Transitivity and homoclinic classes for singular-hyperbolic systems

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Abstract

A *singular hyperbolic set* is a partially hyperbolic set with singularities (all hyperbolic) and volume expanding central direction [**MPP1**]. We study connected, singular-hyperbolic, attracting sets with dense closed orbits *and only one singularity.* These sets are shown to be transitive for most C^r flows in the Baire's second category sence. In general these sets are shown to be either transitive or the union of two homoclinic classes. In the later case we prove the existence of finitely many homoclinic classes. Our results generalize for singular-hyperbolic systems a well known result for hyperbolic systems in [**N**].

1 Introduction

A *singular hyperbolic set* is a partially hyperbolic set with singularities (hyperbolic ones) and volume expanding central direction [**MPP1**]. The singular-hyperbolic sets were introduced in [**MPP1**] to study robust transitive sets for three-dimensional flows. There are some works dealing with such sets [**C**, **M1**, **M2**, **MP1**, **MP2**, **MPP2**, **PPV**, **Y**]. Some of them explore the similarities between the hyperbolic and singular-hyperbolic sets. We further explore such similarities motivated by the following well known fact [**N**].

Proposition 1.1. *Let H be a non-trivial hyperbolic set of a C^r flow X on a closed manifold, $r \geq 1$. Suppose that the following properties hold.*

1. *H is connected.*
2. *H is attracting, i.e. $H = \cap_{t>0} X_t(U)$ for some compact neighborhood U.*
3. *The closed orbits contained in H are dense in H.*

Then, H is a homoclinic class. In particular, H is transitive.

The obviuos question is whether the conclusions above hold replacing hyperbolic by singular-hyperbolic. This question has negative answer [**MP3**]. Nevertheless we shall prove that singular-hyperbolic sets which are connected, attracting, has dense closed orbits and only one singularity are transitive for most C^r three-dimensional flows in the Baire's second category sence. In general the considered sets are shown to be either transitive or the union of two homoclinic classes. In the later case we prove the existence of finitely many homoclinic classes. Our results generalize Proposition 1.1 for singular-hyperbolic systems.

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Let us state our results in a precise way. Throughout M denotes a closed three-manifold and $\mathcal{X}^r$ denotes the space of C^r flows on M endowed with the usual C^r topology, $r \geq 1$. Every flow $X = X_t$ in this work is orientable, namely it is tangent to a C^r vector field still denoted by X.

A subset of $\mathcal{X}^r$ is *residual* if it is a countable intersection of open-dense subsets of $\mathcal{X}^r$. A property holds for generic flows in $\mathcal{X}^r$ if it holds for every flow in a residual subset of $\mathcal{X}^r$. A compact invariant set A of X is: *attracting* if $A = \cap_{t>0} X_t(U)$ for some compact neighborhood U; *non-trivial* if it is not a closed orbit of X; and *transitive* if $A = \omega_X(p)$ for some $p \in A$. Here $\omega_X(p)$ denotes the set of accumulation points of the positive orbit of p (this is the ω-limit set of p).

A *homoclinic class* of X is the closure $H_X(z)$ of the transversal intersection points of the stable and unstable manifolds of a hyperbolic periodic orbit z of X. It is known by the Birkhoff-Smale Theorem that $H_X(z)$ is transitive with dense periodic orbits.

A compact invariant set Λ of X is *partially hyperbolic* if there are an invariant splitting $T\Lambda = E^s \oplus E^c$ and positive constants K, λ such that:

1. E^s *is contracting*, namely
$$|| DX_t/E^s_x || \leq Ke^{-\lambda t}, \quad \forall x \in \Lambda, \ \forall t > 0.$$
2. E^s *dominates* E^c, namely
$$|| DX_t/E^s_x || \cdot || DX_{-t}/E^c_{X_t(x)} || \leq Ke^{-\lambda t}, \quad \forall x \in \Lambda, \ \forall t > 0.$$

The central direction E^c of Λ is said to be *volume expanding* if the additional condition
$$| J(DX_t/E^c_x) | \geq Ke^{\lambda t}$$
holds $\forall x \in \Lambda$, $\forall t > 0$ where $J(\cdot)$ means the jacobian.

Definition 1.2. *A partially hyperbolic set is* singular-hyperbolic *if it has singularities (all hyperbolic) and volume expanding central direction.*

The most representative example of a singular-hyperbolic set is the geometric Lorenz attractor [**ABS**, **GW**]. Our first result is the following.

Theorem A. *The following property holds for generic flows $X \in \mathcal{X}^r$. Let Λ be a singular-hyperbolic set of X. Suppose that the properties below hold.*

1. *Λ is connected.*
2. *Λ is attracting.*
3. *The closed orbits contained in Λ are dense in Λ.*
4. *Λ has a unique singularity.*

Then, Λ is transitive.

Standard C^1-topology arguments [**MP1**, **CMP**], Theorem A and Proposition 1.1 yield the corollary below.

Corollary 1.3. *The following property holds for generic flows $X \in \mathcal{X}^1$. Let Λ be a non-trivial, compact, invariant set of X containing at most one singularity. If Λ is connected, attracting and has dense closed orbits then Λ is a homoclinic class of X.*

The non-generic case of Theorem A is considered below.

Theorem B. *Let Λ be a singular-hyperbolic set of a C^r flow on a closed three-manifold, $r \geq 1$. Suppose that the properties below hold.*

1. *Λ is connected.*
2. *Λ is attracting.*
3. *The closed orbits contained in Λ are dense in Λ.*
4. *Λ has a unique singularity.*

Then, Λ is either

- *transitive or*
- *non-transitive and the union of two homoclinic classes.*

The two alternatives of the above theorem can occur [**MP3**]. Theorem B implies that a singular-hyperbolic set of a three-dimensional flow which is attracting, has dense closed orbits and only one singularity splits in a finite (possibly non-disjoint) union of transitive sets. This provides a sort of spectral decomposition for singular-hyperbolic attracting sets of three-dimensional flows.

Let us make some comments before the idea of the proof. The local product structure of hyperbolic sets is the fundamental tool behind the proof of Proposition 1.1. Indeed this property implies that hyperbolic sets are *tame*, i.e. contain finitely many homoclinic classes [**V**, p. 26]. We cannot use local product structure to prove Theorem B since such a property is not valid for singular-hyperbolic sets. In particular, we don't know whether singular-hyperbolic attracting sets for three-dimensional flows are tame. On the other hand, we know that the homoclinic classes in a singular-hyperbolic attracting set may be non-disjoint [**MP3**].

Let us give the idea of the proofs. Let Λ be a singular-hyperbolic set of a three-dimensional flow X satisfying the hypotheses (1)-(4) of Theorem B. Let σ be the unique singularity of Λ. Then the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of σ are real and satisfy $\lambda_2 < \lambda_3 < 0 < \lambda_1$ and $-\lambda_3 < \lambda_1$ for some order ([**MPP1**]). It follows that σ has a stable manifold $W^s_X(\sigma)$ (tangent to λ_2, λ_3), a unstable manifold $W^u_X(\sigma)$ (tangent to λ_1) and a strong stable manifold $W^{ss}_X(\sigma)$ (tangent to λ_2). Clearly $W^s_X(\sigma)$ is two-dimensional, $W^{ss}_X(\sigma)$ is one-dimensonal and $W^{ss}_X(\sigma) \subset W^s_X(\sigma)$. In particular, $W^s_X(\sigma) \setminus W^{ss}_X(\sigma)$ has two connected components $W^{s,+}_X(\sigma), W^{s,-}_X(\sigma)$. We define $H^+_X(\sigma, \Lambda)$ as the closure of the periodic orbits in Λ whose unstable manifold intersects $W^{s,+}_X(\sigma)$ and similarly we define $H^-_X(\sigma, \Lambda)$.

In Corollary 2.7 we prove $\Lambda = H^+_X(\sigma, \Lambda) \cup H^-_X(\sigma, \Lambda)$. In Theorem 3.1 we prove that if Λ is not transitive then $\omega_X(a)$ is a periodic orbit for all $a \in W^u_X(\sigma) \setminus \{\sigma\}$. The proof uses the arguments in [**MP1**, Theorem 5.2] and Propositions 3.4, 3.5, 3.7. We shall prove Theorem A at the end of Section 3 by using Theorem 3.1 and the Kupka-Smale Theorem [**dMP**]. In Theorem 4.1 we prove that if Λ is not transitive then both $H^+_X(\sigma, \Lambda)$ and $H^-_X(\sigma, \Lambda)$ are homoclinic classes. Theorem B easily follows from Corollary 2.7 and Theorem 4.1.

The proof of Theorem 4.1 goes as follows. By Theorem 3.1 we can fix $a \in W^u_X(\sigma) \setminus \{\sigma\}$ such that $\omega_X(a) = O_0$ for some periodic orbit O_0 of X. In Lemma 4.2 we prove that the expanding eigenvalue of O_0 is positive. Then O_0 divides its unstable manifold in two components denoted by $W^{u,+}_X(p_0), W^{u,-}_X(p_0)$ for some $p_0 \in O_0$ (Definition 4.3). In Proposition 4.6 we prove that $H^+_X(\sigma, \Lambda) = Cl(W^{u,+}_X(p_0))$ and similarly for $-$. Then, in order to prove Theorem 4.1 it suffices to prove that $Cl(W^{u,+}_X(p_0))$ is a homoclinic classs and similarly for $-$. To prove that $Cl(W^{u,+}_X(p_0))$ is a homoclinic classs we use Propositions 4.5 and 4.7. This finishes the proof of Theorem 4.1. Our methods provide a partial solution for a question above:

Corollary 1.4. *Singular-hyperbolic sets for three-dimensional flows which are connected, attracting,* non-transitive, *have dense closed orbits and only one singularity are tame.*

Unfortunately this corollary applies only in the non-generic case by Theorem B. It would be interesting to prove our results when more than one singularity is involved.

2 Preliminary results

In what follows $X = X_t$ is a C^r flow on a closed three-manifold M. The closure of $B \subset M$ is denoted by $Cl(B)$. If A is a compact invariant set of X we denote $Sing_X(A)$ the set of singularites of X in A. We denote by $Per_X(A)$ the union of the periodic orbits of X in A. A compact invariant set H of X is *hyperbolic* if the tangent bundle over H has an invariant decomposition $E^s \oplus E^X \oplus E^u$ such that E^s is contracting, E^u is expanding and E^X is generated by the direction of X [**PT**]. The Stable Manifold Theory [**HPS**] asserts the existence of the stable manifold $W^s_X(p)$ and the unstable manifold $W^u_X(p)$ associated to $p \in H$. These manifolds are respectively tangent to the subspaces $E^s_p \oplus E^X_p$ and $E^X_p \oplus E^u_p$ of T_pM. In particular, $W^s_X(p)$ and $W^u_X(p)$ are well defined if p belongs to a hyperbolic periodic orbit of X. An interesting case is when H is *saddle-type*, i.e. $dim(E^s) = dim(E^u) = 1$. In this case $W^s_X(p)$ and $W^u_X(p)$ are two-dimensional submanifolds of M (recall $dim(M) = 3$). The maps $p \in H \to W^s_X(p)$ and $p \in H \to W^u_X(p)$ are continuous (in compact parts). On the other hand, a compact, singular, invariant set Λ of X is *singular-hyperbolic* if all its singularities are hyperbolic and the tangent bundle over Λ has an invariant decomposition $E^s \oplus E^c$ such that E^s is contracting, E^s dominates E^c and E^c is volume expanding (i.e. the jacobian of DX_t/E^c grows exponentially as $t \to \infty$). See [**MPP1**] for the precise definition. Again the Stable Manifold Theory asserts the existence of the strong stable manifold $W^{ss}_X(p)$ associated to $p \in \Lambda$. This manifold is tangent to the subspace E^s_p of T_pM. For all $p \in \Lambda$ we define $W^s_X(p) = \cup_{t\in \mathbb{R}} W^{ss}_X(X_t(p))$. If p is regular (i.e. $X(p) \neq 0$) then $W^s_X(p)$ is a well defined two-dimensional submanifold of M. The map $p \in \Lambda \to W^s_X(p)$ is continuous (in compact parts) at the regular points p of Λ. The following definition is in [**MPP2**].

Definition 2.1. *A singularity σ of X is* Lorenz-like *if the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of σ are real and satisfy the relation*

$$\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1.$$

A Lorenz-like singularity is hyperbolic, and so, $W^s_X(\sigma)$ and $W^u_X(\sigma)$ do exist. Moreover, the eigenspace of λ_2 is tangent to a one-dimensional invariant manifold $W^{ss}_X(\sigma)$. This manifold is called the *strong stable manifold* of σ. Clearly $W^{ss}_X(\sigma)$ splits $W^s_X(\sigma)$ in two connected components. This fact motivates the following definition.

Definition 2.2. *Let σ be a Lorenz-like singularity of a three-dimensional flow X. We denote by $W^{s,+}_X(\sigma), W^{s,-}_X(\sigma)$ the two connected components of $W^s_X(\sigma) \setminus W^{ss}_X(\sigma)$ (see Figure 1).*

Theorem 2.3. *([***MPP2***]) Let Λ be a singular-hyperbolic set with dense periodic orbits. Then, every $\sigma \in Sing_X(\Lambda)$ is Lorenz-like and satisfies $\Lambda \cap W^{ss}_X(\sigma) = \{\sigma\}$.*

Lemma 2.4. *([***MPP2***]) A compact invariant set without singularities of a singular-hyperbolic set of a three-dimensional flow is hyperbolic saddle-type.*

Lemma 2.5. *Let Λ be a connected, singular-hyperbolic, attracting set with dense periodic orbits of X. For every $p \in Per_X(\Lambda)$ there is $\sigma \in Sing_X(\Lambda)$ such that*

$$W^u_X(p) \cap W^s_X(\sigma) \neq \emptyset.$$

Proof. Fix $p \in Per_X(\Lambda)$. We claim that $Cl(W^u_X(p))$ is singular. Indeed, suppose by contradiction that it is not. Then, $Cl(W^u_X(p))$ is hyperbolic saddle by Lemma 2.4. Using $dim(W^u_X(p)) = 2$ one can prove that $Cl(W^u_X(p))$ is attracting, and so, it is open and closed in Λ. Since Λ is connected we conclude that $\Lambda = Cl(W^u_X(p))$. And this is a contradiction since Λ contains singularities and $Cl(W^u_X(p))$ does not. This contradiction proves the claim. If $\sigma \in Sing_X(Cl(W^u_X(p)))$ we can prove $W^u_X(p) \cap W^s_X(\sigma) \neq \emptyset$ as in the proof of [**MP1**, Theorem 4.1 p. 365]. $\square$

Lemma 2.5 and Theorem 2.3 motivate the following definition.

Definition 2.6. *If Λ is a singular-hyperbolic set of X and $\sigma \in Sing_X(\Lambda)$ is Lorenz-like we define*

- $P^+_X(\sigma, \Lambda) = \{p \in Per_X(\Lambda) : W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset\}$.
- $P^-_X(\sigma, \Lambda) = \{p \in Per_X(\Lambda) : W^u_X(p) \cap W^{s,-}_X(\sigma) \neq \emptyset\}$.
- $H^+_X(\sigma, \Lambda) = Cl(P^+_X(\sigma, \Lambda))$.
- $H^-_X(\sigma, \Lambda) = Cl(P^-_X(\sigma, \Lambda))$.

Lemma 2.5 and Theorem 2.3 easily imply the following.

Corollary 2.7. *Let Λ be a connected, singular-hyperbolic, attracting set with dense periodic orbits and only one singularity σ. Then,*

$$\Lambda = H^+_X(\sigma, \Lambda) \cup H^-_X(\sigma, \Lambda).$$

We finish this section with two technical (but straightforward) lemmas to be used in the sequel. First we state some short definitions. If S is a submanifold we shall denote by T_xS the tangent space at $x \in S$. By *cross-section* of X we mean a compact submanifold Σ transverse to X and diffeomorphic to the two-dimensional square $[0,1]^2$. If Λ is a singular-hyperbolic set of X and $x \in \Sigma \cap \Lambda$, then x is regular and so $W^s_X(x)$ is a two-dimensional submanifold transverse to Σ. In this case we define by $W^s_X(x, \Sigma)$ the connected component of $W^s_X(x) \cap \Sigma$ containing x. We shall be interested in a special cross-section described as follows. Let Λ be a singular-hyperbolic set of a three-dimensional flow X and $\sigma \in Sing_X(\Lambda)$. Suppose that the closed orbits contained in Λ are dense in Λ. Then σ is Lorenz-like by Theorem 2.3. It is then possible to describe the flow of X close to σ by using the Grobman-Hartman Theorem [**dMP**]. Indeed, we can assume that the flow of X around σ is the linear flow given by

$$X = \lambda_1 \partial_{x_1} + \lambda_2 \partial_{x_2} + \lambda_3 \partial_{x_3}$$

in a suitable coordinate system $(x_1, x_2, x_3) \in [-1,1]^3$ around $\sigma = (0,0,0)$. A cross-section Σ of X is *singular* if Σ corresponds to the submanifolds $\Sigma^+ = \{z = 1\}$ or $\Sigma^- = \{z = -1\}$ in the coordinate system (x_1, x_2, x_3). We denote by l^+ and l^- the curves in Σ^+, Σ^- intersecting to $\{y = 0\}$. Note that l^+, l^- are contained in $W^{s,+}_X(\sigma), W^{s,-}_X(\sigma)$ respectively. With these notation we have the following straighforward lemma.

Lemma 2.8. *Let Λ a singular-hyperbolic set with dense periodic orbits of a three-dimensional flow X and $\sigma \in Sing_X(\Lambda)$ be fixed. Then, there are singular cross-sections Σ^+, Σ^- as above such that every orbit of Λ passing close to some point in $W^{s,+}_X(\sigma)$ (resp. $W^{s,-}_X(\sigma)$) intersects Σ^+ (resp. Σ^-). If $p \in \Lambda \cap \Sigma^+$ is close to l^+, then $W^s_X(p, \Sigma^+)$ is a vertical curve crossing Σ^+ as indicated in Figure 1. If $p \in Per_X(\Lambda)$ and $W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset$, then $W^u_X(p)$ contains an interval $J = J_p$ intersecting l^+ transversally. Similarly replacing $+$ by $-$.*

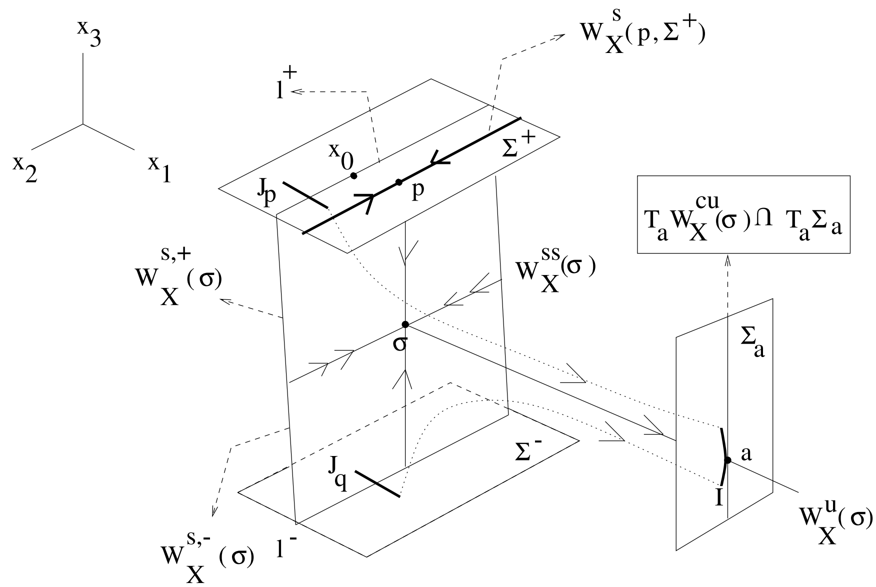

Figure 1: Singular cross-section.

Proof. Let F be a fundamental domain of $W^s_X(\sigma)$ (see [**dMP**]). Denote $F^+ = F \cap W^{s,+}_X(\sigma)$ and $F^- = F \cap W^{s,-}_X(\sigma)$. By Theorem 2.3 we have $(\Lambda \cap F) \cap W^{ss}_X(\sigma) = \emptyset$, and so, $\Lambda \cap F = (\Lambda \cap F^+) \cup (\Lambda \cap F^-)$. On the other hand, both $\Lambda \cap F^+$ and $\Lambda \cap F^-$ are compact and far from $W^{ss}_X(\sigma)$. If follows from the Strong λ-lemma [**D**] that there are a singular cross-sections Σ^+, Σ^- arbitrarily close to σ as above such that every orbit in $\Lambda \cap F^+$ (resp. $\Lambda \cap F^-$) has a point in l^+ (resp. l^-). Then, since F is a fundamental domain, every orbit in $\Lambda \cap W^{s,+}_X(\sigma)$ intersects l^+ (and similarly for $-$). This implies that every orbit of Λ passing close to some point in $W^{s,+}_X(\sigma)$ intersects Σ^+ (and similarly for $-$). Now let $p \in \Lambda \cap \Sigma^+$ be close to l^+. Since the strong unstable manifolds $W^{ss}_X(x)$ have size bounded away from zero ($x \in \Lambda$) we have that if Σ^+ is close to σ, then $W^s_X(x_0, \Sigma^+) = l^+$ ($\forall x_0 \in \Lambda \cap l^+$). The continuity of the set-valued map $p \in \Lambda \to W^s_X(p)$ at the regular points implies that if $p \in \Lambda \cap \Sigma^+$ is close to some point $x_0 \in l^+$ then $W^s_X(p, \Sigma^+)$ is close to $W^s_X(x_0, \Sigma^+)$. It follows that $W^s_X(p, \Sigma^+)$ is a vertical curve as desired. Finally let $p \in Per_X(\Lambda)$ be fixed. We have that $W^u_X(p) \subset \Lambda$ since Λ is attracting, and so, $W^u_X(p) \cap W^{s,+}_X(\sigma) \subset \Lambda \cap W^{s,+}_X(\sigma)$. As already mentioned every orbit in $\Lambda \cap W^{s,+}_X(\sigma)$ intersects l^+. In particular, $W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset$ implies $W^u_X(p) \cap l^+ \neq \emptyset$. Then, we can construct J_p by flow-projecting to Σ^+ nearby $W^u_X(p) \cap l^+$. $\square$

Lemma 2.9. *Let Λ a singular-hyperbolic set with dense periodic orbits of a three-dimensional flow X and $\sigma \in Sing_X(\Lambda)$ be fixed. Let $p, q \in Per_X(\Lambda)$ be such that $W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset$ and $W^u_X(q) \cap W^{s,-}_X(\sigma) \neq \emptyset$. Let $a \in W^u_X(\sigma) \setminus \{\sigma\}$ and Σ_a be a cross-section of X containing a. Then, there is an open interval $I = I_a \subset \Sigma_a$ containing a such that the properties below hold.*

1. *$I \setminus \{a\}$ is formed by two intervals, one of them contained in $W^u_X(p)$ and the other one contained in $W^u_X(q)$. In particular $I \subset \Lambda$.*

2. *$T_x I = E^c_x \cap T_x \Sigma_a$ for all $x \in I$.*

Proof. Let $E^s \oplus E^c$ be the singular-hyperbolic splitting of Λ. As the periodic orbits in Λ are dense in Λ we have that $X(x) \subset E^c_x$ and $dim(E^c_x) = 2$ for all $x \in \Lambda$. By Theorem 2.3 we have that σ is Lorenz-like (recall Definition 2.1). By the Invariant Manifold Theory [**HPS**] there is a centre-unstable manifold $W^{cu}_X(\sigma)$ at σ. This manifold is flow-invariant and tangent to the eigenspace associated to the eigenvalues λ_1, λ_3 of σ. The centre-unstable manifold is not unique but all of them satisfy $W^u_X(\sigma) \subset W^{cu}_X(\sigma)$. As E^s dominates E^c we can easily prove that $T_a W^{cu}_X(\sigma) = E^c_a$. In particular, one has

$$T_a W^{cu}_X(\sigma) \cap T_a\Sigma_a = E^c_a \cap T_a\Sigma_a.$$

See Figure 1. Let Σ^+, Σ^- be the cross-sections given by Lemma 2.8. As $W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset$ and $W^u_X(q) \cap W^{s,-}_X(\sigma) \neq \emptyset$ we can also fix the intervals J_p, J_q in $W^u_X(p) \cap \Sigma^+, W^u_X(q) \cap \Sigma^-$ given in Lemma 2.8. As Λ is attracting we have that J_p and J_q are both contained in Λ. The dominance condition of the splitting implies $T_x J_p \subset E^c_x \cap T_x\Sigma^+$ and $T_{x'} J_q \subset E^c_{x'} \cap \Sigma^-$ for all x, x' in J_p, J_q respectively. It follows from Lemma 2.8 that none of the curves J_p, J_q intersects $W^{ss}_X(\sigma)$. In addition, J_p intersects transversally the curve $l = l^+$ in Lemma 2.8 and similarly for J_q. It follows from the Strong λ-lemma [**D**] that the flow of X carries some interval in J_p into an open interval $I^+ \subset \Sigma_a$, and some interval in J_q into an open interval $I^- \subset \Sigma_a$, such that the union $I = I^+ \cup I^- \cup \{a\}$ satisfies: I is an open interval containing a and

$$T_a I = T_a W^{cu}_X(\sigma) \cap T_a\Sigma_a.$$

Then,

$$T_a I = E^c_a \cap T_a\Sigma_a.$$

As $I \setminus \{a\} = I^+ \cup I^-$ we have the property (1) of the lemma by the invariance of the unstable manifolds. That $x \in I \setminus \{a\}$ satisfies the property (2) of the lemma is clear. That $x = a$ also satisfies the property (2) follows from $T_a I = E^c_a \cap T_a\Sigma_a$. The lemma is proved. □

3 Non-transitive singular-hyperbolic sets

In this section we study non-transitive singular-hyperbolic sets for C^r flows on closed three-manifolds $r \geq 1$. The result is the following.

Theorem 3.1. *Let Λ be a singular-hyperbolic set of a C^r flow X on a closed three-manifold. Suppose that the properties below hold.*

1. *Λ is connected.*
2. *Λ is attracting.*
3. *The closed orbits contained in Λ are dense in Λ.*
4. *Λ has a unique singularity σ.*
5. *Λ is not transitive.*

Then, $\omega_X(a)$ is a periodic orbit of X for every $a \in W^u_X(\sigma) \setminus \{\sigma\}$.

Before the proof we state some preliminars. First of all we observe that the hypothesis (3) of the theorem implies

(H1) $\Lambda = Cl(Per_X(\Lambda))$.

To prove transitivity we shall use the following criterium due to Birkhoff.

Lemma 3.2. *Let T be a compact, invariant set of X such that for all open sets U, V intersecting T there is $s > 0$ such that $X_s(U \cap T) \cap V \neq \emptyset$. Then T is transitive.*

Birkhoff' criterium will be applied together with the following lemma.

Lemma 3.3. *Let Λ be a singular-hyperbolic set of X satisfying (1)-(4) of Theorem 3.1. Let U, V be open sets, $p \in U \cap Per_X(\Lambda)$ and $q \in V \cap Per_X(\Lambda)$. If $W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset$ and $W^u_X(q) \cap W^{s,+}_X(\sigma) \neq \emptyset$, then there are $z \in W^u_X(p)$ arbitrarily close to $W^u_X(p) \cap W^{s,+}_X(\sigma)$ and $t > 0$ such that $X_t(z) \in V$. A similar result holds replacing $+$ by $-$.*

Proof. Let p, q, U, V as in the statement. Fix a cross section $\Sigma = \Sigma^+$ through $W^{s,+}_X(\sigma)$ so that every orbit of Λ passing close to some point in $W^{s,+}_X(\sigma)$ intersects Σ (see Lemma 2.8). By assumption $W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset$ and $W^u_X(q) \cap W^{s,+}_X(\sigma) \neq \emptyset$. Note that these intersections are transversal. Because $W^u_X(q) \cap W^{s,+}_X(\sigma) \neq \emptyset$ there is $z_q \in W^u_X(q) \cap (W^{s,+}_X(\sigma) \cap \Sigma)$. By (H1) there is $q' \in Per_X(\Lambda) \cap \Sigma$ nearby z_q. By Lemma 2.8 the manifolds $W^s_X(x, \Sigma)$ are well defined vertical curves, $\forall x \in \Lambda \cap \Sigma$. Choosing q' nearby z_q, we have that $W^s_X(q', \Sigma)$ crosses Σ as in Figure 2. In particular, as $W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset$, there is $z_p \in W^s_X(q', \Sigma) \cap W^u_X(p)$. Since q' is close to $z_q \in W^u_X(q)$ the negative orbit of q' meets V. Then, as q' is periodic, we have that the positive orbit of q' meets V as well. As $z_p \in W^s_X(q')$ and $q' \in Per_X(\Lambda)$ there is $t' > 0$ such that $X_{t'}(z_p) \in V$. The result follows choosing $z = z_p$ and $t = t'$. $\square$

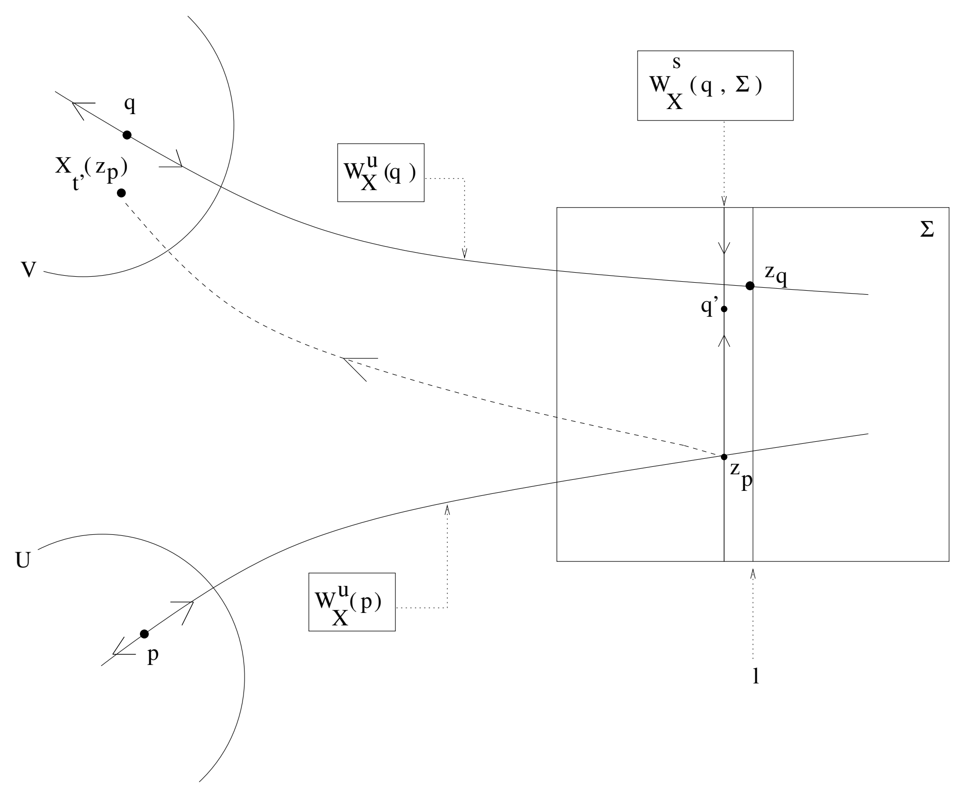

Figure 2: Proof of Lemma 3.3.

The next three propositions give sufficient conditions for the transitivity of singular-hyperbolic sets satisfying (1)-(4) of Theorem 3.1.

Proposition 3.4. *Let Λ be a singular-hyperbolic set of X satisfying (1)-(4) of Theorem 3.1. Suppose that there is a sequence $p_n \in Per_X(\Lambda)$ converging to some point in $W^{s,+}_X(\sigma)$ such that $W^u_X(p_n) \cap W^{s,-}_X(\sigma) \neq \emptyset$, $\forall n$. Then Λ is transitive. Similarly interchanging the roles of $+$ and $-$.*

Proof. To prove that Λ is transitive we only need to prove by Birkohff's criterium that $\forall U, V$ open sets intersecting Λ there is $s > 0$ such that $X_s(U \cap \Lambda) \cap V \neq \emptyset$. For this we proceed as follows: By (H1) there are $p \in Per_X(\Lambda) \cap U$ and $q \in Per_X(\Lambda) \cap V$. Next we proceed according to the following cases:

Case I: $W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset$ and $W^u_X(q) \cap W^{s,+}_X(\sigma) \neq \emptyset$. In this case Lemma 3.3 implies that there are $z \in W^u_X(p)$ and $t > 0$ such that $X_t(z) \in V$. Since $z \in W^u_X(p)$ there is $t' > 0$ such that $X_{-t'}(z) \in U$. As Λ is attracting we have $W^u_X(p) \subset \Lambda$, and so, $z \in \Lambda$. Choose $s = t + t'$ and $w = X_{-t'}(z)$. Then, $s > 0$ and $w \in U \cap \Lambda \cap X_{-s}(V)$. It follows that $X_s(U \cap \Lambda) \cap V \neq \emptyset$ for the chosen $s > 0$ proving the result.

Case II: $W^u_X(p) \cap W^{s,-}_X(\sigma) \neq \emptyset$ and $W^u_X(q) \cap W^{s,-}_X(\sigma) \neq \emptyset$. In this case the proof is similar to the one in Case I replacing $+$ by $-$.

Case III: $W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset$ and $W^u_X(q) \cap W^{s,-}_X(\sigma) \neq \emptyset$. Fix a cross section $\Sigma = \Sigma^+$ through $W^{s,+}_X(\sigma)$ and a open arc $J \subset \Sigma \cap W^u_X(p)$ intersecting $W^{s,+}_X(\sigma)$ transversally (see Lemma 2.8). By Lemma 2.8 we can assume that $p_n \in \Sigma$ for every n. Because the direction E^s of Λ is contracting the size of $W^s_X(p_n)$ is uniformly bounded away from zero. It follows that there is n large so that J intersects $W^s_X(p_n)$ transversally. Applying the Inclination Lemma [**dMP**] to the saturated of $J \subset W^u_X(p)$, and the assumption $W^u_X(p_n) \cap W^{s,-}_X(\sigma) \neq \emptyset$, we conclude that $W^u_X(p) \cap W^{s,-}_X(\sigma) \neq \emptyset$. This implies that Case II holds proving the result.

Case IV: $W^u_X(p) \cap W^{s,-}_X(\sigma) \neq \emptyset$ and $W^u_X(q) \cap W^{s,+}_X(\sigma) \neq \emptyset$. In this case we interchange the roles of p and q in the previous Case III to prove $W^u_X(q) \cap W^{s,-}_X(\sigma) \neq \emptyset$. This implies that Case II holds proving the result. $\square$

Proposition 3.5. *Let Λ be a singular-hyperbolic set of X satisfying (1)-(4) of Theorem 3.1. Suppose that the following property hold : If $p, q \in Per_X(\Lambda)$ then either*

1. *$W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset$ and $W^u_X(q) \cap W^{s,+}_X(\sigma) \neq \emptyset$ or*
2. *$W^u_X(p) \cap W^{s,-}_X(\sigma) \neq \emptyset$ and $W^u_X(q) \cap W^{s,-}_X(\sigma) \neq \emptyset$.*

Then, Λ is transitive.

Proof. Again, by Birkhoff's criterium, we only need to prove that $\forall U, V$ open sets intersecting Λ there is $s > 0$ such that $X_s(U \cap \Lambda) \cap V \neq \emptyset$. For this we proceed as follows: By (H1) there are $p \in Per_X(\Lambda) \cap U$ and $q \in Per_X(\Lambda) \cap V$. First suppose that the alternative (1) holds. Then, by Lemma 3.3, there are $z \in W^u_X(p)$ and $t > 0$ such that $X_t(z) \in V$. As $z \in W^u_X(p)$ we have that $w = X_{-t'}(z) \in U$ for some $t' > 0$. As Λ is an attracting set we have that $w \in \Lambda$. If $s = t + t' > 0$ we conclude that $w \in (U \cap \Lambda) \cap X_{-s}(V)$ and so $X_s(U \cap \Lambda) \cap V \neq \emptyset$. If the alternative (2) of the corollary holds we can find $s > 0$ such that $X_s(U \cap \Lambda) \cap V \neq \emptyset$ in a similar way (replacing $+$ by $-$). The proof follows. $\square$

The following corollary illustrates how Proposition 3.5 will be used. The corollary will not be used to prove the results in this paper. Its statement uses the following short definition. Let Λ a compact invariant set of a three-dimensional flow X and $\sigma \in \Lambda$ a Lorenz-like singularity of X. We say that Λ *accumulated σ by one side* if either $\Lambda \cap W^{s,+}_X(\sigma)$ or $\Lambda \cap W^{s,-}_X(\sigma)$ is $\emptyset$ (otherwise Λ accumulates σ in both sides). A typical example of a singular-hyperbolic, attracting set containing a Lorenz-like singularity accumulated by one side is the geometric Lorenz attractor [**ABS**, **GW**]. It is not difficult to construct singular-hyperbolic attractors with singularities accumulated in both sides. The corollary is the following.

Corollary 3.6. *Let Λ be a singular-hyperbolic set of X satisfying (1)-(4) of Theorem 3.1. If Λ accumulates σ by one side then Λ is transitive.*

Proof. Let $p, q \in \Lambda$ be periodic. Suppose that none of the alternatives of Proposition 3.5 hold for p, q. By Theorem 2.3 and Lemma 2.5 we can assume

$$W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset \quad \text{and} \quad W^u_X(q) \cap W^{s,-}_X(\sigma) \neq \emptyset.$$

Since Λ is attracting we have that $W^u_X(p) \cup W^u_X(q) \subset \Lambda$, and so, Λ intersects both $W^{s,+}_X(\sigma)$ and $W^{s,-}_X(\sigma)$. It follows that Λ accumulates σ in both sides contradicting the assumption. We conclude that one of the two alternatives in Proposition 3.5 holds $\forall p, q \in Per_X(\Lambda)$. It follows from this proposition that Λ is transitive and the result follows. ☐

Proposition 3.7. *Let Λ be a singular-hyperbolic set of X satisfying (1)-(4) of Theorem 3.1. Suppose that there is $a \in W^u_X(\sigma) \setminus \{\sigma\}$ such that $\sigma \in \omega_X(a)$. Then Λ is transitive.*

Proof. Without loss of generality we can assume that there is $z \in \omega_X(a) \cap W^+_X(\sigma)$. If $W^u_X(q) \cap W^{s,-}_X(\sigma) = \emptyset$ for all $q \in Per_X(\Lambda)$, then Theorem 2.3 and Lemma 2.5 imply $W^u_X(q) \cap W^{s,+}_X(\sigma) \neq \emptyset$ for all $q \in Per_X(\Lambda)$. Then Λ would be transitive by Proposition 3.5 since the alternative (1) holds $\forall p, q \in Per_X(\Lambda)$. So, we can assume that there is $q \in Per_X(\Lambda)$ such that $W^u_X(q) \cap W^{s,-}_X(\sigma) \neq \emptyset$. It follows from the dominating condition of the singular-hyperbolic splitting of Λ that the intersection $W^u_X(q) \cap W^{s,-}_X(\sigma)$ is transversal. This allows us to choose a point in $W^u_X(q)$ arbitrarily close to $W^{s,-}_X(\sigma)$ in the side accumulating a. Then, as Λ is attracting and satisfies (H1), it is not difficult to construct a sequence $p_n \in Per_X(\Lambda)$ converging to $z \in W^{s,+}_X(\sigma)$, and p'_n in the orbit of p_n converging to some point in $W^{s,-}_X(\sigma)$. Suppose by contradiction that Λ is not transitive. Then Proposition 3.4 would imply

$$W^u_X(p'_n) \cap W^{s,+}_X(\sigma) = \emptyset \quad \text{and} \quad W^u_X(p_n) \cap W^{s,-}_X(\sigma) = \emptyset$$

for n large. But $W^u_X(p_n) = W^u_X(p'_n)$ since p'_n and p_n are in the same orbit of X. So $W^u_X(p_n) \cap (W^{s,+}_X(\sigma) \cup W^{s,-}_X(\sigma)) = \emptyset$. By Theorem 2.3 one has

$$W^u_X(p_n) \cap W^s_X(\sigma) = \emptyset,$$

a contradiction by Lemma 2.5 since $Sing_X(\Lambda) = \{\sigma\}$. We conclude that Λ is transitive and the proof follows. ☐

Proof of Theorem 3.1: Let Λ be a singular-hyperbolic set of a three-dimensional flow X satisfying (1)-(5) of Theorem 3.1. In particular, Λ has a unique singularity σ. We assume by contradiction that there is $a \in W^u_X(\sigma) \setminus \{\sigma\}$ such that $\omega_X(a)$ is *not* a periodic orbit. The contradiction will follow from Proposition 3.5 once we prove that if $p, q \in Per_X(\Lambda)$ then one of the alternatives hold:

1. $W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset$ and $W^u_X(q) \cap W^{s,+}_X(\sigma) \neq \emptyset$
2. $W^u_X(p) \cap W^{s,-}_X(\sigma) \neq \emptyset$ and $W^u_X(q) \cap W^{s,-}_X(\sigma) \neq \emptyset$.

Indeed, by Theorem 2.3 and Lemma 2.5 we can assume that

$$W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset \text{ and } W^u_X(q) \cap W^{s,-}_X(\sigma) \neq \emptyset. \quad (1)$$

By Lemma 2.9 there is an open interval $I = I_a$, contained in a suitable cross-section $\Sigma = \Sigma_a$ of X containing a, such that (1)-(2) of the lemma hold. See Figure 3. In particular, if I^+, I^- are the connected components of $I \setminus \{a\}$ then $I^+ \subset W^u_X(p)$ and $I^- \subset W^u_X(q)$ (see Figure 3). Furhermore the tangent direction of I is contained in E^c, i.e. $T_xI = E^c_x \cap T_x\Sigma$ for all $x \in I$.

By Proposition 3.7 one has $\sigma \notin \omega_X(a)$ since Λ is not transitive (hypothesis (5) of Theorem 3.1). Then, $H = \omega_X(a)$ is a hyperbolic saddle-type set by Lemma 2.4. As in [**M1**] one proves that H is one-dimensional. Then we can apply the Bowen's Theory of hyperbolic one-dimensional sets [**B**]. More precisely, there are ϵ_0 small and a pairwise disjoint family $\mathcal{S} = \{S_1, \cdots, S_r\}$ of local cross sections of time ϵ_0. Each local cross section in the family has diameter less than $\alpha > 0$. In addition, $H = X_{[-\alpha,0]}(H \cap int(\mathcal{S}')) = X_{[0,\alpha]}(H \cap int(\mathcal{S}'))$, where $\mathcal{S}' = \cup S_i$ and $int(\mathcal{S}')$ denotes the interior of $\mathcal{S}'$. On the other hand, $I \subset \Lambda$ since Λ is attracting. Recall that the tangent direction of I is contained in E^c. Moreover, E^c is volume expanding and H is non-singular ($Sing_X(\Lambda) = \{\sigma\}$). Then, the Poincaré map induced by X on $\mathcal{S}'$ is expanding along I. As in [**MP1**, p. 371] we can find $\delta > 0$ and a open arc sequence $J_n \subset \mathcal{S}'$ in the positive orbit of I satisfying :

(a) the lenght of J_n is uniformly greather than δ;

(b) there is a_n *in the positive orbit of* a contained in the interior of J_n.

We can fix $S = S_i \in \mathcal{S}$ in order to assume that $J_n \subset S$ for every n. Let $x \in S$ be a limit point of the sequence a_n. Then $x \in H \cap int(\mathcal{S}')$. As the interval I is tangent to the central direction E^c we have that the interval sequence J_n converges in the C^1 topology to an interval $J \subset W^u_X(x)$. Observe that J is not trivial by (a).

If $a_n \in W^s_X(x)$ for n large we would obtain that x is periodic [**MP1**, Lemma 5.6] which contradicts the assumption that $\omega_X(a)$ is not periodic. Then $a_n \notin W^s_X(x)$ for every n. As $J_n \to J$ in the C^1 topology and Λ has strong stable manifolds of uniformly size, we have that there is n large and

$$z_n \in (W^s_X(a_{n+1}) \cap S) \cap (J_n \setminus \{a_n\}).$$

For every n let J^+_n and J^-_n denote the two connected components of $J_n \setminus \{a_n\}$ in a way that J^+_n is in the positive orbit of I^+ and J^-_n is in the positive orbit of I^-. The proof breaks in two cases.

Case I: $z_n \in J^+_n$ (see Figure 3). By (H1) there is $v_{n+1} \in Per_X(\Lambda) \cap S$ close to a_{n+1} such that

$$W^s_X(v_{n+1}) \cap J^+_n \neq \emptyset \text{ and } W^s_X(v_{n+1}) \cap J^-_{n+1} \neq \emptyset.$$

As v_{n+1} is periodic, close to a_{n+1} and $W^s_X(v_{n+1}) \cap J^-_{n+1} \neq \emptyset$ we conclude that the orbit of v_{n+1} passes arbitrarily close to some point in $W^{s,-}_X(\sigma)$. Then, $W^u_X(v_{n+1}) \cap W^{s,+}_X(\sigma) = \emptyset$ by Proposition 3.4 since Λ is not transitive. It follows from Theorem 2.3 that

$$W^u_X(v_{n+1}) \cap W^{s,-}_X(\sigma) \neq \emptyset. \quad (2)$$

But $W^s_X(v_{n+1})$ intersects J^+_n transversally. Applying the Inclination Lemma and (2) we conclude that the positive orbit of J^+_n intersects $W^{s,-}_X(\sigma)$. As such a positive orbit is in $W^u_X(p)$ we conclude that

$$W^u_X(p) \cap W^{s,-}_X(\sigma) \neq \emptyset.$$

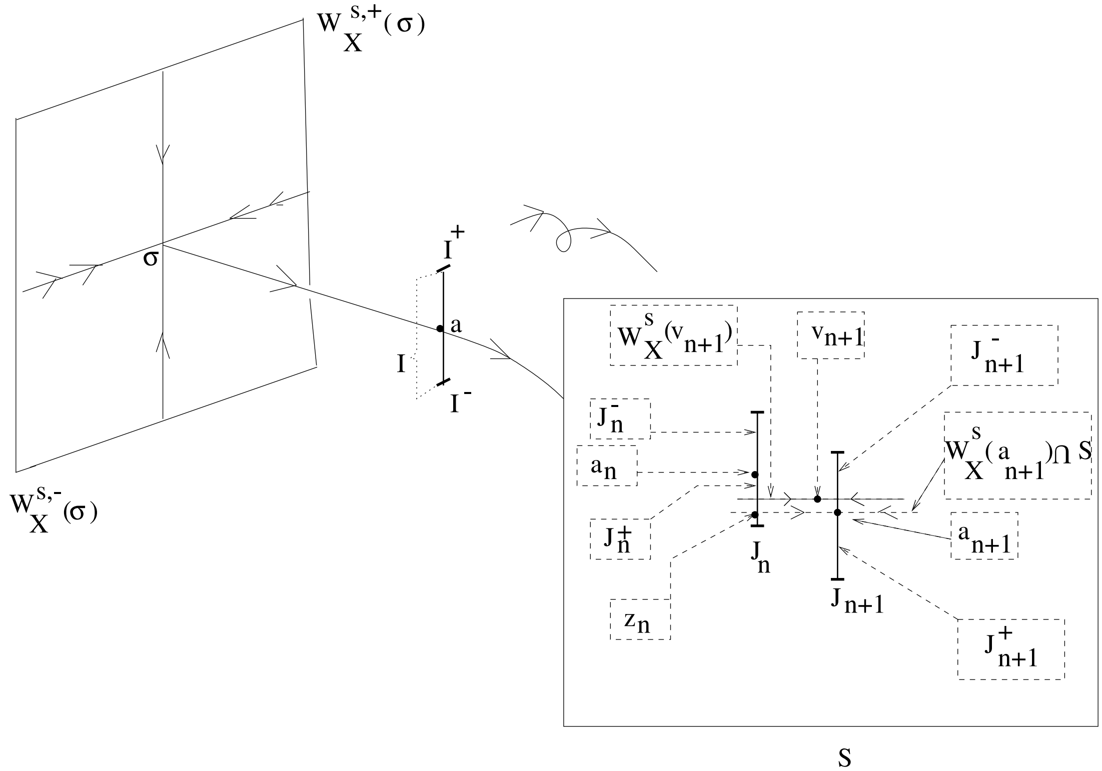

Figure 3: The case $z_n \in J_n^+$.

So, in this case, (1) implies

$$W_X^u(p) \cap W_X^{s,-}(\sigma) \neq \emptyset \quad \text{and} \quad W_X^u(q) \cap W_X^{s,-}(\sigma) \neq \emptyset$$

proving that alternative (1) holds.

Case II: $z_n \in J_n^-$. As in the previous case there is $w_{n+1} \in Per_X(\Lambda) \cap S$ close to a_{n+1} such that

$$W_X^s(w_{n+1}) \cap J_n^- \neq \emptyset \quad \text{and} \quad W_X^s(w_{n+1}) \cap J_{n+1}^+ \neq \emptyset.$$

As before the orbit of w_{n+1} passes close to $W_X^{s,+}(\sigma)$. Since Λ is not transitive we have by Proposition 3.4 and Theorem 2.3 as before that

$$W_X^u(w_{n+1}) \cap W_X^{s,+}(\sigma) \neq \emptyset.$$

Again $W_X^s(w_{n+1})$ intersects J_n^- transversally. Then, the Inclination Lemma as before implies that the positive orbit of J_n^- intersects $W_X^{s,+}(\sigma)$. As such a positive orbit is in $W_X^u(q)$ we conclude that

$$W_X^u(q) \cap W_X^{s,+}(\sigma) \neq \emptyset.$$

So, in this case, applying (1) one has

$$W_X^u(p) \cap W_X^{s,+}(\sigma) \neq \emptyset \quad \text{and} \quad W_X^u(q) \cap W_X^{s,+}(\sigma) \neq \emptyset$$

proving that alternative (2) holds. $\square$

Proof of Theorem A: By the Kupka-Smale Theorem [**dMP**] the set of Kupka-Smale flows $\mathcal{R}$ is residual in $\mathcal{X}^r, \forall r \geq 1$. If X is Kupka-Smale then the unstable manifold of any Lorenz-like singularity σ of X cannot intersect the stable manifold of any hyperbolic saddle periodic orbit of X (because such a intersection is not transversal). So, there is no $a \in W^u_X(\sigma) \setminus \{\sigma\}$ such that $\omega_X(a)$ is a periodic orbit. Then the result follows from Theorem 3.1. □

4 Proof of Theorem B and Corollary 1.4

Theorem B will be a direct consequence of Corollary 2.7 and the following result.

Theorem 4.1. *Let Λ be a singular-hyperbolic set of a C^r flow X on a closed three-manifold, $r \geq 1$. Suppose that the properties below hold.*

1. *Λ is connected.*
2. *Λ is attracting.*
3. *The closed orbits contained in Λ are dense in Λ.*
4. *Λ has a unique singularity σ.*
5. *Λ is not transitive.*

Then, $H^+_X(\sigma, \Lambda)$ and $H^-_X(\sigma, \Lambda)$ are homoclinic classes of X.

Before the proof we state some preliminary results. First of all we shall assume, along this section, that Λ and X satisfies (1)-(5) of Theorem 4.1. It follows from Theorem 3.1 that $\omega_X(a)$ is a periodic orbit for every $a \in W^u_X(\sigma) \setminus \{\sigma\}$. Hereafter we fix $a \in W^u_X(\sigma) \setminus \{\sigma\}$, and O_0 periodic with $\omega_X(a) = O_0$. We also fix $p_0 \in O_0$. It is clear that O_0 is saddle-type. Then, as X is three-dimensional, the Poincaré map induced by X in a suitable cross-section at p_0 has only one eigenvalue with modulus > 1. We shall call such a eigenvalue as the *expanding eigenvalue* of p_0. The lemma below will be used only in Definition 4.3.

Lemma 4.2. *The expanding eigenvalue of p_0 is positive.*

Proof. Suppose by contradiction that the expanding eigenvalue of p_0 is negative. Fix a cross-section Σ of X containing p_0. This section defines a Poincaré map $\Pi : Dom(\Pi) \subset \Sigma \to \Sigma$. By definition the eigenvalues of the derivative $D\Pi(p_0)$ are precisely the ones of p_0. It follows that $D\Pi(p_0)$ has a negative eigenvalue. As $p_0 \in Per_X(\Lambda)$, Theorem 2.3 and Lemma 2.5 imply that $W^u_X(p_0)$ intersects $W^{s,+}_X(\sigma)$ or $W^{s,-}_X(\sigma)$. We shall assume the first case since the proof for the second one is similar. Under such conditions we shall prove that $W^u_X(p) \cap W^{s,+}_X(\sigma) \neq \emptyset$ for all $p \in Per_X(\Lambda)$. For this we proceed as follows. Let $p \in Per_X(\Lambda)$ be fixed. Again by Theorem 2.3 and Lemma 2.5 we have that $W^u_X(p)$ intersects $W^{s,+}_X(\sigma)$ or $W^{s,-}_X(\sigma)$. In the first case we are done. So, we can assume that $W^u_X(p) \cap W^{s,-}_X(\sigma) \neq \emptyset$. It follows from Lemma 2.8 that $W^u_X(p)$ contains an interval $K' = J_p$ as indicated in Figure 4. By the definition of p_0 there is $a \in W^u_X(\sigma) \setminus \{\sigma\}$ such that $\omega_X(a)$ is the full orbit of p_0 (in particular $a \in W^s_X(p)$). We can assume that K' points in the side of a as indicated in the figure. Then, as $a \in W^s_X(p_0)$, the positive iterate of K' first passes through a and then goes to Σ yielding the interval K in Figure 4. Note that the closure of K intersects $W^s_X(p_0, \Sigma)$ transversally at a' as in Figure 4 (observe that a' is in the positive orbit of a). On the other hand, since $W^u_X(p_0) \cap W^{s,+}_X(\sigma) \neq \emptyset$, we have that there is an interval $J \subset W^{s,+}_X(\sigma) \cap \Sigma$ intersecting $W^u_X(p_0, \Sigma)$ transversally as indicated in Figure 4. Since the expanding eigenvalue of $D\Pi(p_0)$ is negative we have by the Inclination

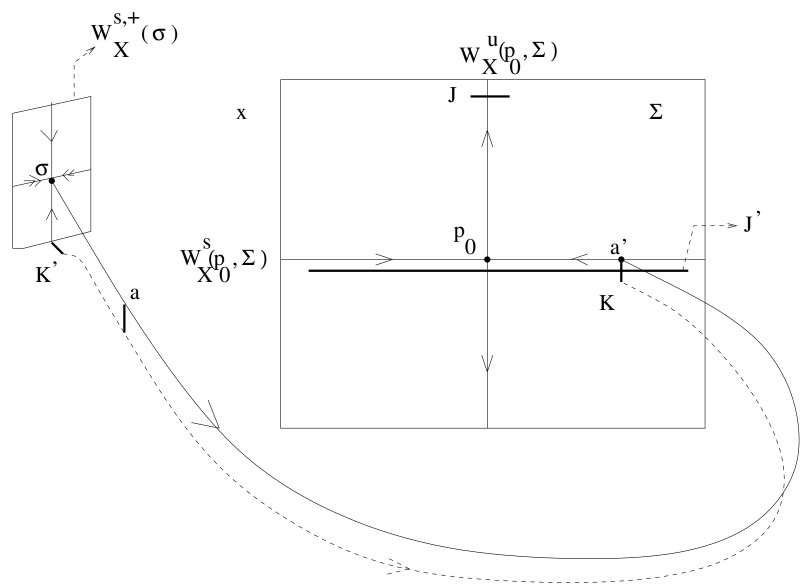

Figure 4: Positive expanding eigenvalues.

Lemma that the negative Π-iterates of J accumulates $W^s_X(p_0,\Sigma)$ *in both sides* of $\Sigma\setminus W^s_X(p_0,\Sigma)$. In particular, as K intersects $W^s_X(p_0,\Sigma)$ transversally at a', there is a negative iterate J' of J intersecting K as indicated in Figure 4. It follows from the invariance of the stable and unstable manifolds that such a intersection belongs to $W^u_X(p)\cap W^{s,+}_X(\sigma)$ since $J\subset W^{s,+}_X(\sigma)$ and $K\subset W^u_X(p)$. This proves $W^u_X(p)\cap W^{s,+}_X(\sigma)\neq\emptyset$ for all $p\in Per_X(\Lambda)$. So, the alternative (1) in Proposition 3.5 holds for all $p,q\in Per_X(\Lambda)$ proving that Λ is transitive, a contradiction. This contradiction shows that the expanding eigenvalue of p_0 is positive. $\square$

Let us explain how Lemma 4.2 applies in the proof of Theorem 4.1. By Proposition 3.5 and (5) of Theorem 4.1 there are $p,q\in Per_X(\Lambda)$ such that

$$W^u_X(p)\cap W^{s,+}_X(\sigma)\neq\emptyset \text{ and } W^u_X(q)\cap W^{s,-}_X(\sigma)\neq\emptyset.$$

By Lemma 2.9 there is an interval $I=I_a$, contained in a suitable cross-section of X and containing a, such that if I^+,I^- are the connected components of $I\setminus\{a\}$ then $I^+\subset W^u_X(p$ and $I^-\subset W^u_X(q)$. In addition I is tangent to the central direction E^c of Λ (see Figure 5). As $a\in W^s_X(p_0)$ and I is tangent to E^c we have that the flow of X carries I to an interval I' transverse to $W^s_X(p_0)$ at a_0 as indicated in Figure 5. Note that the flow carries I^+ and I^- into I^+_0 and I^-_0 respectively. On the other hand, Lemma 4.2 implies that $W^u_X(p_0)$ is a cylinder with generating curve O_0. In particular, O_0 separates $W^u_X(p_0)$ in two connected components. We denote such components by $W^{u,+}_X(p_0)$ and $W^{u,-}_X(p_0)$ according to the following definition.

Definition 4.3. *We denote by $W^{u,+}_X(p_0)$ the connected component of $W^u_X(p_0)\setminus O_0$ which is accumulated (via Inclination Lemma) by the positive orbit of I^+_0. We denote by $W^{u,-}_X(p_0)$ the connected component of $W^u_X(p_0)\setminus O_0$ which is accumulated by the positive orbit of I^-_0.*

Lemma 4.4. *Definition 4.3 does not depend on p,q,J_p,J_q.*

Proof. This lemma is a straightforward consequence of the Inclination Lemma, the Strong λ-lemma and Lemma 4.2. The details are left to the reader. $\square$

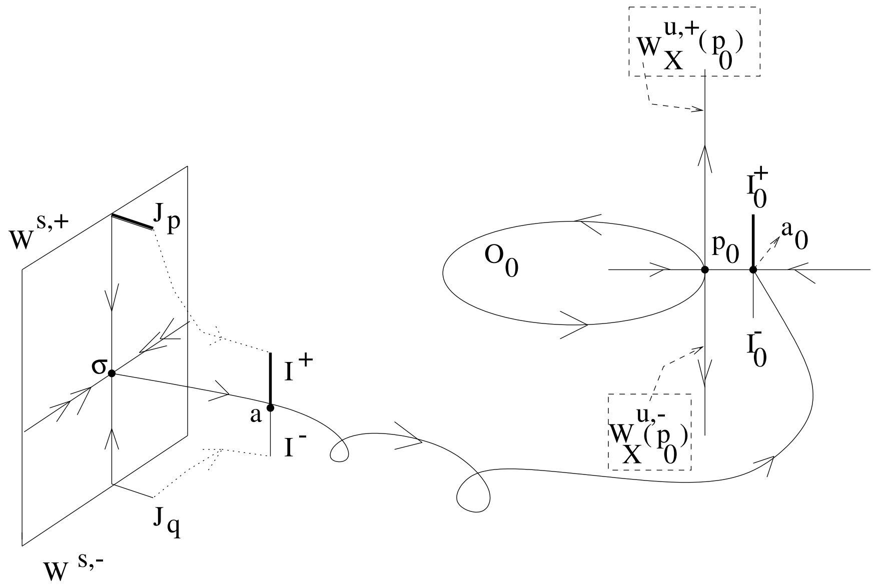

Figure 5: Definition of $W_X^{u,+}(p_0)$ and $W_X^{u,-}(p_0)$.

The next propositions give some properties of $W_X^{u,+}(p_0)$ and $W_X^{u,-}(p_0)$. In particular, Theorem 4.1 will follow from Propositions 4.6 and 4.8.

Proposition 4.5. *$W_X^{u,+}(p_0) \cap W_X^{s,+}(\sigma) \neq \emptyset$ and $W_X^{u,+}(p_0) \cap W_X^{s,-}(\sigma) = \emptyset$. Similarly interchanging the roles of $+$ and $-$.*

Proof. For simplicity denote $W = W_X^{u,+}(p_0)$. First we prove $W \cap W_X^{s,-}(\sigma) = \emptyset$. Suppose by contradiction that $W \cap W_X^{s,-}(\sigma) \neq \emptyset$. As this last intersection is transversal there is an interval $J \subset W_X^{s,-}(\sigma)$ intersecting W transversally. Now, fix a cross-section $\Sigma = \Sigma^+$ as in Lemma 2.8 and let $p \in Per_X(\Lambda)$ be such that $W_X^u(p) \cap W_X^{s,+}(\sigma) \neq \emptyset$. Then, there is an small interval $I \subset W_X^u(p) \cap \Sigma$ transversal to $\Sigma \cap W_X^{s,+}(\sigma)$. By the definition of $W = W_X^{u,+}(p_0)$ (Definition 4.3) we have that the positive orbit of I accumulates on W. As J is transversal to W the Inclination Lemma implies that the positive orbit of I intersects J. This proves $W_X^u(p) \cap W_X^{s,-}(\sigma) \neq \emptyset$ for all $p \in Per_X(\Lambda)$. It would follow that alternative (2) of Proposition 3.5 holds $\forall p, q$ contradicting (5) of Theorem 4.1. This contradiction proves $W \cap W_X^{s,-}(\sigma) = \emptyset$ as desired. Now suppose by contradiction that $W \cap W_X^{s,+}(\sigma) = \emptyset$. As $W \cap W_X^{s,-}(\sigma) = \emptyset$ we would obtain from Theorem 2.3 that $W \cap W_X^s(\sigma) = \emptyset$. But the denseness of the periodic orbits together with the Inclination Lemma imply $W \cap W_X^s(\sigma) \neq \emptyset$. This contradiction proves $W \cap W_X^{s,+}(\sigma) \neq \emptyset$ and the result follows. $\square$

Proposition 4.6. *$H_X^+(\sigma, \Lambda) = Cl(W_X^{u,+}(p_0))$. Similarly replacing $+$ by $-$.*

Proof. Fix $q \in P_X^+(\sigma, \Lambda)$, i.e. $W_X^u(q) \cap W_X^{s,+}(\sigma) \neq \emptyset$. Note that $W_X^{u,+}(p_0) \cap W_X^{s,+}(\sigma) \neq \emptyset$ by Lemma 4.5. Then, as in the proof of Lemma 3.3, one can prove that $W_X^{u,+}(p_0)$ accumulates on q (in this case $p = p_0$ and $z_q \in W_X^{u,+}(p_0)$ at Figure 4). This proves $H_X^+(\sigma, \Lambda) \subset Cl(W_X^{u,+}(p_0))$. Conversely let $x \in W_X^{u,+}(p_0)$ be fixed. Since the periodic orbits are dense in Λ and $W_X^{u,+}(p_0) \subset \Lambda$ there is $z \in Per_X(\Lambda)$ nearby x. Choosing z close to x we assure $W_X^s(z) \cap W_X^{u,+}(p_0) \neq \emptyset$ because stable manifolds have size uniformly bounded away from zero. If $W_X^u(z) \cap W_X^{s,-}(\sigma) \neq \emptyset$

then the Inclination Lemma and $W^s_X(z) \cap W^{u,+}_X(p_0) \neq \emptyset$ would imply $W^u_X(p_0) \cap W^{s,-}_X(\sigma) \neq \emptyset$. This contradicts Proposition 4.5 and so $W^u_X(z) \cap W^{s,-}_X(\sigma) = \emptyset$. By Theorem 2.3 and Lemma 2.5 we conclude that $z \in P^+_X(\sigma, \Lambda)$ and then $x \in H^+_X(\sigma, \Lambda)$ by Definition 2.6. The lemma is proved. $\square$

Proposition 4.7. *If $z \in Per_X(\Lambda)$ and $W^s_X(z) \cap W^{u,+}_X(p_0) \neq \emptyset$, then $Cl(W^s_X(z) \cap W^{u,+}_X(p_0)) = Cl(W^{u,+}_X(p_0))$. Similarly replacing $+$ by $-$.*

Proof. First we show $Cl(W^{u,+}_X(p_0) \cap W^{s,+}_X(\sigma)) = Cl(W^{u,+}_X(p_0))$. By Proposition 4.6 one has $Cl(W^{u,+}_X(p_0)) = H^+_X(\sigma, \Lambda)$. Fix $x \in W^{u,+}_X(p_0)$. Since the periodic orbits are dense in Λ there is $w \in Per_X(\Lambda)$ close to x. Suppose that $W^u_X(w) \cap W^{s,+}_X(\sigma) = \emptyset$. Then $W^u_X(w) \cap W^{s,-}_X(\sigma) = \emptyset$ by Theorem 2.3 and Lemma 2.5. It follows from the Inclination Lemma and Proposition 4.5 that $W^{u,+}_X(p_0) \cap W^{s,-}_X(\sigma) \neq \emptyset$ contradicting Proposition 4.5. This proves $W^u_X(w) \cap W^{s,+}_X(\sigma) \neq \emptyset$. Note that $W^s_X(w) \cap W^{u,+}_X(p_0) \neq \emptyset$ is close to x (as $w \to x$). As $W^u_X(w) \cap W^{s,+}_X(\sigma) \neq \emptyset$ is transversal we can apply the Inclination Lemma in order to find a transverse intersection $W^{u,+}_X(p_0) \cap W^{s,+}_X(\sigma)$ close to x. This proves $Cl(W^{u,+}_X(p_0) \cap W^{s,+}_X(\sigma)) = Cl(W^{u,+}_X(p_0))$. Now we prove $Cl(W^s_X(z) \cap W^{u,+}_X(p_0)) = Cl(W^{u,+}_X(p_0))$. Choose $x \in W^{u,+}_X(p_0)$. Then there is an interval $I_x \subset W^{u,+}_X(p_0)$ arbitrarily close to x such that $I_x \cap W^{s,+}_X(\sigma) \neq \emptyset$. The positive orbit of I_x first passes through a and, afterward, it accumulates on $W^{u,+}_X(p_0)$ (see Figure 5). But $W^s_X(z) \cap W^{u,+}_X(p_0) \neq \emptyset$ by assumption. As this intersection is transversal the Inclination Lemma implies that the positive orbit of I_x intersects $W^s_X(z)$. By taking the backward flow of the last intersection we get $W^s_X(z) \cap I_x \neq \emptyset$. This proves the lemma. $\square$

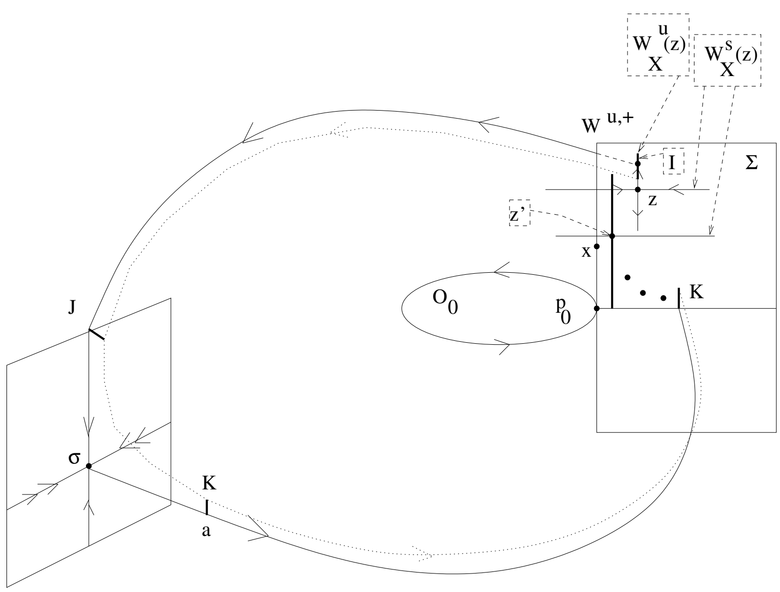

Figure 6: $Cl(W^{u,+}_X(p_0))$ is a homoclinic class.

Given $z \in Per_X(\Lambda)$ we denote by $H_X(z)$ the homoclinic class associated to z.

Proposition 4.8. *If $z \in Per_X(\Lambda)$ is close to a point in $W^{u,+}_X(p_0)$, then $H_X(z) = Cl(W^{u,+}_X(p_0))$. Similarly replacing $+$ by $-$.*

Proof. Let $z \in Per_X(\Lambda)$ be a point close to one in $W^{u,+}_X(p_0)$. It follows from the continuity of the stable manifolds that $W^s_X(z) \cap W^{u,+}_X(p_0) \neq \emptyset$. We claim that $H_X(z) = Cl(W^{u,+}_X(p_0))$. Indeed, by Proposition 4.5 one has $W^{u,+}_X(p_0) \cap W^{s,-}_X(\sigma) = \emptyset$. This equality and the Inclination Lemma imply $W^u_X(z) \cap W^s_X(\sigma) \subset W^{s,+}_X(\sigma)$. By Proposition 4.7 we have that $W^s_X(z) \cap W^{u,+}_X(p_0)$ is dense in $W^{u,+}_X(p_0)$ since $W^s_X(z) \cap W^{u,+}_X(p_0) \neq \emptyset$. Let Σ be a cross-section containing p_0 as in Figure 6 and fix $x \in W^{u,+}_X(p_0)$. We can assume that $x, z \in \Sigma$ are as in Figure 6. Since $W^u_X(z) \cap W^s_X(\sigma) \neq \emptyset$ and $W^u_X(z) \cap W^s_X(\sigma) \subset W^{s,+}_X(\sigma)$ there is an interval $I \subset W^u_X(z)$ intersecting $W^{s,+}_X(\sigma)$. Then, the positive orbit of I yields the interval J in the figure. In addition, the positive orbit of J yields the interval K in that figure. Note that the positive orbit of K *accumulates* $W^{u,+}_X(p_0)$ as in Figure 6 (recall Definition 4.3). As $W^s_X(z) \cap W^{u,+}_X(p_0)$ is dense in $W^{u,+}_X(p_0)$ and $x \in W^{u,+}_X(p_0)$ we have that $W^s_X(z)$ passes close to x as indicated in the figure. The Inclination Lemma applied to the positive orbit of K yields a homoclinic point z' associated to z which is close to x. This proves that $x \in H_X(z)$, and so, $Cl(W^{u,+}_X(p_0)) \subset H_X(z)$. The reversed inclusion is a direct consequence of the Inclination Lemma applied to $W^s_X(z) \cap W^{u,+}_X(p_0) \neq \emptyset$. We conclude that $Cl(W^{u,+}_X(p_0)) = H_X(z)$ proving the result. $\square$

Proof of Theorem 4.1: Let Λ be a singular-hyperbolic set of X satisfying (1)-(5) of Theorem 4.1. Then we can apply the results in this section. To prove that $H^+_X(\sigma, \Lambda)$ is a homoclinic classs it suffices by Proposition 4.6 to prove that $Cl(W^{u,+}_X(p_0))$ is a homoclinic class. By (3) of Theorem 4.1 we can choose $z \in Per_X(\Lambda)$ arbitrarily close to a point in $W^{u,+}_X(p_0)$. Then $Cl(W^{u,+}_X(p_0)) = H_X(z)$ by Proposition 4.8 and the result follows. $\square$

Proof of Corollary 1.4: Let Λ be a singular-hyperbolic set of a three-dimensional flow X which is connected, attracting, have dense closed orbits and only one singularity (denoted by σ). By Theorem 2.3 we have that σ is Lorenz-like, and so, $W^u_X(\sigma) \setminus \{\sigma\}$ consists of two regular orbits. Fix a, a' in each of such orbits. We shall assume that Λ is not transitive. Then, Λ satisfies the hypotheses (1)-(5) of Theorem 3.1. It follows from this theorem that $\omega_X(a) = O_0$ and $\omega_X(a') = O'_0$, where O_0 and O'_0 are periodic orbits of X. On the other hand, Λ satisfies the hypotheses (1)-(5) of Theorem 4.1 too. Then the results of this section can be applied. As before we can fix $p'_0 \in O'_0$ and prove the corresponding version of the results of this section for p'_0. In particular, we can define $W^{u,+}_X(p'_0), W^{u,-}_X(p'_0)$ as in Definition 4.3 and prove as in Proposition 4.8 that if $z \in Per_X(\Lambda)$ is close to a point in $W^{u,+}_X(p'_0)$ then $Cl(W^{u,+}_X(p'_0)) = H_X(z)$ (similarly for $-$).

Now assume by contradiction that Λ is not tame, i.e. it has infinitely many homoclinic classes $H_X(z_n)$. We have two possibilities, namely $\sigma \in Cl(\cup_n H_X(z_n))$ or $\sigma \notin Cl(\cup_n H_X(z_n))$. If $\sigma \notin Cl(\cup_n H_X(z_n))$ then $Cl(\cup_n H_X(z_n))$ is a hyperbolic set by Lemma 2.4. This is a contradiction because hyperbolic sets cannot contain infinitely many homoclinic classes [**N**]. This contradiction proves $\sigma \in Cl(\cup_n H_X(z_n))$. Let Σ^+, Σ^- be the cross-sections of X as in Lemma 2.8 for σ, i.e. every orbit of Λ passing close to some point in $W^{s,+}_X(\sigma)$ (resp. $W^{s,-}_X(\sigma)$) intersects Σ^+ (resp. Σ^-).

Because $\sigma \in Cl(\cup_n H_X(z_n))$ the above property of $\Sigma^\pm$ and Theorem 2.3 imply that the orbit of z_n can be assumed to intersect $\Sigma^+ \cup \Sigma^-$. Replacing z_n by one point in its orbit if necessary we can further assume that $z_n \in \Sigma^+ \cup \Sigma^-$. Recall that Σ^+, Σ^- are respectively equiped with the vertical curves l^+, l^- at Figure 1. Note that $(\Sigma^+ \setminus l^+) \cup (\Sigma^- \setminus l^-)$ is formed by four connected components. The forward orbits starting in two of these components passes close to a (as indicated in Figure 1) while the forward orbits starting in the remaining ones passes close to a'. Since the orbit of z_n are periodic ($\forall n$) we have that $z_n \notin l^+ \cup l^-$. Then, as the sequence z_n is infinite (and $(\Sigma^+ \setminus l^+) \cup (\Sigma^- \setminus l^-)$ has four connected components only), we can assume that z_n is in only one of these components. It follows that the positive orbit of z_n

passes close to either a or a' ($\forall n$). We are going to assume that the orbit of z_n passes close to a ($\forall n$) becase the argument for a' is enterely similar (for this we use the version of the results of this section for a' and p_0'). As we can observe in Figure 5 the positive orbit of a intersects the stable manifold close to p_0 in the point a_0'. The orbit positive of z_n (which passes close to a) will pass close to a_0' too. Clearly such a positive orbit is not in the stable manifold of p_0. By usual fundamental domain arguments [**dMP**] we have that there is a sequence z_n' in the positive orbit of z_n converging to a point in either $W_X^{u,+}(p_0)$ or $W_X^{u,-}(p_0)$. In each case we conclude by Proposition 4.8 that $H_X(z_n')$ is either $Cl(W_X^{u,+}(p_0))$ or $Cl(W_X^{u,+}(p_0))$ for all n (say). Because the sequence z_n' is infinite we can assume that $H_X(z_n') = Cl(W_X^{u,+}(p_0))$ for all n. Because z_n' is in the orbit of z_n we conclude that $H_X(z_n) = Cl(W_X^{u,+}(p_0))$ for all n. This would imply that $H_X(z_n) = H_X(z_m) = Cl(W_X^{u,+}(p_0))$ for all n, m a contradiction. This contradiction proves the result. $\square$

References

[**ABS**] Afraimovich, V. S., Bykov, V. V., Shilnikov, L. P., *On attracting structurally unstable limit sets of Lorenz attractor type*, Trudy Moskov. Mat. Obshch. **44** (1982), 150-212.

[**B**] Bowen, R., *Symbolic dynamics for hyperbolic flows*, Amer. J. Math. **95** (1973), 429-460.

[**CMP**] Carballo, C. M., Morales, C. A., Pacifico, M. J., *Homoclinic classes for generic C^1 vector fields*, Ergodic Theory Dynam. Systems, to appear.

[**C**] Colmenarez, W., *Algumas propriedades de atratores para fluxos em dimensão três*, Thesis (2002), Universidade Federal do Rio de Janeiro (in Portuguese).

[**dMP**] de Melo, W., Palis, J., *Geometric theory of dynamical systems. An introduction*, Translated from the Portuguese by A. K. Manning. Springer-Verlag, New York-Berlin (1982).

[**D**] Deng, B., *The Silnikov problem, exponential expansion, strong λ-lemma, C^1-linearization, and homoclinic bifurcation*, J. Differential Equations **79** (1989), 189-231.

[**GW**] Guckenheimer, J., Williams, R., F., *Structural stability of Lorenz attractors*, Inst. Hautes Études Sci. Publ. Math. **50** (1979), 59-72.

[**HPS**] Hirsch, M, Pugh, C., Shub, M., *Invariant Manifolds*, Springer Lectures Notes **583**.

[**M1**] Morales, C. A., *Singular-hyperbolic sets and topological dimension*, Preprint (2002), to appear.

[**M2**] Morales, C. A., *The explosion of singular-hyperbolic attractors*, Preprint (2002), to appear.

[**MP1**] Morales, C. A., Pacifico, M. J., *Mixing attractors for* 3*-flows*, Nonlinearity **14** (2001), 359-378.

[**MP2**] Morales, C. A., Pacifico, M. J., *A dichotomy for three-dimensional flows*, Ergodic Theory Dynam. Systems, to appear.

[**MP3**] Morales, C. A., Pacifico, M. J., in preparation.

[**MPP1**] Morales, C. A., Pacifico, M. J., Pujals, E. R., *On C^1 robust transitive sets for three-dimensional flows*, C.R. Acad. Sci. Paris, **326**, Série I (1998), 81-86.

[**MPP2**] Morales, C. A., Pacifico, M. J., Pujals, E. R., *Singular hyperbolic systems*, Proc. Amer. Math. Soc. **127** (1999) 3393-3401.

[**N**] Newhouse, S., *Hyperbolic limit sets*, Trans. Amer. Math. Soc. **167** (1972), 125-150.

[**PPV**] Pacifico, M. J., Pujals, E. R., Viana, M., *Sensitiveness and SRB measure for singular hyperbolic attractors*, Preprint (2002) to appear.

[**PT**] Palis, J., Takens, F., *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations. Fractal dimensions and infinitely many attractors*, Cambridge Studies in Advanced Mathematics **35** (1993) Cambridge University Press, Cambridge.

[**V**] Viana, M., *Dynamics: beyond uniform hyperbolicity*, Lecture at Collége de France, March (2002).

[**Y**] Yamanaka, T., *A characterization of dense vector fields in $\mathcal{G}^1(M)$ on 3-manifolds*, Hokkaido Math. J. **31** (2002), 97-105.

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