INITIAL BOUNDARY VALUE PROBLEMS FOR A QUASILINEAR PARABOLIC SYSTEM IN THREE-PHASE CAPILLARY FLOW IN POROUS MEDIA

HERMANO FRID & VLADIMIR SHELUKHIN

ABSTRACT. We study two types of initial boundary value problems for a quasilinear parabolic system motivated by three-phase flows in porous media in the presence of capillarity effects. The first type of problem prescribes a mixed boundary condition, involving a combination of the value of the solution and its normal derivative at the boundary. The second type prescribes the value of the solution at the boundary, which is the so called Dirichlet boundary condition. We prove the existence and uniqueness of smooth solution for the first type of initial boundary value problem, and we obtain the existence of a solution for the second one as a limit case of the first type. The main assumption about the diffusion matrix of the system is that it is triangular with strictly positive diagonal elements. Another interesting feature is concerned specifically with the application to three-phase capillary flow in a porous medium. Namely, we derive an important practical consequence of the assumption that the capillarity matrix is upper triangular, if we further impose that the second diagonal element depends only on the second variable, i.e., the second phase. We show that this mathematical assumption in turn provides an efficient method for the definition of the capillary pressures in the interior of the triangle of saturations through the solution of a well posed boundary value problem for a linear hyperbolic system. Finally, as an example, we include the analysis of a special case of three-phase capillary flow model where the capillarity matrix is degenerate, but we are still able to solve it due to the particular form of the flux functions.

1. Introduction

We consider initial boundary value problems for 2×2 quasilinear parabolic systems of the form

$$u_t + f(u)_x = (B(u)u_x)_x, \quad 0<t<T, \quad x\in\Omega := (-1,1), \tag{1.1}$$

motivated by one-dimensional three-phase capillary flows through a porous medium, say, an oil reservoir. Here,

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

We first explain our abstract results concerning (1.1) and then we discuss the application already mentioned.

We consider two initial boundary value problems for system (1.1):

$$\delta u_\eta + u = u_b, \text{ at } |x|=1, \ t>0, \qquad u|_{t=0} = u_0(x), \ x\in\Omega \tag{1.2}$$

1991 *Mathematics Subject Classification.* Primary: 35K55; Secondary: 35B65, 76S05, 76T05.
Key words and phrases. Porous media, three-phase capillary flows, existence, uniqueness.
.

and

$$u = u_b, \text{ at } |x| = 1, \ t > 0, \qquad u|_{t=0} = u_0(x), \ x \in \Omega, \tag{1.3}$$

where $\delta = \text{const} > 0$ and

$$u_\eta|_{x=\pm 1} = \pm u_x|_{x=\pm 1}, \quad u_b|_{x=\pm 1} = u_\pm(t).$$

Motivated by the application, the functions f_i, B_{ij}, $i, j = 1, 2$, are assumed to be defined and smooth over the closed triangle

$$\Delta := \{u \in \mathbb{R}^2 \ : \ 0 \le u_i, \ u_1 + u_2 \le 1\}, \tag{1.4}$$

and we take initial and boundary data satisfying

$$u_0(x), u_\pm(t) \in \Delta, \ \forall x \in \Omega, t > 0. \tag{1.5}$$

We then seek for solutions to our problems (1.1),(1.2) and (1.1),(1.3) defined on $Q := \Omega \times (0, T)$, for any $T > 0$, which satisfy

$$u(t, x) \in \Delta, \ \forall (x, t) \in Q. \tag{1.6}$$

For the mathematical analysis of the above problems, we need to assume that for certain constant $\nu > 0$ we have

$$B_{ii} \ge \nu \quad i = 1, 2, \ \forall u \in \Delta, \tag{1.7}$$

and that the matrix B is (upper) triangular, i.e.,

$$B_{21}(u) \equiv 0. \tag{1.8}$$

Concerning the behavior of the functions f_i and B_{ij} on the boundary of Δ, we assume the following:

$$f_1|_{u_1=0} = \text{const}, \quad f_2|_{u_2=0} = \text{const}, \quad (f_1 + f_2)|_{u_1+u_2=1} = \text{const}, \tag{1.9}$$

$$B_{12}|_{u_2=0} = 0, \quad (B_{11} - B_{12} - B_{22})|_{u_1+u_2=1} = 0. \tag{1.10}$$

As for the smoothness of the functions f_i, B_{ij} over Δ, we assume

$$f_i, \quad \frac{\partial f_i}{\partial u_j}, \quad B_{ij}, \quad \frac{\partial B_{ij}}{\partial u_k}, \quad \frac{\partial^2 B_{ij}}{\partial u_k \partial u_l} \in H^\beta(\Delta), \tag{1.11}$$

where $H^\beta(\Delta)$ is the space of Hölder continuous functions on Δ with $\beta \in (0, 1)$.

The initial and boundary data are also assumed to be in Hölder spaces:

$$u_0 \in H^{2+\beta}(\overline{\Omega}), \quad u_\pm(t) \in H^{1+\beta}([0, T]). \tag{1.12}$$

We can now state our result concerning the problem (1.1),(1.2).

Theorem 1.1. *Let any $T > 0$ be given. Assume that the data $B(u)$, $f(u)$, $u_0(x)$, and $u_\pm(t)$ satisfy the conditions* (1.5), (1.7), (1.8), (1.9), (1.10), (1.11) *and* (1.12). *Suppose that the compatibility conditions*

$$\pm\delta u_0'(\pm 1) + u_0(\pm 1) = u_\pm(0)$$

are satisfied. Then problem (1.1),(1.2) *has a unique solution $u(t, x)$ such that $u \in H^{2+\beta, 1+\beta/2}(\overline{Q})$. Moreover, u satisfies* (1.6).

The core of the proof is the application of Leray-Schauder's fixed-point theorem, which requires strong a priori estimates in Hölder spaces. This was also the approach followed in [4], where we considered the somewhat simpler case of periodic boundary conditions. It is, actually, the main point for dealing with nonlinear equations in the extensive theory developed by Ladyzhenskaya and Ural'ceva [7] and Ladyzhenskaya, Solonnikov and Ural'ceva [6], which provide the basic tools of our approach. As is well known, the application of a fixed point theorem to obtain a solution to an initial boundary value problem for a nonlinear PDE, in general, requires the definition of an operator in a suitable subset $\mathcal{U}$ of an appropriate space of functions. Such operator is usually given by means of the solution of a linear problem, that is, by associating to each element $v \in \mathcal{U}$ the solution $A(v)$ of a linear problem, and is such that its fixed points, that is, solutions of $A(u) = u$, solve our nonlinear problem. The Hölder spaces provide the standard setting in the context of smooth solutions. The a priori estimates in these spaces play then a twofold fundamental role. First, in order to provide bounds for the possible fixed points of the operator mentioned above, which will help us to conveniently define its domain. Second, these a priori estimates will also hold (with similar proof) for the solutions of the linear problems defining the mentioned operator, which in turn imply the properties required for the application of the fixed point theorem. In our application of Leray-Schauder's theorem, as in [6], we need a priori estimates in the space of Hölder continuos functions, in space and time, for the solution u and its space derivative u_x. Roughly speaking, after obtaining the a priori estimate for u, to obtain the estimate for u_x, we differentiate the system with respect to x and regard the resulting equations as a linear system for $w = u_x$. In this way, the boundary condition (1.2) gives us the Hölder continuity of u_x at the boundary as a consequence of the regularity of u_b and the Hölder continuity of u in $\bar{Q}$, obtained in the first run. This is what turns problem (1.1),(1.2) easier to handle than problem (1.1),(1.3). The idea is then to obtain the solution of the latter as a limit case of the first one, when $\delta \to 0$. But, in doing that, we have to give up the context of classical solutions (up to the boundary) and content ourselves with a weaker notion of solution, which turns out to leave the question of uniqueness, for the moment, still open.

We now state our result concerning problem (1.1),(1.3).

Theorem 1.2. *Let $B(u)$ and $f(u)$ be as in Theorem 1.1. Let the functions $u_0(x)$ and $u_\pm(t)$ satisfy* (1.5) *and $u_0 \in L^2(\Omega)$, $u_\pm \in W^{1,1}(0,T)$. Then problem* (1.1),(1.3) *has a solution $u(t,x)$ in the sense that u is a classical solution of* (1.1) *in $Q := (0,T) \times (-1,1)$, and u satisfies* (1.3) *in the sense of $L^2(0,T)$, for the boundary condition, and $L^2(-1,1)$, for the initial condition. Moreover, if $u_\pm(t) \equiv 0$, $t \in [0,T]$, and $u_0 \in H^{2+\beta}(\overline{\Omega})$, with $u_0(\pm 1) = 0$, then $u(t,x)$ is a classical solution of* (1.1) *in Q, $u \in H^{\beta,\beta/2}(\overline{Q})$ and the initial and boundary conditions are assumed in the usual sense for continuous functions. In any case the solution u satisfies* (1.6).

We prove Theorem 1.1, in sections 2 and 3. Theorem 1.2 is proved in section 4. Concerning correlated works on the theory of quasilinear parabolic systems, we recall first that, in the well known book of Ladyzhenskaya, Solonnikov and Ural'ceva [6], the theory developed for linear and quasilinear equations is applied to quasilinear systems of the type (1.1) (in the multidimensional case, for any number of equations) where the diffusion matrix is a scalar multiple of the identity matrix. We also recall the results of H. Amann [2] which, in the case of 2×2

one-dimensional systems, require $\frac{\partial f_2}{\partial u_1} \equiv 0$ and $\frac{\partial B_{22}}{\partial u_1} \equiv 0$, besides (1.7) and (1.8), and assume uniform boundedness of a certain Hölder norm of the local solution, in order to extent it to all values of time $t > 0$.

In section 5 we recall the basic facts about capillary multiphase flow in a porous medium. In this context one studies the evolution in time and space of the saturations of three immiscible fluid phases, say, oil, gas and water, denoted by u_1, u_2, u_3, through a reservoir, which is assumed to be a (porous) solid cylinder with homogeneous cross sections. By saturation of a phase we mean the percentage of this phase in an infinitesimal pore, so that we must have $u_1 + u_2 + u_3 = 1$. In section 5, we describe how a system like (1.1) for the saturations u_1 and u_2 is derived from the mass conservation equations together with Darcy's basic law for flows in porous media. The dependent vector variable $u = (u_1, u_2)$ then must assume values in the triangle Δ defined in (1.4). By the formulas for the functions f_i and B_{ij}, derived from the mentioned basic facts, we easily check the validity of (1.9) and (1.10), when (1.8) holds. The physical diffusion matrix is diagonalizable with nonnegative eigenvalues everywhere in Δ, as shown in [1]. However, it may, in general, become singular at the boundary of Δ. So, to apply Theorems 1.1 and 1.2 one should, in general, add an artificial viscosity to the actual system.

Also in section 5, we discuss the mobility laws and capillary pressure laws which yield (1.8) and also the additional restriction

$$\frac{\partial B_{22}}{\partial u_1} = 0, \tag{1.13}$$

not needed in Theorems 1.1 and 1.2. The mobilities play a role in Darcy's law analogous to the coefficient of heat conduction in Fourier's law (see section 5). The capillary pressures, denoted here by P_1 and P_2, are defined by $P_1 = p_1 - p_3$ and $P_2 = p_2 - p_3$, where p_i is the pressure in the i-th phase.

Actually, the main point of section 5 is a remarkable consequence of the mathematical assumptions (1.8) and (1.13) on the capillarity matrix. It is related with the problem of defining the capillary pressures in the interior of the triangle of saturations. As is well known by reservoir engineers, mobilities and capillary pressures can be plotted as functions of the saturations only in the case of flows where just two phases are present [10, 5], that is, when $u \in \partial\Delta$. While there are some widely accepted models which artificially prescribe the mobility functions in the interior of Δ, such as the one proposed by H. Stone [13], the same is no longer true for the functions representing the capillary pressures. Now, the constraints (1.8) and (1.13) amount to a linear hyperbolic system of partial differential equations for the capillary pressures P_1 and P_2, whose coefficients involve the mobility functions, for which we set the boundary conditions

$$P_i(u)|_{u_j=0} = \pi_i(u_i), \quad i,j = 1,2, \quad j \neq i, \tag{1.14}$$

enforcing compatibility with the two-phase flow case, where the functions π_i can be obtained from two-phase flows experiments, as already mentioned. The result is a consistent recipe for defining the capillary pressures in the interior of the triangle of saturations, given the mobility functions everywhere defined therein! We call this procedure for determining P_1 and P_2 in the interior of Δ the *method of physical interpolation for capillary pressures*. As an example, we study in detail the case when the mobilities are linear functions of the corresponding phase saturation.

Finally, in section 6, the techniques developed in sections 2 to 5 are applied to the study of a particular degenerate reservoir fluid flow model. This model yields a Dirichlet initial boundary value problem for a degenerate system of the form (1.1) where (1.8), (1.13) and yet $\frac{\partial f_2}{\partial u_1} = 0$ hold. In this case the second equation in (1.1) reads

$$\frac{\partial u_2}{\partial t} + \frac{\partial f_2(u_2)}{\partial x} = \frac{\partial}{\partial x}(B_{22}(u_2)\frac{\partial u_2}{\partial x}). \tag{1.15}$$

Since (1.6) must hold, (1.15) is not completely decoupled from the first equation in the corresponding system (1.1). Here, f and B in (1.1) have the form

(1.16)
$$f = \frac{1}{ku_2+1}\begin{pmatrix} u_1 \\ (1+k)u_2 \end{pmatrix}, \quad B = \begin{pmatrix} \alpha k_1(1-\xi)\pi_1'(\xi) & \xi(B_{11}-B_{22}) \\ 0 & \frac{k_1k_2u_2(1-u_2)\pi_2'(u_2)}{k_2u_2+k_1(1-u_2)} \end{pmatrix},$$

where k_1, k_2 are positive constants, $k = \frac{k_2}{k_1} - 1$, $\xi = u_1(1-u_2)^{-1}$, and π_1, π_2 are given functions satisfying $\pi_1'(\xi) \geq 0$, $\pi_2'(u_2) \geq 0$.

The main difficulty here is that the matrix B is degenerate at $\partial\Delta$. The introduction of the variable ξ, so called relative saturation, plays a decisive role in overcoming this difficulty. We then obtain the following result proved in section 6.

Theorem 1.3. *Assume $\pi_1(\xi), \pi_2(u_2)$ are given smooth functions of $\xi, u_2 \in (0,1)$, with $\pi_1'(\xi) \geq 0$, $\pi_2'(u_2) \geq 0$, $0 < \xi < 1$, $0 < u_2 < 1$, and*

$$u_0 \in L^\infty(\Omega), \quad u_\pm \in W^{1,1}(0,T), \quad 0 < \delta \leq u_{2,\pm}(t) \leq 1-\delta, \quad \delta \leq \xi_\pm(t) \leq 1-\delta,$$
$$\delta \leq u_{2,0}(x) \leq 1-\delta, \quad \delta \leq \xi_0(x) \leq 1-\delta,$$

for some $\delta \in (0,1)$, where

$$\xi_0 = \frac{u_{1,0}}{1-u_{2,0}}\ , \quad \xi_\pm = \frac{u_{1,\pm}}{1-u_{2,\pm}}\ .$$

Then, with f and B given by (1.16), problem (1.1),(1.3) has a weak solution $u(t,x)$ such that

$$u \in L^\infty(Q), \quad u_x \in L^2(Q), \quad u_t \in L^2(0,T;W^{-1,2}(\Omega)),$$

and the estimates

$$0 < \delta \leq u_2 \leq 1-\delta, \quad \delta \leq \xi \leq 1-\delta, \tag{1.17}$$

hold.

2. A mixed initial boundary value problem

In this section, for given $\varepsilon, \delta > 0$, we consider the initial boundary value problem

$$u_t + f(u)_x = (B(u)u_x)_x + \varepsilon h, \quad (t,x) \in Q \equiv (0,T)\times\Omega, \quad \Omega = (-1,1), \tag{2.1}$$

$$\delta u_\eta + u = u_{b,\varepsilon} \quad \text{at} \quad |x| = 1, \quad u|_{t=0} = u_{0,\varepsilon}(x). \tag{2.2}$$

Here

$$u_\eta|_{x=\pm1} = \pm u_x, \quad u_b|_{x=\pm1} = u_\pm(t),$$
$$u_{i\pm,\varepsilon} = (1-\varepsilon)(\frac{\varepsilon}{2} + u_{i\pm}), \quad u_{i0,\varepsilon} = (1-\varepsilon)(\frac{\varepsilon}{2} + u_{i0}), \qquad i = 1,2.$$

The function $h = (h_1, h_2)$ is any smooth map satisfying $h(\omega)\cdot\nu(\omega) < 0$, for all $\omega \in \partial\Delta$, with $\nu(\omega)$ the unit outer normal to $\partial\Delta$, defined everywhere in $\partial\Delta$, except at the vertices. For example, $h(u) = u_* - u$, where u_* is any point in the interior of Δ.

The problem (2.1),(2.2) should be considered as a perturbation of problem (1.1),(1.2). The inclusion of the perturbation term εh and the slight change in the initial and boundary data are necessary for the proof of the positive invariance of the triangle Δ in Lemma 2.1 below. However, we remark in advance that in all the remaining lemmas in this section, the constants c in the estimates do not depend on ε, which will easily allow to make $\varepsilon \to 0$ in the conclusion of the proof of Theorem 1.1, in the next section. In Lemmas 2.1 to 2.10 below, we omit the subscript ε when referring to the functions $u_{0,\varepsilon}(x)$, $u_{\pm,\varepsilon}(t)$.

Lemma 2.1. *The solution u of* (2.1),(2.2) *takes values in* $\operatorname{int}(\Delta)$ *whenever u_0 and $u_\pm$ take values in Δ.*

Proof. We follow the method of positively invariant regions [12] (see also [11]). Let us denote $u_3 = 1 - u_1 - u_2$ and $G_i(u) = -u_i$, $i = 1, 2, 3$. Setting $z_i = G_i(u)$, we prove that $z_i < 0$ for each i. Clearly,

$$\max_{x\in\Omega} z_i(0, x) < 0, \quad i \in \{1, 2, 3\}.$$

Suppose, there is a first time $t_1 > 0$ such that

$$\max_{x\in\Omega} z_i(t_1, x) = z_i(t_1, x_0) = 0$$

for some i. We consider separetely the two cases: $|x_0| < 1$ and $|x_0| = 1$. First, we observe that $x_0 = 1$ is impossible. Indeed, it follows from the equality

$$\delta z_{ix} + z_i = -u_{i,+\varepsilon} \tag{2.3}$$

that $z_{ix}(t_1, 1) < 0$, which gives a contradiction since $z_i(t_1, x) \le 0$, $x \in [-1, 1]$. Similarly, we prove that $x_0 = -1$ cannot hold either. So, the case $|x_0| = 1$ is ruled out.

Let us consider the case $|x_0| < 1$. Multiplying (2.1) by $\nabla_u G_i$ and using (1.9), (1.10), one arrives at

$$z_{it} + \alpha_i z_{ix} = (\mu_i z_{ix})_x + \varepsilon h \cdot \nabla_u G_i \quad \text{at} \quad (t_1, x_0), \tag{2.4}$$

where α_i and $\mu_i \ge 0$ are scalar functions of $u \in \{G_i(u) = 0\} \cap \Delta$. By the assumption,

$$z_i(t_1, x_0) = \max_{\substack{0\le\tau\le t_1 \\ |y|\le 1}} z_i(\tau, y).$$

Hence, we must have

$$z_{ix}(t_1, x_0) = 0, \quad z_{ixx}(t_1, x_0) \le 0, \quad z_{it}(t_1, x_0) \ge 0. \tag{2.5}$$

Due to the choice of h, we have

$$h \cdot \nabla_u G_i < 0 \quad \text{at} \quad (t_1, x_0).$$

Now, it follows from (2.4) that $z_{it}(t_1, x_0) < 0$, contradicting (2.5). ∎

Lemma 2.2. *The estimate*

$$\|u_x\|_{L^2(Q)} + \delta \sum_{\pm} \int_0^T |u_x(t, \pm 1)|^2 dt \le c \tag{2.6}$$

holds with a constant c depending on ν and the norms $\|\dot{u}_\pm\|_{L^1(0,T)}$ and $\|h\|_{L^1(Q)}$. In particular, by (2.1)*, it follows that*

$$\|u_t\|_{L^2(0,T;W^{-1,2}(\Omega))} \le c \tag{2.7}$$

uniformly in ε and δ.

Proof. Denote

$$w = \frac{1-x}{2}u_- + \frac{1+x}{2}u_+ \,, \quad v = u - w.$$

Then we have from equation $(2.1)_2$ that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v_2^2 dx + \int_\Omega B_{22}|v_{2x}|^2 dx = v_2(B_{22}(v_{2x}+w_{2x}) - f_2)|_{-1}^{+1}$$
$$+ \int_\Omega v_{2x}(f_2 - B_{22}w_{2x}) - w_{2t}v_2 + \varepsilon h_2 v_2 \; dx.$$

Since

$$v|_{x=\pm 1} = \mp\delta(v_x + w_x)|_{x=\pm 1}$$

and $B_{22} \ge \nu$, estimate (2.6) for u_2 follows by the Cauchy inequality.

From equation $(2.1)_1$, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v_1^2 dx + \int_\Omega B_{11}|v_{1x}|^2 dx = v_1(B_{11}(v_{1x}+w_{1x}) + B_{12}u_{2x} - f_1)|_{-1}^{+1}$$
$$- \int_\Omega v_{1x}(B_{11}w_{1x} + B_{12}u_{2x} - f_1) - w_{1t}v_1 + \varepsilon h_1 v_1 \; dx.$$

By the same argument, one can derive the claim of the lemma for the function u_1, using estimate (2.6) for u_2. □

Let $|u|_Q^{(\alpha)}$ denote the norm of function $u(x,t)$ in the Hölder space $H^{\alpha,\alpha/2}(\overline{Q})$ (cf. [6]):

$$|u|_Q^{(\alpha)} = \sup_{\overline{Q}} |u(x,t)| + \sup_{x_1,x_2\in\overline{\Omega},t\in[0,T]} \frac{|u(x_1,t) - u(x_2,t)|}{|x_1 - x_2|^\alpha}$$
$$+ \sup_{x\in\overline{\Omega},t_1 t_2\in[0,T]} \frac{|u(x,t_1) - u(x,t_2)|}{|t_1 - t_2|^{\alpha/2}}.$$

The following estimates depend, in general, on δ.

Lemma 2.3. *There are constants c and $\alpha \in (0,1)$ such that*

$$|u_2|_Q^{(\alpha)} \le c. \tag{2.8}$$

Moreover, if $u_\pm \equiv 0$, the estimate (2.8) *holds uniformly in δ.*

Proof. Let $\zeta(t,x)$ be a test function with values between 0 and 1 and that is different from zero only for $x \in K_\rho$, the ball of radius ρ centered at $x^0 \in \overline{\Omega}$. Denote

$$\Omega_\rho = \overline{\Omega} \cap K_\rho = [x_-^0, x_+^0], \quad x_+^0 = \min\{1, x_0 + \rho\}, \quad x_-^0 = \max\{-1, x_0 - \rho\}.$$

Given $\delta' > 0$, we multiply equation $(2.1)_2$ by

$$\zeta^2 \max\{u_2 - k, 0\} \equiv \zeta^2 u_2^{(k)}, \quad k \ge -\delta',$$

and integrate over Ω_ρ. We have

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_\rho}\zeta^2|u_2^{(k)}|^2\,dx+\int\limits_{\Omega_\rho}\zeta^2B_{22}|u_{2x}^{(k)}|^2\,dx=\zeta^2B_{22}u_{2x}u_2^{(k)}|_{x_-^0}^{x_+^0}-\zeta^2f_2u_2^{(k)}|_{x_-^0}^{x_+^0}$$

$$-\int\limits_{\Omega_\rho}2\zeta\zeta_xB_{22}u_{2x}u_2^{(k)}-\zeta\zeta_t|u_2^{(k)}|^2-f_2(2\zeta\zeta_xu_2^{(k)}+\zeta^2u_{2x}^{(k)})-\varepsilon h_2\zeta^2u_2^{(k)}\,dx.$$

Observe that

$$\delta u_x|_{x=\pm1}=\pm(u_\pm-u)|_{x=\pm1},$$

$$\zeta^2B_{22}u_{2x}u_2^{(k)}|_{x_-^0}^{x_+^0}\le\frac{1}{\delta}\zeta^2B_{22}u_2^{(k)}u_{2+}|_{x=1}+\frac{1}{\delta}\zeta^2B_{22}u_2^{(k)}u_{2-}|_{x=-1}\ ,$$

$$|\zeta^2v^{(k)}|_{|x|=1}\le|\int\limits_{\Omega_\rho}\zeta^2v_x^{(k)}+2\zeta\zeta_xv^{(k)}\,dx| \tag{2.9}$$

for small ρ. Thus, using $B_{22}\ge\nu$,

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_\rho}\zeta^2|u_2^{(k)}|^2\,dx+\nu\int\limits_{\Omega_\rho}\zeta^2|u_{2x}^{(k)}|^2\,dx\le\frac{\nu}{2}\int\limits_{\Omega_\rho}\zeta^2|u_{2x}^{(k)}|^2\,dx \tag{2.10}$$

$$+c_1\int\limits_{\Omega_\rho}|u_2^{(k)}|^2(|\zeta_x|^2+|\zeta\zeta_t|)+\zeta^2\mathbf{1}_{A_{k,\rho}(t)}\,dx,$$

where $A_{k,\rho}(t)$ is the intersection of the support of $u_2^{(k)}$ with K_ρ, and $\mathbf{1}_A$ is the characteristic function of the set A. Proceeding in an analogous way, we prove that (2.10) also holds with u_2 replaced by $-u_2$, for $k\le1+\delta'$. These inequalities imply that u_2 belongs to a class $\mathcal{B}_2(Q\cup\Gamma,M,\gamma,r,\delta',\kappa)$ (see [6], Chapter II, §7 and §8), with $\Gamma=\partial Q\setminus\{t=T\}$, $M=1$, $r=6$, $\kappa=2$, and, hence, $u_2\in H^{\alpha,\alpha/2}(\overline{Q})$ for some $\alpha\in(0,1)$.

As for the last statement, indeed, in this case,

$$\zeta^2B_{22}u_{2x}u_2^{(k)}|_{x_-^0}^{x_+^0}=-\sum_{x=\pm1}\frac{1}{\delta}\zeta^2B_{22}u_2u_2^{(k)}\le0,$$

and the constant c_1 in (2.10) does not depend on δ. □

Lemma 2.4. *There is a constant c such that*

$$\max_{0\le t\le T}\{\int\limits_{\Omega}u_{2x}^2\,dx+\delta(\sum_{x=\pm1}u_{2x}^2)\}+\int\limits_{Q}u_{2xx}^2+u_{2x}^4+u_{2t}^2\,dx\,dt\le c\ . \tag{2.11}$$

Moreover, if $u_\pm\equiv0$, the constant c in (2.11) *does not depend on δ.*

Proof. Let $\zeta(x)$ be the test function like above. We multiply equation $(2.1)_2$ by $(\zeta^2u_{2x})_x$ and integrate over Ω_ρ. Using the equality

$$\delta\dot u_\eta+\dot u=\dot u_b\quad\text{at}\quad x=\pm1,$$

and the Young inequality, we obtain that

$$\frac{1}{2}\frac{d}{dt}\{\int\limits_{\Omega_\rho}\zeta^2u_{2x}^2\,dx+\delta(\sum_{x=x_\pm^0}\zeta^2u_{2x}^2)\}+\nu\int\limits_{\Omega_\rho}\zeta^2u_{2xx}^2\,dx\le J,$$

$$J \equiv \|\dot{u}_b\|_{C([0,T])} \times \sum_{x=x_\pm^0} |\zeta^2 u_{2x}| + \frac{\nu}{4} \int\limits_{\Omega_\rho} \zeta^2 u_{2xx}^2 \, dx$$

$$+c_* \int\limits_{\Omega_\rho} \zeta^2 u_{2x}^4 dx + c \int\limits_{\Omega_\rho} \zeta^2 (u_{2x}^2 + u_{1x}^2) + u_{2x}^2(\zeta_x^2 + \zeta^2) + \zeta^2 \, dx.$$

We estimate the first term in the righ-hand side of the above identity using an inequality like (2.9) and, again, Young's inequality, in order to obtain

$$\|\dot{u}_b\|_{C([0,T])} \times \sum_{x=x_\pm^0} |\zeta^2 u_{2x}| \le \frac{\nu}{4} \int_{\Omega_\rho} \zeta^2 u_{2xx}^2 \, dx + c_\nu \int_{\Omega_\rho} u_{2x}^2 \zeta_x^2 + \zeta^2 \, dx.$$

Finally, we recall the inequality ([6], Ch. II, lemma 5.4)

$$\int\limits_{K_\rho} \zeta^2 v_x^4 dx \le 16 \mathrm{osc}^2\{v, K_\rho\} \int\limits_{K_\rho} 2\zeta^2 v_{xx}^2 + \zeta^2 v_x^2 \, dx. \tag{2.12}$$

By Lemma 2.3,

$$\mathrm{osc}^2\{u_2, K_\rho\} \le c\rho^{\alpha_1}, \quad \alpha_1 < \alpha.$$

Now, the assertion of the lemma follows if we take ρ such that $32 c_* \rho^\alpha < \nu/4$, where α is the constant from Lemma 2.3. □

Lemma 2.5. *There are constants c and $\alpha \in (0,1)$ such that $|u_1|_Q^{(\alpha)} \le c$. Moreover, if $u_\pm \equiv 0$, the constant c in Lemma 2.5 does not depend on δ.*

Proof. Let $\zeta(t,x)$ be a function like in Lemma 2.3. Then, given $\delta' > 0$, for $k \ge -\delta'$,

$$\frac{1}{2}\frac{d}{dt} \int\limits_{\Omega_\rho} \zeta^2 |u_1^{(k)}|^2 \, dx + \int\limits_{\Omega_\rho} \zeta^2 B_{11} |u_{1x}^{(k)}|^2 \, dx = J_1 + J_2 \ , \quad J_1 = \zeta^2 B_{11} u_{1x} u_1^{(k)} |_{x_-^0}^{x_+^0} ,$$

$$J_2 = \int\limits_{\Omega_\rho} \zeta\zeta_t u_1^{(k)} - 2\zeta\zeta_x B_{11} u_{1x} u_1^{(k)} + u_1^{(k)} \zeta^2 [(B_{12} u_{2x})_x - \frac{\partial f_1}{\partial x} + \varepsilon h_1] \, dx.$$

We have

$$J_1 \le \frac{1}{\delta} \sum_{x=\pm 1} \zeta^2 B_{11} u_{1,\pm} u_1^{(k)} \overset{(2.9)}{\le} \int\limits_{\Omega_\rho} \frac{\nu}{4} \zeta^2 |u_{1x}^{(k)}|^2 + c\,\zeta_x^2 |u_1^{(k)}|^2 + \zeta^2 \mathbf{1}_{A_{k,\rho}(t)} \, dx,$$

$$J_2 \le \int\limits_{\Omega_\rho} |u_1^{(k)}|^2 (|\zeta\zeta_t| + \zeta^2) \, dx + c(\int\limits_{\Omega_\rho} \zeta \mathbf{1}_{A_{k,\rho}(t)} \, dx)^{1/2} (\int\limits_{\Omega_\rho} \mathbf{1}_{A_{k,\rho}(t)} (1 + u_{2x}^4 + u_{2xx}^2) \, dx)^{1/2}.$$

As in Lemma 2.3, we can prove analogously that the above inequality also holds with u_1 replaced by $-u_1$ and $k \le 1 + \delta'$. Hence, (see [6] Ch. II, §7 and §8) the function u_1 belongs to the class $\mathcal{B}_2(Q \cup \Gamma, M, \gamma, r, \delta', \kappa)$, for some $\gamma > 0$, with $M = 1$, $r = 6$, $\kappa = 2$, and the lemma is proved. □

Lemma 2.6. *There is a constant c such that*

$$\max_{0 \le t \le T} \{ \int\limits_{\Omega} u_{1x}^2 \, dx + \delta \sum_{x=\pm 1} |u_{1x}|^2 \} + \int\limits_{Q} u_{1xx}^2 + u_{1x}^4 + u_{1t}^2 \, dx \, dt \le c.$$

Moreover, if $u_\pm \equiv 0$, the constant c in the above lemma does not depend on δ.

Proof. Let $\zeta(x)$ be a test function like in Lemma 2.5. Then it follows from equation $(2.1)_1$ that

$$\frac{1}{2}\frac{d}{dt}\{\int_{\Omega_\rho} \zeta^2 u_{1x}^2\,dx + \delta(\sum_{x=x_\pm^0} \zeta^2 u_{1x}^2)\} + \nu \int_{\Omega_\rho} \zeta^2 u_{1xx}^2\,dx \le cJ,$$

$$J = \|\dot{u}_b\|_{C([0,T])} \sum_{x=x_\pm^0} |\zeta^2 u_{1x}| + \frac{\nu}{2}\int_{\Omega_\rho} \zeta^2 u_{1xx}^2\,dx + \int_{\Omega_\rho} \zeta_x^2(u_{2x}^2 + u_{1x}^2 + u_{2xx}^2 + u_{2x}^4)\,dx$$

$$+ \int_{\Omega_\rho} \zeta^2(u_{1x}^4 + u_{2x}^4 + u_{1x}^2\zeta_x^2 + u_{2xx}^2 + u_{1x}^2 + 1 + u_{2x}^4\zeta_x^4 + u_{2x}^2\zeta_x^2).$$

Applying inequality (2.12), one arrives at the conclusion of the lemma. □

Lemma 2.7. *There is a constant c such that*

$$\int_Q |u_{ix}|^6\,dx\,dt \le c, \quad \int_Q |u_{ixx}|^3\,dx\,dt \le c, \quad \int_Q |u_{ix}u_{ixx}|^2\,dx\,dt \le c. \tag{2.13}$$

Proof. We start with the simple inequality

$$\int_Q |u_{ix}|^6\,dx\,dt \le \int_0^T \max_{x\in\Omega} |u_{ix}|^4 \int_\Omega |u_{ix}|^2\,dx\,dt.$$

Observe that for any x and y,

$$v_x^2(x) - v_x^2(y) = 2\int_x^y v_{xx}v_x dz\ ,$$

so

$$\max_{|x|<1} v_x^4 \le \|v_x\|^2_{L^2(\Omega)} + 8\|v_{xx}\|^2_{L^2(\Omega)}\|v_x\|^2_{L^2(\Omega)}.$$

Hence,

$$\|u_{ix}\|^6_{L^6(Q)} \le \|u_{ix}\|^4_{L^\infty(0,T;L^2(\Omega))}(1 + 8\|u_{ixx}\|^2_{L^2(Q)}) \le c,$$

and the first estimate of the lemma is proved.

Let us write equation $(2.1)_2$ as

$$\begin{cases} u_{2t} = B_{22}u_{2xx} + F, \\ F = \frac{\partial B_{22}}{\partial u_2}|u_{2x}|^2 - (\frac{\partial f_2}{\partial u_1}u_{1x} + \frac{\partial f_2}{\partial u_2}u_{2x}) + \frac{\partial B_{22}}{\partial u_1}u_{1x}u_{2x} + \varepsilon h_2 \end{cases}. \tag{2.14}$$

By Lemmas 2.4 and 2.6, $\|F\|_{L^3(Q)} \le c$. Now, the theory of linear parabolic equations with smooth coefficients (see [6], Ch. IV,§9) can be applied to derive the estimate

$$\int_Q |u_{2xx}|^3\,dx\,dt \le c, \tag{2.15}$$

which is the second estimate in (2.13) for u_2. The third estimate in (2.13) for u_2 follows because of the inequality

$$\int_Q |uv|^2\,dx\,dt \le (\int_Q |u|^6\,dx\,dt)^{1/3}(\int_Q |v|^3\,dx\,dt)^{2/3}. \tag{2.16}$$

Equation $(2.1)_1$ writes

$$u_{1t} = B_{11} u_{1xx} + F, \tag{2.17}$$

$$F = u_{1x}(\frac{\partial B_{11}}{\partial u_1} u_{1x} + \frac{\partial B_{11}}{\partial u_2} u_{2x}) + u_{2x}(\frac{\partial B_{12}}{\partial u_1} u_{1x} + \frac{\partial B_{12}}{\partial u_2} u_{2x}) + B_{12} u_{2xx}$$
$$-(\frac{\partial f_1}{\partial u_1} u_{1x} + \frac{\partial f_1}{\partial u_2} u_{2x}) + \varepsilon h_1,$$

where, as above, $\|F\|_{L^3(Q)} \leq c$. Hence, the estimate (2.15) is also valid for u_{1xx}, which concludes the verification of the second estimate in (2.13). Now, the third estimate in (2.13) for u_1 follows also due to inequality (2.16).

□

Lemma 2.8. *There are constants c and $\alpha \in (0,1)$ such that $|u_{2x}|_Q^{(\alpha)} \leq c$.*

Proof. Let us differentiate equation $(2.1)_2$ with respect to x. The function $v := u_{2x}$ solves the linear equation

$$v_t = (B_{22} v_x)_x + F_2 + g_x,$$
$$F_2 = \frac{\partial^2 B_{22}}{\partial u_2^2}(u_{2x})^3 + 2\frac{\partial B_{22}}{\partial u_2} u_{2x} u_{2xx} + \frac{\partial B_{22}}{\partial u_1}(u_{1xx} u_{2x} + u_{1x} u_{2xx})$$
$$+ \frac{\partial^2 B_{22}}{\partial u_1^2} u_{1x}^2 u_{2x} + 2\frac{\partial^2 B_{22}}{\partial u_1 \partial u_2} u_{1x} u_{2x}^2,$$
$$g = -\frac{\partial f_2}{\partial u_1} u_{1x} - \frac{\partial f_2}{\partial u_2} u_{2x} + \varepsilon h_2.$$

By the above lemmas,

$$\|F_2\|_{q,r,Q} \equiv (\int (\int_\Omega F_2^q)^{r/q} dt)^{1/r} \leq c, \quad \|g^2\|_{q,r,Q} \leq c,$$

when $q = 2$, $r = 2$. Clearly, the constants q and r satisfy the conditions

$$\frac{1}{r} + \frac{1}{2q} = 1 - \kappa, \quad 0 < \kappa < \frac{1}{2}, \quad q \in [1, \infty], \quad r \in [\frac{1}{1-\kappa}, \frac{2}{1-2\kappa}],$$

with $\kappa = 1/4$. Moreover, it follows from the boundary conditions (2.2) and Lemma 2.3, that

$$\|v(t, \pm 1)\|_{H^{\alpha/2}([0,T])} \leq c.$$

Thus, by the theory of linear parabolic equations ([6], Ch.III,§10)

$$|v|_Q^{(\alpha')} \leq c$$

for some $\alpha' \leq \alpha$. □

Lemma 2.9. *There are constants c and $\alpha \in (0,1)$ such that $|u_{1x}|_Q^{(\alpha)} \leq c$.*

Proof. Denoting $u_{1x} = v$ and differentiating equation $(2.1)_1$ with respect to x, we have

$$v_t = (B_{11}(u_1, u_2) v_x)_x + F_1 + g_{1x}, \tag{2.18}$$

$$F_1 = u_{1xx}(\frac{\partial B_{11}}{\partial u_1} u_{1x} + \frac{\partial B_{11}}{\partial u_2} u_{2x}) + u_{1x}\{\frac{\partial B_{11}}{\partial u_1} u_{1xx} + u_{1x}(\frac{\partial^2 B_{11}}{\partial u_1^2} u_{1x} + \frac{\partial^2 B_{11}}{\partial u_1 \partial u_2} u_{2x})$$
$$+ \frac{\partial B_{11}}{\partial u_2} u_{2xx} + u_{2x}(\frac{\partial^2 B_{11}}{\partial u_1 \partial u_2} u_{1x} + \frac{\partial^2 B_{11}}{\partial u_2^2} u_{2x})\},$$

$$g_1 = u_{2x}(\frac{\partial B_{12}}{\partial u_1}u_{1x} + \frac{\partial B_{12}}{\partial u_2}u_{2x}) + B_{12}u_{2xx} - (\frac{\partial f_1}{\partial u_1}u_{1x} + \frac{\partial f_1}{\partial u_2}u_{2x}) + \varepsilon h_1.$$

By Lemma 2.7, $\|F_1\|_{L^2(Q)} \le c$. With the estimate of Lemma 2.8 for u_{2x} at hand, the function F in equation (2.14) meets the estimate $\|F\|_{L^4(Q)} \le c$. It implies that

$$\|u_{2xx}\|_{L^4(Q)} \le c, \quad \|g_1\|_{L^4(Q)} \le c.$$

Now, one may treat (2.18) as a linear parabolic equation for v, with

$$\|F_1, g_1^2\|_{2,2,Q} \le c, \quad \|v(t, \pm 1)\|_{H^{\alpha/2}([0,T])} \le c.$$

By the same argument as in Lemma 2.8, we conclude that

$$|u_{1x}|_Q^{(\alpha')} \le c$$

for some $\alpha^{'} < \alpha$. □

Lemma 2.10. *Let*

$$u_0 \in H^{2+\beta}(\overline{\Omega}), \quad u_\pm \in H^{1+\beta/2}([0,T]), \quad 0 < \beta < 1,$$

and the compatibility conditions

$$\pm\delta u_0^{'}(\pm 1) + u_0(\pm 1) = u_\pm(0) \tag{2.19}$$

be satisfied. Then there is a constant c *such that solutions to problem* (2.1),(2.2) *satisfy the estimate* $|u|_Q^{(2+\beta)} \le c$. *The constant* c *does not depend on* ε *but depends on* T, $\|u_0|H^{2+\beta}(\overline{\Omega})\|$, $\|u_\pm|H^{1+\beta/2}([0,T])\|$, *and the* L^∞*–norms of* $f(u)$, $\nabla_u f$, $B_{ij}(u)$, $\nabla_u B_{ij}$, *and* $\nabla_u^2 B_{ij}$.

Proof. First, we observe that the data $u_{0\varepsilon}$ and $u_{b\varepsilon}$ also satisfy the compatibility conditions (2.19). We know from the above lemmas that there are constants c and $\alpha \in (0,1)$ such that $|u_i, u_{ix}|_Q^{(\alpha)} \le c$. If $\gamma = \min\{\alpha, \beta\}$, it follows from the linear equation (2.14) that $|u_2|_Q^{(2+\gamma)} \le c$ ([6], Ch.IV, §5). By the same argument, we conclude from equation (2.17) that $|u_1|_Q^{(2+\gamma)} \le c$. To increase γ up to β, one should return to problem (2.14), which now ensures that $|u_2|_Q^{(2+\beta)} \le c$. Next, one should pass to equation (2.17) to make sure that $|u_1|_Q^{(2+\beta)} \le c$. □

3. Existence and uniqueness

To prove the solvability of problem (2.1),(2.2), we apply a fixed point argument in the form of the Leray-Schauder principle as in [6]. Let $\mathcal{B}$ be a Banach space of vector-functions $u(t,x) \in \mathbb{R}^2$, having the bounded norm

$$\|u\|_{\mathcal{B}} = |u|_Q^{(\beta)} + |u_x|_Q^{(\beta)}.$$

Given $v \equiv (v_1, v_2) \in \mathcal{B}$ and $\lambda \in [0,1]$, we define $u = (u_1, u_2)$ as a solution to the linear problem

$$\begin{aligned} &u_{1t} + \lambda[\frac{\partial f_1(v)}{\partial x} - (B_{11}(v)u_{1x})_x - (B_{12}(v)u_{2x})_x - \varepsilon h_1(\mathbf{a})] = (1-\lambda)u_{1xx}, \\ &u_{2t} + \lambda[\frac{\partial f_2(v)}{\partial x} - (B_{22}(v)u_{2x})_x - \varepsilon h_2(v)] = (1-\lambda)u_{2xx}, \end{aligned} \tag{3.1}$$

$$\delta u_\eta + u = u_{b\varepsilon}, \quad u|_{t=0} = u_{0\varepsilon} = (1-\varepsilon)(\frac{\varepsilon}{2} + u_{01}, \frac{\varepsilon}{2} + u_{02}). \tag{3.2}$$

The second equation does not involve the function u_1. So, by the theory of linear parabolic equations, given $\lambda \in [0,1]$, the operator A_λ, which associates with each

$v \in \mathcal{B}$ the solution $A_\lambda(v)$ of (3.1),(3.2), is well-defined, and its fixed points are solutions to problem (2.1),(2.2) when $\lambda = 1$.

In order to apply Leray-Schauder theorem, as in [6], one must choose appropriately a domain $\mathcal{U}$ for the operators A_λ, $\lambda \in [0,1]$, and verify each of the conditions of the theorem. By repeating the arguments of the lemmas in Section 2, one arrives at the a priori estimates for the fixed points u_λ of the operator A_λ:

$$u_\lambda \in \Delta, \quad |u_\lambda, u_{\lambda x}|_Q^{(\beta)} \leq M,$$

where the constants M, M_1 are independent of λ. We choose as the domain of the operators A_λ the set

$$\mathcal{U} = \{u \in \mathcal{B} \,:\, u(x,t) \in \Delta', \quad |u_\lambda, u_{\lambda x}|_Q^{(\beta)} \leq M'\},$$

where $\Delta' \subseteq \mathbb{R}^2$ is a closed set satisfying $\mathrm{int}(\Delta') \supset \Delta$ and $M' > M$. Clearly, $\mathcal{U}$ is the closure of a bounded connected open set in $\mathcal{B}$, and all the fixed-points u_λ of A_λ are in the interior $\mathcal{U}$. This means that we have verified the first and most difficult of the following conditions for the application of Leray-Schauder theorem:

1. The boundary of $\mathcal{U}$ does not contain solutions of $A_\lambda(u) = u$;
2. The set $\cup_{\lambda\in[0,1]} A_\lambda(\mathcal{U})$ is compact in $\mathcal{B}$;
3. The mapping $(\lambda, v) \mapsto A_\lambda(v)$ is continuous from $\mathcal{U} \times [0,1]$ to $\mathcal{B}$;
4. The family of maps $\{v \mapsto A_\lambda(v)\}_{\lambda\in[0,1]}$ is equicontinuous on $\mathcal{U}$;
5. The operator A_0 has a unique fixed point in the interior of $\mathcal{U}$, and the mapping $v \mapsto v - A_0(v)$ has an inverse near this fixed point.

The verification of the conditions (2) to (4) above is immediate from all the preceding discussion in this section and section 2, while (5) follows immediately from the fact that $A_0(v)$ is in fact independent of v, that is, it is one and the same element of $\mathcal{B}$ for all v. Hence, problem (2.1),(2.2) has at least one solution in the Hölder space $H^{2+\beta,1+\beta/2}(\bar{Q})$. Uniqueness obtained in a standard way, by using the theory of linear equations with smooth coefficients (see [6], Ch. IV). Thus, we have proved the following.

Theorem 3.1. *Let the functions $f(u)$, $\nabla_u f$, $B_{ij}(u)$, $\nabla_u B_{ij}$, $\nabla_u^2 B_{ij}$, and the function $h(u)$ be Hölder continuous with the Hölder exponent $\beta \in (0,1)$. Let the conditions of Lemma 2.10 be satisfied. Then problem (2.1),(2.2) has a unique solution $u(t,x) \in H^{2+\beta,1+\beta/2}(\bar{Q})$ such that $u(t,x) \in \Delta$ for each $(t,x) \in Q$.*

Proof of Theorem 1.1. Since the estimate of Lemma 2.10 does not depend on ε there is a sequence $\varepsilon_k \downarrow 0$, such that the corresponding sequence u_k of solutions of problem (2.1),(2.2) converges to a function $u \in H^{2+\beta,1+\beta/2}(\overline{Q})$ in the norm $|\cdot|_Q^{(2+\gamma)}$ for any $\gamma < \beta$. Clearly, u solves the problem (1.1),(1.2). Thus, Theorem 1.1 is proved. □

4. Dirichlet boundary conditions

Proof of Theorem 1.2. Let us consider the Dirichlet problem (1.1),(1.3). The estimates (2.6) and (2.7) are uniform with respect to $\delta \downarrow 0$. By the Aubin-Lions compactness theorem [8], they imply that there is a sequence u_k, $\delta_k \downarrow 0$, of solutions to problem (1.1),(1.2) and a function u such that

$$u \in L^\infty(Q), \quad u_x \in L^2(Q), \quad u_t \in L^2(0,T; W^{-1,2}(\Omega)), \tag{4.1}$$

$$u_k \to u \quad \text{in} \quad L^2(Q), \quad u(t,x) \in \Delta \quad \text{for each} \quad (t,x) \in Q.$$

Clearly, the function u solves equation (1.1) weakly. Now, given any open set Q' with $\overline{Q'} \subseteq Q$, we have that u_k is uniformly bounded in $H^{2+\beta,1+\beta/2}(\overline{Q'})$, so that $u \in H^{2+\beta,1+\beta/2}(\overline{Q'})$. In particular, u is a classical solution of (1.1) in Q. Due to estimate (2.6), the boundary condition $u|_{x=\pm 1} = u_\pm$ holds in $L^2(0,T)$. The inclusions (4.1) imply that $u \in C(0,T;L^2(\Omega))$, so the function u satisfies the initial condition $u|_{t=0} = u_0$ weakly in $L^2(\Omega)$. This proves the first part of Theorem 1.2. The last part is a consequence of the fact that, when $u_b = 0$, the estimates of Lemmas 2.2 to 2.5 are uniform in δ. Therefore, in this case we have $u \in H^{\beta,\beta/2}(\overline{Q})$ and, so, the last assertion follows. □

5. Basic equations of three-phase flow

For the reader's convenience we recall the underlying laws of multiphase flows in a porous medium [1]. We consider one-dimensional horizontal flows of three incompressible immiscible fluids formed in phases. The balance of masses is governed by the mass conservation equations

$$\frac{\partial}{\partial t}(mu_i\rho_i) + \frac{\partial}{\partial x}(\rho_i v_i) = 0, \quad \rho_i = \text{const}, \tag{5.1}$$

where m denotes porosity of the porous medium, u_i, ρ_i, and v_i are the saturation, density, and seepage velocity of the i-th phase. The functions u_i satisfy the volume-balance equation

$$u_1 + u_2 + u_3 = 1. \tag{5.2}$$

The theory of multiphase flows in porous media is based on the following form of Darcy's law

$$v_i = -k\lambda_i p_{ix}, \tag{5.3}$$

where k stands for the absolute permeability, p_i is the pressure of the i-th phase and λ_i is the mobility of the i-th phase and it is assumed to be a function of $u = (u_1, u_2)$, that is, $\lambda_i = \lambda_i(u_1, u_2)$, $i = 1,2,3$. Experimentally, only the functions $\lambda_1(u_1, 0)$, $\lambda_2(0, u_2)$, $\lambda_3(u_1, 0)$ and $\lambda_3(0, u_2)$ are known, which correspond to mobilities in two-phase flows. But there are widely accepted models prescribing the mobility functions everywhere in Δ, based on the knowledge of them for two-phase flows, such as the one of H. Stone [13].

The capillary pressures are defined as pressure differences (cf.,e.g.,[10, 1]), and we assume here that they are functions of the saturations u_1, u_2, that is,

$$P_1(u_1, u_2) = p_1 - p_3, \quad P_2(u_1, u_2) = p_2 - p_3. \tag{5.4}$$

Similarly to what happens with the mobility functions, the capillary pressure functions are only known in practice in the case of two-phase flows. However, as opposed to the case of the mobility functions, so far there is no widely accepted way of defining these functions everywhere in Δ, assuming them to be known for two-phase flows. One of the main points in this section is the introduction of a new method to define the capillary pressures everywhere in Δ, assuming that they are known for two-phase flows and that the mobility functions are known everywhere in Δ.

Denote

$$\lambda = \sum_1^3 \lambda_i, \quad f_i = \frac{\lambda_i}{\lambda}, \qquad i = 1,2,3. \tag{5.5}$$

For

$$v = \sum_1^3 v_i,$$

we find from (5.1) and (5.2) that $v_x = 0$, so v depends on t only. We assume for simplicity that $k = m = 1$ as well.

Eliminating the pressure derivative p_{3x}, we have from (2.3)

$$v_1 = f_1(v(t) + \lambda_2 P_{2x} - (\lambda_2 + \lambda_3)P_{1x}), \quad v_2 = f_2(v(t) + \lambda_1 P_{1x} - (\lambda_1 + \lambda_3)P_{2x}).$$

If we substitute these velocities into the first two equations in (5.1) and if we pass to the new time-variable $t := \int_0^t v(s)ds$, we reduce the function $v(t)$ to 1 and obtain a system of the form (1.1), where the 2×2-matrix B is given by

(5.6)

$$B_{11} = \frac{\lambda_1(\lambda_2+\lambda_3)}{\lambda}\frac{\partial P_1}{\partial u_1} - \frac{\lambda_1\lambda_2}{\lambda}\frac{\partial P_2}{\partial u_1}, \; B_{12} = -\frac{\lambda_1\lambda_2}{\lambda}\frac{\partial P_2}{\partial u_2} + \frac{\lambda_1(\lambda_2+\lambda_3)}{\lambda}\frac{\partial P_1}{\partial u_2},$$

$$B_{21} = \frac{\lambda_2(\lambda_1+\lambda_3)}{\lambda}\frac{\partial P_2}{\partial u_1} - \frac{\lambda_1\lambda_2}{\lambda}\frac{\partial P_1}{\partial u_1}, \quad B_{22} = -\frac{\lambda_1\lambda_2}{\lambda}\frac{\partial P_1}{\partial u_2} + \frac{\lambda_2(\lambda_1+\lambda_3)}{\lambda}\frac{\partial P_2}{\partial u_2}.$$

Since u is a saturation vector, we must have $u \in \Delta$. We assume that the mobilities λ_i satisfy the natural conditions (see, e.g., [1])

$$\lambda_i \geq 0, \quad \lambda_i|_{u_i=0} = 0, \quad i \in \{1,2,3\}. \tag{5.7}$$

Let us describe our method for defining the capillary pressures everywhere in Δ, asuming that they are known for two-phase flows. It is based on the assumption of a special structure for the matrix B given by (5.6).

From the preceding discussion, in the system of the form (1.1) for the saturations derived above, the functions $P_i(u_1, u_2)$, $i = 1, 2$, appear only in the matrix B. We now impose that

$$B_{21} = 0, \quad B_{22} = B_{22}(u_2) \geq 0, \quad B_{11} \geq 0 \quad \text{in} \quad \Delta. \tag{5.8}$$

The hypothesis (5.8) means that the first and the third phases are not responsible for the amount of diffusion in the equation for the second phase. The first two conditions in (5.8) read

$$A\frac{\partial P_1}{\partial u_1} = \frac{\partial P_2}{\partial u_1}, \quad \frac{\partial P_2}{\partial u_2} = A\frac{\partial P_1}{\partial u_2} + \frac{\lambda B_{22}}{\lambda_2(\lambda_1+\lambda_3)}, \quad A = \frac{\lambda_1}{\lambda_1+\lambda_3}. \tag{5.9}$$

We study these equations for $P_i(u_1, u_2)$ in the case when the mobilities λ_i are linear functions:

$$\lambda_i = k_i u_i, \quad k_i = \text{const}. \tag{5.10}$$

A symmetry group analysis (see [9]), performed for system (5.9), suggests to look for solutions of the form

$$P_i = q_i(\xi) + Q_i(u_2), \quad \xi = \frac{u_1}{1-u_2} \equiv \frac{u_1}{u_1+u_3}. \tag{5.11}$$

It follows from (5.9) that the functions q_i and Q_i solve the system

$$q_2'(\xi) = q_1'(\xi)A(\xi), \quad A = \frac{k_1\xi}{(k_1-k_3)\xi + k_3}, \quad Q_1'(u_2) = -\frac{k_0 B_{22}(u_2)}{1-u_2},$$

$$Q_2'(u_2) = B_{22}(u_2)\left(\frac{1}{k_3(1-u_2)} + \frac{1}{k_2 u_2}\right), \quad k_0 = \frac{k_3-k_1}{k_1k_3}. \tag{5.12}$$

Assume that the capillary pressure $P_1(u)$ is a given function of u_1 at the part of the boundary of the triangle Δ where $u_2 = 0$:

$$P_1|_{u_2=0} = \pi_1(u_1).$$

Assume also that the capillary pressure $P_2(u)$ is a given function of u_2 at the edge where $u_1 = 0$ of the triangle Δ:

$$P_2|_{u_1=0} = \pi_2(u_2).$$

It follows from (5.11) that

$$\pi_1(u_1) = q_1(u_1) + Q_1(0), \quad \pi_2(u_2) = q_2(0) + Q_2(u_2).$$

It is naturally to set

$$q_1(\xi) = \pi_1(\xi), \quad Q_2(u_2) = \pi_2(u_2).$$

Then the other functions $Q_1(u_2)$ and $q_2(\xi)$ are defined from (5.12) as follows:

$$q_2(\xi) = \int_0^\xi A(\xi)\pi_1^{'}(\xi)d\xi, \quad B_{22}(u_2) = \frac{k_2k_3u_2(1-u_2)\pi_2^{'}(u_2)}{k_2u_2 + k_3(1-u_2)},$$

$$Q_1^{'}(u_2) = -\frac{k_0}{1-u_2}B_{22}(u_2).$$

Thus, we arrive at the formulas for the capillary pressures:

$$\begin{cases} P_1(u_1, u_2) = \pi_1(\xi) - \int_0^{u_2} \frac{k_0k_2k_3u_2\pi_2^{'}(u_2)}{k_2u_2+k_3(1-u_2)}\, du_2 + \text{const}, \\ P_2(u_1, u_2) = \int_0^{\xi} A(\xi)\pi_1^{'}(\xi)\, d\xi + \pi_2(u_2) + \text{const}. \end{cases} \tag{5.13}$$

We call the procedure yielding formulas (5.13) *the method of physical interpolation* since these formulas define the capillary pressures P_1 and P_2 in Δ from their values when $u_2 = 0$ and $u_1 = 0$, respectively.

Substituting (5.10) and (5.13) in (5.6), we get

$$B_{11} = k_1(1 - A(\xi))\pi_1^{'}(\xi), \quad B_{22} = \frac{k_2k_3u_2(1-u_2)\pi_2^{'}(u_2)}{k_2u_2 + k_3(1-u_2)},$$

$$B_{21} = 0, \quad B_{12} = \xi(B_{11} - B_{22}). \tag{5.14}$$

When

$$\pi_1^{'}(\xi) \geq 0, \quad \pi_2^{'}(u_2) \geq 0,$$

system (1.1) is parabolic in Δ but degenerate at $\partial\Delta$.

Finally, we remark that the matrix B given by (5.14) satisfies

$$B_{12}|_{u_1=0} = 0, \quad B_{21} \equiv 0, \quad (B_{11} - B_{12} - B_{22})|_{u_1+u_2=1} = 0.$$

More generally, it is easy to verify that (5.7) implies that the functions B_{ij} of the general capillarity matrix given by (5.6) satisfy

$$B_{12}|_{u_1=0} = 0, \quad B_{21}|_{u_2=0} = 0, \quad (B_{11} + B_{21} - B_{21} - B_{22})|_{u_1+u_2=1} = 0. \tag{5.15}$$

It is also immediate to verify that (5.7) implies that the functions f_i defined in (5.5), for the general three-phase flow model, satisfy

$$f_1|_{u_1=0} = 0, \quad f_2|_{u_2=0} = 0, \quad (f_1 + f_2)|_{u_1+u_2=1} = 1. \tag{5.16}$$

Both (5.15) and (5.16) imply sufficient conditions in order to have $u(x,t) \in \Delta$ for any smooth solution of (1.1), which behaves nicely for $x \in \partial\Omega$.

6. A degenerate problem

Here, we study a particular system arising in petroleum reservoir fluid flows. We assume that mobilities are linear functions

$$\lambda_i = k_i u_i, \quad i \in \{1,2,3\}, \quad k_1 = k_3,$$

and the capillary pressures are given by the formulas (5.13). In this case, the flow is governed by the degenerate parabolic system

$$\begin{aligned} u_{1t} + f_1(u_1,u_2)_x &= (B_{11}(u_1,u_2)u_{1x})_x + (B_{12}(u_1,u_2)u_{2x})_x, \\ u_{2t} + f_2(u_2)_x &= (B_{22}(u_2)u_{2x})_x, \end{aligned} \tag{6.1}$$

with

$$f_1 = \frac{u_1}{ku_2+1}, \quad f_2 = (1+k)\frac{u_2}{ku_2+1}, \quad k = \frac{k_2}{k_1} - 1,$$

and B_{11}, B_{22}, B_{12} are given by (5.14), with $k_1 = k_3$ (see (1.16)). Equations $(6.1)_1$ and $(6.1)_2$ are coupled through the condition $u(t,x) \in \Delta$, which can be written as

$$0 \le u_i(t,x) \le 1, \quad u_2(t,x) \le 1 - u_1(t,x). \tag{6.2}$$

We consider the Dirichlet initial-boundary value problem

$$u|_{x=\pm 1} = u_\pm(t), \quad u|_{t=0} = u_0(x). \tag{6.3}$$

Proof of Theorem 1.3. Consider the approximate non-degenerate problem

$$\begin{cases} u_t + f(u)_x = (B^\nu(u)u_x)_x, \\ \nu u_\eta + u = u_b^\nu \quad \text{at} \quad |x| = 1, \quad u|_{t=0} = u_0^\nu(x), \end{cases} \tag{6.4}$$

with

$$B_{11}^\nu = \nu + \chi_\nu(u_2)B_{11}, \quad B_{22}^\nu = \nu + \chi_\nu(u_2)B_{22}, \quad B_{12}^\nu = \chi_\nu(u_2)\xi(B_{11}^\nu - B_{22}^\nu),$$
$$u_0^\nu \in H^{2+\beta}(\overline{\Omega}), \quad u_0^\nu(x) \in \Delta, \quad u_\pm^\nu \in H^{1+\beta/2}([0,T]), \quad u_\pm^\nu(t) \in \Delta,$$
$$\pm\nu u_0^\nu(\pm 1) + u_0^\nu(\pm 1) = u_\pm^\nu(0),$$
$$\|u_\pm^\nu - u_\pm\|_{W^{1,1}(0,T)} \to 0, \quad \|u_0^\nu - u_0\|_{L^2(\Omega)} \to 0, \quad \text{as} \quad \nu \downarrow 0.$$

Here, $\chi_\nu(u_2)$ is a smooth function such that

$$\chi_\nu(u_2) = 1 \quad \text{if} \quad 0 \le u_2 \le 1 - \nu, \quad \chi_\nu(u_2) = 0, \quad \text{if} \quad 1 - \frac{\nu}{2} \le u_2 \le 1.$$

Clearly, the matrix B^ν satisfies the hypotheses of Theorem 1.1, and so we have the unique solvability of problem (6.4). We also observe that, under the conditions of Theorem 1.3 on the data u_0^ν and $u_\pm^\nu$, any smooth solution of problem (6.4) satisfies the a priori estimate

$$\delta \le u_2(t,x) \le 1 - \delta. \tag{6.5}$$

With this estimate at hand, the matrix B^ν, for small ν, reads

$$B_{11}^\nu = \nu + B_{11}, \quad B_{22}^\nu = \nu + B_{22}, \quad B_{12}^\nu = \xi(B_{11}^\nu - B_{22}^\nu). \tag{6.6}$$

We then have $B_{22}^\nu(u^\nu) \ge \delta^2$ uniformly for $\nu \downarrow 0$. Thus, the constant c in Lemma 2.2 does not depend on ν, and

$$\|u_{2x}^\nu\|_{L^2(Q)} \le c, \quad \|u_{2t}^\nu\|_{L^2(0,T;W^{-1,2}(\Omega))} \le c, \tag{6.7}$$

uniformly in ν.

Taking into account the last equality in (6.6) for the entry B_{12}^{ν}, one can calculate from equations (6.4) that the function $\xi = u_1^{\nu}/(1-u_2^{\nu})$ solves the problem

$$\xi_t + \frac{\xi_x}{ku_2^{\nu}+1} = (B_{11}^{\nu}\xi_x)_x - \frac{\xi_x u_{2x}^{\nu}(B_{11}^{\nu}+B_{22}^{\nu})}{1-u_2^{\nu}} ,$$

$$\frac{\nu(1-u_2^{\nu})}{1-u_{\pm}^{\nu}}\xi_\eta + \xi = \xi_{\pm} \quad \text{at} \quad x = \pm 1, \quad \xi|_{t=0} = \xi_0(x).$$

As we saw in section 5, the variable ξ appears naturally when one is looking for invariant solutions of the homogeneous system corresponding to (5.9). It enables us to apply the maximum principle. By the latter,

$$\delta \le \xi(t,x) \le 1-\delta , \tag{6.8}$$

uniformly in ν. Now, it is a consequence of (6.5) and (6.8) that

$$\delta^2 \le u_1^{\nu}(t,x) \le (1-\delta)^2, \quad B_{11}^{\nu} \ge \delta^2.$$

By the same argument as in Lemma 2.2, we have

$$\|u_{1x}^{\nu}\|_{L^2(Q)} \le c, \quad \|u_{1t}^{\nu}\|_{L^2(0,T;W^{-1,2}(\Omega))} \le c, \tag{6.9}$$

uniformly in ν.

Estimates (6.7) and (6.9) imply by the Aubin-Lions compactness theorem that there are a sequence $u^n \equiv u^{\nu_n}$ and a function u such as described in Theorem 1.3 and such that

$$u^n(t,x) \to u(t,x) \quad \text{a.e. in} \quad Q, \quad u_x^n \to u_x \quad \text{weakly in} \quad L^2(Q).$$

Clearly, u is a weak solution of problem (6.1)-(6.3). Theorem 1.3 is proved. □

Acknowledgements

The research of H. Frid was partially supported by CNPq through the grants 352871/96-2, 46.5714/00-5, 479416/2001-0, and FAPERJ through the grant E-26/152.192/2002. The research of V. Shelukhin was partially supported by Russian Fund of Fundamental Researches, grant 03-05-65299, INTAS, grant 01-868, and CNPq, grants 46.5714/00-5, 479416/2001-0.

References

[1] Allen M.B., Behie J.B., Trangenstein J.A.. "Multiphase Flows in Porous Media: Mechanics, Mathematics and Numerics". Springer-Verlag Lecture Notes in Engineering, **No. 34** (1988)

[2] Amann H., *Dynamic theory of quasi-linear parabolic systems III. Global existence.* Math. Z. **202**, (1989), 219-250.

[3] Chen Z., Ewing R.E., *Comparison of various formulations of three-phase flow in porous media*, Journal of Computational Physics **132**, (1997), 362-373.

[4] Frid H., Shelukhin V.,*A quasilinear parabolic system for three-phase capillary flow in porous media.* To appear in SIAM Jr. Math. Anal.

[5] Hassanizadeh S.M.,Gray W.G. *Theormodynamic basis of capillary pressure in porous media.* Water Resources Research **29**, No. 10,(1993),3389-3405.

[6] Ladyzhenskaya O.A., Solonnikov V.A., Ural'ceva N.N., "Linear and Quasi-linear Equations of Parabolic Type", Providence, R.I.: Amer. Math. Soc. 1968.

[7] Ladyzhenskaya O.A., Ural'ceva, N.N., "Linear and Quasilinear Equations of Elliptic Type". Academic Press, New York, 1968.

[8] J.L. Lions, "Quelques Methods de Resolution des Problemes aux Limits Non Lineaires", Dunod, Paris, 1969.

[9] Ovsiannikov L.V., "Group Analysis of Differential Equations", Academic Press, New York, London, 1982.
[10] Peaceman, D.W., "Fundamentals of Numerical Reservoir Simulation". Elsevier Scientific Publishing Company, Amsterdam-Oxford-New York, 1977.
[11] Serre D., "Systèmes de Lois de Conservation", vol. II. Diderot Editeur, Arts et Sciences, Paris, New York, Amsterdam, (1996).
[12] Smoller J., "Shock Waves and Reaction-Diffusion Equations", Springer-Verlag, New-York, Heidelberg, Berlin, (1983).
[13] Stone H.L., *Probability model for estimating three-phase relative permeability*, J. of Petroleum Technology, February, (1970), 214-218.
[14] Varchenko A.N., Zazovskii A.F., *Three-phase filtration of immiscible fluids.* Itogi Nauki i Tekhniki, Seriya Kompleksnie i spetsial'nie Razdely Mekhaniki **4**, (1991), 98-154 (in Russian).

(H. Frid) Instituto de Matemática Pura e Aplicada - IMPA, Estrada Dona Castorina, 110, Rio de Janeiro, RJ 22460-320, Brazil
E-mail address: hermano@impa.br

(V. Shelukhin) Lavrentyev Institute of Hydrodynamics, Lavrentyev pr. 15, Novosibirsk, 630090, Russia
E-mail address: shelukhin@hydro.nsc.ru