta < \frac{\pi}{4}. \tag{6.5}$$

Taking this into account and from the fact that $\tan\epsilon\eta < \epsilon^2\tan\eta$, for $0 < \eta < \epsilon\eta < \pi/4$, we conclude that

$$\sigma_\eta \le C_4 e^{\overline{\overline{\sigma}}}\left[\frac{\tan\frac{\gamma-1}{2\sqrt{\gamma}}\epsilon\eta_0}{\tan\frac{\gamma-1}{2\sqrt{\gamma}}\eta_0}\right]^{(\gamma+1)/(\gamma-1)} \le C_5 e^{\overline{\overline{\sigma}}}\epsilon^{2(\gamma+1)/(\gamma-1)},$$

for some absolute constant C_5. Since $\sigma_\eta \ge 0$, as can be shown from hypothesis A_2, we conclude the proof. ☐

Let $C(\overline{\overline{\sigma}}) = \hat{C}e^{\overline{\overline{\sigma}}}$ and

$$c_2 = \frac{1}{2k}\log\frac{C(\overline{\overline{\sigma}})\left(\frac{k+1}{k-1}\right)^{2\frac{\gamma+1}{\gamma-1}}+1}{C(\overline{\overline{\sigma}})\left(\frac{k+1}{k-1}\right)^{2\frac{\gamma+1}{\gamma-1}}-1}. \tag{6.6}$$

Combining Lemma 6.2 and 6.3 we have

Lemma 6.4. *The image under the map $T(a,\theta)$ of the shock curve $\sigma = R_i(\eta;\sigma_0,\eta_0)$ or $\sigma = L_i(\eta;\sigma_0,\eta_0)$, $i = 1,2$ in $\tilde{W}(a,k) \cap \{\eta < \frac{c_2}{\theta}\}$ satisfies A_2.*

Proof. By using Lemma 6.3 we have

$$\frac{1}{2\theta}\log\frac{\sigma_\eta + (-1)^i}{\sigma_\eta - (-1)^i} \ge \frac{1}{2\theta}\log\frac{C\left(\frac{k+1}{k-1}\right)^{2\frac{\gamma+1}{\gamma-1}}+1}{C\left(\frac{k+1}{k-1}\right)^{2\frac{\gamma+1}{\gamma-1}}-1}.$$

Since $W(a,k)$ defined as $|\sigma - a| < k\eta$ we conclude upon the use of Lemma 6.2. ☐

As was noted before, the shock curves do not satisfy DiPerna's condition B_4 when considered in the plane of the classical Riemann invariants. In fact the inverse of B_4 holds.

Lemma 6.5. *Let z, w be the classical Riemann invariants. Then*
i. Let $(\hat{z}_0, \hat{w}_0) \in R_1(z_0, w_0)$. If $z = L_2(w; z_0, w_0)$, $\hat{z} = L_2(\hat{w}; \hat{z}_0, \hat{w}_0)$ and $\Delta\hat{w} = \Delta w$ then $\Delta\hat{z} \le \Delta z$,
ii. Let $(\hat{z}_0, \hat{w}_0) \in L_2(z_0, w_0)$. If $z = R_1(w; z_0, w_0)$, $\hat{z} = R_1(\hat{w}; \hat{z}_0, \hat{w}_0)$ and $\Delta\hat{z} = \Delta z$ then $\Delta\hat{w} \le \Delta w$.

Proof. We prove only part (i). The proof of (ii) is similar. Suppose that the observer system moves with such a speed that $\sigma_0 = z_0 + w_0 = 0$. Consider two L_2 curves originating from (z_0, w_0) and $(\hat{z}_0, \hat{w}_0)$, as in part $(i.)$ of the lemma (Figure 2). By Lemma 2.6 shock curves are identical up to translations along σ-axis. Thus, the proof will be completed once it is shown that Δz decreases in η_0, with Δw being fixed. To formalize the argument, let us change η_0 for η and choose $\epsilon(\eta) > 0$ such that

$$\frac{L_2(\eta+\epsilon; 0, \eta) - \epsilon}{\sqrt{2}} + \sqrt{2}\epsilon = \Delta w. \tag{6.7}$$

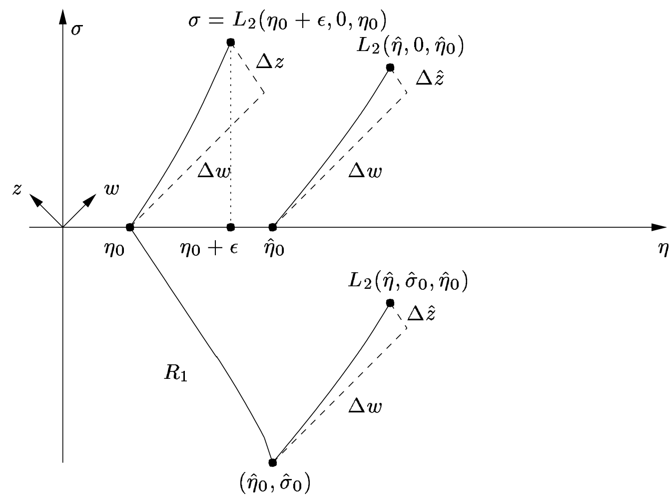

FIGURE 2

Then, we have to show that

$$\frac{d}{d\eta}\Delta z = \frac{d}{d\eta}\frac{L_2(\eta+\epsilon;0,\eta)-\epsilon}{\sqrt{2}} < 0. \tag{6.8}$$

Using the condition (6.7) we can write (6.8) as

$$\frac{\partial_\eta L_2(\eta+\epsilon;0,\eta)+\partial_{\eta_0}L_2(\eta+\epsilon;0,\eta_0)}{1+\partial_\eta L_2(\eta+\epsilon;0,\eta)} < 0.$$

By Lemma 2.2 $\partial_\eta L_2(\eta+\epsilon;0,\eta) \geq 0$, and the last inequality is equivalent to $\partial_\eta L_2(\eta+\epsilon;0,\eta)+\partial_{\eta_0}L_2(\eta+\epsilon;0,\eta_0) < 0$. The parametrization of a shock in $(\rho,\,\rho)$ plane is given in Lemma 2.7. Using formulae (2.5) we can write $L_2(\eta+\epsilon;0,\eta) = \sigma\circ v(\rho(\eta+\epsilon),\rho(\eta))$. Consequently,

$$\begin{aligned}
\partial_\eta L_2(\eta+\epsilon;0,\eta) &= \frac{d\sigma}{dv}\frac{\partial v}{\partial\rho}(\rho(\eta+\epsilon),\rho(\eta))\frac{d\rho}{d\eta}(\eta+\epsilon),\\
\partial_{\eta_0} L_2(\eta+\epsilon;0,\eta) &= \frac{d\sigma}{dv}\frac{\partial v}{\partial\rho_R}(\rho(\eta+\epsilon),\rho(\eta))\frac{d\rho}{d\eta}(\eta),
\end{aligned}$$

where $\frac{d\sigma}{dv} = \frac{2c}{c^2-v^2} > 0$, and $\frac{d\rho}{d\eta} = \frac{p(\rho)+\rho c^2}{c\sqrt{p'(\rho)}}$. Note that $L_2(\eta_1;0,\eta_2) = L_2(\eta_2;0,\eta_1)$ for any η_1, η_2. We conclude that $\partial_{\eta_0}L_2(\eta+\epsilon;0,\eta) = \partial_\eta L_2(\eta;0,\eta+\epsilon)$ and $\partial_\eta L_2$ was computed in the previous lemma. We set $\rho = \rho(\eta+\epsilon)$, $\rho_0 = \rho(\eta)$ and use shorthand notations introduced in the last lemma.

$$\begin{aligned}\partial_\eta L_2(\eta+\epsilon;0,\eta) &+ \partial_\eta L_2(\eta;0,\eta+\epsilon)\\ &= P\left[p'(p_0+\rho c^2)+\frac{\delta p}{\delta\rho}(p+\rho_0c^2)\right]\rho_\eta(\eta+\epsilon)(p_0+\rho_0c^2)\\ &- P\left[p_0'(p+\rho_0c^2)+\frac{\delta p}{\delta\rho}(p_0+\rho c^2)\right]\rho_\eta(\eta)(p+\rho c^2),\\ P &= \frac{c^3}{c^2-v^2}\sqrt{\frac{1}{\delta p\delta\rho}}\frac{\delta\rho}{(p+\rho_0c^2)^{3/2}(p_0+\rho c^2)^{3/2}}>0.\end{aligned}$$

Substituting the expressions for ρ_η derived from (2.5) and setting $P_1 = P\frac{(p_0+\rho_0c^2)(p+\rho c^2)}{c}$ we have

$$\begin{aligned}\partial_\eta L_2(\eta+\epsilon;0,\eta) &+ \partial_\eta L_2(\eta;0,\eta+\epsilon)\\ &= P_1\left[p'(p_0+\rho c^2)+\frac{\delta p}{\delta\rho}(p+\rho_0c^2)\right]\frac{1}{\sqrt{p'}}\\ &- P_1\left[p_0'(p+\rho_0c^2)+\frac{\delta p}{\delta\rho}(p_0+\rho c^2)\right]\frac{1}{\sqrt{p_0'}}\\ &= P_1(p+\rho_0c^2)\left[\frac{1}{\sqrt{p'}}\frac{\delta p}{\delta\rho}-\sqrt{p_0'}\right]\\ &- P_1(p_0+\rho c^2)\left[\frac{1}{\sqrt{p_0'}}\frac{\delta p}{\delta\rho}-\sqrt{p'}\right].\end{aligned}\tag{6.9}$$

Then, from the fact that $p+\rho_0c^2 < p_0+\rho c^2$, since $\frac{\delta p}{\delta\rho} < c^2$ and $\frac{\delta p}{\delta\rho} - \sqrt{p'p_0'} > \frac{\delta p}{\delta\rho} - \frac{p'+p_0'}{2} > 0$, when $1\le\gamma\le 2$ we have

$$\begin{aligned}\partial_\eta L_2(\eta+\epsilon;0,\eta) &+ \partial_\eta L_2(\eta;0,\eta+\epsilon)\\ &\le P_1(p_0+\rho c^2)(\sqrt{p'}-\sqrt{p_0'})\left[\sqrt{p'p_0'}-\frac{\delta p}{\delta\rho}\right]<0.\end{aligned}$$

□

Now we are ready to derive the sufficient condition on the shock curves to make their images under $T(a,\theta)$ satisfy B_4. Define maps T_1, $T_2 : \mathbb{R}^2\to\mathbb{R}^2$.

$$\begin{aligned}T_1(a,\theta) &: (z,w)\to(z',w), \quad z' = 1-\exp 2\theta(a/2-z),\\ T_2(a,\theta) &: (z,w)\to(z,w'), \quad w' = -1+\exp 2\theta(w-a/2).\end{aligned}$$

Obviously, $T(a,\theta) = T_1\circ T_2$. Moreover we have the simple lemma.

Lemma 6.6 (see [5], Lemma 3.14). *$T(a,\theta)$ maps shocks curves in some region U onto shock curves satisfying B_4 in the image $T[U]$ iff $T_1(a,\theta)$ and $T_2(a,\theta)$ map shock curves in U onto shock curves satisfying $B_4.1$ and $B_4.2$ in $T_1[U]$ and $T_2[U]$, respectively.*

Lemma 6.7. *Suppose that*

$$\left|\frac{d}{d\eta}\log\left(L_2(\eta+\epsilon;0,\eta)-\epsilon\right)\right| \le \theta, \quad \forall\,\eta,\,\eta+\epsilon\in(\eta_1,\,\eta_2),\ \epsilon>0,\tag{6.10}$$

$$\left|\frac{d}{d\eta}\log\left(R_1(\eta+\epsilon;0,\eta)+\epsilon\right)\right| \le \theta, \quad \forall\,\eta,\,\eta+\epsilon\in(\eta_1,\,\eta_2),\ \epsilon>0,\tag{6.11}$$

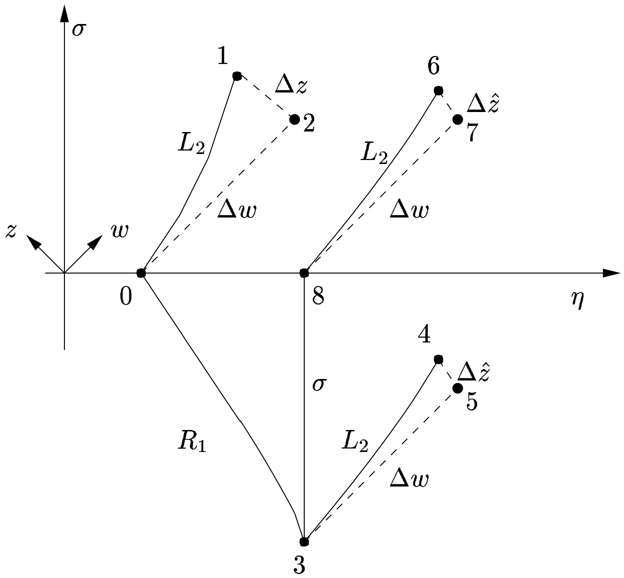

FIGURE 3

for some $\eta_1 < \eta_2$. Then, (6.10) implies that the images under the map T_1 of the shock curves in $\mathbb{R} \times \{\eta_1 < \eta < \eta_2\}$ satisfy B_4.1, and (6.11) implies that the images under the map T_1 of the shock curves in $\mathbb{R} \times \{\eta_1 < \eta < \eta_2\}$ satisfy B_4.2.

Proof. We proof only the implication following from (6.10); the proof of another part of the lemma is similar. Throughout the proof we refer to Figure 3. As before we can assume that $\sigma_0 = z_0 + w_0 = 0$. Fix $\Delta w > 0$ and $\delta > 0$. Let us, as in hypothesis B_4.1, consider points

$$\begin{aligned} 0:\ &(\sigma_0, \eta_0), & 1: &(L_2(\eta_0+\epsilon_1;\sigma_0,\eta_0),\eta_0+\epsilon_1),\\ 2:\ &(\frac{\Delta w}{\sqrt{2}}, \eta_0 + \frac{\Delta w}{\sqrt{2}}), & 3: &(\hat\sigma_0,\hat\eta_0) = (R_1(\eta_0+\delta;\sigma_0,\eta_0),\eta_0+\delta),\\ 4:\ &(L_2(\hat\eta_0+\epsilon_2;\hat\sigma_0,\hat\eta_0),\hat\eta_0+\epsilon_2), & 5: &(\hat\sigma_0 + \frac{\Delta w}{\sqrt{2}}, \hat\eta_0 + \frac{\Delta w}{\sqrt{2}}),\\ 6:\ &(L_2(\hat\eta_0+\epsilon_2;0,\hat\eta_0+\epsilon_2),\hat\eta_0+\epsilon_2), & 7: &(0,\eta_0+\delta+\frac{\Delta w}{\sqrt{2}}),\\ 8:\ &(0,\eta_0+\delta), & \sigma &= -\hat\sigma_0, \end{aligned}$$

where ϵ_i, $i = 1, 2$ are determined by Δw. Since T_1 does not depend on w and $\frac{d}{dz}z' \geq 0$ we have to verify that $z'_4 - z'_5 \geq z'_1 - z'_2$, where z'_i stands for z' coordinate of the image under T_1 of point i. This is equivalent to each of the following

$$\begin{aligned} e^{-\theta(\sigma_5-\eta_5)} - e^{-\theta(\sigma_4-\eta_4)} &\geq e^{-\theta(\sigma_2-\eta_2)} - e^{-\theta(\sigma_1-\eta_1)},\\ e^{\theta\sigma}\left[e^{-\theta(\sigma_7-\eta_7)} - e^{-\theta(\sigma_6-\eta_6)}\right] &\geq e^{-\theta(\sigma_2-\eta_2)} - e^{-\theta(\sigma_1-\eta_1)}, \end{aligned}$$

$$e^{\theta\sigma}\left[e^{-\theta(-\eta_0-\delta)} - e^{-\theta(S_2-\eta_0-\delta)}\right] \geq e^{-\theta(-\eta_0)} - e^{-\theta(S_1-\eta_0)}. \tag{6.12}$$

Here

$$\begin{aligned} S_1 &= L_2(\eta_0+\epsilon_1,\eta_0)-\epsilon_1>0, \\ S_2 &= L_2(\eta_0+\delta+\epsilon_2,\eta_0+\delta)-\epsilon_2>L_2(\eta_0+\delta+\epsilon_1,\eta_0+\delta)-\epsilon_1>0, \end{aligned}$$

where we used the fact that $\epsilon_2>\epsilon_1$, which follows from Lemma 6.5, and star-like property of the curve L_2. The inequality (6.12) will hold provided

$$\frac{d}{d\eta}\left[e^{\theta\eta}-e^{\theta(\eta-L_2(\eta+\epsilon;0,\eta)+\epsilon)}\right]\geq 0,$$

for $\eta_1<\eta<\eta+\epsilon<\eta_2$. The condition (6.10) will suffice. □

By the next lemma we investigate the validity of conditions (6.10) and (6.11).

Lemma 6.8. *There is a constant* $c_1(\overline{\overline{\sigma}},k,\delta_0)$ *such that* (6.10) *and* (6.11) *holds for shocks in* $\tilde{W}(a,k)\cap\{\frac{c_1}{\theta}<\eta\}$.

Proof. Inequality (6.11) is verified in a similar way as (6.10), which we prove below. Equality (6.9) implies

$$\left|\frac{dL_2(\eta+\epsilon;0,\eta)}{d\eta}\right| = \left|P_1(p+\rho_0c^2)\left[\frac{1}{\sqrt{p'}}\frac{\delta p}{\delta\rho}-\sqrt{p_0'}\right]-P_1(p_0+\rho c^2)\left[\frac{1}{\sqrt{p_0'}}\frac{\delta p}{\delta\rho}-\sqrt{p'}\right]\right|$$

As in previous lemma we have $p+\rho_0c^2<p_0+\rho c^2$, $p_0'<\frac{\delta p}{\delta\rho}<p'$ and $\frac{\delta p}{\delta\rho}>\sqrt{p'p_0'}$. This allows us to write

$$\left|\frac{dL_2(\eta+\epsilon,0,\eta)}{d\eta}\right|\leq P_1(p_0+\rho c^2)\delta\sqrt{p'}\left(1+\sqrt{\frac{p'}{p_0'}}\right)\leq 2P_1(p_0+\rho c^2)\delta\sqrt{p'}\sqrt{\frac{p'}{p_0'}}.$$

Reminding that

$$P_1 = \frac{1}{1-(v/c)^2}\frac{1}{\sqrt{\delta p\delta\rho}}\frac{\delta\rho(p_0+\rho_0c^2)(p+\rho c^2)}{(p+\rho_0c^2)^{3/2}(p_0+\rho_0c^2)^{3/2}},$$

and using Lemma 2.1 we can estimate

$$2P_1(p_0+\rho c^2)\delta\sqrt{p'}\sqrt{\frac{p'}{p_0'}} \leq C_6e^{\overline{\overline{\sigma}}}\sqrt{\frac{\delta p}{\delta\rho}}\delta\sqrt{p'}\frac{(p_0+\rho_0c^2)(p+\rho c^2)}{(p+\rho_0c^2)^{3/2}(p_0+\rho c^2)^{1/2}}\sqrt{\frac{p'}{p_0'}}.$$

It is convenient to write the last estimates in terms of the ratio ρ/ρ_0. We have

$$\begin{aligned} \frac{dL_2(\eta+\epsilon;0,\eta)}{d\eta} &\leq C_7e^{\overline{\overline{\sigma}}}\sqrt{\frac{\rho_0}{p_0}}\sqrt{\frac{\left(\frac{\rho}{\rho_0}-1\right)}{\left(\frac{p}{p_0}-1\right)}}\frac{\sqrt{p_0'}\left(\sqrt{\frac{p'}{p_0'}}-1\right)\sqrt{\frac{\rho}{\rho_0}}}{\left(\frac{\rho}{\rho_0}\frac{p}{\rho c^2}+1\right)^{3/2}}\sqrt{\frac{p'}{p_0'}} \\ &\leq C_8e^{\overline{\overline{\sigma}}}\sqrt{\frac{\left(\frac{\rho}{\rho_0}-1\right)}{\left(\frac{p}{p_0}-1\right)}}\frac{\left(\frac{\rho}{\rho_0}\right)^{1/2}\left(\sqrt{\frac{p'}{p_0'}}-1\right)}{\left(\frac{\rho}{\rho_0}\frac{p}{\rho c^2}+1\right)^{3/2}}\sqrt{\frac{p'}{p_0'}}. \end{aligned}$$

We set

$$\eta+\epsilon=\tau\eta,\quad n=\frac{\gamma-1}{2\sqrt{\gamma}}\eta,\quad g=\frac{\rho}{\rho_0}=\left[\frac{\tan\tau n}{\tan n}\right]^{\frac{2}{\gamma-1}}\geq 1.$$

In the above notation we estimate

$$\begin{aligned}\left|\frac{dL_2(\eta+\epsilon;0,\eta)}{d\eta}\right| &\le C_8 e^{\overline{\overline{\overline{\sigma}}}}\frac{\sqrt{g}(g-1)^{1/2}(g^{(\gamma-1)/2}-1)}{(g^\gamma-1)^{1/2}\left(g\frac{p(\underline{\underline{\rho}})}{\underline{\underline{\rho}}c^2}+1\right)^{3/2}}g^{\frac{\gamma-1}{2}}\\
&\le C_8 e^{\overline{\overline{\overline{\sigma}}}} g^{\frac{\gamma}{2}}(g^{(\gamma-1)/2}-1) = C_8 e^{\overline{\overline{\overline{\sigma}}}}\left[\frac{\tan\tau n}{\tan n}\right]^{\frac{\gamma}{\gamma-1}}\left(\frac{\tan\tau n}{\tan n}-1\right)\\
&\le C_8 e^{\overline{\overline{\overline{\sigma}}}}\tau^{\frac{2\gamma}{\gamma-1}}\left(\frac{\tan\tau n}{\tan n}-1\right) \le C_9 e^{\overline{\overline{\overline{\sigma}}}}\tau^{\frac{2\gamma}{\gamma-1}}(\tau-1). \qquad (6.13)\end{aligned}$$

The last inequality is true since $0 < n < \tau n < \pi/4$. Consider

$$\begin{aligned}L_2(\eta+\epsilon;0,\eta) &= \sigma\circ v = \log\frac{c+v}{c-v} \ge \frac{2}{c}v \qquad (6.14)\\
&= \frac{2}{c}c^2\sqrt{\frac{\delta p\delta\rho}{(p+\rho_0c^2)(p_0+\rho c^2)}} = \frac{2}{c}\sqrt{\frac{p_0(\rho/\rho_0-1)(p/p_0-1)}{\rho(\frac{p}{\rho_0c^2}+1)(\frac{p_0}{\rho c^2}+1)}}\\
&= \frac{2}{c}\sqrt{\frac{p_0}{\rho_0}}\sqrt{\frac{(g-1)(g^\gamma-1)}{g(g\tan^2\tau n+1)(g^{-1}\tan^2 n+1)}}\\
&= \frac{2\tan n}{\kappa}\sqrt{\frac{p_0}{\rho_0}}\sqrt{\frac{(g-1)(g^\gamma-1)}{g(g\tan^2\tau n+1)(g^{-1}\tan^2 n+1)}}.\end{aligned}$$

Then,

$$\begin{aligned}&\frac{1}{\eta}L_2(\eta+\epsilon;0,\eta)-(\tau-1)\\
&\ge \frac{2\tan n}{\kappa\eta}\sqrt{\frac{(g-1)(g^\gamma-1)}{g(g\tan^2\tau n+1)(g^{-1}\tan^2 n+1)}}-(\tau-1)\\
&\ge \frac{(\gamma-1)}{\kappa\sqrt{\gamma}}\sqrt{\frac{(g-1)(g^\gamma-1)}{g(g\tan^2\tau n+1)(g^{-1}\tan^2 n+1)}}-(\tau-1)\\
&\ge (\gamma-1)\sqrt{\frac{(g-1)^2}{g(g\tan^2\tau n+1)(g^{-1}\tan^2 n+1)}}-(\tau-1)\\
&= \frac{(\gamma-1)(g-1)}{\sqrt{g(g\tan^2\tau n+1)(g^{-1}\tan^2 n+1)}}-(\tau-1)\\
&\ge \frac{2\left(\frac{\tan\tau n}{\tan n}-1\right)}{\sqrt{g(g\tan^2\tau n+1)(g^{-1}\tan^2 n+1)}}-(\tau-1)\\
&\ge (\tau-1)\left(\frac{2}{\sqrt{g(g\tan^2\tau n+1)(g^{-1}\tan^2 n+1)}}-1\right).\end{aligned}$$

Let us write

$$\tau = z^{(\gamma-1)}, \quad 1 < z < z_0,$$

for some z_0 to be chosen later. From (4.1) and formula $\tan n = \frac{\sqrt{p'}}{c}$ we have $\tan n < 1 - \delta_0$. Also, since $g^2 = \left[\frac{\tan \tau n}{\tan n}\right]^{\frac{4}{\gamma-1}} \leq \tau^{\frac{8}{\gamma-1}} = z^8 \leq z_0^8$, it follows

$$\begin{aligned}(g\tan^2 \tau n + 1)(\tan^2 n + g) &= g^2 \tan^2 \tau n + g(1+\tan^2 n) + \tan^2 n \\ &\leq g^2 + 2g + 1 - \delta_0 \leq 3g^2 + 1 - \delta_0 \\ &\leq 3z_0^8 + 1 - \delta_0 < 4,\end{aligned}$$

when z_0 close to 1. In this way we obtain

$$\frac{1}{\eta}L_2(\eta+\epsilon;0,\eta) - (\tau-1) \geq C_{13}(z_0)(\tau-1), \qquad 1 < \tau < z_0^{(\gamma-1)}. \tag{6.15}$$

Now

$$L_2(\eta+\epsilon;0,\eta) - \epsilon = \eta\left(\frac{1}{\eta}L_2(\eta+\epsilon;0,\eta) - (\tau-1)\right) \geq C_{13}\eta(\tau-1) = C_{13}\epsilon. \tag{6.16}$$

Note that shocks are starlike in (σ, η) plane. It implies that for fixed η $L_2(\eta + \epsilon; 0, \eta)/\epsilon - 1$ is nondecreasing in ϵ, which in turn implies that (6.16) holds for any $\epsilon > 0$ $(\tau > 1)$. Combining (6.13) and (6.16) we obtain

$$\left|\frac{d}{d\eta}\log\left(L_2(\eta+\epsilon;0,\eta) - \epsilon\right)\right| \leq C_{14}e^{\overline{\overline{\sigma}}}\frac{\tau^{\frac{2\gamma}{\gamma-1}}}{\eta}. \tag{6.17}$$

Take $c_1 = C_{14}e^{\overline{\overline{\sigma}}}\tau^{\frac{2\gamma}{\gamma-1}}$, $\tau = \frac{1+k}{1-k}$. □

The proof of Theorem 4.1 will be complete upon using Lemma 6.4 and Lemma 6.8 if we show that there is $k = k$, $\overline{\overline{\sigma}}$ such that properties (4.3) hold. Since we know the explicit dependence of c_2 and c_1 on $k, \gamma, \overline{\overline{\sigma}}$, this result will follow by straightforward but technical computation given in the next lemma.

Lemma 6.9. *There are $\iota > 0$ and $\beta > 0$ such that if $k = z(\gamma)^{\gamma-1} - 1$, where $z(\gamma) = (\gamma-1)^{-\iota}$, and $\overline{\overline{\sigma}} = \bar{\sigma} + \beta\log(\gamma-1)^{-1}$, then* (4.3) *holds. Moreover, ι, β can be chosen to satisfy $\beta/\iota < 2\arctan\frac{\kappa}{c}$.*

Proof. In the proof we denote by $C_i > 0$, $i = 1..11$ some absolute constants. We have $z(\gamma)^{\gamma-1} \to 1$ and $z(\gamma) \to \infty$. To simplify the notations let us write z for $z(\gamma)$ and $C = C_1 e^{\overline{\overline{\sigma}}}$. Then

$$\begin{aligned} c_2(k) &= \frac{1}{2(z^{\gamma-1}-1)}\log\frac{Cz^{2(\gamma+1)} + \left(1-(z^{\gamma-1}-1)\right)^{2\frac{\gamma+1}{\gamma-1}}}{Cz^{2(\gamma+1)} - \left(1-(z^{\gamma-1}-1)\right)^{2\frac{\gamma+1}{\gamma-1}}} \\ \text{(6.18)} &= \frac{1}{2(z^{\gamma-1}-1)}\log\left[1 + \frac{2\left(1-(z^{\gamma-1}-1)\right)^{2\frac{\gamma+1}{\gamma-1}}}{Cz^{2(\gamma+1)}\left(1-\left(1-(z^{\gamma-1}-1)\right)^{2\frac{\gamma+1}{\gamma-1}}C^{-1}z^{-2\gamma-2}\right)}\right].\end{aligned}$$

Consider function

$$\begin{aligned}\left(1-(z^{\gamma-1}-1)\right)^{2\frac{\gamma+1}{\gamma-1}} &= \left(1-(z^{\gamma-1}-1)\right)^{\frac{1}{z^{\gamma-1}-1}\frac{z^{\gamma-1}-1}{\gamma-1}2(\gamma+1)} \\ &= \left(\left(1-(z^{\gamma-1}-1)\right)^{\frac{1}{z^{\gamma-1}-1}}\right)^{\frac{z^{\gamma-1}-1}{\gamma-1}2(\gamma+1)} \\ &= \left(e^{-1+O(z^{\gamma-1}-1)}\right)^{\frac{z^{\gamma-1}-1}{\gamma-1}2(\gamma+1)} \\ &= e^{-\frac{z^{\gamma-1}-1}{\gamma-1}2(\gamma+1)}e^{O(z^{\gamma-1}-1)\frac{z^{\gamma-1}-1}{\gamma-1}2(\gamma+1)}.\end{aligned} \tag{6.19}$$

In the above computation we used the fact that $(1-x)^{\frac{1}{x}} = e^{-1+O(x)}$. Also, since

$$O(z^{\gamma-1}-1)\frac{z^{\gamma-1}-1}{\gamma-1}2(\gamma+1) \le C_1\frac{(z^{\gamma-1}-1)^2}{\gamma-1}2(\gamma+1) \le C_2(\gamma-1)\log^2 z(\gamma) \to 0,$$

then

$$C_3 e^{-\frac{z^{\gamma-1}-1}{\gamma-1}2(\gamma+1)} \le \left(1-(z(\gamma)^{\gamma-1}-1)\right)^{2\frac{\gamma+1}{\gamma-1}} \le C_4 e^{-\frac{z^{\gamma-1}-1}{\gamma-1}2(\gamma+1)} \to 0. \tag{6.20}$$

The limit holds since $\frac{z^{\gamma-1}-1}{(\gamma-1)\log z(\gamma)} \to 1$. (6.20) and the fact that $C \to \infty$ imply that the argument of the log in (6.18) is close to 1. We obtain

$$C_5\frac{1}{z^{\gamma-1}-1}\frac{e^{-\frac{z^{\gamma-1}-1}{\gamma-1}2(\gamma+1)}}{Cz^{2(\gamma+1)}} \le c_2(k) \le C_6\frac{1}{z^{\gamma-1}-1}\frac{e^{-\frac{z^{\gamma-1}-1}{\gamma-1}2(\gamma+1)}}{Cz^{2(\gamma+1)}}. \tag{6.21}$$

Finally, since $\frac{z^{\gamma-1}-1}{(\gamma-1)\log z(\gamma)} \to 1$, we get

$$\begin{aligned} c_2 &\ge C_7\frac{1}{(\gamma-1)\log z}\frac{1}{Cz^{5(\gamma+1)}} \ge C_7\frac{1}{(\gamma-1)\log z}\frac{1}{Cz^{10}} \\ &\ge C_7\frac{1}{\gamma-1}\frac{1}{Cz^{11}} \\ &\ge C_8\frac{1}{(\gamma-1)^{1-\beta-11\iota}} \to +\infty, \end{aligned} \tag{6.22}$$

if

$$1-\beta-11\iota > 0. \tag{6.23}$$

(6.22) and (6.17) can be used to derive

$$\frac{c_1}{c_2} \le C_9 C^2(\gamma-1)z^{15} \le C_{10}(\gamma-1)^{1-2\beta-15\iota} \to 0,$$

if

$$1-2\beta-15\iota > 0. \tag{6.24}$$

The right side of (6.21) reads

$$c_2(k)k \le C_{11}\frac{1}{Cz^{2(\gamma+1)}} \to 0.$$

There are $\iota > 0$, $\beta > 0$ which solve (6.23), (6.24) and satisfy $\beta/\iota < 2\arctan\frac{\kappa}{c}$. □

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