Sufficient conditions for robustness of attractors

C. A. Morales, M. J. Pacifico [*]

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Abstract

A recent problem in dynamics is to determinate whether an attractor Λ of a C^r flow X is C^r robust transitive or not. By *attractor* we mean a transitive set to which all positive orbits close to it converge. An attractor is C^r robust transitive (or C^r *robust* for short) if it exhibits a neighborhood U such that the set $\cap_{t>0} Y_t(U)$ is transitive for every flow Y C^r close to X. We give sufficient conditions for robustness of attractors based on the following definitions. An attractor is *singular-hyperbolic* if it has singularities (all hyperbolic) and is partially hyperbolic with volume expanding central direction [**MPP**]. An attractor is C^r *critically-robust* if it exhibits a neighborhood U such that $\cap_{t>0} Y_t(U)$ is in the closure of the closed orbits is every flow Y C^r close to X. We show that on compact 3-manifolds all C^r critically-robust singular-hyperbolic attractors with only one singularity are C^r robust.

1 Introduction

A recent problem in dynamics is to determinate whether an attractor Λ of a C^r flow X is C^r robust transitive or not. By *attractor* we mean a transitive set to which all positive orbits close to it converge. An attractor is C^r robust transitive (or C^r *robust* for short) if it exhibits a neighborhood U such that the set $\cap_{t>0} Y_t(U)$ is transitive for every flow Y C^r close to X. We give sufficient conditions for robustness of attractors based on the following definitions. An attractor is *singular-hyperbolic* if it has singularities (all hyperbolic) and is partially hyperbolic with volume expanding central direction [**MPP**]. An attractor is C^r *critically-robust* if it exhibits a neighborhood U such that $\cap_{t>0} Y_t(U)$ is in the closure of the closed orbits is every flow Y C^r close to X. We show that on

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compact 3-manifolds all C^r critically-robust singular-hyperbolic attractors with only one singularity are C^r robust. Let us state our result in a precise way.

Hereafter X_t is a flow induced by a C^r vector field X on a compact 3-manifold M. The ω-limit set of $p \in M$ is the accumulation point set $\omega_X(p)$ of the positive orbit of p. An invariant set is *transitive* if it is $\omega_X(p)$ for some point p on it. An *attracting set* is a compact set realizing as as $\cap_{t>0} X_t(U)$ for some neighborhood U and an *attractor* is a transitive attracting set. See [**Mi**] where several definitions of attractor are discussed. The central definition of this paper is the following.

Definition 1.1. *An attractor of a C^r flow X is C^r robust transitive (or C^r* robust *for short) if it exhibits a neighborhood U such that $\cap_{t>0} Y_t(U)$ is a transitive set of Y for every flow Y C^r close to X.*

Very recently [**B**, **P**] introduce the problem of finding sufficient conditions for robustness of attractors. To attact this problem we introduce the following definitions. A compact invariant set Λ of X is *partially hyperbolic* if there are an invariant splitting $T\Lambda = E^s \oplus E^c$ and positive constants K, λ such that:

1. E^s *is contracting*, namely
$$|| DX_t/E^s_x ||\leq Ke^{-\lambda t}, \quad \forall x \in \Lambda,\ \forall t > 0.$$
2. E^s *dominates* E^c, namely
$$|| DX_t/E^s_x || \cdot || DX_{-t}/E^c_{X_t(x)} ||\leq Ke^{-\lambda t}, \quad \forall x \in \Lambda,\ \forall t > 0.$$

The central direction E^c of Λ is said to be *volume expanding* if the additional condition
$$| J(DX_t/E^c_x) |\geq Ke^{\lambda t}$$
holds $\forall x \in \Lambda$, $\forall t > 0$ where $J(\cdot)$ means the jacobian.

Definition 1.2. *([**MPP**]) An attractor is* singular-hyperbolic *if it has singularities (all hyperbolic) and is partially hyperbolic with volume expanding central direction.*

The most representative example of a C^r robust singular-hyperbolic attractor is the geometric Lorenz attractor [**GW**]. The main result in [**MPP**] claims that C^1 robust non-trivial attractors on compact 3-manifolds are singular-hyperbolic. The converse is false, namely there are singular-hyperbolic attractors on compact 3-manifolds which are not C^r robust [**MPu**]. The following definition gives a further sufficient condition for robustness.

Definition 1.3. *An attractor of a C^r flow X is C^r* critically-robust *if there is a neighborhood U of it such that $\cap_{t>0} Y_t(U)$ is in the closure of the closed orbits of Y is every flow Y C^r close to X.*

Hyperbolic attractors on compact manifolds are C^r robust and C^r critically-robust for all r. The geometric Lorenz attractor [**GW**] is an example of a singular-hyperbolic attractor with only one singularity which is also C^r robust and C^r critically-robust. In general singular-hyperbolic attractors with only one singularity may be neither C^r robust nor C^r critically-robust [**MPu**]. Nevertheless we shall prove that on compact 3-manifolds C^r critically-robustness implies C^r robustness among singular-hyperbolic attractors with only one singularity. More precisely one has the following.

Theorem A. *C^r critically-robust singular-hyperbolic attractors with only one singularity on compact 3-manifolds are C^r robust.*

This theorem gives explicit sufficient conditions for robustness of attractors *depending on the perturbed flow*. E. Pujals is interested in conditions *depending on the unperturbed flow only*. It would be also interesting to determinate whether the conclusion of Theorem A holds interchanging the roles of robust and critically-robust in the statement. The proof of Theorem A relies on the results in the recent work [**MP2**]. We reproduce these results in Section 2 for the sake of completeness. The proof of Theorem A is in Section 3.

2 Singular-hyperbolic attracting sets

In this section we describe the results in [**MP2**]. Some proofs will be omitted. We refer to [**MP2**] for further details. Hereafter X is a C^r flow on a closed 3-manifold M. The closure of B will be denoted by $Cl(B)$. If A is a compact invariant set of X we denote $Sing_X(A)$ the set of singularites of X in A. We denote by $Per_X(A)$ the union of the periodic orbits of X in A. A compact invariant set H of X is *hyperbolic* if the tangent bundle over H has an invariant decomposition $E^s \oplus E^X \oplus E^u$ such that E^s is contracting, E^u is expanding and E^X is generated by the direction of X [**PT**]. The Stable Manifold Theory [**HPS**] asserts the existence of the stable manifold $W^s_X(p)$ and the unstable manifold $W^u_X(p)$ associated to $p \in H$. These manifolds are respectively tangent to the subspaces $E^s_p \oplus E^X_p$ and $E^X_p \oplus E^u_p$ of T_pM. In particular, $W^s_X(p)$ and $W^u_X(p)$ are well defined if p belongs to a hyperbolic periodic orbit of X. If O is an orbit of X we denote $W^s_X(O) = W^s_X(p)$ and $W^u_X(O) = W^u_X(p)$ for some $p \in O$. We observe that $W^{s(u)}_X(O)$ does not depend on $p \in O$. When $dim(E^s) = dim(E^u) = 1$ we say that H is *saddle-type*. In this case $W^s_X(p)$ and $W^u_X(p)$ are two-dimensional submanifolds of M. The maps $p \in H \to W^s_X(p)$ and $p \in H \to W^u_X(p)$ are continuous (in compact parts). On the other hand, a compact, singular, invariant set Λ of X is *singular-hyperbolic* if all its singularities are hyperbolic and the tangent bundle over Λ has an invariant decomposition $E^s \oplus E^c$ such that E^s is contracting, E^s dominates E^c and E^c is volume expanding (i.e. the jacobian of DX_t/E^c grows exponentially as $t \to \infty$). Again the Stable Manifold Theory asserts the existence of the strong stable manifold $W^{ss}_X(p)$ associated to $p \in \Lambda$. This manifold is tangent to the subspace E^s_p of T_pM. For all $p \in \Lambda$ we define $W^s_X(p) = \cup_{t\in I\!R} W^{ss}_X(X_t(p))$. If p is regular (i.e. $X(p) \neq 0$) then $W^s_X(p)$ is a well defined two-dimensional submanifold of M. The map $p \in \Lambda \to W^s_X(p)$ is continuous (in compact parts) at the regular points p of Λ. A singularity σ of X is *Lorenz-like* if its eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are real and satisfy

$$\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$$

up to some order. A Lorenz-like singularity σ is hyperbolic, and so, $W^s_X(\sigma)$ and $W^u_X(\sigma)$ do exist. Moreover, the eigenspace of λ_2 is tangent to a one-dimensional invariant manifold $W^{ss}_X(\sigma)$. This manifold is called the *strong stable manifold* of σ. Clearly $W^{ss}_X(\sigma)$ splits $W^s_X(\sigma)$ in two connected components. We denote by $W^{s,+}_X(\sigma), W^{s,-}_X(\sigma)$ the two connected components of $W^s_X(\sigma) \setminus W^{ss}_X(\sigma)$.

When Λ is a singular-hyperbolic set with dense periodic orbits we know that every $\sigma \in Sing_X(\Lambda)$ is Lorenz-like and satisfies $\Lambda \cap W^{ss}_X(\sigma) = \{\sigma\}$ (see [**MPP**]). It follows also from [**MPP**] that any compact invariant set without singularities of Λ is hyperbolic saddle-type. When the periodic orbits are dense in Λ attracting we have that for every $p \in Per_X(\Lambda)$ there is $\sigma \in Sing_X(\Lambda)$ such that

$$W^u_X(p) \cap W^s_X(\sigma) \neq \emptyset.$$

This fact is proved using the methods in [**MP1**].

If Λ is a singular-hyperbolic set and $\sigma \in Sing_X(\Lambda)$ is Lorenz-like we define

- $P^+ = \{p \in Per_X(\Lambda) : W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset\}$.
- $P^- = \{p \in Per_X(\Lambda) : W^u_X(p) \cap W^{s,-}_X(\sigma) \neq \emptyset\}$.
- $H^+_X = Cl(P^+)$.
- $H^- = Cl(P^-)$.

These sets will play important role in the sequel.

Lemma 2.1. *Let Λ be a connected, singular-hyperbolic, attracting set with dense periodic orbits and only one singularity σ. Then, $\Lambda = H^+ \cup H^-$.*

Next we state a technical lemma to be used later. If S is a submanifold we shall denote by T_xS the tangent space at $x \in S$. A *cross-section* of X is a compact submanifold Σ transverse to X and diffeomorphic to the two-dimensional square $[0,1]^2$. If Λ is a singular-hyperbolic set of X and $x \in \Sigma \cap \Lambda$, then x is regular and so $W^s_X(x)$ is a two-dimensional submanifold transverse to Σ. In this case we define by $W^s_X(x,\Sigma)$ the connected component of $W^s_X(x) \cap \Sigma$ containing x. We shall be interested in a special cross-section described as follows. Let Λ be a singular-hyperbolic set of a three-dimensional flow X and $\sigma \in Sing_X(\Lambda)$. Suppose that the closed orbits contained in Λ are dense in Λ. Then σ is Lorenz-like [**MPP**], and so, it is possible to describe the flow using the Grobman-Hartman Theorem [**dMP**]. Indeed, we can assume that the flow of X around σ is the linear flow $\lambda_1\partial_{x_1} + \lambda_2\partial_{x_2} + \lambda_3\partial_{x_3}$ in a suitable coordinate system $(x_1,x_2,x_3) \in [-1,1]^3$ around $\sigma = (0,0,0)$. A cross-section Σ of X is *singular* if Σ corresponds to the submanifolds $\Sigma^+ = \{z = 1\}$ or $\Sigma^- = \{z = -1\}$ in the coordinate system (x_1,x_2,x_3). We denote by l^+ and l^- the curves in Σ^+,Σ^- intersecting to $\{y = 0\}$. Note that l^+,l^- are contained in $W^{s,+}_X(\sigma), W^{s,-}_X(\sigma)$ respectively. We state without proof the following straighforward lemma.

Lemma 2.2. *Let Λ a singular-hyperbolic set with dense periodic orbits of a three-dimensional flow X and $\sigma \in Sing_X(\Lambda)$ be fixed. Then, there are singular cross-sections Σ^+,Σ^- as above such that every orbit of Λ passing close to some point in $W^{s,+}_X(\sigma)$ (resp. $W^{s,-}_X(\sigma)$) intersects Σ^+ (resp. Σ^-). If $p \in \Lambda \cap \Sigma^+$ is close to l^+, then $W^s_X(p,\Sigma^+)$ is a vertical curve crossing Σ^+. If $p \in Per_X(\Lambda)$ and $W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset$, then $W^u_X(p)$ contains an interval $J = J_p$ intersecting l^+ transversally. Similarly replacing $+$ by $-$.*

To prove transitivity we shall use the following criterium of Birkhoff.

Lemma 2.3. *Let T be a compact, invariant set of X such that for all open sets U,V intersecting T there is $s > 0$ such that $X_s(U \cap T) \cap V \neq \emptyset$. Then T is transitive.*

Birkhoff's criterium is used together with the following lemma.

Lemma 2.4. *Let Λ be a connected, singular-hyperbolic, attracting set with dense periodic orbits and only one singularity σ. Let U,V be open sets, $p \in U \cap Per_X(\Lambda)$ and $q \in V \cap Per_X(\Lambda)$. If $W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset$ and $W^u_X(q) \cap W^{s,+}_X(\sigma) \neq \emptyset$, then there are $z \in W^u_X(p)$ arbitrarily close to $W^u_X(p) \cap W^{s,+}_X(\sigma)$ and $t > 0$ such that $X_t(z) \in V$. A similar result holds replacing $+$ by $-$.*

Hereafter we let Λ be a singular-hyperbolic set of $X \in \mathcal{X}^r$ satisfying:

1. Λ is connected.

2. Λ is attracting.

3. The closed orbits contained in Λ are dense in Λ.

4. Λ has only one singularity σ.

We note that (3) above implies

(H1) $\Lambda = Cl(Per_X(\Lambda))$.

Proposition 2.5. *Suppose that the following property hold : If $p, q \in Per_X(\Lambda)$ then either*

1. $W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset$ *and* $W^u_X(q) \cap W^{s,+}_X(\sigma) \neq \emptyset$ *or*

2. $W^u_X(p) \cap W^{s,-}_X(\sigma) \neq \emptyset$ *and* $W^u_X(q) \cap W^{s,-}_X(\sigma) \neq \emptyset$.

Then, Λ is transitive.

Proof. By the Birkhoff's criterium we only need to prove that $\forall U, V$ open sets intersecting Λ $\exists s > 0$ such that $X_s(U \cap \Lambda) \cap V \neq \emptyset$. For this we proceed as follows: By (H1) there are $p \in Per_X(\Lambda) \cap U$ and $q \in Per_X(\Lambda) \cap V$. First suppose that the alternative (1) holds. Then, by Lemma 2.4, there are $z \in W^u_X(p)$ and $t > 0$ such that $X_t(z) \in V$. As $z \in W^u_X(p)$ we have that $w = X_{-t'}(z) \in U$ for some $t' > 0$. As Λ is an attracting set we have that $w \in \Lambda$. If $s = t + t' > 0$ we conclude that $w \in (U \cap \Lambda) \cap X_{-s}(V)$ and so $X_s(U \cap \Lambda) \cap V \neq \emptyset$. If the alternative (2) of the corollary holds we can find $s > 0$ such that $X_s(U \cap \Lambda) \cap V \neq \emptyset$ in a similar way (replacing $+$ by $-$). The proof follows. $\square$

Proposition 2.6. *If there is a sequence $p_n \in Per_X(\Lambda)$ converging to some point in $W^{s,+}_X(\sigma)$ such that $W^u_X(p_n) \cap W^{s,-}_X(\sigma) \neq \emptyset$ for all n, then Λ is transitive. Similarly interchanging the roles of $+$ and $-$.*

Proof. Let $p, q \in Per_X(\Lambda)$ be fixed. Suppose that $W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset$ and $W^u_X(q) \cap W^{u,-}_X(\sigma) \neq \emptyset$. By Lemma 2.2 we can fix a cross section $\Sigma = \Sigma^+$ through $W^{s,+}_X(\sigma)$ and a open arc $J \subset \Sigma \cap W^u_X(p)$ intersecting $W^{s,+}_X(\sigma)$ transversally. Again by Lemma 2.2 we can assume that $p_n \in \Sigma$ for every n. Because the direction E^s of Λ is contracting we have that the size of $W^s_X(p_n)$ is uniformly bounded away from zero. It follows that there is n large so that J intersects $W^s_X(p_n)$ transversally. Applying the Inclination Lemma [**dMP**] to the saturated of $J \subset W^u_X(p)$, and the assumption $W^u_X(p_n) \cap W^{s,-}_X(\sigma) \neq \emptyset$, we conclude that $W^u_X(p) \cap W^{s,-}_X(\sigma) \neq \emptyset$. So, the alternative (2) of Proposition 2.5 holds. It follows from this proposition that Λ is transitive. $\square$

Proposition 2.7. *If there is $a \in W^u_X(\sigma) \setminus \{\sigma\}$ such that $\sigma \in \omega_X(a)$, then Λ is transitive.*

Proof. Without loss of generality we can assume that there is $z \in \omega_X(a) \cap W^+_X(\sigma)$. If $W^u_X(q) \cap W^{s,-}_X(\sigma) = \emptyset$ for all $q \in Per_X(\Lambda)$, then $W^u_X(q) \cap W^{s,+}_X(\sigma) \neq \emptyset$ for all $q \in Per_X(\Lambda)$ ([**MPP**]). Then Λ would be transitive by Proposition 2.5 since the alternative (1) holds $\forall p, q \in Per_X(\Lambda)$. So, we can assume that there is $q \in Per_X(\Lambda)$ such that $W^u_X(q) \cap W^{s,-}_X(\sigma) \neq \emptyset$. It follows from the dominating condition of the singular-hyperbolic splitting of Λ that the intersection $W^u_X(q) \cap W^{s,-}_X(\sigma)$ is transversal. This allows us to choose a point in $W^u_X(q)$ arbitrarily close to $W^{s,-}_X(\sigma)$ in the side accumulating a. As Λ is attracting and satisfies (H1), it is not hard to find a sequence $p_n \in Per_X(\Lambda)$ converging to $z \in W^{s,+}_X(\sigma)$ such that for all n there is p'_n in the orbit of p_n such that the sequence p'_n converges to some point in $W^{s,-}_X(\sigma)$. Now, suppose by contradiction that Λ is not transitive. Then Proposition 2.6 would imply

$$W^u_X(p'_n) \cap W^{s,+}_X(\sigma) = \emptyset \quad \text{and} \quad W^u_X(p_n) \cap W^{s,-}_X(\sigma) = \emptyset$$

for n large. But $W^u_X(p_n) = W^u_X(p'_n)$ since p'_n and p_n are in the same orbit of X. So $W^u_X(p_n) \cap (W^{s,+}_X(\sigma) \cup W^{s,-}_X(\sigma)) = \emptyset$. However

$$W^u_X(p_n) \cap W^s_X(\sigma) = \emptyset,$$

a contradiction since $Sing_X(\Lambda) = \{\sigma\}$. We conclude that Λ is transitive and the proof follows. $\square$

Theorem 2.8. *If Λ is not transitive, then for all $a \in W^u_X(\sigma) \setminus \{\sigma\}$ there is a periodic orbit O with positive expanding eigenvalues of X such that $a \in W^s_X(O)$.*

Proof. We fix $a \in W^u_X(\sigma) \setminus \{\sigma\}$. By contradiction we assume that $\omega_X(a)$ is *not* a periodic orbit. We obtain a contradiction once we prove that if $p, q \in Per_X(\Lambda)$ then p, q satisfy one of the two alternatives in Proposition 2.5. To prove this we proceed as follows. As note before we have that both $W^u_X(p)$ and $W^u_X(q)$ intersect $W^s_X(\sigma)$ ([**MP1**]). Then we can assume

$$W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset \ \text{ and } \ W^u_X(q) \cap W^{s,-}_X(\sigma) \neq \emptyset. \tag{1}$$

By using (1) and the linear coordinate around σ it is easy to construct an open interval $I = I_a$, contained in a suitable cross-section $\Sigma = \Sigma_a$ of X containing a, such that $I \setminus \{a\}$ is formed by two intervals $I^+ \subset W^u_X(p)$ and $I^- \subset W^u_X(q)$. Observe that the tangent vector of I is contained in $E^c \cap T\Sigma_a$. Proposition 2.7 implies that $\sigma \notin \omega_X(a)$ since Λ is not transitive. It follows that $H = \omega_X(a)$ is a hyperbolic saddle-type set ([**MPP**]). As in [**M**] one proves that H is one-dimensional, and so, the Bowen's Theory of hyperbolic one-dimensional sets [**Bw**] can be applied. In particular we can choose a family of cross-sections $\mathcal{S} = \{S_1, \cdots, S_r\}$ of small diameter such that H is the flow-saturated of $H \cap int(\mathcal{S}')$, where $\mathcal{S}' = \cup S_i$ and $int(\mathcal{S}')$ denotes the interior of $\mathcal{S}'$. On the other hand, $I \subset \Lambda$ since Λ is attracting. Recall that the tangent direction of I is contained in E^c. Since E^c is volume expanding and H is non-singular we have that the Poincaré map induced by X on $\mathcal{S}'$ is expanding along I. As in [**MP1**, p. 371] we can find $\delta > 0$ and a open arc sequence $J_n \subset \mathcal{S}'$ in the positive orbit of I with lenght bounded away from 0 such that there is a_n *in the positive orbit of* a contained in the interior of J_n. We can fix $S = S_i \in \mathcal{S}$ in order to assume that $J_n \subset S$ for every n. Let $x \in S$ be a limit point of a_n. Then $x \in H \cap int(\mathcal{S}')$. Because I is tangent to E^c the interval sequence J_n converges to an interval $J \subset W^u_X(x)$ in the C^1 topology ($W^u_X(x)$ exists since $x \in H$ and H is hyperbolic). J is not trivial since the lenght of J_n is bounded away from 0. If $a_n \in W^s_X(x)$ for n large we would obtain that x is periodic [**MP1**, Lemma 5.6], a contradiction since $\omega_X(a)$ is not periodic. We conclude that $a_n \notin W^s_X(x)$, $\forall n$. As $J_n \to J$ and Λ has strong stable manifolds of uniformly size, we have that

$$\exists z_n \in (W^s_X(a_{n+1}) \cap S) \cap (J_n \setminus \{a_n\})$$

for all n large. For every n we let J_n^+ and J_n^- denote the two connected components of $J_n \setminus \{a_n\}$ in a way that J_n^+ is in the positive orbit of I^+ and J_n^- is in the positive orbit of I^-. Clearly we have either $z_n \in J_n^+$ or $z_n \in J_n^-$.

If $z_n \in J_n^+$ there is $v_{n+1} \in Per_X(\Lambda) \cap S$ close to a_{n+1} such that

$$W^s_X(v_{n+1}) \cap J_n^+ \neq \emptyset \text{ and } W^s_X(v_{n+1}) \cap J_{n+1}^- \neq \emptyset.$$

Since v_{n+1} is periodic [**MPP**] implies that $W^u_X(v_{n+1})$ intersects either $W^{s,+}_X(\sigma)$ or $W^{s,-}_X(\sigma)$. The choice of v_{n+1} implies that its orbits passes close to a point in $W^{s,-}_X(\sigma)$. Since Λ is not transitive we conclude that $W^u_X(v_{n+1})$ intersects $W^{s,-}_X(\sigma)$ Since $W^s_X(v_{n+1}) \cap J_n^+$ is transversal the Inclination Lemma then imply $W^u_X(p) \cap W^{s,-}_X(\sigma) \neq \emptyset$. Hence

$$W^u_X(p) \cap W^{s,-}_X(\sigma) \neq \emptyset \text{ and } W^u_X(q) \cap W^{s,-}_X(\sigma) \neq \emptyset.$$

If $z_n \in J_n^-$ we can prove by similar arguments that

$$W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset \text{ and } W^u_X(q) \cap W^{s,+}_X(\sigma) \neq \emptyset.$$

These alternatives yield the desired contradiction. We conclude that $\omega_X(a) = O$ for some periodic orbit O of X. To finish let us prove that the expanding eigenvalue of O is positive. Suppose by contradiction that it is not so. Fix a cross-section Σ intersecting O in a single point p_0. This section defines a Poincaré map

$\Pi : Dom(\Pi) \subset \Sigma \to \Sigma$ of which p_0 is a hyperbolic fixed point. The assumption implies that $D\Pi(p_0)$ has negative expanding eigenvalue. Because $p_0 \in Per_X(\Lambda)$, we have that $W^u_X(p_0)$ intersects either $W^{s,+}_X(\sigma)$ or $W^{s,-}_X(\sigma)$ ([**MPP**]). We shall assume the first case since the proof for the second one is similar. We claim that $W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset$ for all $p \in Per_X(\Lambda)$. Indeed, let $p \in Per_X(\Lambda)$ be fixed. Again $W^u_X(p)$ intersects $W^{s,+}_X(\sigma)$ or $W^{s,-}_X(\sigma)$. In the first case we are done. So, we can assume that $W^u_X(p) \cap W^{s,-}_X(\sigma) \neq \emptyset$. By flow-saturating this intersection we obtain an interval $K \subset W^u_X(p) \cap \Sigma$ intersecting $W^s_X(p_0, \Sigma)$ transversally. On the other hand, there is an interval $J \subset W^{s,+}_X(\sigma) \cap \Sigma$ intersecting $W^u_X(p_0, \Sigma)$ transversally. Since the expanding eigenvalue of $D\Pi(p_0)$ is negative we have by the Inclination Lemma that the backward iterates $\Pi^{-n}(J)$ of J accumulates $W^s_X(p_0, \Sigma)$ *in both sides*. Because K has transversal intersection with $W^s_X(p_0\Sigma)$ we conclude that one of such backward iterates intersects K. This intersection point yields $W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset$ as desired. This proves the claim. The claim together with Proposition 2.5 implies that Λ is transitive, a contradiction. This contradiction proves the result. $\square$

Hereafter we shall assume that Λ is not transitive. Let $a \in W^s_X(\sigma) \setminus \{\sigma\}$ be fixed. By Theorem 2.8 we have that $a \in W^s_X(O)$ for some periodic orbit O with positive expanding eigenvalue. This last property implies that the unstable manifold $W^u_X(O)$ of O is a cylinder with generating curve O. Then O separates $W^u_X(O)$ in two connected components. Such components are denoted by $W^{u,+}, W^{u,-}$ according the following (see Figure 1): There is an interval $I = I_a$, contained in a suitable cross-section of X and containing a, such that if I^+, I^- are the connected components of $I \setminus \{a\}$ then $I^+ \subset W^u_X(p)$ and $I^- \subset W^u_X(q)$ for some periodic points $p, q \in \Lambda$ (recall that Λ is not transitive). In addition I is tangent to the central direction E^c of Λ (see Figure 1). As $a \in W^s_X(O)$ and I is tangent to E^c we have that the flow of X carries I to an interval I' transverse to $W^s_X(O)$ at a. Note that the flow carries I^+ and I^- into I^+_0 and I^-_0 respectively.

Definition 2.9. *We denote by $W^{u,+}$ the connected component of $W^u \setminus O$ which is accumulated (via Inclination and Strong-λ-lemmas [*__dMP__*,* __D__*]) by the positive orbit of I^+_0. Similarly we denote $W^{u,-}$ the connected component of $W^u_X(O) \setminus O$ which is accumulated by the positive orbit of I^-_0.*

The above definition does not depend on p, q, J_p, J_q (this can be easily proved using the Strong λ-lemma [**D**]).

Proposition 2.10. *$W^{u,+} \cap W^{s,-}_X(\sigma) = \emptyset$ and $W^{u,+} \cap W^{s,+}_X(\sigma) \neq \emptyset$. Similarly interchanging the roles of $+$ and $-$.*

Proof. For simplicity denote $W = W^{u,+}$. First we prove $W \cap W^{s,-}_X(\sigma) = \emptyset$. Suppose by contradiction that $W \cap W^{s,-}_X(\sigma) \neq \emptyset$. As this last intersection is

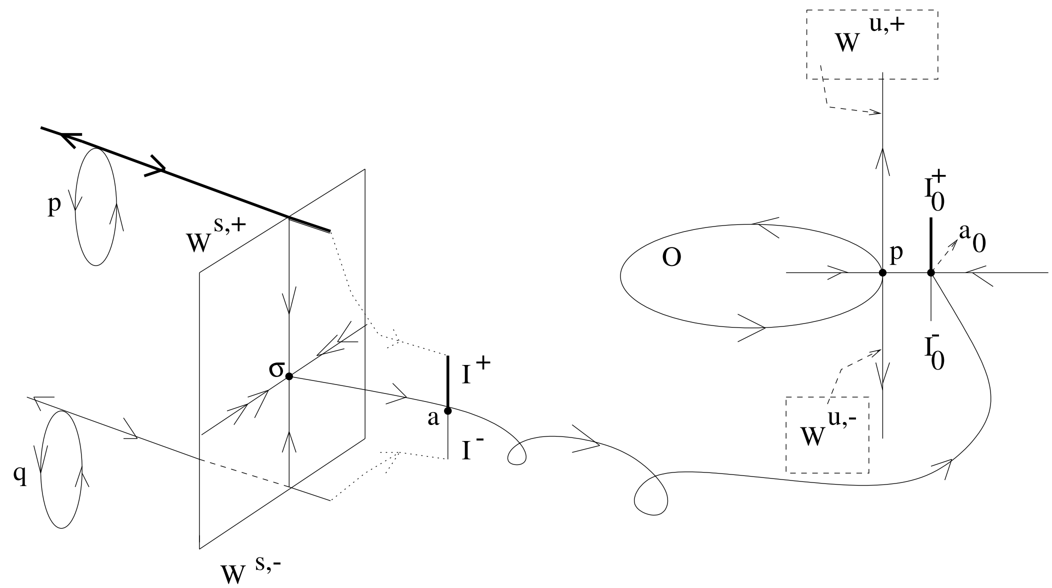

Figure 1: Definition of $W^{u,+}$ and $W^{u,-}$.

transversal there is an interval $J \subset W^{s,-}_X(\sigma)$ intersecting W transversally. Now, fix a cross-section $\Sigma = \Sigma^+$ as in Lemma 2.2 and let $p \in Per_X(\Lambda)$ be such that $W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset$. Then, there is an small interval $I \subset W^u_X(p) \cap \Sigma$ transversal to $\Sigma \cap W^{s,+}_X(\sigma)$. By the definition of $W = W^{u,+}$ (Definition 2.9) we have that the positive orbit of I accumulates on W. As J is transversal to W the Inclination Lemma implies that the positive orbit of I intersects J. This proves $W^u_X(p) \cap W^{s,-}_X(\sigma) \neq \emptyset$ for all $p \in Per_X(\Lambda)$. It would follow that alternative (2) of Proposition 2.5 holds $\forall p, q$ contradicting the non-transitivity of Λ. This contradiction proves $W \cap W^{s,-}_X(\sigma) = \emptyset$ as desired. Now suppose by contradiction that $W \cap W^{s,+}_X(\sigma) = \emptyset$. As $W \cap W^{s,-}_X(\sigma) = \emptyset$ we would obtain $W \cap W^s_X(\sigma) = \emptyset$ ([**MPP**]). But the denseness of the periodic orbits together with the Inclination Lemma imply $W \cap W^s_X(\sigma) \neq \emptyset$. This is a contradiction which proves $W \cap W^{s,+}_X(\sigma) \neq \emptyset$. The result follows. ☐

Proposition 2.11. *$H^+ = Cl(W^{u,+})$ and $H^- = Cl(W^{u,-})$.*

Proof. Fix $q \in P^+$, i.e. $W^u_X(q) \cap W^{s,+}_X(\sigma) \neq \emptyset$. Note that $W^{u,+} \cap W^{s,+}_X(\sigma) \neq \emptyset$ by Lemma 2.10. By using (H1) and the Inclination Lemma it is not hard to prove that $W^{u,+}$ accumulates on q. This proves $H^+ \subset Cl(W^{u,+})$. Conversely let $x \in W^{u,+}$ be fixed. By (H1) and $W^{u,+} \subset \Lambda$ there is $z \in Per_X(\Lambda)$ nearby x. Choosing z close to x we assure $W^s_X(z) \cap W^{u,+} \neq \emptyset$ because stable manifolds have size uniformly bounded away from zero. If $W^u_X(z) \cap W^{s,-}_X(\sigma) \neq \emptyset$ then the Inclination Lemma and $W^s_X(z) \cap W^{u,+} \neq \emptyset$ would imply $W^u \cap W^{s,-}_X(\sigma) \neq \emptyset$. This contradicts Proposition 2.10 and so $W^u_X(z) \cap W^{s,-}_X(\sigma) = \emptyset$. By [**MP1**] we obtain

$z \in P^+$ proving $x \in H^+$. The lemma is proved. $\square$

Proposition 2.12. *If $z \in Per_X(\Lambda)$ and $W^s_X(z) \cap W^{u,+} \neq \emptyset$, then $Cl(W^s_X(z) \cap W^{u,+}) = Cl(W^{u,+})$. Similarly replacing $+$ by $-$.*

Proof. Previously we show $Cl(W^{s,+}_X(\sigma) \cap W^{u,+}) = Cl(W^{u,+})$. Fix $x \in W^{u,+}$. By (H1) there is $w \in Per_X(\Lambda)$ close to x. In particular $W^s_X(w) \cap W^{u,+} \neq \emptyset$. If $W^u_X(w) \cap W^{s,+}_X(\sigma) = \emptyset$ then $W^u_X(w) \cap W^{s,-}_X(\sigma) \neq \emptyset$ by [**MP1**]. It would follow from the Inclination Lemma that $W^{u,+} \cap W^{s,-}_X(\sigma) \neq \emptyset$ contradicting Proposition 2.10. We conclude that $W^u_X(w) \cap W^{s,+}_X(\sigma) \neq \emptyset$. Note that $W^s_X(w) \cap W^{u,+} \neq \emptyset$ is close to x as $w \to x$. As $W^u_X(w) \cap W^{s,+}_X(\sigma) \neq \emptyset$ is transversal we can apply the Inclination Lemma in order to find a transverse intersection $W^{u,+} \cap W^{s,+}_X(\sigma)$ close to x. This proves $Cl(W^{s,+}_X(\sigma) \cap W^{u,+}) = Cl(W^{u,+})$. Finally we prove $Cl(W^s_X(z) \cap W^{u,+}) = Cl(W^{u,+})$. Choose $x \in W^{u,+}$. As $Cl(W^{s,+}_X(\sigma) \cap W^{u,+}) = Cl(W^{u,+})$ there is an interval $I_x \subset W^{u,+}$ arbitrarily close to x such that $I_x \cap W^{s,+}_X(\sigma) \neq \emptyset$. The positive orbit of I_x first passes through a and, afterward, it accumulates on $W^{u,+}$. But $W^s_X(z) \cap W^{u,+} \neq \emptyset$ by assumption. As this intersection is transversal the Inclination Lemma implies that the positive orbit of I_x intersects $W^s_X(z)$. By taking the backward flow of the last intersection we get $W^s_X(z) \cap I_x \neq \emptyset$. This proves the lemma. $\square$

Given $z \in Per_X(\Lambda)$ we denote by $H_X(z)$ the homoclinic class associated to z.

Proposition 2.13. *If $z \in Per_X(\Lambda)$ is close to a point in $W^{u,+}$, then $H_X(z) = Cl(W^{u,+})$. Similarly replacing $+$ by $-$.*

Proof. Let $z \in Per_X(\Lambda)$ be a point close to one in $W^{u,+}$. It follows from the continuity of the stable manifolds that $W^s_X(z) \cap W^{u,+} \neq \emptyset$. We claim that $H_X(z) = Cl(W^{u,+})$. Indeed, by Proposition 2.10 one has $W^{u,+} \cap W^{s,-}_X(\sigma) = \emptyset$. This equality and the Inclination Lemma imply $W^u_X(z) \cap W^s_X(\sigma) \subset W^{s,+}_X(\sigma)$. By Proposition 2.12 we have that $W^s_X(z) \cap W^{u,+}$ is dense in $W^{u,+}$ since $W^s_X(z) \cap W^{u,+} \neq \emptyset$. Let Σ be a cross-section containing p_0 and fix $x \in W^{u,+}$. We can assume $x, z \in \Sigma$. Since $W^u_X(z) \cap W^s_X(\sigma) \neq \emptyset$ and $W^u_X(z) \cap W^s_X(\sigma) \subset W^{s,+}_X(\sigma)$ there is an interval $I \subset W^u_X(z)$ intersecting $W^{s,+}_X(\sigma)$. Then, the positive orbit of I yields the interval J in the figure. In addition, the positive orbit of J yields the interval K in that figure. Note that the positive orbit of K *accumulates* $W^{u,+}$ (recall Definition 2.9). As $W^s_X(z) \cap W^{u,+}$ is dense in $W^{u,+}$ and $x \in W^{u,+}$ we have that $W^s_X(z)$ passes close to x as indicated in the figure. The Inclination Lemma applied to the positive orbit of K yields a homoclinic point z' associated to z which is close to x. This proves that $x \in H_X(z)$, and so, $Cl(W^{u,+}) \subset H_X(z)$. The reversed inclusion is a direct consequence of the Inclination Lemma applied to $W^s_X(z) \cap W^{u,+} \neq \emptyset$. We conclude that $Cl(W^{u,+}) = H_X(z)$ proving the result. $\square$

Theorem 2.14. *Let Λ be a singular-hyperbolic set of a C^r flow X on a closed three-manifold, $r \geq 1$. Suppose that the properties below hold.*

1. *Λ is connected.*
2. *Λ is attracting.*
3. *The closed orbits contained in Λ are dense in Λ.*
4. *Λ has a unique singularity σ.*
5. *Λ is not transitive.*

Then, H^+ and H^- are homoclinic classes of X.

Proof. Let Λ be a singular-hyperbolic set of X satisfying (1)-(5) of Theorem 2.14. Then we can apply the results in this section. To prove that H^+ is a homoclinic classs it suffices by Proposition 2.11 to prove that $Cl(W^{u,+})$ is a homoclinic class. By (3) of Theorem 2.14 we can choose $z \in Per_X(\Lambda)$ arbitrarily close to a point in $W^{u,+}$. Then $Cl(W^{u,+}) = H_X(z)$ by Proposition 2.13 and the result follows. $\square$

3 Proof of Theorem A

First we introduce some notations. Hereafter M is a compact 3-manifold and $\mathcal{X}^r$ is the space of C^r flows in M equiped with the C^r topology, $r \geq 1$. The *nonwandering set* of $X \in \mathcal{X}^r$ is the set $\Omega(X)$ of points $p \in M$ such that for all neighborhood U of p and $T > 0$ there is $t > T$ such that $X_t(U) \cap U \neq \emptyset$. An attracting Λ with isolating block U has a continuation $\Lambda(Y)$ for Y C^r close to X defined by $\Lambda(Y) = \cap_{t>0} Y_t(U)$. This continuation is then definied when Λ is an attractor. A compact invariant set is *non-trivial* if it is not a closed orbit of X. Transitive sets for flows are always connected. The proof of Theorem A is based on the following result.

Theorem 3.1. *Let Λ be a singular-hyperbolic set of $X \in \mathcal{X}^r$, $r \geq 1$. Suppose that the following properties hold.*

1. *Λ is connected.*
2. *Λ is attracting.*
3. *The closed orbits contained in Λ are dense in Λ.*
4. *Λ has only one singularity.*
5. *Λ is not transitive.*

Then, for every neighborhood U of Λ there is a flow Y C^r close to X such that

$$\Lambda(Y) \not\subset \Omega(Y).$$

To prove this theorem we shall use the following definitions and facts. Let $X \in \mathcal{X}^r$ and Λ be a singular-hyperbolic set of X satisfying (1)-(5) of Theorem 3.1. Let σ be the unique singularity of Λ. As mentioned in the previous section σ is Lorenz-like. As in Section 2 we have that $W^{ss}_X(\sigma)$ divides $W^s_X(\sigma)$ in two connected components which we denote by $W^{s,+}_X(\sigma), W^{s,-}_X(\sigma)$. For simplicity we denote such components by $W^{s,+}, W^{s,-}$ respectively. Recall that $Per_X(\Lambda)$ denotes the union of the periodic orbits of X in Λ. A point in $Per_X(\Lambda)$ is said to be periodic. Fix such $a \in W^u_X(\sigma) \setminus \{\sigma\}$. By Theorem 2.8 we have that $\omega_X(a) = O$ for some periodic orbit with positive expanding eigenvalues of X. In particular, $W^{u,+}$ and $W^{u,-}$ are defined (Definition 2.9).

Lemma 3.2. *$Cl(W^{u,+}) \cap W^{s,-} = \emptyset$.*

Proof. Suppose by contradiction that $Cl(W^{u,+}) \cap W^{s,-} \neq \emptyset$. Choose a cross-section Σ^- intersecting $W^{s,-}_X(\sigma)$ such that every orbit's sequence in Λ converging to some point in $W^{s,-}$ intersect Σ^- (Lemma 2.2). Let $q \in \Lambda$ be periodic such that $W^u_X(q) \cap W^{s,-}_X(\sigma) \neq \emptyset$. As $Cl(W^{u,+}) \cap W^{s,-} \neq \emptyset$ we have that $Cl(W^{u,+}) \cap \Sigma^- \neq \emptyset$. Because the closed orbits are dense we can prove that $q \in Cl(W^{u,+})$. It follows that $H^- \subset Cl(W^{u,+})$, and so, $\Lambda = Cl(W^{u,+})$ by Lemma 2.1. On the other hand, since Λ is not transitive, we have that $H^+ = Cl(W^{u,+})$ (Proposition 2.11) and H^+ is a homoclinic class (Theorem 2.14). Since homoclinic classes are transitive sets we conclude that Λ is transitive, a contradiction. This contradiction proves the result. $\square$

Lemma 3.3. *Let D be a fundamental domain of $W^{uu}_X(p_0)$ contained in $W^{u,+}$. Then there are a neighborhood V of D and a cross-section Σ^- of X intersecting $W^{s,-}$ satisfying the following properties.*

1. *Every X-orbit's sequence in Λ converging to some point in $W^{s,-}$ intersects Σ^-.*

2. *Every positive X-orbit with initial point in V does not interset Σ^-.*

Proof. Fix a fundamental domain F of $W^s_X(\sigma)$ and define $F^- = F \cap W^{s,-}_X(\sigma)$ Then there is a compact interval $F' \subset F^-$ such that $\Lambda \cap F^- \subset F'$. By Lemma 3.2 there is $\epsilon > 0$ such that $B_\epsilon(Cl(W^{u,+})) \cap B_\epsilon(F') = \emptyset$. Clearly we can choose a cross-section Σ^- of X inside $B_\epsilon(F')$ such that every X-orbit's sequence in Λ converging to some point in $W^{s,-}$ intersects Σ^-. On the other hand, as $W^{u,+}$ is invariant and $Cl(W^{u,+}) \cap B_\epsilon(F') = \emptyset$ we have that every positive orbit with initial point in D cannot intersect Σ^-. By using the contracting foliation of Λ we have the same property for every positive trajectory with initial point in a neighborhood V of D. This proves the result. $\square$

Now we define a perturbation (pushing) close to a point $a \in W^u_X(\sigma) \setminus \{\sigma\}$. For this end we fix the following cross-sections:

1. Σ_a containing a in its interior.
2. $\Sigma' = X_1(\Sigma)$.
3. Σ_0 intersecting O in a single interior point.
4. Σ^+, Σ^- which intersect $W^{s,+}, W^{s,-}$ and pointing to the side of a respectively.

Every X-orbit intersecting $\Sigma^+ \cup \Sigma^-$ will intersect Σ. Note that there is a well defined neighborhood $\mathcal{O}$ given by

$$\mathcal{O} = X_{[0,1]}(\Sigma_a).$$

This neighborhood will be the support of the pushing described in the Figures 2 and 3. The pushing in $\mathcal{O}$ yielding the perturbed flow Y of X is obtained in the standard way (see [**dMP**]).

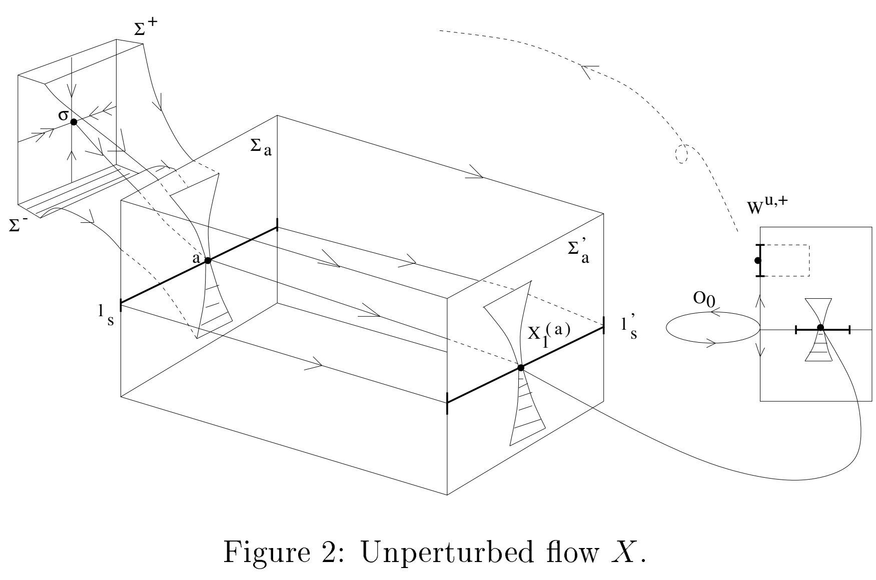

Figure 2: Unperturbed flow X.

We have to prove that $\Lambda(Y) \not\subset \Omega(Y)$ for the perturbed flow Y. For this purpose we observe that by (5) of Theorem 3.1 and Proposition 2.5 we can assume that there q periodic in U such that $W^u_X(q) \cap W^{s,-}_X(\sigma) \neq \emptyset$. We obtain in this way an interval K in $\Sigma^- \cap W^u_X(q)$ crossing Σ^- as in Figure 3. The Y-flow carries

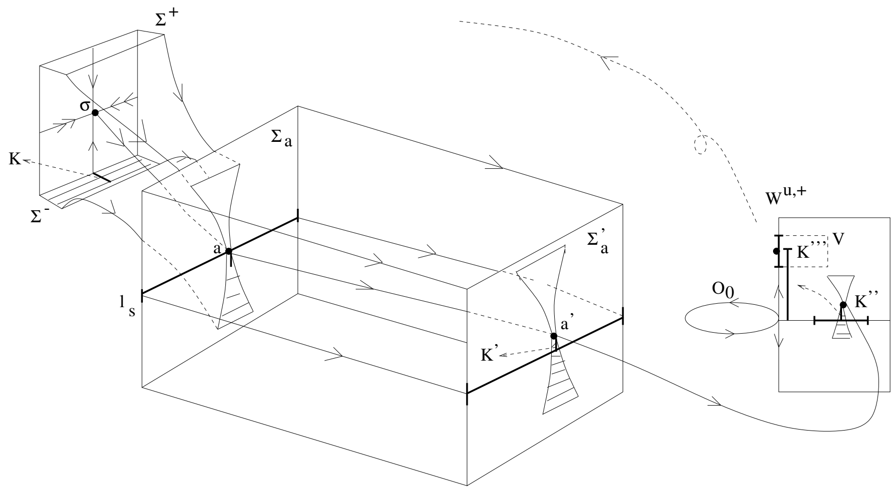

Figure 3: Perturbed flow Y.

K to an interval K'' as in Figure 3. Let $q(Y), W^u_Y(q(Y)), K''(Y), \sigma(Y)$ denote the continuation of these objects for the perturbed flow Y. We observe that $K(Y) \subset \Lambda(Y)$ since $\Lambda(Y)$ is an attracting set, $q(Y) \in \Lambda(Y)$ and $K \subset W^u_Y(q(Y))$. Then Theorem 3.1 will follow from the lemma below.

Lemma 3.4. $K(Y) \not\subset \Omega(Y)$.

Proof. Suppose by contradiction that $K(Y) \subset \Omega(Y)$ and pick $x \in Int(K(Y))$ (= the interior of the interval $K(Y)$). The Y's flow carries the points nearby x to the neighborhood V depicted in Figure 3. This neighborhood is obtained by saturation a fundamental domain in $W^{u,+}$ by the strong stable manifolds [**HPS**]. Note that there are points x' nearby x which back close to x by the positive Y-flow ($x \in K(Y) \subset \Omega(Y)$). In particular, *the positive Y-orbit of x' returns to* Σ^-. On the other hand, Lemma 3.3-(2) implies that every X-orbit starting in V doesn't intersect Σ^-. Since $X = Y$ outside $\mathcal{O}$ we conclude that the positive Y-orbit of x' intersects Σ^+. Afterward this positive orbit passes throught the box $\mathcal{O}$ and arrives to V. By repeating the argument we conclude that *the positive Y-orbit of x' never return to* Σ^-, a contradiction. The lemma is proved. □

Proof of Theorem A: Let Λ be a singular-hyperbolic attractor of a C^r flow X on a compact 3-manifold M. Assume that Λ is C^r critically-robust and has a unique singularity σ. Denote by $\Lambda(Y) = \cap_{t>0} Y_t(U)$ the continuation of Λ in a neighborhood U of Λ for Y close to X. Denote by $C(Y)$ the union of the closed orbits of a flow Y. As Λ is C^r critically-robust it follows that there is a neighborhood U of Λ such that $\Lambda(Y) \cap C(Y)$ is dense in $\Lambda(Y)$ for every flow Y C^r close to X. Clearly $\Lambda(Y)$ is a singular-hyperbolic set of Y for all Y close to X.

Because Λ has a unique singularity we have that $\Lambda(Y)$ has a unique singularity as well. Because Λ is transitive we have that Λ is connected. Then the neighborhood U above can be arranged connected, and so, $\Lambda(Y)$ is also connected. Summarizing we have that $\Lambda(Y)$ is a singular-hyperbolic set of Y satisfying (1)-(4) of Theorem 3.1. If Λ were not C^r robust, then it would exist Y C^r close to X such that $\Lambda(Y)$ is not transitive. In this case $\Lambda(Y)$ satisfies (1)-(5) of Theorem 3.1. It follows that there is Y' C^r close to Y such that $\Lambda(Y') \not\subset \Omega(Y')$. And this is a contradiction since $\Lambda(Y') \subset \Omega(Y')$ (as $\Lambda(Y') \cap C(Y')$ is dense in $\Lambda(Y')$). This proves the result. $\square$

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C. A. Morales, M. J. Pacifico
Instituto de Matemática
Universidade Federal do Rio de Janeiro
C. P. 68.530, CEP 21.945-970
Rio de Janeiro, R. J. , Brazil
e-mail: *morales@impa.br*, *pacifico@impa.br*