Maximal entropy measures for Viana maps

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Abstract

In this note we construct measures of maximal entropy for a certain class of maps with critical points called Viana maps. The main ingredients of the proof are the non-uniform expansion features and the slow recurrence (to the critical set) of generic points with respect to the natural candidates for attaining the topological entropy.

1 Introduction

Nowadays, it is well-known that a dynamical system can be understood from the study of its invariant measures (this study is called *ergodic theory*). However, there are in general many invariant measures for a given system, and it is necessary to make a selection of some "interesting" measures. Motivated by statistical physics, some candidates called *equilibrium states*, are the measures which satisfy a variational principle.

The theory of equilibrium states in the case of uniformly hyperbolic systems (developed by Bowen, Ruelle, Sinai among others) is now a classical theory with completely satisfactory results. But, in the non-uniform (higher dimensional) context only few results are known (see, for instance, Oliveira [O] for a recent progress in this direction; for a random version of it, see Arbieto, Matheus, Oliveira [AMO]).

The central theme of this note is to prove the existence of maximal entropy measures for a class of non-uniformly expanding maps with critical set introduced by Viana [V]. Some features of these maps (which we would like to call *Viana maps*) include the existence of a unique absolutely continuous invariant measure μ_0 with non-uniform expansion, slow recurrence to the critical set and positive Lyapounov exponents (see [V]). Moreover, μ_0 is stochastically stable[1].

On the other hand, an obstruction to construct general equilibrium states for Viana maps (among other non-uniformly expanding maps) is the *a priori* dependence on the Lebesgue measure as reference in the concepts of non-uniform expansion and slow recurrence to the critical set. To overcome this difficult we apply the basic strategy of Oliveira [O]:

- We select a certain set $\mathcal{K}$ of invariant measures which are natural candidates to realize the variational principle (for nearly constant potentials);
- We prove that any measure outside $\mathcal{K}$ can not be an equilibrium measure and any measure inside $\mathcal{K}$ is *expanding* (see the definition 3.9 below);
- We show that any expanding measure admits generating partitions. This leads us to the semicontinuity of the entropy, and hence, by a standard argument, to the existence of equilibrium states for nearly constant potentials.

However, it is worth pointing out an extra difficulty of the case of Viana maps: the presence of critical points is an obstacle to obtaining infinitely many times of uniform expansion directly from the non-uniform expansion condition (indeed, some slow recurrence to the critical set should be proven).[2] Since the abundance of hyperbolic times is fundamental to prove that expanding measures admit generating partitions (see lemma 3.11), we need to solve this problem.

Fortunately, the slow recurrence to the critical set can be obtained from the integrability of the distance function to the critical set $\mathcal{C}$. Because the Viana maps *behave like a power of the distance* to $\mathcal{C}$, it is reasonable that this integrability problem is related to the regularity of the Lyapounov exponents

[1] Here, the stochastic stability is restricted to the random perturbations with the same critical set $\mathcal{C}$ and whose derivative at any point $x \notin \mathcal{C}$ (i.e., in the complement of the critical set) is equal to the derivative of the unperturbed system at the same point x (see Alves, Araújo [AA])

[2] In the context of Oliveira [O], this problem does not occur since the transformations are local diffeomorphisms, and so infintely many hyperbolic times can be obtained directly from non-uniform expansion.

(i.e., the Lyapounov exponents are bounded away from $-\infty$). Now, the regularity of the Lyapounov exponents of measures with a non-trivial chance to attain the supremum of the variational principle (i.e., whose entropy is not $-\infty$) is an easy consequence of Ruelle's inequality. Thus, the presence of critical points is not a great trouble and we still can use the basic strategy of Oliveira [O].

Now we are going to state our main result. In order to do this, let us recall some definitions and notations.

In general, given a continuous map $f : M \to M$ of a compact metric space M and a continuous function $\phi : M \to \mathbb{R}$, we call an f-invariant Borel probability measure μ an *equilibrium state* of (f, ϕ) if μ realizes the variational principle

$$h_\mu(f) + \int \phi d\mu = \sup_{\eta \in \mathcal{I}} \left(h_\eta(f) + \int \phi d\eta \right),$$

where $\mathcal{I}$ is the set of f-invariant Borel probabilities.

Now we are in position to state our main result:

Theorem A. *Viana maps admit equilibrium states for any nearly constant potential ϕ (in the sense of the condition (3) below). In particular, Viana maps have maximal entropy measure (since they are equilibrium states for the constant potential $\phi = 0$). Moreover, any equilibrium state of the Viana maps (associated to nearly constant potentials) are hyperbolic measures with all Lyapounov exponents greater than some $\lambda_0 > 0$.*

To close the introduction, let us comment about the organization of the paper: section 2 contains the definition of Viana maps, some of its properties and a precise statement of what does "ϕ is nearly constant" means. In section 3 we present the proof of the theorem A. Finally, in section 4, we point out some generalizations and questions concernig the theorem A.

2 Viana maps

Let $a_0 \in (1, 2)$ such that $x = 0$ is pre-periodic for the quadratic map $h(x) = a_0 - x^2$. Denote by $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and $b : \mathbb{S}^1 \to \mathbb{R}$ a Morse function, e.g., $b(\theta) = \sin(2\pi\theta)$. Fix some $\alpha > 0$ sufficiently small and put

$$\widetilde{f}(\theta, x) := (g(\theta), a(\theta) - x^2) := (g(\theta), Q(\theta, x)),$$

where g is the expanding map $g(\theta) = d\theta$ of $\mathbb{S}^1$, for some integer $d \geq 16$ and $a(\theta) = a_0 + \alpha b(\theta)$. [3] Since $a_0 < 2$, it is easy to check that there is $I \subset (-2, 2)$ such that the closure of $\widetilde{f}(\mathbb{S}^1 \times I)$ is contaned in the interior of $\mathbb{S}^1 \times I$. Indeed, $h(x) = a_0 - x^2$ has a unique fixed point $x_0 = \frac{1}{2}(-1 - \sqrt{1 + 4a_0}) < 0$ which is a repeller. In particular, if we take $\beta > 0$ slightly smaller than $-x_0$, the interval $I = [-\beta, \beta]$ satisfies $h(I) \subset \text{int}(I)$ and $|h'| > 1$ on $\mathbb{R} \setminus \text{int}(I)$. Hence, if α is sufficienlty small we still have $\widetilde{f}(\mathbb{S}^1 \times I) \subset \text{int}(\mathbb{S}^1 \times I)$. Notice also that, if f is C^0-close to $\widetilde{f}$ then f also has $\mathbb{S}^1 \times I$ as a forward invariant region, i.e., any f close to $\widetilde{f}$ may be considered as a map $f : \mathbb{S}^1 \times I \to \mathbb{S}^1 \times I$.

In what follows, we consider the parameters $d \geq 16$ and $a_0 \in (1, 2)$ fixed and α is sufficiently small depending on d and a_0. Under these conditions, Viana proved that

Theorem 2.1 (Viana [V]). *For α is sufficiently small, there exists a positive constant $c_0 > 0$ such that any map f sufficiently close to $\widetilde{f}$ in the C^3 topology has both Lyapounov exponents greater than c_0 at Lebesgue almost every point, that is,*

$$\liminf_{n \to \infty} \frac{1}{n} \log \|Df^n(\theta, x)\| > c_0, \qquad \textit{for Lebesgue a.e. } (\theta, x) \in \mathbb{S}^1 \times I.$$

A very common fact in smooth ergodic theory is: "the positivity of Lyapounov exponents acts as a strong evidence of the existence of absolutely continuous invariant measures". Indeed, this general principle works in several examples of maps with positive Lyapounov exponents, including Viana maps:

Theorem 2.2 (Alves [A]). *Any map f sufficiently C^3 close to $\widetilde{f}$ admits an unique absolutely continuous invariant measure μ_0.*

Next, let us study the tangent bundle dynamics of the Viana maps. Denote by f_0 the product map $f_0 = g \times h$, i.e., $f_0 : \mathbb{S}^1 \times I \to \mathbb{S}^1 \times I$, $f_0(\theta, x) = (d\theta, a_0 - x^2)$. Since $Df_0 \cdot \frac{\partial}{\partial\theta} = d \cdot \frac{\partial}{\partial\theta}$, $Df_0 \cdot \frac{\partial}{\partial x} = -2x\frac{\partial}{\partial x}$, $d \geq 16 > 4$ and $I \subset (-2, 2)$, it is not hard to see that the splitting $T_{(\theta,x)}(\mathbb{S}^1 \times I) = E^u \oplus E^c$ is a dominated decomposition of the tangent bundle of $\mathbb{S}^1 \times I$ into two invariant (one-dimensional) subbundles where E^u is uniformly expanding and

[3] The assumption $d \geq 16$ doesn't play crucial role in our results and it is present here only for sake of simplicity. Indeed, the results of Buzzi, Sester and Tsujii [BST] can be applied to replace $d \geq 16$ by $d \geq 2$, at least when we consider the C^∞-topology. See the section 4 for details.

E^c is dominated by E^u. In other words, f_0 is a partially hyperbolic map of type $E^u \oplus E^c$. From the theory of partial hyperbolicity we know that, if α is sufficiently small, $\widetilde{f}$ (as any C^1 nearby map) also admits a partially hyperbolic splitting of type $E^u \oplus E^c$ which varies continuously. As a consequence, we get

Proposition 2.3. *Given $\epsilon > 0$, the Lyapounov exponent*

$$\lambda^u(p,f) := \lim_{n\to\pm\infty} \frac{1}{n} \log \|Df^n(p)|_{E^u}\|$$

associated to E^u at every point p verifies

$$\log(d-\epsilon) \leq \lambda^u(p,f) \leq \log(d+\epsilon), \tag{1}$$

for any f close to $\widetilde{f}$ (if α is sufficiently small).

Proof. Since E^u is a continuous function of f and the derivative of f_0 expands $\frac{\partial}{\partial\theta}$ by the constant factor d, it is clear that the proposition holds for any f sufficiently C^1-close to f_0, which is certainly true in the context of Viana maps, provided α is small. □

Before ending this section, we apply the previous proposition to obtain a lower bound on the entropy of the SRB measure μ_0. Using Pesin's formula we know that the entropy $h_{\mu_0}(f)$ of μ_0 is the integrated sum of the positive Lyapounov exponents of μ_0. Hence, if we use theorem 2.1 and proposition 2.3, it follows that

$$h_{\mu_0}(f) \geq \log(d-\epsilon) + c_0. \tag{2}$$

Once this notation is established, we are able to state precisely our condition on the potential ϕ.

Definition 2.4. We say that the potential ϕ is *nearly constant* if

$$\max\phi - \min\phi < \frac{1}{2}(c_0 - \frac{\log(d+\epsilon)}{\log(d-\epsilon)}). \tag{3}$$

Note that the right-hand side of (3) is positive for $\epsilon > 0$ sufficiently small depending on d and c_0, the lower bound on the Lyapounov exponents of the Viana maps.

After these preparation, we are ready to prove our main result.

3 Proof of theorem A

From now on, we consider Viana maps f, which are, by definition, all C^3 close maps to $\widetilde{f}$ for any small α.

We start with the control of the recurrence (to the critical set $\mathcal{C}$) of generic points of invariant ergodic measures whose Lyapounov exponents are regular. The first step is to prove the following lemma:

Lemma 3.1. *Let η be an ergodic measure such that $\lambda^c(\eta) > -\infty$, where λ^c is the Lyapounov exponent of η associated to E^c. Then,*

$$\int |\log \; dist\,(p, \mathcal{C})| d\eta < \infty.$$

Proof. Since η is ergodic, $\lambda^c(\eta) = \int \log \|Df|_{E^c}\| d\eta > -\infty$. On the other hand, the definition of f_0 and α sufficiently small implies $\frac{1}{3}\|Df(p)|_{E^c}\| \leq dist(p, \mathcal{C}) \leq 3\|Df(p)|_{E^c}\|$. These two facts together finish the proof. $\square$

A consequence of this lemma is the slow recurrence (to $\mathcal{C}$) of generic points of ergodic measures with regular Lyapounov exponents:

Corollary 3.2. *Let η be an ergodic measure with $\lambda^c(\eta) > -\infty$. Then, for any $\gamma > 0$, there exists $\delta > 0$ such that*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \; dist_\delta(f^j(p), \mathcal{C}) \leq \gamma, \qquad \textit{for } \eta - \textit{a.e. point } p. \tag{4}$$

Here $dist_\delta$ is the δ-truncated distance defined by

$$dist_\delta(p, \mathcal{C}) = \begin{cases} 1 & \textit{if } \; dist\;(p, \mathcal{C}) \geq \delta, \\ dist\;(p, \mathcal{C}) & \textit{otherwise} \end{cases} \tag{5}$$

Proof. By the lemma 3.1, the function $|\log(dist(p, \mathcal{C}))|$ is η-integrable. In particular, $\eta(\mathcal{C}) = 0$. Using these facts together with the definition of the δ-truncated distance $dist_\delta$, we have that for any $\gamma > 0$, there exists some $\delta > 0$ with

$$\int -\log dist_\delta(p, \mathcal{C})\, d\eta \leq \gamma.$$

Because η is ergodic, a simple application of Birkhoff's theorem to the function $-\log(dist_\delta(p, \mathcal{C}))$ completes the proof. $\square$

At this point, we define

Definition 3.3. Let K be the set of ergodic measures μ whose central Lyapounov exponent $\lambda^c(\mu)$ are greater than $\frac{1}{4}\big(c_0 - \frac{\log(d+\epsilon)}{\log(d-\epsilon)}\big)$. We define $\mathcal{K}$ as the set of invariant measures μ whose ergodic decomposition (μ_p) belongs to K for μ-almost every p. The sets K and $\mathcal{K}$ are not empty since μ_0 belongs to both of them. See theorem 2.1.

It is interesting to consider $\mathcal{K}$ since it contains any measure liable of satisfying the variational principle:

Lemma 3.4. *There exists a constant $\kappa_0 > 0$ such that every measure $\eta \notin \mathcal{K}$ satisfies*

$$h_\eta(f) + \int \phi \, d\eta + \kappa_0 < \sup_\mu \left(h_\mu(f) + \int \phi \, d\mu \right). \tag{6}$$

Proof. The idea is to compare $h_\eta(f)$ with $h_{\mu_0}(f)$, where μ_0 is the unique SRB measure of f (see theorem 2.2). Without loss of generality, we can suppose that η is ergodic. In this case, $\eta \notin \mathcal{K}$ means $\lambda^c(\eta) \leq \frac{1}{4}\big(c_0 - \frac{\log(d+\epsilon)}{\log(d-\epsilon)}\big)$. Ruelle's inequality combined with the estimate (1) of proposition 2.3 gives

$$h_\eta(f) \leq \log(d+\epsilon) + \frac{1}{4}\left(c_0 - \frac{\log(d+\epsilon)}{\log(d-\epsilon)}\right).$$

Hence, the condition on the potential (3) and estimate (2) imply

$$h_\eta(f) + \int \phi \, d\eta < h_{\mu_0}(f) + \int \phi \, d\mu_0 - \frac{1}{4}\log(d+\epsilon).$$

Taking $\kappa_0 = \frac{1}{4}\log d < \frac{1}{4}\log(d+\epsilon)$, for instance, concludes the proof. □

Before proceeding further in the proof of theorem A, we recall the concept of hyperbolic time for maps with critical points. The setting of this definition is as follows.

Consider $f : M \to M$ a C^2 map which is local diffeomorphism except at a zero Lebesgue measure set $\mathcal{C} \subset M$. Assume f *behaves like a power of the distance* to the critical set $\mathcal{C}$, i.e., there are constants $B > 1$ and $\ell > 0$ such that, for every $p, q \notin \mathcal{C}$ with $2\,dist(p,q) < dist(p,\mathcal{C})$ and $v \in T_pM$ we have:

- $\frac{1}{B}\; dist(p,\mathcal{C})^\ell \leq \frac{\|Df(p)v\|}{\|v\|} \leq B\; dist(p,\mathcal{C})^\ell$;

- $|\log\|Df(p)^{-1}\| - \log\|Df(q)^{-1}\|| \le B\frac{dist(p,q)}{dist(p,\mathcal{C})^{\ell}}$;
- $|\log|\det Df(p)^{-1}| - \log|\det Df(q)^{-1}| \le B\frac{dist(p,q)}{dist(p,\mathcal{C})^{\ell}}$,

Remark 3.5. As the reader can easily verify, Viana maps satisfy all the previous assumptions, since these conditions are C^2-open and $\widetilde{f}$ satisfies them.

For the definition of hyperbolic times, we fix $0 < b < \min\{1/2, 1/(2\beta)\}$.

Definition 3.6. Given $0 < \sigma < 1$ and $\delta > 0$, we say that n is a (σ,δ)-hyperbolic time for p if, for all $1 \le k \le n$,

$$\prod_{j=n-k}^{n-1} \|Df(f^j(p))^{-1}\| \le \sigma^k \quad \text{and } dist_\delta(f^{n-k}(p),\mathcal{C}) \ge \sigma^{bk}.$$

The usefulness of hyperbolic times is the key property:

Proposition 3.7. *Given $\sigma < 1$ and $\delta > 0$, there exists $\delta_1 > 0$ such that if n is a (σ,δ)-hyperbolic time of p, then there exists a neighbourhood V_p of p such that:*

- *f^n maps V_p diffeomorphically onto the ball $B_{\delta_1}(f^n(p))$;*
- *For every $1 \le k \le n$ and $y, z \in V_p$,*

$$dist(f^{n-k}(y), f^{n-k}(z)) \le \sigma^{k/2} dist(f^n(y), f^n(z)).$$

Proof. See Alves, Bonatti and Viana [ABV, p.377]. □

Next, we recall the following criterion to prove the existence of infinitely many hyperbolic times.

Notation. For a fixed $\sigma < 1$, let $H(\sigma)$ be the set of points $p \in M$ with the following two properties:

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log\|Df(f^j(p))^{-1}\| \le 3\log\sigma < 0$$

and, for any $\gamma > 0$ there is some $\delta > 0$ satisfying

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} -\log\big(dist_\delta(f^j(p),\mathcal{C})\big) \le \gamma.$$

Proposition 3.8. *Given $\sigma < 1$, there exist $\nu > 0$ and $\delta > 0$ depending only on σ and f such that, given any $p \in H(\sigma)$ and $N \geq 1$ sufficiently large, there are (σ, δ)-hyperbolic times $1 \leq n_1 < \cdots < n_l \leq N$ for p with $l \geq \nu N$.*

Proof. See Alves, Bonatti and Viana [ABV, p.379] □

Coming back to the proof of theorem A, we introduce the following definition:

Definition 3.9. μ is called a σ^{-1}-*expanding measure* if $p \in H(\sigma)$ for μ-almost every point p.

We now prove that any measure $\mu \in \mathcal{K}$ is expanding:

Lemma 3.10. *Any $\mu \in \mathcal{K}$ is a σ^{-1}-expanding with $\sigma = \exp(-\frac{1}{12}\zeta)$, where $\zeta := c_0 - \frac{\log(d+\epsilon)}{\log(d-\epsilon)} > 0$.*

Proof. Without loss of generality we can assume $\mu \in \mathcal{K}$ ergodic. This implies that

$$\lambda^c(p) = \lambda^c(\mu) = \int \log \|Df|_{E^c}\| d\mu \geq \frac{1}{4}\left(c_0 - \frac{\log(d+\epsilon)}{\log(d-\epsilon)}\right)$$

for μ-a.e. p. Because μ is ergodic and the central direction E^c is one-dimensional, this is equivalent to

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(p))^{-1}\| \leq -3\frac{1}{12}\left(c_0 - \frac{\log(d+\epsilon)}{\log(d-\epsilon)}\right) = 3\log\sigma < 0.$$

This fact and the corollary 3.2 gave us the assumptions in the definition of $H(\sigma)$ at μ-almost every point. □

An easy consequence of the expanding features of a given measure is the existence of generating partitions, which turns out to be a relevant property when studying entropy and equilibrium states:

Lemma 3.11. *Given $\sigma < 1$, there exists $\delta_1 > 0$ such that any partition $\mathcal{P}$ with diameter less than δ_1 is a generating partition for any σ^{-1}-expanding measure μ.*

Proof. Define

$$A_\epsilon(p) := \{y \in M : dist(f^j(y), f^j(p)) \leq \epsilon \quad \text{for every } n \geq 0\}.$$

First we prove that $A_\epsilon(p) = \{p\}$ for μ-a.e. p and then we show how this can be used to finish the proof.

Claim 3.12. *For any σ^{-1}-expanding measure μ, there exists $\delta_1 > 0$ such that for any $\epsilon < \delta_1$ and μ-a.e. p,*

$$A_\epsilon(p) = \{p\}.$$

Proof of the claim. Proposition 3.8 guarantees that μ-a.e. p has infinitely many hyperbolic times $n_i(p)$, since $\mu \in \mathcal{K}$. Hence, applying the proposition 3.7 we conclude that there exists some $\delta_1 > 0$ such that if $z \in A_\epsilon(p)$ with $\epsilon < \delta_1$ then for any n_i we have

$$dist(z,p) \leq \sigma^{n_i/2} dist(f^{n_i}(z), f^{n_i}(p)) \leq \sigma^{n_i/2}\epsilon.$$

Since $n_i \to \infty$ as $i \to \infty$, we deduce that $z = p$. □

It is now a more or less standard matter to show that the previous claim implies that any finite partition $\mathcal{P} = \{P_1, \ldots, P_l\}$ with $\operatorname{diam}(\mathcal{P}) < \delta_1$ is a generating partition. Indeed, given any measurable set A and given $\delta > 0$, consider $K_1 \subset A$ and $K_2 \subset M - A$ compact sets with $\mu(A \setminus K_1) \leq \delta$ and $\mu(A^c \setminus K_2) \leq \delta$. Let $r := dist(K_1, K_2) > 0$. Claim 3.12 shows that, for n large enough, $\operatorname{diam}\mathcal{P}^{(n)}(p) \leq r/2$ for every p in a set of μ-measure greater than $1 - \delta$, where

$$\mathcal{P}^{(n)}(p) := \{C^{(n)} := P_{i_1} \cap \ldots f^{-n+1}(P_{i_n}) \text{ with } p \in C^{(n)}\}.$$

Consider the sets $C_1^{(n)}, \ldots, C_m^{(n)} \in \mathcal{P}^{(n)}$ that intersect K_1. Then, it is not difficult to see that

$$\mu(\cup C_i^{(n)} \Delta A) \leq 3\delta.$$

Since A is an arbitrary measurable set and $\delta > 0$ is an arbitrary positive real number, this proves that $\mathcal{P}$ is a generating partition. □

Finally, we conclude the proof of theorem A:

Proof of theorem A. We take a sequence μ_k of invariant measures such that $h_{\mu_k}(f) + \int \phi d\mu_k \to \sup_\eta(h_\eta(f) + \int \phi d\eta)$. when $k \to \infty$. Taking a subsequence if necessary, we also assume $\mu_k \to \mu$ as $k \to \infty$. By the lemma 3.4 we may assume that $\mu_k \in \mathcal{K}$.

We are going to prove that μ is an equilibrium measure and every equilibrium measure belongs to $\mathcal{K}$. Clearly, these claims finish the proof.

Fix a finite partition $\mathcal{P}$ with diameter less than $\delta_1 > 0$, the constant of lemma 3.11 (with σ being the constant of lemma 3.10) such that $\mu(\partial P) = 0$

for any $P \in \mathcal{P}$. Observe that Kolmogorov-Sinai's theorem implies $h_{\mu_k}(f) = h_{\mu_k}(f,\mathcal{P})$ by the lemmas 3.11 and 3.10, since $\mu_k \in \mathcal{K}$. Because the function $\nu \to h_\nu(f,\mathcal{P})$ is upper-semicontinuous at every measure ν with $\nu(\partial\mathcal{P}) = 0$, we get

$$\begin{aligned}
\sup_\eta (h_\eta(f) + \int \phi d\eta) &= \limsup_k (h_{\mu_k}(f) + \int \phi d\mu_k) \\
&= \limsup_k (h_{\mu_k}(f,\mathcal{P}) + \int \phi d\mu_k) \\
&= h_\mu(f,\mathcal{P}) + \int \phi d\mu \\
&\leq h_\mu(f) + \int \phi d\mu.
\end{aligned}$$

In other words, μ is an equilibrium state. Finally, the fact that every equilibrium state η belongs to $\mathcal{K}$ follows directly from the lemma 3.4. This concludes the proof. □

4 Remarks

Let us make few comments about the hypothesis $d \geq 16$ in the definition of the Viana maps, about the generalization of the theorem A in the context of random perturbations and about some natural questions associated to its statement.

The hypothesis $d \geq 16$ in the definition of the maps $\widetilde{f}$ was used by Viana to ensure that C^3 small perturbations have two positive Lyapounov exponents. However, Buzzi, Sester and Tsujii [BST] noticed that if one is willing to consider C^∞ small perturbations (instead of C^3), then the theorems of Viana and Alves (see theorems 2.1, 2.2) still hold. Also, it is proved in [BST] that the vertical expansion on the dynamical strip I of the derivative of f is uniformly less than 2. In particular, our comments on the tangent bundle dynamics of f are true even if $2 \leq d \leq 16$. Summarizing,

Theorem A also holds for C^∞ small perturbations of Viana maps with $d \geq 2$

(instead of stronger requirement $d \geq 16$).

Also, it is not difficult to check that the arguments of this note apply to the context of small random perturbations of Viana maps. Indeed, the techniques of Alves and Araújo [AA] allow one to consider hyperbolic times, expanding measures, slow recurrence and non-uniform expansion in the non-deterministic setting [4] , and prove the analogues (in the random context) of Viana's result (see theorem 2.1) and Alves' result (see theorem 2.2) about the existence of a unique physical measure with all Lyapounov exponents larger that some positive number. We obtain a version of theorem A for small random perturbations of Viana maps by considering the modified definition of the set of measures $\mathcal{K}$ (as defined in Arbieto, Matheus, Oliveira [AMO]).

Finally, a natural question motivated by the existence of equilibrium measures is the uniqueness and ergodic propoerties, including decay of correlations, for the equilibrium state obtained in theorem A. Since the approach of Young towers is not easy in the context of general equilibrium measures[5] and Ruelle's thermodynamical formalism via the spectral properties of the transfer operator is delicate in the presence of critical points, this question presents an interesting problem for the extension of the theory of equilibrium states beyond uniform hyperbolicity. For results in this direction in one dimension, see [PS1], [PS2].

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[5] This occurs essentially because the known constructions of Young towers depends on the Lebesgue measure as a reference measure, but the general equilibrium measures are singular with respect to Lebesgue, even in the uniformly hyperbolic case.

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