AUTOMORPHISMS AND NON-INTEGRABILITY

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ABSTRACT. On this note we prove that a holomorphic foliation of the projective plane with rich, but finite, automorphism group does not have invariant algebraic curves.

1. INTRODUCTION

On the monograph [2], Jouanolou proves that there exists a residual set on the space of holomorphic foliations of the projective plane of degree d, $d \geq 2$, whose elements do not have any invariant algebraic curve. The hard part of the proof is to exhibit an explicit example having such property. Jouanolou shows that the foliations induced by the homogeneous 1-forms

$$\Omega_d = \det \begin{pmatrix} x & y & z \\ y^d & z^d & x^d \\ dx & dy & dz \end{pmatrix} ,$$

where $d \geq 2$, have such properties. His proof explores the richness of the automorphism groups of such examples.

In [5] and [6], Zoladek, construct new examples of foliations of the projective plane without invariant algebraic curves. His examples also have rich automorphism groups.

In [3], the four authors prove that some higher dimensional analogues of Jouanolou's example does not have invariant algebraic hypersurfaces. Again the automorphism group of these foliations play a major role on the curse of the proof.

Besides taking advantage of a rich automorphism group the above mentioned works make use of the explicit form of the equations investigated and impose some restrictions on the singularities of the equations.

The purpose of this note is to settle a general result for foliations of the projective plane with *rich* automorphism group. Unfortunately we cannot completely get ride of the hypothesis on the singular set of the foliation, although our assumptions are on the eigenvalues of just one of the singularities. More precisely we prove the following

Theorem 1. *Let $\mathcal{F}$ be a holomorphic foliations foliations of $\mathbb{P}^2$. Suppose that $\mathcal{F}$ has at least one singularity with invertible linear part and whose quotient of eigenvalues is not rational nor a root of a unity. If the automorphism group of $\mathcal{F}$ acts without nontrivial fixed points on the space of cofactors then $\mathcal{F}$ does not admit any invariant algebraic curve.*

For a definition of the space of cofactors and the action of automorphism group of the foliation on it the reader should consult section 2.

2. COFACTOR REPRESENTATION

On this paper a *degree d singular holomorphic foliation* of $\mathbb{P}^2$ will be given by a 1-form ω on $\mathbb{C}^3$ killed by the radial vector field with homogeneous coefficients of degree $d+1$. More explicitly if $\omega = Adx + Bdy + Cdz$ then A, B and C are homogenous polynomials of degree $d+1$ and $xA + yB + zC = 0$. We will denote the foliation induced by ω by $\mathcal{F}_\omega$.

The *automorphism group* of ω is the subgroup of $\mathrm{SL}(3,\mathbb{C})$ defined by

$$\mathrm{Aut}(\omega) = \{\phi \in \mathrm{SL}(3,\mathbb{C}) | \phi^*\omega \wedge \omega = 0\}.$$

The *automorphism group* of $\mathcal{F}_\omega$ is the image of $\mathrm{Aut}(\omega)$ by the natural projection $\mathrm{SL}(3,\mathbb{C}) \to \mathrm{PSL}(3,\mathbb{C})$, i.e., $\mathrm{Aut}(\mathcal{F}_\omega) = \mathrm{P}(\mathrm{Aut}(\omega))$.

Note that if $\phi \in \mathrm{Aut}(\omega)$ then there exists a non zero complex number $\lambda(\phi)$ such that $\phi^*\omega = \lambda(\phi) \cdot \omega$. Moreover,

$$\begin{aligned} \lambda: \quad \mathrm{Aut}(\omega) &\to \quad \mathbb{C}^* \\ \phi &\to \quad \lambda(\phi) \end{aligned}$$

is a homomorphism of groups.

If $F \in \mathbb{C}[x,y,z]$ is a homogeneous polynomial describing an invariant algebraic curve for $\mathcal{F}$ then

$$dF \wedge \omega = F\Theta_F,$$

where Θ_F is a 3-form satisfying the following

Lemma 1. *There exists a unique 3-form β_F such that*

$$\Theta_F = \frac{\deg(F)}{d+2} \cdot dw + i_R\beta_F.$$

proof. Applying i_R to both sides of $dF \wedge \omega = F\Theta_F$ we have

$$i_R\Theta_F = \deg(F) \cdot \omega.$$

Follows from Euler's formula that $i_R(d\omega) = (d+2)\omega$ and consequently

$$i_R\left(\Theta_F - \frac{\deg(F)}{d+2} \cdot d\omega\right) = 0.$$

The existence and uniqueness of β_F follows from the exacteness of the Kozsul's complex. $\qquad\square$

From the above lemma follows that for every automorphism ϕ of ω the following identity holds

$$\beta_{\phi^*F} = \frac{\phi^*\beta_F}{\lambda(\phi)}.$$

If $\mathcal{F}$ is a foliation of degree d then β_F has degree $d-1$. Denote by Ω^3_{d-1} the space of homogeneous 3-forms on $\mathbb{C}^3$ of degree $d-1$. Hence we have a representation $\Phi_\omega : \mathrm{Aut}(\omega) \to \mathrm{Hom}(\Omega^3_{d-1}, \Omega^3_{d-1})$ defined by

$$\Phi_{\mathcal{F}}(\phi) = \left(\beta \mapsto \frac{\phi^*\beta}{\lambda(\phi)}\right).$$

Since $\Phi_\omega(\mu \cdot \mathrm{Id}) = \mathrm{Id}$ for every $\mu \in \mathbb{C}^*$, it follows that $\Phi_\omega = \Phi_{\mathcal{F}_\omega} \circ \pi$ where $\pi : \mathrm{Aut}(\omega) \to \mathrm{Aut}(\mathcal{F}_\omega)$ is the natural projection and

$$\Phi_{\mathcal{F}_\omega} : \mathrm{Aut}(\mathcal{F}_\omega) \to \mathrm{Hom}(\Omega^3_{d-1}, \Omega^3_{d-1})$$

is, by definition, the *cofactor representation*. We will say that the *automorphism group of $\mathcal{F}$ acts without nontrivial fixed points on the space of cofactors* if $\Phi_{\mathcal{F}}$ has no eigenvector with eigenvalue 1, i.e., for every $\phi \in \mathrm{Aut}(\mathcal{F}_\omega)$ and every $\beta \in \Omega^3_{d-1}$, $\beta \neq 0$, we have that $\Phi_{\mathcal{F}}(\phi)(\beta) \neq \beta$.

Proposition 1. *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$. If the automorphism group of $\mathcal{F}$ acts without nontrivial fixed points on the space of cofactors then either $\mathcal{F}$ admits liouvillian first integral or $\mathcal{F}$ does not admit an invariant algebraic curve*

proof. Suppose that $\mathcal{F}$ admits an invariant algebraic curve C given by homogeneous polynomial F. If we consider the homogeneous polynomial

$$G = \prod_{\sigma \in \mathrm{Aut}(\mathcal{F})} \sigma^* F,$$

we see that G is invariant by both $\mathcal{F}$ and $\mathrm{Aut}(\mathcal{F})$. Consequently the cofactor associated to G is an eigenvector for $\Phi_{\mathcal{F}}$ with eigenvalue 1 and therefore, by hypothesis, equal to zero. Consequently, it follows from lemma 1 that

$$d\omega = \frac{d+2}{\deg(G)} \cdot \frac{dG}{G} \wedge \omega \,.$$

To conclude one has to observe that the multivalued 1–form $G^{-\frac{d+2}{\deg(G)}} \cdot \omega$ is closed and induces a closed multivalued one-form on the projective plane. In particular $\mathcal{F}$ admits a liouvillian first integral. $\qquad\square$

Example 1. Let G be the subgroup of $\mathrm{Aut}(\mathbb{P}^2)$ generated by $\phi_1[x : y : z] = [-x : -y : z]$ and $\phi_2[x : y : z] = [-x : y : -z]$. Among the degree 2 foliations invariant by G there exists a $\mathbb{P}^2 \subset \mathrm{Fol}(2) = \mathbb{P}\mathrm{H}^0(\mathbb{P}^2, \Omega^1(4))$ whose elements are induced by the 1-forms

$$\omega_{(a,b,c)} = (axy^2 + bxz^2)dx + (-ax^2y + cyz^2)dy + (-bx^2z - cy^2z)dz \,.$$

The cofactor representation is given by

$$\begin{aligned}
\Phi(\phi_1)\left((\alpha x + \beta y + \gamma z)dx \wedge dy \wedge dz\right) &= (-\alpha x - \beta y + \gamma z)dx \wedge dy \wedge dz \,, \\
\Phi(\phi_2)\left((\alpha x + \beta y + \gamma z)dx \wedge dy \wedge dz\right) &= (-\alpha x + \beta y - \gamma z)dx \wedge dy \wedge dz \,.
\end{aligned}$$

In particular the cofactor representation does not have any eigenvector with eigenvalue 1. Although the lines $\{ay^2 + bz^2 = 0\}$ and $\{-ax^2 - cz^2 = 0\}$ are invariant by both $\mathcal{F}_{(a,b,c)}$ and $\mathrm{Aut}(\mathcal{F}_{a,b,c})$. Consequently

$$\frac{d\left((ay^2 + bz^2) \cdot (-ax^2 - cz^2)\right)}{(ay^2 + bz^2) \cdot (-ax^2 - cz^2)} \wedge \omega = d\omega \,,$$

and $\mathcal{F}_{(a,b,c)}$ have a liouvillian first integral as predicted by proposition 1. In fact one can easily verify that $\mathcal{F}_{(a,b,c)}$ have a rational first integral.

3. Proof of the Main Result

Before starting the proof of the main result let us state a simple lemma

Lemma 2. Let X be a holomorphic vector field on a neighborhood of $0 \in \mathbb{C}^2$ and $\omega = i_X dx \wedge dy$ a 1-form dual to X. Assume that 0 is a singularity of X. If there exists a 1-form η such that $d\omega = \eta \wedge \omega$ then the trace of $DX(0)$ is zero.

proof. Since $\omega = i_X dx \wedge dy$ we have $d\omega = div(X)dx \wedge dy = (\text{Tr}(DX(0)) + h.t.o)dx \wedge dy$. Now as 0 is a singularity of X we have:

$$(\eta \wedge \omega)(0) = 0 = \text{Tr}(DX(0))dx \wedge dy \,.$$

$\square$

Let p be the reduced singularity of $\mathcal{F}$ whose quotient of eigenvalues is not rational. If $\mathcal{F}$ admits invariant algebraic curve C we can suppose without loss of generality that C is reduced and invariant by $\text{Aut}(\mathcal{F})$. Moreover we can also assume that $\text{Aut}(\mathcal{F})$ acts transitively on the set of irreducible components of C.

Since $\phi_{\mathcal{F}}$ has no nontrivial eigenvector it follows from Lemma 1 that

$$(1) \qquad d\omega = \frac{d+2}{\deg(F)} \cdot \frac{dF}{F} \wedge \omega,$$

where F is a reduced homogeneous polynomial defining C.

From Lemma 2 we have $p \in C$. Since p is reduced, we can choose local coordinates (x, y) where $p = (0,0)$ and $\mathcal{F}$ is defined in these coordinates by

$$(2) \qquad \omega = \lambda x dy - y dx + \text{h.o.t..}$$

Suppose that C is smooth at p and locally defined by $\{x = 0\}$ or $\{y = 0\}$. From (1) and (2) we obtain in both cases that λ is rational contradicting our hypothesis. Therefore C will have two branches passing through p. In this case

$$\frac{d(xy)}{xy} \wedge \omega = (\lambda + 1)dx \wedge dy + \text{h.o.t.} = d\omega + \text{h.o.t..}$$

Comparing this last equality with (1) we deduce that $d + 2 = \deg(F)$ and that the foliation is given by the closed 1-form $\frac{\omega}{F}$ with simple poles along C. By [1, Thm. 2.1, p. 37] it follows that $\mathcal{F}$ is defined by a closed meromorphic 1-form

$$\sum_{i=1}^{k} \alpha_i \frac{dF_i}{F_i},$$

where F_i are the irreducible factors of F, $F = F_1 F_2 \cdots F_k$. Writing the closed 1-form above in local coordinates (x, y) this implies that x and y divide locally the distinct polynomials F_i, since otherwise the quotient of eigenvalues at p would be 1.

Without loss of generality suppose that x divides F_1 and y divides F_2. Thus we obtain that

$$0 = \left(\sum_{i=1}^{k} \alpha_i \cdot \frac{dF_i}{F_i} \right) \wedge \omega = (\lambda \alpha_1 + \alpha_2)dx \wedge dy + \text{h.o.t..}$$

Consequently we have $\lambda = -\frac{\alpha_2}{\alpha_1}$.

We can suppose without loss generality, that there exists $\phi \in \text{Aut}(\mathcal{F})$ such that $\phi^* \frac{dF_1}{F_1} = \frac{dF_2}{F_2}$. Applying ϕ to $\frac{\omega}{F}$ we see that there exists a root of unity β such that

$$\phi^* \left(\sum_{i=1}^{k} \alpha_i . \frac{dF_i}{F_i} \right) = \beta . \left(\sum_{i=1}^{k} \alpha_i . \frac{dF_i}{F_i} \right).$$

Comparing the coefficients of $\frac{dF_2}{F_2}$ in both sides of the equation we infer that $\alpha_1 = \beta \alpha_2$. Therefore $\lambda = -\beta^{-1}$ contrary to our hypothesis. $\square$

4. Some examples

4.1. The examples of Jouanolou.
As stated in the introduction the examples of Jouanolou are the foliations of $\mathbb{P}^2$ induced by the 1-forms $\omega_d = i_R i_{X_d} dx \wedge dy \wedge dz$, where $d \geq 2$ and $X_d = y^d \frac{\partial}{\partial x} + z^d \frac{\partial}{\partial y} + x^d \frac{\partial}{\partial z}$. We will denote the foliation induced by ω_d by $\mathcal{J}_d$. The automorphisms group of $\mathcal{J}_d$ is isomorphic to a semidirect product of $\mathbb{Z}_3$ and $\mathbb{Z}_{d^2+d+1}$. In particular it has order $3 \cdot (d^2 + d + 1)$. It is generated by $\phi_1[x : y : z] = [z : x : y]$ and $\phi_2[x : y : z] = [\delta x : \delta^d y : \delta^{d^2} z]$ where δ is a $d^2 + d + 1$-root of the unity.

The cofactor representation is given by

$$
\begin{aligned}
\Phi_d(\phi_1)\left(F(x,y,z)dx \wedge dy \wedge dz\right) &= F(z,x,y)dx \wedge dy \wedge dz\,, \\
\Phi_d(\phi_2)\left(F(x,y,z)dx \wedge dy \wedge dz\right) &= F(\delta x, \delta^d y, \delta^{d^2} z)dx \wedge dy \wedge dz\,.
\end{aligned}
$$

Lemma 3. *If $d \not\equiv 1 \mod 3$ then automorphism group of $\mathcal{J}_d$ acts without nontrivial fixed points on the space of cofactors.*

proof. The eigenvalue of $\Phi_d(\phi_2) : \Omega_{d-1}^3 \to \Omega_{d-1}^3$ corresponding to the eigenvector $x^i y^j z^k dx \wedge dy \wedge dz$, $i + j + k = d - 1$, is given by $\delta^{i+jd+kd^2}$. To prove the lemma it is sufficient to show that for $d \not\equiv 1 \mod 3$ the system

$$
\begin{aligned}
i + j + k &= d - 1 \\
i + dj + d^2 k &\equiv 0 \mod (d^2 + d + 1)\,,
\end{aligned}
$$

has no solutions with i, j and k nonnegative integers.

Subtracting the first equation from the second we obtain that

$$(d-1)(j + (d+1)k) \equiv -(d-1) \mod (d^2 + d + 1)\,.$$

Since $d^2 + d + 1 = (d+2)(d-1) + 3$ we see that the greatest common divisor of $d^2 + d + 1$ and $d - 1$ is 1 whenever $d \not\equiv 1 \mod 3$. Therefore from our assumptions we have that

$$j + (d+1)k \equiv -1 \mod (d^2 + d + 1)\,,$$

and $j, k \geq 0$ implies that $j + (d+1)k \geq d^2 + d$.

Since $j, k \leq d - 1$ we obtain that $j + (d+1)k \leq d^2 + d - 2$. This contradiction is sufficient to settle the lemma. $\square$

From the lemma above and Theorem 1 we obtain a proof that $\mathcal{J}_d$ has non algebraic solutions when $d \not\equiv 1 \mod 3$. Note that this result also holds for every integer d greater than or equal to 2, see [2].

4.2. **One of Zoladek's examples.** Denote by $\mathcal{F}_{[a:b:c]}$, $[a : b : c] \in \mathbb{P}^2$, the degree 3 foliations of $\mathbb{P}^2$ defined by the 1-forms

$$\omega_{a,b,c} = (ax^2y^2 + bxz^3)dx + (-ax^3y + cy^2z^2)dy + (-bx^2z^2 - cy^3z)dz\,.$$

These foliations are invariant by the automorphism

$$A[x : y : z] = [\delta x : \delta^2 y : \delta^4 z]$$

where, δ is the seventh root of unity.

A simple computation shows that A acts without non-trivial fixed points in Ω_2^3.

When $a \cdot b \cdot c \neq 0$, the singularities distinct from $\{[0 : 0 : 1], [0 : 1 : 0], [1 : 0 : 0]\}$ have as quotient of eigenvalues $\frac{11+5\sqrt{3}i}{14}$. Therefore from Theorem 1 the foliation $\mathcal{F}_{[a:b:c]}$ has no algebraic solutions when $a \cdot b \cdot c \neq 0$.

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