STATISTICAL STABILITY FOR DIFFEOMORPHISMS WITH DOMINATED SPLITTING

by

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Abstract. — In [**2**] Alves Bonatti and Viana proved that SRB measures exist for partially hyperbolic diffeomorphisms with mostly expanding center-unstable direction. The main tool used there is the existence of Gibbs *cu*-states. In this paper we prove many properties of such states, especially with respect to their relationship with the SRB measures. In particular, we prove that the Gibbs *cu*-states vary continuously with the diffeomorphism. As a consequence we obtain existence of finitely many SRB measures and statistical stability for many open classes of diffeomorphisms with dominated splitting.

1. Introduction

A central topic in dynamical systems is the study of statistical properties of the system. In this direction, we can consider the average along the orbits and then compare it with the average of the system in the ambient space. Given any ergodic invariant measure for the system it is well-known that for almost every point with respect to this measure the temporal and spatial averages coincide. In many cases, the invariant measure is a singular measure, so it may be physically very difficult to find a point satisfying the property above. An SRB measure is an invariant measure for the system for which the time average coincides with the spatial average in a positive Lebesgue measure subset of the ambient space.

A program towards a global theory of diffeomorphisms has been proposed a few years ago by Palis [**14**]. The core of his conjecture is that every dynamical system can be approximated by one having only finitely many attractors, all of which have finitely many SRB measures which are robust with respect to small perturbations of the system.

Statistical stability tells us much about how the system varies under small deterministic perturbations.

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The question of the existence of SRB measures has an affirmative answer in the setting of uniformly hyperbolic systems [**23, 7, 6, 21**], as well as of systems with certain weak forms of hyperbolicity [**4, 2**].

Uniformly expanding smooth maps are well-known to be statistically stable, as are Axiom A diffeomorphisms [**22**] restricted to the basin of their attractors. On the other hand, [**1, 3**] proved statistical stability for a large class of transformations exhibiting non-uniformly expanding behavior. Statistical stability for a certain open class of diffeomorphisms having partially hyperbolic attractors whose central direction is mostly contracting was proved in [**10, 9**] .

In [**2**] it was proved that SRB measures exist for diffeomorphisms having dominated splitting with mostly expanding center-unstable direction and other technical conditions. The main tool used there, is the existence of Gibbs *cu*-states. Gibbs *cu*-states are the non-uniform version of the Gibbs u-states introduced by Pesin and Sinai [**16**]. Several other properties of Gibbs u-states are proved in [**5**].

In this paper we extend this properties to Gibbs *cu*-states, especially with respect to their relationship with SRB measures.

1.1. Statement of results. — Let us consider diffeomorphisms $f : M \to M$ defined over a compact Riemannian boundaryless manifold M. We denote by $\|\cdot\|$ the induced norm on TM and by m a fixed normalized Riemannian volume form on M and we call it the *Lebesgue measure* on M.

The *time average* of a continuous function $\varphi : M \to \mathbb{R}$ along the orbit of $x \in M$ is:

$$\tilde{\varphi}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)).$$

If μ is an invariant measure, the *basin* of μ is the set

$$B(\mu) = \{x \in M \ : \ \tilde{\varphi}(x) = \int_M \varphi d\mu, \ \forall \varphi \in C(M;\mathbb{R})\}.$$

An invariant measure μ is an *SRB measure* or *physical measure* if $B(\mu)$ has positive Lebesgue measure.

Let $f : M \to M$ be a C^2-diffeomorphism. Let $U \subseteq M$ be a neighborhood such that $\overline{f(U)} \subseteq U$ and $\Lambda = \cap_{n\geq 1} f^n(U)$ is an attractor.

The attractor Λ has a *dominated splitting* if there is a continuous Df-invariant decomposition $T_\Lambda M = E^{cu} \oplus E^{cs}$ of the tangent bundle of M over Λ and constants $C \geq 0$ and $0 < \lambda < 1$ satisfying

$$\|Df^n|E^{cs}_x\| \|(Df^{-n}|E^{cu}_{f^n(x)})\| \leq C\lambda^n,$$

for every $x \in \Lambda$, and for every $n \geq 1$. The subbundle E^{cs} is *uniformly contracting* if

$$\|Df^n|E^{cs}_x\| \leq C\lambda^n,$$

for every $x \in \Lambda$ and $n \geq 1$. In this case, we denote $E^{cs} = E^s$ and we say that the attractor Λ is *partially hyperbolic*.

The diffeomorphism f has *non-uniform expansion along the center-unstable direction* if there exists a constant $c_0 > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log \|Df^{-1}|E^{cu}_{f^j(x)}\| \leq -c_0 < 0. \tag{1}$$

for all x in a full Lebesgue measure subset of U. Under these conditions Alves, Bonatti and Viana [**2**] proved:

Theorem 1.1. — *If $f \in \mathrm{Diff}^2(M)$ has an attractor Λ which is partially hyperbolic with non-uniformly expansion along the center-unstable direction, then there exist finitely many ergodic SRB measures and the union of their basins covers a full Lebesgue measure subset of the basin of Λ.*

The main tool used in the proof of Theorem 1.1 is the construction of Gibbs *cu*-states. Denote by $u = \dim E^{cu}$ and $s = \dim E^{cs}$.

Definition 1. — *An invariant measure μ supported in Λ is a* Gibbs *cu*-state *if the u larger Lyapunov exponents are positive μ-almost everywhere and the conditional measures of μ along the corresponding local strong-unstable manifolds are almost everywhere absolutely continuous with respect to Lebesgue measure on these manifolds.*

Theorem 1.1 is a direct consequence of the following result also proved in [**2**] and the uniformly contracting condition on the center-stable direction.

Theorem 1.2. — *If $f \in \mathrm{Diff}^2(M)$ has an attractor Λ which admits a dominated splitting with non-uniform expansion along the center-unstable direction, then there exist ergodic Gibbs cu-states supported on Λ.*

Alves, Bonatti and Viana give a constructive proof of the existence of Gibbs *cu*-states (see [**2**] or Subsection 3.1 for more details).

We first study the Gibbs *cu*-state in the setting of diffeomorphisms with dominated splitting and obtain a general description of such measures.

A *cylinder* $\mathcal{C}$ is a diffeomorphic image of $B^u \times B^s$ where B^u and B^s are balls in $\mathbb{R}^u$ and $\mathbb{R}^s$ respectively and "disk" D is a diffeomorphic image of B^u . We say that a C^1-disk D *crosses*

$\mathcal{C}$ if it is contained in $\mathcal{C}$ and is a graph over B^u. We also denote $B(x,\delta)$ the ball around $x \in M$ with radius $\delta > 0$.

Theorem A. — *Let $f \in \mathrm{Diff}^2(M)$ exhibit an attractor with dominated splitting and let μ be a Gibbs cu-state for f. Then*

1. *for μ-almost every point x and every $\delta > 0$ small enough, there exists a cylinder containing x, $\mathcal{C}(x,\delta) \subseteq B(x,\delta)$, and a family $\mathcal{K}(x,\delta)$ of disjoint unstable disks crossing the cylinder $\mathcal{C}(x,\delta)$ such that their union $K(x,\delta)$ has positive μ-measure;*
2. *denoting by ρ_z the density of the conditional measure μ_D along the disk $D = W^u_{loc}(z)$ through z, then, for μ-almost every $z \in \operatorname{supp}\mu$ and for every x, $y \in D$, we have*

$$\frac{\rho_z(x)}{\rho_z(y)} = \prod_{k=0}^{\infty} \frac{\det(Df^{-1}|E^{cu}_{f^{-k}(x)})}{\det(Df^{-1}|E^{cu}_{f^{-k}(y)})}; \tag{2}$$

   the densities are Hölder continuous functions bounded away from zero and infinity;
3. *the support of μ contains global unstable manifolds whose union has full μ-measure;*
4. *every ergodic component of μ is a Gibbs cu-state.*

The main tools used in the proof are Pesin's theory and distortion properties given by the dominated splitting.

If we add the hypothesis of non-uniform expansion along the center unstable direction, our main result is Theorem C below. But an essential ingredient in its proof is the following fact: the construction of Gibbs *cu*-states done in [**2**] provides all the possible Gibbs *cu*-states.

We denote by $\mathcal{G}(f)$ the class of Gibbs *cu*-states for f constructed in Theorem 1.2.

Theorem B. — *If $f \in \mathrm{Diff}^2(M)$ has a dominated splitting which is non-uniformly expanding along the E^{cu} direction, then every ergodic Gibbs cu-state supported in Λ is in $\mathcal{G}(f)$.*

Another ingredient in the proof of Theorem C are the uniform bounds obtained from [**2**] and the properties of Gibbs *cu*-states given by Theorem A.

Theorem C. — *Consider the set of pairs (f,μ) where $f \in \mathrm{Diff}^2(M)$ has an attractor with dominated splitting and non-uniform expansion along the center-unstable direction with uniform c_0 and μ is a Gibbs cu-state for f. Then this set is closed.*

As a corollary of Theorem C we obtain the following relationship between SRB measures and Gibbs *cu*-states:

Corollary D. — *If $f \in \mathrm{Diff}^2(M)$ has an attractor with dominated splitting which is non-uniformly expanding along the E^{cu} direction, then every ergodic SRB measure is a Gibbs cu-state.*

Another consequence of Theorem C is related to the statistical stability of partially hyperbolic diffeomorphisms.

Definition 2. — *We say that* $f_0 \in \mathrm{Diff}^r(M)$ *is* C^r-statistically stable *if for every sequence* $f_n \in \mathrm{Diff}^r(M)$ *converging to* f_0 *in the* C^r*-topology, and for every sequence* μ_n *of SRB measures for* f_n*, the weak* accumulation measures of* $(\mu_n)_n$ *are in the convex hull of finitely many SRB measures for* f_0*.*

Corollary E. — *If* $f \in \mathrm{Diff}^2(M)$ *has an attractor* Λ *which is partially hyperbolic and non-uniformly expanding along the center-unstable direction with* c_0 *uniform in a neighborhood of* f*, then* f *is* C^k*-statistically stable,* $k \geq 2$*.*

An interesting consequence of Theorem C is the following

Theorem F. — *Let* $f \in \mathrm{Diff}^2(M)$ *have an attractor* Λ *exhibiting a dominated splitting with non-uniform expansion along the center-unstable direction. Let us suppose that* f *satisfies*

$$\limsup_{n\to+\infty} \frac{1}{n}\|Df^n|E^{cs}_x\| < 0 \tag{3}$$

for each disk D *contained in every unstable local manifold and for a positive Lebesgue measure subset of points* $x \in D$*. Then* f *has finitely many SRB measures and the union of their basins covers a full Lebesgue measure subset of the basin of* Λ*. In addition, if non-uniform expansion along the central-unstable direction holds in a* C^k*-neighborhood of* f *with uniform* c_0 *and every diffeomorphism in such neighborhood satisfies (3) , then* f *is* C^k*-statistically stable,* $k \geq 2$*.*

This paper is organized as follows. In Section 2 we study the Gibbs *cu*-states using only the hypothesis of dominated splitting on the attractor and we prove Theorem A. In Section 3 we add the hypothesis of non-uniform expansion along the center-unstable direction. We first outline the proof of Theorem 1.2 and then we prove Theorem B and Theorem C. Finally, Section 4 is dedicated to studying the relationship between Gibbs *cu*-states and SRB measures. There we prove Corollary D, Corollary E and Theorem F, and we present an example of an open class of diffeomorphisms of the torus $\mathbb{T}^n$, $n \geq 4$, with dominated splitting, but not partially hyperbolic, exhibiting non-uniform expansion along the center-unstable direction, and admitting a unique SRB measure whose basin has full Lebegue measure in $\mathbb{T}^n$.

2. Gibbs cu-states

In this section we assume that $f : M \to M$ is a C^2-diffeomorphism exhibiting an attractor with dominated splitting and that there is a Gibbs cu-state μ for f supported on Λ. Our main goal is to prove Theorem A.

2.1. Preliminary results. — We first recall some well know properties for dominated splitting, Pesin's theory and conditional measures to be used along this work.

2.1.1. Attractors with dominated splitting. — Let $f : M \to M$ be a C^2-diffeomorphism. Let $U \subseteq M$ be a neighborhood such that $\overline{f(U)} \subseteq U$ and $\Lambda = \cap_{n\geq 1} f^n(U)$ is an attractor.
The attractor Λ is a compact set, maximal invariant for f in U. We call the *basin* of Λ the set

$$\mathcal{B}(\Lambda) = \bigcup_{n>0} f^{-n}(U)$$

of points whose future orbits accumulate on Λ.

We are assuming the attractor Λ has *dominated splitting*: there is a continuous Df-invariant decomposition $T_\Lambda M = E^{cu} \oplus E^{cs}$ of the tangent bundle of M over Λ and constants $C \geq 0$ and $0 < \lambda < 1$ satisfying

$$\|Df^n | E^{cs}_x\| \|(Df^{-n} | E^{cu}_{f^n(x)})\| \leq C\lambda^n,$$

for every $x \in \Lambda$, and for every $n \geq 1$.

It follows from the definition that the angles between the subbundles E^{cu} and E^{cs} are uniformly bounded away from zero.

Given $0 < a < 1$, we define the *center-unstable cone field* $C^{cu}_a = (C^{cu}_a(x))_{x\in\Lambda}$ of width a by

$$C^{cu}_a(x) = \{v_1 + v_2 \in E^{cs}_x \oplus E^{cu}_x \ : \ \|v_1\| \leq a\|v_2\|\}. \tag{4}$$

This cone field is positively invariant under Df. We define the center-stable cone field $C^{cs}_a = (C^{cs}_a(x))_{x\in\Lambda}$ of width a in a similar way, just reversing the roles of the subbbundles in (4).

Of course, we may extend continuously the two subbundle to U (it is not necessary for the extension to preserve the invariance of the splitting under Df) and so extend the cone fields to the neighborhood U also (Now, the positive invariance of C^{cs} is preserved). So the domination property is open in $\mathrm{Diff}^2(M)$ as consequence of the existence of cone fields. Both the splitting and the cone fields vary continuously with the diffeomorphism in $\mathrm{Diff}^2(M)$.

Given a disk D we denote by dist_D the Riemannian metric induced on D by the Riemannian metric on the manifold M. This is the distance "inside" the disk D. In the same way, we denote by m_D the Lebesgue measure induced by m on the disk D.

An essential ingredient here is the Hölder control of the Jacobian given by the domination.

Lemma 2.1. — *There exists $\xi > 0$ such that, given $L > 0$ and any C^2 disk $D \subseteq U$ transverse to the center stable direction E^{cs}, then there exists $C_1 > 0$ such that*

$$x \mapsto \log |\det(Df|T_x f^n(D))|$$

is (C_1, ξ)-Hölder on every domain of diameter L inside any $f^n(D)$, $n \geq 1$.

We refer the reader to the proof in [**2**], Section 2. We observe that this constant depends only on the diffeomorphism f.

2.1.2. Pesin's Theory. — Now we assume that $f \in \text{Diff}^r(M)$, $r > 1$, has an attractor with dominated splitting. Let μ be an f-invariant measure supported on Λ.

Oseledets theorem [**13, 12**] ensures that for μ-almost every point $x \in \Lambda$ there exist unique Lyapunov exponents and Lyapunov subspaces forming a splitting of T_xM, depending measurably of x.

Assume that μ has $u = \dim E^{cu}$ positive Lyapunov exponents and denote by $\lambda(x)$ the smallest of them. Then, the direct sum of Lyapunov space associated to these Lyapunov exponents is equal to E^{cu}_x. This follows from the domination condition.

In general, the angles between Lyapunov subspaces are not bounded away from zero, but the angles may decrease at most subexponentialy fast. However, by the domination condition the angle between E^{cu}_x and the direct sum of complementary Lyapunov subspaces is bounded away from zero.

Pesin's theory [**15, 19, 11, 18**] ensures that x has a local unstable manifold $W^u_{loc}(x)$ which is a C^r-embedded disk through x and there exists a constant C_x, depending measurably on x, such that:

[PT1] $W^u_{loc}(x)$ is tangent to E^{cu}_x at x;
[PT2] $\text{dist}_{W^u_{loc}(f^{-n}(x))}(f^{-n}(x_1), f^{-n}(x_2)) \leq C_x e^{-n\lambda(x)} \text{dist}_{W^u_{loc}(x)}(x_1, x_2)$, for all $x_1, x_2 \in W^u_{loc}(x)$ and for all $n \geq 1$;
[PT3] $f^{-1}(W^u_{loc}(x)) \subseteq W^u_{loc}(f^{-1}(x))$;

The size of $W^u_{loc}(x)$ also depends measurably on x. From this measurable dependence follows the existence of *hyperbolic blocks* $(\Lambda_n)_{n\in\mathbb{N}}$ [**15, 19, 11, 18**] . An hyperbolic block $\Lambda_n \subseteq \Lambda$ is a compact subset of Λ such that:

[HB1] $C_x < n$ and $\lambda_x > 1/n$ for every $x \in \Lambda_n$
[HB2] $\Lambda_n \subseteq \Lambda_{n+1}$ and $f(\Lambda_n) \subseteq \Lambda_{n+1}$ for all $n \in \mathbb{N}$ and $\mu(\Lambda_n) \to 1$ as $n \to \infty$;

[HB3] The C^r disks $W^u_{loc}(x)$ varies continuously with $x \in \Lambda_n$, in particular the size of $W^u_{loc}(x)$ is bounded away from zero for $x \in \Lambda_n$.

Using the properties above we can describe the support of every measure with u Lyapunov exponents.

Lemma 2.2. — *Let μ be a f-invariant measure such that μ-almost every $x \in \Lambda$ has u positive Lyapunov exponents in the E^{cu}_x direction. For μ-almost every point x and every $\delta > 0$ small enough, there exists a cylinder containing x, $\mathcal{C}(x,\delta) \subseteq B(x,\delta)$, and a family $\mathcal{K}(x,\delta)$ of disjoint unstable disks crossing the cylinder $\mathcal{C}(x,\delta)$ such that their union $K(x,\delta)$ has positive μ-measure.*

Proof. — Let μ be a f-invariant measure such that μ-almost every $x \in \Lambda$ has u positive Lyapunov exponents in the E^{cu}_x direction, for instance a Gibbs cu-state. For such generic x, there exist an hyperbolic block Λ_n, such that $x \in \Lambda_n$, and a unique C^1-embedded disk $W^u_{loc}(x)$ tangent to E^{cu}_x at x, such that the diameter of $f^n(W^u_{loc}(x))$ converges exponentially fast to zero as $n \to \infty$. The C^1-disk $W^u_{loc}(x)$ depends in a continuous way on the point x in Λ_n. In particular, there exists a uniform lower bound on the size of $W^u_{loc}(x)$ in Λ_n: there is $\delta_n > 0$ such that the pre-image of $W^u_{loc}(x)$ under the exponential map of M at x contains the graph of a C^1 map defined from $B(0,\delta_n) \cap E^{cu}_x$ to E^{cs}_x.

Given any $0 < \delta < \delta_n/4$ and $x \in \Lambda_n$ we can define the tubular neighborhood $\mathcal{C}(x,\delta)$ of $W^u_{loc}(x)$ as the image under the exponential map of M at y of all the vectors of norm less than $\delta > 0$ in the orthogonal complement of E^{cu}_y in T_yM, for all $y \in W^u_{loc}(x)$. If $\delta > 0$ is small enough then this neighborhood $\mathcal{C}(x,\delta)$ is a cylinder and it comes equipped with the canonical projection π onto $W^u_{loc}(x)$ which is a C^1 map. We denote by $\mathcal{K}(x,\delta)$ the family of local strong-unstable manifolds at points of Λ_n that cross $\mathcal{C}(x,\delta)$ and by $K(x,\delta)$ the union of the disks in the family $\mathcal{K}(x,\delta)$.

There exist $y_1,..,y_k \in \Lambda_n$ such that $\Lambda_n \subseteq \cup_{j=1}^k \mathcal{C}(y_j,\delta)$, because Λ_n is compact. We may suppose that each of these cylinders has positive μ-measure, and we obtain a covering (μ mod 0) of Λ_n. As a consequence, for all $j = 1,..,k$ we have $\mu(\Lambda_n \cap \mathcal{C}(y_j,\delta)) > 0$. On the other hand, for each $z \in \Lambda_n \cap \mathcal{C}(y_j,\delta)$, we have that $W^u_{\delta_n}(z)$ crosses $\mathcal{C}(y_j,\delta)$, because $\delta < \delta_n/4$. Then, for all $j = 1,..,k$,

$$\mu(K(y_j,\delta)) > \mu(\Lambda_n \cap \mathcal{C}(y_j,\delta)) > 0.$$

We consider the set of $x \in \operatorname{supp}\mu$ such that $x \in \Lambda_n$ for some $n \geq 1$. This set has full μ-measure. For $\delta > 0$, there exists $y \in \Lambda_n$ such that $x \in \mathcal{C}(y,\delta)$. Since x is in $\operatorname{supp}\mu$, $\mathcal{C}(y,\delta)$ must have positive μ measure. It is clear that δ can be chosen arbitrary small. To obtain the statement, we write $\mathcal{C}(x,\delta) = \mathcal{C}(y,\delta)$ and $\mathcal{K}(x,\delta) = \mathcal{K}(y,\delta)$.

□

Let $\mathcal{W}^u = \{W^u_{loc}(x)\}$ be the Pesin's unstable lamination. Then the associated holonomy maps preserve zero Lebesgue measure sets [**15, 19**]. More precisely, let x be in some hyperbolic block Λ_n, and x_1, x_2 be points of $W^u_{loc}(x)$. Let Σ_1 and Σ_2 be small smooth disks transverse to $W^u_{loc}(x)$ at x_1 and x_2. Given any point $y \in \Lambda_n$ close to x the local unstable manifold $W^u_{loc}(y)$ intersects each cross section Σ_i at exactly one point y_i. This defines the holonomy map

$$H^u : y_1 \to y_2$$

which is an homeomorphism between subsets of Σ_1 and Σ_2, respectively. Moreover, H^u is absolutely continuous, that is, it maps zero Lebesgue measure sets to zero Lebesgue measure sets.

Remark 1: Analogous properties hold in the case of negative Lyapunov exponents.

2.1.3. Conditional measures along the unstable foliation. — Let $x \in \operatorname{supp}\mu$ be fixed and let $C(x,\delta)$, $\mathcal{K}(x,\delta)$ and $K(x,\delta)$ be the sets defined in Lemma 2.2 above. Because $\mu(K(x,\delta)) > 0$ we can define $\mu|K(x,\delta)$, the restriction measure of μ on $K(x,\delta)$, by

$$\mu|K(x,\delta)(B) = \frac{\mu(K(x,\delta) \cap B)}{\mu(K(x,\delta))}$$

for all measurable subsets $B \subseteq K(x,\delta)$.

Let $\mathcal{P}$ be the coarsest partition of $K(x,\delta)$ such that every local unstable manifold $W^u_{loc}(x)$ is entirely contained in some atom of $\mathcal{P}$. In such case the atoms are embedded manifolds corresponding to local unstable manifolds. Then $\mathcal{P}$ is a measurable partition in the sense of Rokhlin [**20**]: it may be written as the product

$$\mathcal{P} = \bigvee_{n=1}^{\infty} \mathcal{P}_n$$

of an increasing sequence of finite partitions $\mathcal{P}_n$ of $K(x,\delta)$. Moreover, to each atom of $\mathcal{P}$, there corresponds a unique disk D which is an unstable local manifold, so we may identify $\mathcal{P} = \mathcal{K}(x,\delta)$.

Let $\pi_s : K(x,\delta) \to B^s$ be the projection on B^s along the center-unstable leaves. We can induce a measure $\hat{\mu}$ on B^s given by

$$\hat{\mu}(A) = \mu(\pi_s^{-1}(A) \cap K(x,\delta))$$

for measurable $A \subseteq B^s$.

By Rokhlin's disintegration theorem [**20**], there exists a family $\{\mu_D \ : \ D \in \mathcal{K}(x,\delta)\}$ of conditional probability measures satisfying

[CP1] $\mu_D(D) = 1$ for $\hat{\mu}$-almost every $D \in \mathcal{K}(x, \delta)$.
[CP2] $\mu(E) = \int \mu_D(E)\, d\hat{\mu}(D)$ for every measurable subset E of $K(x, \delta)$.

The family is essentially unique: given another choice $\{\mu'_D : D \in \mathcal{K}(x, \delta)\}$ of measures satisfying [CP1,CP2] then $\mu_D = \mu'_D$ for $\hat{\mu}$-almost every $D \in \mathcal{K}(x, \delta)$.

Given $D \in \mathcal{K}(x, \delta)$, let $P_n \in \mathcal{P}$ be the element of the partition $\mathcal{P}_n$ that contains D then

$$\lim_{n\to\infty} \frac{1}{\nu(P_n)} \int_{P_n} \varphi\, d\mu = \int \varphi\, d\mu_D,$$

for every continuous function $\varphi : M \to \mathbb{R}$.

By definition, conditional measures of a Gibbs *cu*-state are absolutely continuous along the unstable manifolds, that is, for μ-almost every x and every $\delta > 0$, the conditional measures μ_D defined above are absolutely continuous with respect to Lebesgue measure m_D in D, for $\hat{\mu}$-almost every $D \in \mathcal{K}(x, \delta)$.

2.2. Proof of Theorem A. — We begin by studying the relations between a Gibbs *cu*-state and its ergodic components.

Let ξ be an invariant measure for f. Let $R(f)$ be the set of *regular points* of f, that is the set of points in M such that the Birkhoff averages exist and

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \lim_{n\to-\infty} \frac{1}{n} \sum_{j=1}^{n+1} \varphi(f^j(x))$$

for all $\varphi \in C^0(M; \mathbb{R})$. It is well-known that this set has full measure with respect to any f-invariant measure ξ.

Given a point x let us denote by ξ_x the probability measure given by the time average along the orbit of x

$$\int \varphi\, d\xi_x = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x)$$

for every continuous $\varphi : M \to \mathbb{R}$. According to the Ergodic Decomposition Theorem (cf. [**12**]) ξ_x is well defined and ergodic for every x in a set $\Sigma(f) \subseteq M$ that has full measure with respect to any invariant measure. Moreover, for every bounded measurable function $\varphi : M \to \mathbb{R}$ we can write

$$\int \varphi\, d\xi = \int \int \varphi\, d\xi_x\, d\xi(x),$$

for every such φ the integral $\int \varphi\, d\xi_x$ coincides with the time average ξ-almost everywhere, and $x \in \operatorname{supp} \xi_x$ for ξ-almost all x.

Let μ be a Gibbs cu-state. Given a point in the support of μ and $\delta > 0$, we fix a cylinder $\mathcal{C}$, a family of disks $\mathcal{K}$ crossing $\mathcal{C}$ and the union of such disks K as in Lemma 2.2.

Fix B a measurable subset of M such that

$$m_D(B \cap D) = 0 \quad \text{for every} \quad D \in \mathcal{K},$$

and B is maximal among all measurable sets with this property. Observe that $\mu(B) = 0$, because μ is absolutely continuous along the leaves on $\mathcal{K}$. Let $\tilde{Z} = K \cap \Sigma(f) \cap R(f) \setminus B$. Then $\mu(\tilde{Z}) > 0$ and let $(\mu|\tilde{Z})$ be the restriction of μ to $\tilde{Z}$.

Let A be any measurable subset of $\tilde{Z}$ such that $m_D(A \cap D) = 0$ for every $D \in \mathcal{K}$. Then $\mu(A)$ must be zero, since we took $\mu(B)$ maximal. This means that $(\mu|\tilde{Z})$ is absolutely continuous with respect to the product $m_D \times \hat{\mu}$, where $\hat{\mu}$ stands for the quotient measure induced by $(\mu|\tilde{Z})$ on $\mathcal{K}$. As a consequence, the conditional measures $(\mu|\tilde{Z})_D$ of $(\mu|\tilde{Z})$ on the disks $D \in \mathcal{K}$ are absolutely continuous with respect to Lebesgue measure m_D for $\hat{\mu}$-almost all $D \in \mathcal{K}$.

On the other hand, for any measurable set $A \subseteq \tilde{Z}$,

$$\mu(A) = \int \mu_x(A)\, d\mu(x),$$

where the integral is taken over $\Sigma(f) \subseteq M$.

Let us denote by $\mathbf{1}_A$ the characteristic function of the measurable subset A. Then we have (as already mentioned)

$$\mu_x(A) = \int \mathbf{1}_A \, d\mu_x = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbf{1}_A(f^j(x))$$

μ^*-almost everywhere. So $\mu_x(A)$ can be non-zero only if x has some iterate in $A \subseteq \tilde{Z}$, for μ-almost every point x. Let $k(z)$ denote the first backward return time to $\tilde{Z}$ of a point $z \in \tilde{Z}$, this means $k(z)$ is the smallest positive integer such that $f^{-k(z)}(z) \in \tilde{Z}$. This is defined μ-almost everywhere, by Poincare's recurrence theorem. Observing also that $\mu_z = \mu_{f^j(z)}$ for every z and every integer $j \in \mathbb{Z}$, we have

$$\mu(A) = \int_{\Sigma(f)} \mu_x(A)\, d\mu(x) = \int_{\tilde{Z}} k(z)\mu_z(A)\, d\mu(z)$$

for any measurable subset A of $\tilde{Z}$.

Lemma 2.3. — *Let λ be a finite measure on a measure space Z, with $\lambda(Z) > 0$. Let $\mathcal{K}$ be a measurable partition of Z, and $(\lambda_z)_{z\in Z}$ be a family of finite measures on Z such that*

1. *the function $z \to \lambda_z(A)$ is measurable, and constant on each element of $\mathcal{K}$, for any measurable set $A \subset Z$.*

2. $\{w : \lambda_z = \lambda_w\}$ *is a measurable set with full λ_z-measure, for every $z \in Z$.*

Assume that $\lambda(A) = \int l(z)\lambda_z(A)\, d\lambda$ for some measurable function $l : Z \to \mathbb{R}_+$ and any measurable subset A of Z. Let $\{\tilde{\lambda}_\gamma : \gamma \in \mathcal{K}\}$, and $\{\tilde{\lambda}_{z,\gamma} : \gamma \in \mathcal{K}\}$, be the disintegrations of λ and λ_z, respectively, into conditional probability measures along the elements of the partition $\mathcal{K}$. Then

$$\tilde{\lambda}_{z,\gamma} = \tilde{\lambda}_\gamma$$

for λ-almost every $z \in Z$ and $\hat{\lambda}_z$-almost every γ, where $\hat{\lambda}_z$ is the quotient measure induced by λ_z on $\mathcal{K}$.

The reader can see the proof in [**2**], Lemma 6.2.

Lemma 2.4. — *If μ is a Gibbs cu-state of f, then every ergodic component of μ is a Gibbs cu-state.*

Proof. — If we take $Z = \tilde{Z}$, $\lambda = (\mu|\tilde{Z})$, $\mathcal{K} = \mathcal{K}$, $\lambda_z = (\mu_z|\tilde{Z})$ and $l(z) = k(z)$ for each $z \in \tilde{Z}$ in Lemma 2.3, then the conditional probability measures $(\mu_z|\tilde{Z})_D$ of $(\mu_z|\tilde{Z})$ along the disks $D \in \mathcal{K}$ coincide almost everywhere with the corresponding conditional measures μ_D of $(\mu|\tilde{Z})$. Recall that we had already shown that the latter are almost everywhere absolutely continuous with respect to Lebesgue measure on the corresponding disks $D \in \mathcal{K}$. Moreover, the u-largest Lyapunov exponents are positive μ_D almost everywhere.

□

Remark 2: If μ is an invariant measure with ergodic decomposition $(\mu_x)_{x\in\Sigma(f)}$, and μ_x is a Gibbs *cu*-state for μ-almost every point x, then μ is a Gibbs *cu*-state also. In fact, μ-almost every point x is in a disk D contained in a local unstable manifold, and so, its larger Lyapunov exponents are positive. Since D is completely contained in an unique ergodic component, so $\mu_D = \mu_{x,D}$. But $\mu_{x,D}$ is absolutely continuous respect to Lebesgue measure, so μ_D is absolutely continuous also.

In what follows we assume μ is an ergodic Gibbs *cu*-state. Given $z \in \operatorname{supp}\mu$ we denote by μ_D the conditional measure of μ in $D = W^u_{loc}(z)$ and we denote by ρ_z the density of the conditional measure μ_D of μ along the unstable disk D through z. Our next goal is to characterize this density.

Lemma 2.5. — *For every x and y in the same local unstable manifold, the product*

$$\prod_{k=0}^{\infty} \frac{\det(Df^{-1}|E^{cu}_{f^{-k}(x)})}{\det(Df^{-1}|E^{cu}_{f^{-k}(y)})}$$

converges and is bounded away from zero and infinity.

Proof. — Let $x, y \in W^u_\varepsilon(z)$ and set $J^u_k(x) = |\det Df^{-1}|E^{cu}_{f^k(x)}|$, $k \geq 0$. Lemma 2.1 implies that the map $x \to \log(J^u_k(x)^{-1})$ is (C_1, ξ)-Hölder. Let $\lambda > 0$ be the smallest Lyapunov exponent for z in the E^{cu}-direction. Then, for $N \geq 1$,

$$\begin{aligned}
\left|\log \prod_{k=0}^{N} \frac{J^u_k(x)}{J^u_k(y)}\right| &\leq \sum_{k=0}^{N} |\log J^u_k(x) - \log J^u_k(y)| \\
&\leq \sum_{k=0}^{N} C_1 \mathrm{dist}_{f^k(W^u_\varepsilon(z))}(f^k(x), f^k(y))^\xi \\
&\leq \sum_{k=0}^{N} C_1 C^\xi e^{-k\lambda\xi} \mathrm{dist}_{W^u_\varepsilon(z)}(x, y)^\xi
\end{aligned}$$

The symmetry of the product (we may exchange x and y) and the convergence of the series imply that the product converges for all $x, y \in W^u_\varepsilon(z)$ and is non-zero. Moreover the convergence is absolute and Hölder with respect to x and y, so the product is bounded away from zero and infinity.

□

Remark 3: The convergence of the product depends only on f and on the smallest Lyapunov exponent along the center-unstable disks.

Proposition 2.1. — *For μ-almost every $z \in \operatorname{supp}\mu$ and for every $x,\ y \in W^u_{loc}(z)$,*

$$\frac{\rho_z(x)}{\rho_z(y)} = \prod_{k=0}^{\infty} \frac{\det(Df^{-1}|E^{cu}_{f^{-k}(x)})}{\det(Df^{-1}|E^{cu}_{f^{-k}(y)})}.$$

The densities are Hölder continuous and bounded away from zero and infinity.

Proof. — We fix a generic z and $W^u_{loc}(z) = D$. Since μ_z is absolutely continuous with respect to Lebesgue measure in D, there exists some $\rho : D \to \mathbb{R}$ which is measurable and positive μ_z-almost everywhere such that

$$\mu_z(B) = \int_B \rho \, dm_D$$

for all Borelean subsets $B \subseteq D$. Let ρ_n be the density of $\mu_{f^{-n}(z)}$ on $f^{-n}(D)$. By change of variables we have for $x \in W^u_{loc}(z)$ that

$$\rho(x) = C\rho_n(f^{-n}(x)) \prod_{k=0}^{n-1} J^u_k(x) \tag{5}$$

for any $n \geq 0$ where $C > 0$ is a constant of normalization depending of z and n. Then, for every $x,\ y \in W^u_{loc}(z)$,

$$\frac{\rho(x)}{\rho(y)} = \frac{\rho_n(f^{-n}(x))}{\rho_n(f^{-n}(y))} \prod_{k=0}^{n-1} \frac{J^u_k(x)}{J^u_k(y)}.$$

By Lemma 2.5 the right hand product converges to a non-zero value, so the quotient $\rho_n(f^{-n}(x))/\rho_n(f^{-n}(y))$ also converges. We claim there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\frac{\rho_{n_k}(f^{-n_k}(x))}{\rho_{n_k}(f^{-n_k}(y))} \to 1$$

when $k \to \infty$. This claim completes the proof.

For $\delta > 0$, let $\mathcal{C}(z,\delta)$, $\mathcal{K}(z,\delta)$ and $K(z,\delta)$ be the sets defined in Lemma 2.2. Let $\varepsilon > 0$ be fixed and let Λ_ε be a compact subset of $K(z,\delta)$ such that $\mu(K(z,\delta) \setminus \Lambda_\varepsilon) < \varepsilon$ and the map

$$\begin{aligned} \Lambda_\varepsilon \subseteq B^u \times B^s &\to \mathbb{R} \\ (x,w) &\mapsto \rho_w(x) \end{aligned}$$

is continuous. In particular, we may assume the following: given $k \in \mathbb{N}$, there exists $\delta_k > 0$, not depending on $w \in \Lambda_\varepsilon$, such that

$$\left| \frac{\rho_w(x)}{\rho_w(y)} - 1 \right| < \frac{1}{k}$$

for all $x, y \in W^u_{loc}(z) \cap \Lambda_\varepsilon$ satisfying $\mathrm{dist}_{W^u_{loc}(w)}(x,y) < \delta_k$. As $\mu(\Lambda_\varepsilon) > 0$, Poincare's recurrence theorem together with the properties [PT1, PT2] of the unstable Pesin lamination (see subsection 2.1.2) imply that for μ-almost every $w \in \Lambda_\varepsilon$ there exists a n_k large enough such that:

1. there exists $\tilde{w} \in \Lambda_\varepsilon$ such that $f^{-n_k}(W^u_{loc}(w)) \subseteq W^u_{loc}(\tilde{w})$,
2. diam $f^{-n_k}(W^u_{loc}(w)) \leq \delta_k$.

Since μ is f-invariant, from item 1 it follows that $\rho_{n_k,w} = \rho_{\tilde{w}}$ for $\mu_{f^{-n_k}(w)}$-almost every point in $f^{-n_k}(W^u_{loc}(w))$. Now item 2 implies that for every $x, y \in W^u_{loc}(w) \cap \Lambda_\varepsilon$

$$\mathrm{dist}_{W^u_{loc}(\tilde{w})}(f^{-n_k}(x), f^{-n_k}(y)) = \mathrm{dist}_{f^{-n_k}(W^u_{loc}(w))}(f^{-n_k}(x), f^{-n_k}(y)) \leq \delta_k$$

and so,

$$\left| \frac{\rho_{n_k,w}(f^{-n_k}(x))}{\rho_{n_k,w}(f^{-n_k}(y))} - 1 \right| = \left| \frac{\rho_{\tilde{w}}(f^{-n_k}(x))}{\rho_{\tilde{w}}(f^{-n_k}(y))} - 1 \right| < \frac{1}{k}$$

and this finishes the proof.

□

If x has a local strong unstable manifold $W^u_{loc}(x)$, we define the *global unstable manifold* of x as the set

$$W^u(x) = \bigcup_{n\geq 0} f^n(W^u_{loc}(f^{-n}(x)).$$

Next result follows from the fact that the densities ρ are bounded away from zero and infinity.

Corollary 2.1. — *If μ is a Gibbs cu-state of f, then the support of μ contains global unstable manifolds whose union has full μ-measure.*

Proof. — Let Λ_n be a hyperbolic block. Then for all $x \in \Lambda_n$ there exists $W^u_\delta(x)$ with $\delta > 0$ uniform on x. Moreover $\mu(\Lambda_n) > 1 - \varepsilon_n$, with $\varepsilon_n \to 0$ as $n \to \infty$. It is sufficient to prove that for μ-almost every $x \in \Lambda$, one has $W^u_\delta(x) \subseteq \operatorname{supp}\mu$.

For each $x \in \operatorname{supp}\mu$, we can construct a cylinder $\mathcal{C}$ that contains $W^u_\delta(x)$ and such that if $z \in \Lambda_n \cap \mathcal{C}$, then $W^u_\delta(z)$ crosses $\mathcal{C}$. Suppose there is $y \in W^u_\delta(x)$ such that $y \notin \operatorname{supp}\mu$. Then there exists a small neighborhood $y \in V \subseteq \mathcal{C}$ such that $\mu(V) = 0$. By the disintegration of μ we have

$$\mu(V) = \int \mu_z(V \cap W^u_\delta(z))\, d\hat{\mu}(z),$$

but each μ_z has strictly positive density. Then there exists a neighborhood of x having zero $\hat{\mu}$-measure, which contradicts the fact that x is in the support of μ.

□

Here we conclude the proof of Theorem A. Lemma 2.2 and Proposition 2.1 correspond to statements 1 and 2 of Theorem A. Corollary 2.1 and Lemma 2.4 correspond to statements 3 and 4. Next lemma plays an important role in the following sections:

Lemma 2.6. — *Let μ be a Gibbs cu-state for f. For μ-almost every $x \in \Lambda$ and every $\delta > 0$ small enough, there exists a cylinder $\mathcal{C}(x,\delta)$ such that $\hat{\mu}$-almost every disk $D \in \mathcal{K}(x,\delta)$ satisfies $B(\mu) \cap R(f) \cap D$ has full Lebesgue measure in D.*

Proof. — We observe that $\mu(B(\mu) \cap R(f)) = 1$. For almost every $x \in B(\mu) \cap R(f) \cap \operatorname{supp}\mu$ and every $\delta > 0$ small enough, there exists a cylinder $\mathcal{C}$ and a family $\mathcal{K}$ of disk crossing $\mathcal{C}$ such that the union of those disks has positive μ-measure. Let us consider the probability measure $(\mu|K)$. Then

$$(\mu|K)(B(\mu) \cap R(f) \cap K) = 1.$$

By disintegration, $\hat{\mu}|K$-almost every disk $D \in \mathcal{K}$ satisfies

$$\mu_D(D \cap B(\mu) \cap R(f)) = \mu_D(D).$$

Because μ_D is absolutely continuous with respect to Lebesgue measure and the density ρ_D is bounded from zero and infinity, it follows that $m_D(B(\mu) \cap R(f) \cap D) = m_D(D)$. □

3. Gibbs cu-states and the non-uniform expansion condition

3.1. Building Gibbs cu-states. — Let $f : M \to M$ be a C^2-diffeomorphism having an attractor Λ with a dominated splitting and non-uniform expansion along the E^{cu} direction. The goal of this subsection is to briefly review the construction of Gibbs cu-states (cf. Theorem 1.2 [**2**]).

A disk $D \subset U$ is *tangent to the center-unstable cone field* C^{cu} if the tangent subspace to D at each point $x \in D$ is contained in the corresponding cone $C^{cu}(x)$. We fix a C^2 disk D tangent to the center-unstable cone field such that:

1. The set of points in D having non-hyperbolic behavior has full Lebesgue measure in the disk. This is possible because we assume that almost every point in U satisfies (1).
2. There are fixed $\xi > 0$ and $C_1 > 0$ as in Lemma 2.1 such that the functions J_k defined on $f^k(D) \subset U$ by $J_k(x) = \log|\det Df|T_x f^k(D)|$, for $k = 1, .., n$ are (C_1, ξ)-Hölder. These constants depend only on f.

Definition 3. — *Given* $0 < \sigma < 1$*, we say that* n *is a* σ-hyperbolic time *for a point* $x \in U$ *if*

$$\prod_{j=n-k+1}^{n} \|Df^{-1}|E^{cu}_{f^j(x)}\| \leq \sigma^k$$

for all $1 \leq k \leq n$*.*

Conditions 1 and 2 above imply that there exist many (positive density at infinity) σ-hyperbolic times for points $x \in D$ satisfying (1) with $\sigma < e^{-c_0/3}$. The rate depends on c_0 and f. This follows from an adapted version of Pliss' Lemma [**17**] also proved in [**12**] and [**2**]:

Proposition 3.1. — *Given any* $x \in \tilde{D}$ *and any sufficiently large* $N \geq 1$*, there exist* σ*-hyperbolic times* $1 \leq n_1 < .. < n_l \leq N$ *for* x *with* $l \geq \frac{-|\log\sigma|}{\sup|\log\|Df^{-1}|E^{cu}\| - 2|\log\sigma|}N$*.*

Remark 4: Hyperbolic times can not depend continuously on the diffeomorphism f, but the rate

$$\theta = \frac{-|\log\sigma|}{\sup|\log\|Df^{-1}|E^{cu}\| - 2|\log\sigma|}$$

depends continuously on f (in the C^1-topology) and c_0.

As a consequence of the existence of σ-hyperbolic times, we obtain backward uniform contraction and bounded distortion properties. More precisely (see [**2**]) :

Proposition 3.2. — *There exist $C_2 > 0$ and $\delta_1 > 0$ such that for all $x \in D$, for all σ-hyperbolic times n and for every $y \in D$ such that $\mathrm{dist}_{f^n(D)}(f^n(x), f^n(y)) \le \delta_1$, we have*

$$\mathrm{dist}_{f^{n-k}(D)}(f^{n-k}(x), f^{n-k}(y)) \le \sigma^{k/2}\mathrm{dist}_{f^n(D)}(f^n(x), f^n(y)), \tag{6}$$

$$\frac{1}{C_2} \le \frac{|\det Df^n|T_yD|}{|\det Df^n|T_xD|} \le C_2. \tag{7}$$

The constant C_2 in (7) above depends on σ, δ_1 and depends on the Hölder constant C_1. We remark that (7) is similar to the quotient factor in Proposition 2.1.

For each $j \ge 1$, let $\hat{H}_j$ be a finite set of $x \in D$ such that j is an σ-hyperbolic time for x. For $\delta = \delta_1/4$, we denote by $\Delta_j(x,\delta)$ the δ-neighborhood of $f^j(x)$ inside $f^j(D)$. We choose $\hat{H}_j$ such that the balls $\Delta_j(x,\delta)$ are pairwise disjoint. We denote by Δ_j the union of such balls.

We can choose $\hat{H}_j$ satisfying the following (see [**2**] Proposition 3.3 and Lemma 3.4): there exists a constant $\tau > 0$, depending only on f, such that for any j

$$f^j_* m_D(\Delta_j \cap f^j(U)) \ge f^j_* m_D(\Delta_j \cap f^j(\hat{H}_j)) \ge \tau m_D(\hat{H}_j)$$

Consider the set of accumulation points of $(\Delta_j)_j$:

$$\Delta_\infty = \bigcap_{n=1}^{\infty} \overline{\bigcup_{j\ge n} \Delta_j}.$$

Observe that $\Delta_j \subseteq f^j(D) \subseteq f^j(U)$. Then, since U is positively invariant,

$$\overline{\bigcup_{j\ge n} \Delta_j} \subseteq \overline{f^n(U)} \subseteq f^{n-1}(U)$$

and so $\Delta_\infty \subseteq \Lambda$.

Given $y \in \Delta_\infty$ there exist a sequence $(j_i)_i \to \infty$, disks $D_i = \Delta(x_i,\delta) \subseteq \Delta_{j_i}$ and points $y_i \in D_i$, $y_i \to y$ as $i \to \infty$. By passing to a subsequence if necessary, we may suppose that the centers x_i converge to some point x and, by Arzela-Ascoli theorem, that the D_i converge to a disk $D(x)$ of radius δ around x. Then y is in the closure $\overline{D(x)}$ of $D(x)$, and $\overline{D(x)} \subseteq \Delta_\infty$ and the points x are in the set

$$\hat{H}_\infty = \bigcap_{n=1}^{\infty} \overline{\bigcup_{j \geq n} f^j(\hat{H}_j)}.$$

Observe that D_i is contained in the j_i-iterate of D, which was taken tangent to the center-unstable cone field. So the domination property implies that the angle between D_i and E^{cu} goes to zero as $i \to \infty$. By Proposition 3.2, given $k \geq 1$ then f^{-k} is a $\sigma^{k/2}$-contraction on D_i, for every large i. Passing to the limit when $i \to \infty$, we get that every f^{-k} is a $\sigma^{k/2}$-contraction on $D(x)$, and that $D(x)$ is tangent to the center-unstable subbundle at every point of $D(x) \subseteq \Lambda$, including x.

In particular we have shown that the subspace E^{cu}_x is indeed uniformly expanding for Df. The domination property means that any expansion that Df exhibits along the complementary direction is weaker than this. Then, see [**15**], there exists a unique strong-unstable manifold $W^u_{loc}(x)$ tangent to E^{cu} which is contracted by negative iterates of f at a rate of at least $\sigma^{k/2}$, when k gets large. Moreover $D(x)$ is contained in $W^u(x)$ because it is contracted by every f^{-k}, $k \geq 1$, and all its negative iterates are tangent to the center-unstable cone field. Summing up, we have

Proposition 3.3. — *The family of disks $D(x)$, with $x \in \hat{H}_\infty$, constructed as above satisfies:*

1. *the radius of $D(x)$ is $\delta_1/4$ uniformly in $x \in \hat{H}_\infty$;*
2. *for every $y \in \Delta_\infty$ there exists $x \in \hat{H}_\infty$ such that $y \in \overline{D(x)}$;*
3. *for all $x \in \hat{H}_\infty$, the subspace E^{cu}_x satisfies*

$$\|Df^{-k}|E^{cu}_x\| \leq \sigma^{k/2}, \quad \textit{for all} \;\; k \geq 0;$$

4. *$D(x)$ is contained in the corresponding strong-unstable manifold $W^u_{loc}(x)$;*
5. *$D(x)$ is tangent to the center-unstable subbundle at every point of $\Lambda \cap D(x)$.*

We now consider the sequence of averages of push-forwards of Lebesgue measure restricted to such a disk D

$$\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} f^j_* \, m_D.$$

Remark 5: The argument that follows does not change if we consider φm_D instead of m_D, where φ is a measurable function bounded away from zero and infinity, m_D- almost everywhere.

We decompose μ_n as a sum of two measures ν_n and η_n, where

$$\nu_n = \frac{1}{n} \sum_{j=0}^{n-1} f_*^j \, m_D | \Delta_j$$

and $\eta_n = \mu_n - \nu_n$. Observe that the support of ν_n is $\cup_{j=0}^{n-1} \Delta_j$.

Now, we consider any subsequence $(n_k)_k$ such that μ_{n_k} and ν_{n_k} converge to μ and ν respectively. Then the support of ν is contained in the set

$$\Delta_\infty = \bigcap_{n=1}^{\infty} \overline{\bigcup_{j \geq n} \Delta_j} \subseteq \Lambda$$

of accumulation points of $(\Delta_j)_j$. Proposition 3.3 gives a characterization of the support of ν. Moreover ([**2**] Proposition 3.5 and Remark 3.6), there is $\alpha_1 = \alpha_1(c_0, f) > 0$ such that, for all $n \geq 1$ and $k \geq n$ large enough,

$$\nu_k(f^n(U)) \geq \nu_k(f^k(U)) \geq \alpha_1.$$

This is because $\overline{f^k(U)} \subseteq f^n(U)$. Then, $\nu(f^n(U)) \geq \alpha_1$ and so

$$\nu(\Lambda) = \nu(\bigcap_{n \geq 1} f^n(U)) = \liminf_{n \to \infty} \nu(f^n(U)) \geq \alpha_1.$$

Recall from Proposition 3.3 that, given any $y \in \Delta_\infty$, there exist a point $x \in \hat{H}_\infty$ and a disk $D(x)$ of size $\delta_1/4$ around x such that $y \in \overline{D(x)} \subseteq \Delta_\infty$. For any such x and $r > 0$ small, let $C_r(x)$ be the tubular neighborhood of $\overline{D(x)}$, defined as the union of the images under the exponential map at each point $z \in \overline{D(x)}$ of all vectors orthogonal to $\overline{D(x)}$ at z with norm less than or equal to r. We take r to be sufficiently small, so that $C_r(x)$ is a cylinder endowed with the canonical projection $\pi : C_r(x) \to \overline{D(x)}$. We may suppose that the boundary of $C_r(x)$ has zero ν-measure (observe that r depends on the size of the domain of the exponential map, and so depends continuously on f).

For any $\varepsilon > 0$, we can fix a cover of $\overline{D(x)}$ by finitely many domains $D_{x,l} \subseteq \overline{D(x)}$, $l = 1, .., N(\varepsilon)$, small enough so that the intersection of each $C_{x,l} = \pi^{-1}(D_{x,l})$ with any smooth disk γ tangent to the center-unstable cone field has diameter less than ε inside γ. We choose the cover with the least possible $N(\varepsilon)$ and take the $D_{x,l}$ diffeomorphic to the compact ball B^u, so that every $C_{x,l}$ is a cylinder.

We say that a disk γ crosses $C_{x,l}$ if π maps $\gamma \cap C_{x,l}$ diffeomorphicaly onto $D_{x,l}$. For each $j \geq 0$, let $K_j(x,l)$ be the union of the intersections of $C_{x,l}$ with all the disks in Δ_j that cross $C_{x,l}$ and let $K_\infty(x,l)$ be the union of the intersections of $C_{x,l}$ with all the disks in Δ_∞ that cross $C_{x,l}$. Fixing a small enough ε for at least one of the cylinders $C_{x,l}$ the part of the measure ν that is carried by the disks in $K_\infty(x,l)$ has positive mass $\alpha > 0$, depending on the rate of hyperbolic times, and so depending on c_0 and f (See [**2**] Lemma 4.2 and Lemma 4.3).

In the following we write $\mathcal{C} = C_{x,l}$, $\tilde{D} = D_{x,l}$ and $\mathcal{K}_j$, $0 \leq j \leq \infty$, the family of disks whose union is $K_j = K_j(x,l)$.

Observe that from the construction of μ_n the measure $f_*^j m_D | \Delta_j$ is absolutely continuous with respect to Lebesgue measure along $f^j(D)$. Moreover, from Proposition 3.2(7) the density of the normalization of this measure is uniformly bounded from below and from above. The construction preserves this property for ν_n and ν.

Let us introduce $\hat{K} = \cup_{0 \leq j \leq \infty} K_j \times \{j\}$. In this space, we consider the sequence of finite measures $\hat{\nu}_n$ defined by

$$\hat{\nu}_n(B_0 \times \{0\} \cap .. \cap B_{n-1} \times \{n-1\}) = \frac{1}{n} \sum_{j=0}^{n-1} f_*^j m_D(B_j),$$

and $\hat{\nu}_n(B_n) = 0$ whenever B is in $\cup_{n \leq j \leq \infty} K_j \times \{j\}$. We also consider a sequence of partitions $\mathcal{P}_k$ in $\hat{K}$ constructed as follows. Fix an arbitrary point $z \in \tilde{D}$ and let V be the inverse image $\pi^{-1}(z)$ under the canonical projection. Fix a sequence $\mathcal{V}_k$, $k \geq 1$, of increasing partitions of V with diameter going to zero. Then, by definition, two points (x, m), $(y, n) \in \hat{K}$ are in the same atom of the partition $\mathcal{P}_k$ if

– the disk in Δ_m containing x and the disk in Δ_n containing y intersect some common element of $\mathcal{V}_k$;
– either $m \geq k$ and $n \geq k$, or $m = n < k$.

It is clear from the construction that for any point $\xi \in K_j$ and every $0 \leq j \leq \infty$, one has

$$\mathcal{P}_1(\xi) \supset .. \supset \mathcal{P}_k(\xi) \supset \dots,$$

and $\cap_{k=1}^{\infty} \mathcal{P}_k(\xi)$ coincides with the intersection of the cylinder $\mathcal{C}$ with the disk in Δ_j that contains ξ. We define $\hat{\pi} : \hat{K} \to \tilde{D}$ by $\hat{\pi}(x, j) = \pi(x)$.

Clearly, any weak* accumulation measure of the sequence $\hat{\nu}_n$ must be supported in $K_\infty \times \{\infty\}$. We have chosen a sequence $(n_k)_k$ such that ν_{n_k} converges to the measure ν. It is easy to see that this is just the same as saying that $\hat{\nu}_k$ converges to the measure $\hat{\nu}$ defined by $\hat{\nu}(B \times \{\infty\}) = \nu(B)$ for any Borel set $B \subseteq \mathcal{C}$, so ν and $\hat{\nu}$ are naturally identified.

Proposition 3.4. — *There exist $C_3 > 1$, depending on f only, and a family of conditional measures $(\nu_\gamma)_\gamma$ of $\nu | \mathcal{K}_\infty$ along the disks $\gamma \in \mathcal{K}_\infty$ such that ν_γ is absolutely continuous with respect to the Lebesgue measure m_γ on γ, with*

$$\frac{1}{C_3} m_\gamma(B) \leq \nu_\gamma(B) \leq C_3 m_\gamma(B) \tag{8}$$

for any Borel set $B \subseteq \gamma$.

The reader can see the proof in [**2**] Section 4. The constant C_3 depends on the Lebesgue measure along the disks in the cylinder (so depends on f) and depends on the distortion bound C_2 obtained in Proposition 3.2.

The construction of Gibbs *cu*-states concludes as follows: there exists an ergodic component μ_z of μ having positive measure on $\mathcal{K}_\infty$ which is absolutely continuous along the disks.

Each disk $D \in \mathcal{K}_\infty$ is completely contained in some ergodic component because it is contained in some local-unstable manifold. In particular, all Lyapunov exponents of μ_z in the center unstable direction are larger than $-\log\sigma > 0$. The domination condition implies that all the other exponents are less than $-\log\sigma + \log\lambda < -\log\sigma$. Again, by Pesin theory, μ_z-almost every point has a local strong-unstable manifold which is an embedded disk whose backward orbits contract at the exponential rate $\log\sigma$. Moreover the disks $D \in \mathcal{K}_\infty$ contain the local strong-unstable manifolds of points in its interior.

Summing up this section, we have the following

Theorem 3.1. — [Alves, Bonatti, Viana [**2**]] *Any diffeomorphism f with a dominated splitting which is non-uniformly expanding along the center unstable direction has an ergodic Gibbs cu-state. More precisely: there exist a cylinder $\mathcal{C} \subseteq M$ and a family $\mathcal{K}_\infty$ of disjoint disks contained in $\mathcal{C}$ which are graphs over B^u, and a ergodic invariant probability measure μ supported on Λ such that:*

1. *the cylinder contains a ball whose radius is uniformly bounded away from zero, depending continuously on the diffeomorphism f;*
2. *there exists $\alpha > 0$ such that the union of all disks in K_∞ has μ-measure larger than α, depending on f and c_0;*
3. *μ has absolutely continuous conditional measures along the disk in $\mathcal{K}_\infty$. The densities of the conditional measures are bounded away from zero and infinity by a constant depending on f and c_0;*
4. *the $u = \dim E^{cu}$ largest Lyapunov exponents are larger than $-\log\sigma > 0$.*

3.2. Proofs of Theorems B and C. — We start by proving Theorem B. Let $f \in \mathrm{Diff}^2(M)$ and Λ be an attractor having a dominated splitting which is mostly expanding along the E^{cu} direction. Let $\mathcal{G}(f)$ be the class of Gibbs *cu*-states constructed in Subsection 3.1.

Proof of Theorem B: Let μ be an ergodic Gibbs *cu*-state for f supported on Λ. Let also $D \subseteq W^u_{loc}(x)$ be in the support of μ, such that $D \cap B(\mu)$ has full μ_D-measure in D (cf. Lemma 2.6). We may assume that D satisfies condition 2 in Subsection 3.1 taking an iterate of D if necessary ([**2**] Corollary 2.4, Proposition 2.9). Condition 1 is satisfied by ergodicity: consider the function $\varphi(x) = \log \|Df^{-1}|E^{cu}_x\|$. Birkhoff's Ergodic Theorem implies

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df^{-1}|E^{cu}_{f^j(x)}\| = \int \log \|Df^{-1}|E^{cu}_y\| \, d\mu(y) \leq -c_0 < 0$$

for μ_D-almost every point in D, where c_0 depends on the Lyapunov exponents of D in the E^{cu} direction. But $\mu_D = \rho_D m_D$ where ρ_D is a measurable function bounded away from zero and infinity, so the claim above holds Lebesgue-almost everywhere in D.

Let $\tilde{\mu}$ be a ergodic Gibbs cu-state obtained as a weak* accumulation measure of

$$\frac{1}{n} \sum_{j=0}^{n-1} f^j_* \left(\frac{\mu_D}{\mu_D(D)} \right).$$

Of course, $\tilde{\mu} \in \mathcal{G}(f)$ because μ_D is absolutely continuous with respect to Lebesgue measure on D.

Observe that for every continuous $\varphi : M \to \mathbb{R}$ we have

$$\frac{1}{n} \sum_{j=0}^{n-1} f^j_* \left(\frac{\mu_D(\varphi)}{\mu_D(D)} \right) = \frac{1}{\mu_D(D)} \int_D \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x) \, d\mu_D. \tag{9}$$

Denote by F_n the average $\frac{1}{n}\sum_{j=0}^{n-1} \varphi \circ f^j$. Each F_n is μ_D-integrable and bounded by $\|\varphi\|$. Also F_n converges pointwisely to $\int \varphi \, d\mu$ because $B(\mu)$ has full μ_D-measure on the disk D. The dominated convergence theorem implies that the right hand side of (9) converges to $\int \varphi \, d\mu$.

On the other hand the left hand side of (9) was assumed to have an accumulation measure $\tilde{\mu}$, so it converges to $\int \varphi \, d\tilde{\mu}$. As a consequence $\mu = \tilde{\mu}$.

Let (f_n) be a sequence of diffeomorphisms converging to f in the C^k-topology, $k \geq 2$. We assume that each f_n exhibits a dominated splitting with non-uniform expansion along the $E^{cu}(f_n)$ direction with constants C, α and c_0 not depending on $n \geq 0$ (cf. Subsection 1.1). Let μ_n be an ergodic Gibbs cu-state of f_n. We will assume that μ_n tends to a probability measure μ^* in the weak* topology (taking a subsequence if necessary). To prove Theorem C, we need to prove that μ^* is a cu-Gibbs state for f.

Is clear that μ^* is f-invariant. By Theorem B each μ_n is a Gibbs cu-state in $\mathcal{G}(f)$. Then, for each $n \geq 1$, there exist $(\mathcal{C}_n)_n$ and $(\mathcal{K}^n_\infty)_n$ cylinders and families of disks associated to (f_n, μ_n). From Subsection 3.1 we may assume that:

(a) the size of the disks is uniformly bounded from below;

(b) there exists $\alpha > 0$ such that, for all $n \geq 0$, we have

$$\mu_n(K_\infty^n) \geq \alpha > 0, \tag{10}$$

where K_∞^n is the union of the disks in $\mathcal{K}_\infty^n$, since we are assuming that c_0 is uniform (cf. Proposition 3.1);

(c) there exists $C_3 > 1$ such that for all $n \geq 1$ the family of conditional measures $(\mu_{n,D})_{D \in \mathcal{K}_\infty^n}$ of $\mu_n | K_\infty^n$ along the disks $D \in \mathcal{K}_\infty^n$ satisfies:

$$\frac{1}{C_3} m_D(B) \leq \mu_{n,D}(B) \leq C_3 m_D(B) \tag{11}$$

for any Borel set $B \subseteq D$.

We prove that μ^* is a Gibbs cu-state by completing the following steps.

1. We construct a cylinder $\mathcal{C}^*$ and a family $\mathcal{K}_\infty^*$ of disjoint disks contained in $\mathcal{C}^*$ which are graphs over B^u such that all the disks in $\mathcal{K}_\infty^*$ are local uniformly expanding manifolds under f.
2. The union K_∞^* of all disks in $\mathcal{K}_\infty^*$ has positive μ^*-measure.
3. The restriction of μ^* to that union has absolutely continuous conditional measures along the disks in $\mathcal{K}_\infty^*$.
4. Almost every ergodic component of μ^* is a Gibbs cu-state.

Of course, by remark 2.2, μ^* must be a Gibbs cu-state, because almost all of its ergodic components are Gibbs cu-states.

We prove these steps in the following lemmas:

Lemma 3.1. — *There exist a cylinder $\mathcal{C}^*$ and a family $\mathcal{K}_\infty^*$ of disjoint disks contained in $\mathcal{C}^*$ which are graphs over B^u such that all disks in $\mathcal{K}_\infty^*$ are local unstable manifolds.*

Proof. — Let $(\mathcal{C}_n)_n$ and $(\mathcal{K}_\infty^n)_n$ be sequences of cylinders and families of disks associated to μ_n respectively. By the compactness of M and considering a subsequence if necessary, we may suppose that $\mathcal{C}_n$ converges to $\mathcal{C}^*$.

We claim that $\mathcal{C}^*$ is a cylinder. Indeed, the $\mathcal{C}_n$ are diffeomorphic images of $B^u \times B^s$ where B^u and B^s are compact balls in $\mathbb{R}^u$ and $\mathbb{R}^s$ respectively corresponding to $\mathcal{C}_n$, $n \geq 1$. Let $(B_n^u)_n$ and $(B_n^s)_n$ be the diffeomorphic images of B^u and B^s in M, respectively. By the Arzela-Ascoli Theorem $(B_n^u)_n$ converges to a disk B_*^u and $(B_n^s)_n$ converges to a disk B_*^s. So, for $\mathcal{C}^*$ to be a cylinder, it must satisfy:

(i) the diameters of B_n^u and B_n^s do not go to zero, when n tends to infinity.
(ii) the angle between B_n^u and B_n^s does not go to zero, when n tends to infinity.

On the one hand by construction , each $\mathcal{C}_n$ contains balls with radius uniformly bounded away from zero from Theorem 3.1, so (i) is fullfilled. On the other hand, by the domination property of the family (f_n), (ii) must hold.

Now, we consider the family $\mathcal{K}^*_\infty$ of disks D^u contained in $\mathcal{C}^*$ which are accumulated by sequences $(D^u_n)_n$ of disks, $D^u_n \in \mathcal{K}^n_\infty$, $n \geq 1$. Observe that every disk $D^u_n \in \mathcal{K}^n_\infty$ is tangent to the center-unstable cone field of f_n; by continuity of the splitting with respect to the diffeomorphism, $D^u \in \mathcal{K}^*_\infty$ must be tangent to the center-unstable cone field of f. For any $x, y \in D^u$ let $(x_n)_n$ and $(y_n)_n$ be two sequences of points in D^u_n converging to x and y respectively. By Proposition 3.3, for all $k \geq 0$ fixed we have

$$\mathrm{dist}(f_n^{-k}(x_n), f_n^{-k}(y_n)) \leq \sigma^{-k/2}\mathrm{dist}(x_n, y_n).$$

Passing to the limit when $n \to \infty$, we obtain

$$\mathrm{dist}(f^{-k}(x), f^{-k}(y)) \leq \sigma^{-k/2}\mathrm{dist}(x, y),$$

for all $x, y \in D^u$ and all $k \geq 0$. We conclude that every f^k is an $\sigma^{k/2}$-contraction on $D(x)$, and $D(x)$ is tangent to the center-unstable subbundle at every point in $\Lambda \cap D(x)$ (including x).

In particular we have shown that the subspace E^{cu}_x is indeed uniformly expanding for Df. The domination property means that any expansion Df may exhibit along the complementary direction is weaker than this. Then, there exists a unique strong-unstable manifold $W^u_{loc}(x)$ tangent to E^{cu} which is contracted by negative iterates of f by a rate of at least $\sigma^{k/2}$, when k gets large, see [**15**]. Moreover $D(x)$ is contained in $W^u(x)$ because it is contracted by every f^{-k}, $k \geq 1$, and all its negative iterates are tangent to the center-unstable cone field.

□

Lemma 3.2. — *The union K^*_∞ of all disks in $\mathcal{K}^*_\infty$ has positive μ^*-measure.*

Proof. — Recall from Subsection 3.1 that there exists $\alpha > 0$ such that, for all $n \geq 0$, we have $\mu_n(K^n_\infty) \geq \alpha > 0$. Let us fix $\delta > 0$. Then there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $K^n_\infty \subseteq B(K^*_\infty, \delta)$. On the other hand, we know that $K^*_\infty = \bigcap_{\delta > 0} B(K^*_\infty, \delta)$. Choosing $\delta > 0$ such that $\mu^*(\partial B(K^*_\infty, \delta)) = 0$, we have

$$\mu^*(B(K^*_\infty, \delta)) = \lim_{n\to\infty} \mu_n(B(K^*_\infty, \delta)) \geq \alpha > 0,$$

and so,

$$\mu^*(K^*_\infty) = \liminf_{\delta\to 0} \mu^*(B(K^*_\infty, \delta)) \geq \alpha > 0.$$

□

Lemma 3.3. — *There exist a constant $C_1 > 0$ and a family of conditional measures $(\mu_D^*)_D$ of $\mu^*|K_\infty^*$ along the disks $D \in \mathcal{K}_\infty^*$ such that μ_D^* is absolutely continuous with respect to Lebesgue measure m_D on D, with*

$$\frac{1}{C_3} m_D(B) \le \mu_D^*(B) \le C_3 m_D(B)$$

for every Borel set $B \subseteq D$.

Proof. — We start by taking the cylinder $\mathcal{C}^*$ from the construction proving Lemma 3.1. By the compactness of $\mathcal{C}^*$, for any $\xi \in D$ where D is any disk in $\mathcal{K}_\infty^*$, the exponential map is well defined in a ball of radius $\tilde{r} > 0$ around ξ. For any Borel set $B \subset D$ we define the set $\tilde{B}$ as the tubular neighborhood of B, that is, the union of the images under the exponential map at each point $\xi \in B$ of all vectors orthogonal to D at ξ.

We fix a sequence of partitions $\mathcal{P}_k$ of $\mathcal{K}_\infty^*$ constructed as follows. Let V be the image of $\{0\} \times B_*^s$ under the diffeomorphism between $B_*^u \times B_*^s$ and $\mathcal{C}^*$. Fix a sequence $\mathcal{V}_k$, $k \ge 1$, of increasing partitions of V with positive diameter less than $\tilde{r}$ and going to zero. Then, we say that two points x, $y \in \hat{K}$ are in the same atom of the partition $\mathcal{P}_k$ if the disk D_1 containing x and the disk D_2 containing y intersect the same element of $\mathcal{V}_k$. It is clear from the construction that for any point $\xi \in K_\infty^*$,

$$\mathcal{P}_1(\xi) \supset .. \supset \mathcal{P}_k(\xi) \supset \ldots$$

and $\cap_{k=1}^\infty \mathcal{P}_k(\xi)$ coincides with the disk in D that contains ξ.

For any Borelean set $B \subseteq D$ we have, from Proposition 3.4,

$$\frac{1}{C_3} m(B_n)\mu_n(\mathcal{P}_k(\xi)) \le \mu_n(\tilde{B} \cap \mathcal{P}_k(\xi)) \le C_3 m(B_n)\mu_n(\mathcal{P}_k(\xi))$$

where C_3 does not depend on n, $B_n = \tilde{B} \cap D_n$ and D_n is a disk in $\mathcal{K}_\infty^n$ near D, $n \ge 1$. By construction $m(B_n)$ converges to $m(B)$ and so passing to the limit when $n \to \infty$ we have

$$\frac{1}{C_3} m(B)\mu^*(\mathcal{P}_k(\xi)) \le \mu^*(\tilde{B} \cap \mathcal{P}_k(\xi)) \le C_3 m(B)\mu^*(\mathcal{P}_k(\xi)).$$

Now, by the Radon-Nikodym Theorem, we have that the disintegration of μ^* along the disk $\cap_{k=1}^\infty \mathcal{P}_k(\xi)$ is absolutely continuous with respect to Lebesgue measure in this disk, and the densities are almost everywhere bounded from above by C_3 and from below by $1/C_3$.

□

Remark 6: The densities of μ_D^* are uniformly (with respect to D) bounded away from zero and infinity.

Fix B, a measurable subset of M, such that

$$m_\gamma(B \cap \gamma) = 0 \quad \text{for every} \quad \gamma \in \mathcal{K}^*_\infty,$$

and $\mu^*(B)$ is maximal among all measurable sets with this property. Observe that $\mu^*(B) = 0$, because μ^* is absolutely continuous along the leaves on $\mathcal{K}^*_\infty$ (cf. Lemma 3.3). Let $Z_\infty = K^*_\infty \cap \Sigma(f) \cap R(f) \setminus B$. Then $\mu^*(Z_\infty) > 0$ and let $(\mu^*|Z_\infty)$ be the restriction of μ^* to Z_∞.

Let A be any measurable subset of Z_∞ such that $m_\gamma(A \cap \gamma) = 0$ for every $\gamma \in \mathcal{K}^*_\infty$. Then $\mu^*(A)$ must be zero, since we took $\mu^*(B)$ maximal. This means that $(\mu^*|Z_\infty)$ is absolutely continuous with respect to the product $m_\gamma \times \hat{\mu}^*$, where $\hat{\mu}^*$ stands for the quotient measure induced by $(\mu^*|Z_\infty)$ on $\mathcal{K}^*_\infty$. As a consequence, the conditional measures $\tilde{\mu}^*_\gamma$ of $(\mu^*|Z_\infty)$ on the disks $\gamma \in \mathcal{K}^*_\infty$ are absolutely continuous with respect to Lebesgue measure m_γ for $\hat{\mu}^*$-almost all $\gamma \in \mathcal{K}^*_\infty$. On the other hand, by the Ergodic Decomposition Theorem (cf. [**12**]), for any measurable set $A \subseteq Z_\infty$,

$$\mu^*(A) = \int \mu^*_x(A)\, d\mu^*(x),$$

where the integral is taken over $\Sigma(f) \subseteq M$.

Let us denote by $\mathbf{1}_A$ the characteristic function of the measurable subset A. Then we have (as in Section 2.2)

$$\mu^*_x(A) = \int \mathbf{1}_A \, d\mu^*_x = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbf{1}_A(f^j(x))$$

μ^*-almost everywhere. So $\mu^*_x(A) > 0$ only if x has some iterate in $A \subseteq Z_\infty$, for μ^*-almost every point x. Let $k(z)$ denote the first backward return time to Z_∞ of a point $z \in Z_\infty$: $k(z)$ is the smallest positive integer such that $f^{-k(z)}(z) \in Z_\infty$. This is defined μ^*-almost everywhere, by Poincare's recurrence theorem. As in Section 2.2, we have

$$\mu^*(A) = \int \mu^*_x(A)\, d\mu^*(x) = \int_{Z_\infty} k(z)\mu^*_z(A)\, d\mu^*(z)$$

for any measurable subset A of Z_∞.

Lemma 3.4. — *The measure μ^*_x is a Gibbs cu-state, for μ^*-almost every point $x \in K^*_\infty$.*

Proof. — Let $x \in K^*_\infty \cap \Sigma \cap R(f)$. Observe that $K^*_\infty \cap \Sigma \cap R(f)$ has full μ^*-measure on K^*_∞ and that

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df^{-n}|E^{cu}_{f^n(x)}\| \le -\frac{1}{2}\log\sigma < 0$$

by the uniform contraction of E^{cu}_x for Df^{-1} (see the proof of Lemma 3.1). This implies that the u largest Lyapunov exponents are positive.

Now, if we take $Z = Z_\infty$, $\lambda = (\mu^*|Z_\infty)$, $\mathcal{K} = \mathcal{K}^*_\infty$, $\lambda_z = (\mu^*_z|Z_\infty)$ and $l(z) = k(z)$ for each $z \in Z_\infty$ in Lemma 2.3, the conditional probability measures $\tilde{\mu}^*_{z,D}$ of $(\mu^*_z|Z_\infty)$ along the disks $D \in \mathcal{K}^*_\infty$ coincide almost everywhere with the corresponding conditional measures $\tilde{\mu}^*_D$ of $(\mu^*|Z_\infty)$. Recall that we had already shown that the latter are almost everywhere absolutely continuous with respect to Lebesgue measure on the corresponding disks $D \in \mathcal{K}^*_\infty$.

□

Now, let K_0 be the set $\cup_{j=0}^{\infty} f^{-j}(K^*_\infty)$ and define G to be the set of all $x \in \Sigma(f) \cap R(f) \cap K_0$ such that μ^*_x is a Gibbs cu-state and set

$$\nu^* = \int_G \mu^*_x \, d\mu^*(x).$$

Since μ^*_x is ergodic for μ^*-almost every x and for all $j \in \mathbb{N}$, then $\mu^*_x(G) = 1$ if, and only if, $x \in G$, so supp$\nu^* = \overline{G}$ ([**25**], Theorem 1.5). Moreover μ^*_x is a Gibbs cu-state for all $x \in G$, and so is $\frac{\nu^*}{\nu^*(M)}$, see remark 2.2 (Notice that since $K^*_\infty \subseteq G$ we have $\nu^*(G) \geq \mu^*(K^*_\infty) \geq \alpha > 0$).

For $n \in \mathbb{N}$, each μ_n is ergodic and $\mu_n(K^n_\infty) \geq \alpha > 0$, so ([**25**], Theorem 1.5)

$$\mu_n\Big(\bigcup_{j=0}^{\infty} f_n^{-j}(K^n_\infty)\Big) = 1. \tag{12}$$

Set $A^j_n = \{x \in K^n_\infty \ : \text{j is the first return time of } x\}$. These sets are pairwise disjoints and $K^n_\infty = \cup_{j \in \mathbb{N}} A^j_n$ up to a zero μ_n-measure subset. So,

$$\mu_n(K^n_\infty) = \sum_{j=1}^{\infty} \mu_n(A^j_n).$$

Lemma 3.5. — *The serie* $\sum_{j=1}^{\infty} \mu_n(A^j_n)$ *converge uniformly respect to* $n \in \mathbb{N}$.

Proof. — Let us fix $j \in \mathbb{N}$. By invariance of μ_n and disintegration, we have

$$\mu_n(A^j_n) = \mu_n(f^j_n(A^j_n)) = \int_{K^n_\infty} \int_D \mathbf{1}_{f^j_n(A^j_n)}(x) \rho^n_D(x) \, dm_D(x) \, d\hat{\mu}_n(D).$$

From Proposition 3.4, there is a uniform (respect to n) constant $C_3 \geq 0$ such that, for $\hat{\mu}_n$-almost every $D \in K^n_\infty$, we have

$$\int_D \mathbf{1}_{f^j_n(A^j_n)}(x) \rho^n_D(x) \, dm_D(x) = \int_{f_n^{-j}(D)} \mathbf{1}_{A^j_n}(x) \rho^n_{f_n^{-j}(D)}(x) \, dm_{f_n^{-j}(D)}(x) \leq C_3 m_{f_n^{-j}(D)}(A^j_n),$$

where $\rho^n_{f_n^{-j}(D)}$ is the density of μ_n in the disk $f_n^{-j}(D)$. But D is an unstable disk for f_n which smallest Lyapunov exponent is bounded for below by $\lambda = -\log\sigma > 0$ (see Theorem 3.1 part 4). This together with [PT2] implies that there exists a constant C_4 such that

$$m_{f_n^{-j}(D)}(A_n^j) \leq m_{f_n^{-j}(D)}(f_n^{-j}(D)) \leq C_4 e^{-\lambda j}$$

for all $n \geq 0$. Hence, we conclude that

$$\int_D \mathbf{1}_{f_n^j(A_n^j)}(x)\rho_D^n(x)\, dm_D(x) \leq C_3 C_4 e^{-\lambda j}$$

for every $n \geq 0$ and so we may conclude that the series

$$\sum_{j=1}^{\infty} \mu_n(A_n^j) \leq \sum_{j=1}^{\infty} C_3 C_4 e^{-\lambda j}$$

converge uniformly respect to $n \in \mathbb{N}$. $\square$

Remark 7: If we define $A^j = \{x \in K_\infty^* \ : \text{j is the first return time of } x\}$ as above, the serie $\sum_{j=1}^{\infty} \mu^*(A^j)$ converge. Moreover, from Lemma 3.3, Lemma 3.4 and [PT2], we have

$$\sum_{j=1}^{\infty} \mu^*(A^j) \leq \sum_{j=1}^{\infty} C_3 C_4 e^{-\lambda j}.$$

Let $k(x,n)$ the smallest positive integer such that $f_n^{k(x,n)}(x) \in K_\infty^n$. From (12), this number is defined for μ_n-almost every point x. For $N \in \mathbb{N}$, let us denote by K_n^N the set

$$K_n^N = \{x \in M \ : \ k(x,n) = N\}.$$

For all $N \in \mathbb{N}$, we define $\mu_n^N = (\mu_n | K_n^N)$. Follows from Lemma 3.5 that, for any $n \geq 0$ fixed, μ_n^N converge in the weak* topology to μ_n as N goes to infinity, uniformly from n. In particular that means, for each $n \in \mathbb{N}$, for every $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$, not depending from n, such that

$$\left| \int \varphi \, d\mu_n^N - \int \varphi \, d\mu_n \right| < \varepsilon,$$

for all $N \geq N(\varepsilon)$, for every $\varphi : M \to \mathbb{R}$ continuous.
In the same way, let $k(x)$ be the smallest positive integer such that $f^{k(x)}(x) \in K_\infty^n$. For $N \in \mathbb{N}$, let us denote by K_0^N the set

$$K_0^N = \{x \in M \ : \ k(x,0) = N\}.$$

For all $N \in \mathbb{N}$, we define $\nu_N^* = (\nu^* | K_0^N)$ and clearly we have $\nu_N^* = (\mu^* | K_0^N)$. As above, ν_N^* converge in the weak* topology to ν^* as N goes to infinity, that means, for every $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$, such that

$$\left| \int \varphi \, d\nu_N^* - \int \varphi \, d\nu^* \right| < \varepsilon,$$

for all $N \geq N(\varepsilon)$, for every $\varphi : M \to \mathbb{R}$ continuous.

On the other hand, observing that $K_n^N \subseteq f_n^{-N}(K_\infty^n)$ and $K^N \subseteq f^{-N}(K_\infty^*)$, it is clear that, for any $N \in \mathbb{N}$ fixed, μ_n^N converges to ν_N^* when n goes to infinity. That means, for every $N \in \mathbb{N}$ fixed, for every $\varepsilon > 0$, there exists $n(\varepsilon, N) \in \mathbb{N}$, such that

$$\left| \int \varphi \, d\nu_N^* - \int \varphi \, d\mu_n^N \right| < \varepsilon,$$

for all $n \geq n(\varepsilon, N)$, for every $\varphi : M \to \mathbb{R}$ continuous.

Lemma 3.6. — $\mu^* = \nu^*$

Proof. — Let $\varphi : M \to \mathbb{R}$ continuous and let $\varepsilon > 0$. Fix $N \in \mathbb{N}$ such that

$$|\int \varphi d\nu^* - \int \varphi d\nu_N^*| < \varepsilon \text{ and } |\int \varphi d\mu_n - \int \varphi d\mu_n^N| < \varepsilon,$$

for every $n \geq 1$. Fixed N, let $n \in \mathbb{N}$ such that

$$|\int \varphi d\mu_n - \int \varphi d\mu^*| < \varepsilon \text{ and } |\int \varphi d\nu_N^* - \int \varphi d\mu_n^N| < \varepsilon,$$

and finally,

$$\begin{aligned} |\int \varphi d\nu^* - \int \varphi d\mu^*| &\leq |\int \varphi d\nu^* - \int \varphi d\nu_N^*| + |\int \varphi d\nu_N^* - \int \varphi d\mu_n^N| \\ &\quad + |\int \varphi d\mu_n^N - \int \varphi d\mu_n| + |\int \varphi d\mu_n - \int \varphi d\mu^*| \\ &< 4\varepsilon. \end{aligned}$$

□

We conclude that $\mu^* = \nu^*$ and so μ^* is a Gibbs cu-state, concluding the proof of Theorem C.

3.2.1. Convergence of the densities. — Let D_n and D be disks in K_∞^n and K_∞^* respectively such that D is accumulated by D_n as n goes to infinity. Let ρ_n and ρ be the densities of μ_n and μ^* defined in D_n and D respectively. We may assume, for n large enough that $D_n = D$.

From Proposition 2.1 we know that ρ_n and ρ are Hölder continuous functions with the same Hölder constants and uniformly bounded away from zero and infinity.

Proposition 3.5. — *ρ_n converge to ρ uniformly.*

Proof. — Since ρ_n and ρ are uniformly bounded away from zero and infinity, then $\log \rho_n$ and $\log \rho$ are uniformly bounded.

Denoted by $J_{k,n}^u(x) = |\det Df_n^{-1}|E^{cu}_{f_n^k(x)}|$ and $J_k^u(x) = |\det Df^{-1}|E^{cu}_{f^k(x)}|$, $k \geq 0$. From Proposition 2.1 we have

$$\frac{\rho_n(x)}{\rho_n(y)} = \prod_{k=0}^{\infty} \frac{J_{k,n}^u(x)}{J_{k,n}^u(y)} \text{ and } \frac{\rho(x)}{\rho(y)} = \prod_{k=0}^{\infty} \frac{J_k^u(x)}{J_k^u(x)},$$

for all $x, y \in D$, and for $n \geq 1$. Then, repeating the proof of Lemma 2.5 we have, for every $n \geq 1$,

$$\begin{aligned} |\log \rho_n(x) - \log \rho_n(y)| &\leq \sum_{k=0}^{\infty} |\log J_{k,n}^u(x) - \log J_{k,n}^u(y)| \\ &\leq \sum_{k=0}^{\infty} C_1 \text{dist}_{f_n^k(D)}(f_n^k(x), f_n^k(y))^\xi \\ &\leq \sum_{k=0}^{\infty} C_1 C^\xi e^{-k\lambda\xi} \text{dist}_D(x,y)^\xi. \end{aligned}$$

The same inequality hold for $|\log \rho(x) - \log \rho(y)|$. In consequence, the set $\{\rho_n, \rho\}$ is equicontinuous. Arzela-Ascoli theorem implies the result.

□

4. Gibbs cu-states and SRB measures

In this section we study the relationship between Gibbs cu-states and SRB measures and conclude with some applications of our result to the study of statistical stability for partially hyperbolic systems. First, our goal is to prove Corollary D.

We assume that f is a C^2-diffeomorphism with a topological attractor Λ with a dominated splitting which is non-uniformly expanding along the center-unstable direction. Consider a disk D transverse to the center-stable direction. First, we prove that the constructions of the previous section can be done replacing the disk D by a positive Lebesgue measure subset E.

Proposition 4.1. — *Given a center-unstable domain D and any positive Lebesgue measure set $E \subseteq D$, every weak* accumulation point of*

$$\mu_{n,E} = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j \left(\frac{m_E}{m_D(E)}\right)$$

has an ergodic component which is a Gibbs cu-state of f.

Proof. — Given any $\delta > 0$ we may find pairwise disjoint domains $D_1, ..., D_s$ in D such that the relative Lebesgue measure on E inside each D_i is larger than $1-\delta$, and the total measure of E outside the union of the D_i is less than $\delta m_D(E)$. Then, for any $j \geq 1$, we have

$$\begin{aligned} f_*^j \left(\frac{m_E}{m_D(E)}\right) &= \sum_{i=1}^{s} \frac{m_D(D_i)}{m_D(E)} f_*^j \left(\frac{m_{D_i}}{m_D(D_i)}\right) \\ &\quad + \frac{1}{m_D(E)} f_*^j m_{(E \setminus \cup_{i=1}^s D_i)} - \frac{1}{m_D(E)} \sum_{i=1}^{s} f_*^j m_{D_i \setminus E}. \end{aligned}$$

The total masses of both the second and the third term do not depend on j, and are less than δ. Therefore, every accumulation point of $\mu_{n,E}$ differs from an accumulation point of

$$\sum_{i=1}^{s} \frac{m_D(D_i)}{m_D(E)} \frac{1}{n} \sum_{j=0}^{n-1} f_*^j \left(\frac{m_{D_i}}{m_D(D_i)}\right)$$

by a measure whose total mass is less than δ. Applying Theorem 1.2 to each domain D_i, every point of accumulation of this last sequence has an ergodic component which is a Gibbs *cu*-state whose densities are uniformly bounded and satisfy the ratio relation of Proposition 2.1. Making δ go to zero and applying Theorem A, Theorem B and Theorem C we get that every weak* accumulation point of $\mu_{n,E}$ has an ergodic component which is a Gibbs *cu*-state. □

Proof of Corollary D: Let μ be an ergodic SRB measure for f supported in Λ. Consider any disk D inside U, where U is a neighborhood of Λ as in Subsection 1.1. Let us suppose the D is transverse to the center-stable subbundle and intersecting the basin of μ on a positive Lebesgue measure subset D_0. On one hand,

$$\frac{1}{n}\sum_{j=0}^{n-1} f_*^j \left(\frac{m_{D_0}}{m_D(D_0)}\right) = \frac{1}{m_D(D_0)} \int_{D_0} \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)}\, dm_D(x)$$

converges to μ when $n \to \infty$. On the other hand, from Proposition 4.1 and the hypothesis of ergodicity it follows that μ must be a Gibbs *cu*-state.

4.1. Statistical Stability. — Now we present several applications of our results to the study of statistical stability for systems with a weak form of hyperbolicity.

4.1.1. The partially hyperbolic case. — Now we assume that f has a partially hyperbolic attractor Λ with splitting $T_\Lambda M = E^{cu} \oplus E^s$. The measure μ constructed in Section 3 has an ergodic component μ_*, with support contained in Λ, which is a Gibbs cu-state. Then there exists a disk $D_\infty \in \mathcal{K}_\infty$ such that $m_{D_\infty}(B(\mu_*)) > 0$ ([**2**], lemma 4.5). Because the strong-stable foliation is absolutely continuous [**8**], $m(B(\mu_*))$ must be positive, so μ_* is an SRB measure. Moreover, a full Lebesgue measure subset of U is contained in the union of finitely many SRB measures supported in Λ ([**2**], corollary 4.6)).

Proof of Corollary E: Let $(\mu_n)_n$ be a sequence of ergodic SRB probability measures for f_n, converging to μ in the weak* topology. By Corollary D each μ_n is a Gibbs cu-state for f_n and, by Theorem C, μ is also a Gibbs cu-state for f. By Theorem A, there exist a cylinder $\mathcal{C}^*$ and a family $\mathcal{K}^*_\infty$ of disjoint disks contained in $\mathcal{C}^*$ which are graphs over B^u and local uniformly expanding manifolds. Moreover, $\mu(\mathcal{K}^*_\infty) \geq \alpha > 0$ and μ is absolutely continuous with respect to Lebesgue measure along these disks.

Now if we take a tubular neighborhood of $D_\infty \in \mathcal{K}_\infty$ given by Lemma 2.6 using the stable foliation, then since this foliation is Hölder continuous we have a positive Lebesgue measure set of points in $B(\mu)$, so μ is an SRB measure. By Theorem 1.1 there are finitely many ergodic SRB measures and the union of their basins covers a full Lebesgue measure subset of U, so μ must be in the convex hull of such measures.

4.1.2. The dominated splitting case. — In the setting where f has an attractor with dominated splitting $E^{cs} \oplus E^{cu}$ and with $s = \dim E^{cs}$ Lyapunov exponent of μ all negatives, μ is in fact a SRB measure. This is a consequence of the absolute continuity property of μ and the absolute continuity of the stable lamination [**15**]: the union of the stable manifolds through the point whose time averages are given by μ is a positive Lebesgue measure subset of M.

Proof of Theorem F: First we prove the existence of SRB measures. Let μ be an ergodic Gibbs cu-state for f. It exists by Theorem 1.2. Let D_∞ be a disk such that $D_\infty \cap B(\mu) \cap R(f)$ has full Lebesgue measure on D_∞. Such a disk exists by Lemma 2.6, and is contained in some local unstable manifold. By hypothesis, a positive Lebesgue measure subset of D_∞ satisfies (3). So the set A of points in $D_\infty \cap B(\mu) \cap R(f)$ satisfying (3) has positive Lebesgue measure on D_∞.

For $\varepsilon > 0$, we denote by $D_\infty(\varepsilon)$ the tubular neighborhood of radius ε around D_∞, defined as the image under the exponential map of M of all the vectors of norm less that $\varepsilon > 0$ in the orthogonal complement of E^{cu}_x, for all $x \in D_\infty$. If $\varepsilon > 0$ is small enough then $D_\infty(\varepsilon)$ is a cylinder.

For every point in $x \in A$, there exists a C^1 embedded disk $W^s_{loc}(x)$ tangent to E^{cs}_x at x such that the diameter of $f^n(W^s_{loc}(x))$ converges exponentially fast to zero as $n \to \infty$. These disks $W^s_{loc}(x)$ depend in a measurable way on the point x, and the lamination $\{W^s_{loc}(x) : x \in A\}$ is absolutely continuous. Since $A \subseteq B(\mu)$ every $y \in W^s_{loc}(x)$ is in $B(\mu)$ also.

The domination condition on the splitting together the absolute continuity of the stable lamination implies that every disk D tangent to the E^{cu} direction crossing $D_\infty(\varepsilon)$, and close enough to D_∞, intersects the lamination $\{W^s_{loc}(x) : x \in A\}$ in a positive Lebesgue measure subset. Finally, Fubini's Theorem implies that the Lebesgue measure of $B(\mu)$ is positive.

Now we prove that there are finitely many ergodic SRB measures. Suppose otherwise. Let (μ_n) be a sequence of ergodic SRB measures of f converging in the weak* topology to a measure μ. By Corollary D, each μ_n must be a Gibbs cu-state for f. Theorem C implies that μ is a Gibbs cu-state. By the argument used above, μ must be an SRB measure also.

Observe from Theorem B that $\mu_n \in \mathcal{G}(f)$ for each n, so there is a sequence $\mathcal{C}_n$ of hyperbolic blocks associated to μ_n converging to $\mathcal{C}$, a hyperbolic block associated to μ. Moreover, the size of the disks crossing the cylinder is uniformly bounded from below.

Let D_∞ crossing $\mathcal{C}$, $D_\infty(\varepsilon)$ and A be the sets defined above for μ. Let D_n be the corresponding disk defined for μ_n in the block $\mathcal{C}_n$, given by Lemma 2.2. For $n \geq 1$ large enough, the disk D_n crosses $D_\infty(\varepsilon)$. The argument above implies that D_n intersects the lamination $\{W^s_{loc}(x) : x \in A\}$ in a subset with positive Lebesgue measure on D_n. Each D_n is contained in some local unstable manifold, so if there exists some point in the basin of μ then every point in these manifolds is in the basin too. But there exists a positive Lebesgue measure subset of D_n contained in the basin of μ_n, so $B(\mu) = B(\mu_n)$ for all $n > 1$ large enough, and then $\mu = \mu_n$.

Let $\mu_1, \dots \mu_n$ be the finitely many SRB measures for f supported in Λ. Now we prove that $m(B(\Lambda) \setminus \cup_{i=1}^n B(\mu_i)) = 0$. Suppose that $m(U \setminus \cup_{i=1}^n B(\mu_i)) > 0$. Then there exists a C^2-disk D tangent to the center-unstable cone field such that conditions 1 and 2 of Section 3.1 hold and $m_D(D \cap \cup_{i=1}^n B(\mu_i)) = 0$. Let $\mu = \mu_i$ be a Gibbs cu-state constructed from the iterates of Lebesgue measure on D as in Section 3.1.

From this construction, given $k \geq 1$ large enough, the Lebesgue measure of $f^n(D) \cap B(\mu)$ on $f^n(D)$ is bounded from below away from zero, thus the Lebesgue measure of $D \cap B(\mu)$ on D is also bounded from below and away from zero. This is a contradiction.

In order to prove statistical stability, consider $(f_n)_n$ a sequence of C^2-diffeomorphisms converging to f in the C^k-topology, $k \geq 2$. Assume that $(\mu_n)_n$ is a sequence of ergodic SRB measures for $(f_n)_{n\geq 1}$ and that μ is a weak* accumulation measure of this sequence. By Corollary D, each μ_n must be a Gibbs cu-state for f_n. Theorem C implies that μ is a Gibbs cu-state. By Theorem A every ergodic component of μ is a Gibbs cu-state. Applying to each ergodic component of μ the argument used above, every ergodic component of μ must also be an SRB measure, and so it must be in the convex hull of finitely many ergodic SRB measures.

Example : Bonatti and Viana [**4**] constructed an open class of C^k-diffeomorphisms $\mathcal{N}$, $k \geq 2$, defined on $\mathbb{T}^n$, $n \geq 4$ such that every $f \in \mathcal{N}$ satisfies:

(a) f has a dominated splitting but is not partially hyperbolic,

(b) f is non-uniformly expanding in the center-unstable direction.

They also proved there exist SRB measures for such f. After this, Tahzibi [**24**] proved that

(c) the SRB measure is unique.

In this case, the SRB measure corresponds to a unique Gibbs *cu*-state and by Theorem C this SRB measure is C^k-statistically stable, $k \geq 2$.

Remark 8: However, it is not known whether there are general conditions ensuring the uniqueness of the SRB measure for partial hyperbolic diffeomorphisms or for diffeomorphisms with dominated splitting.

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