THE BERNOULLI PROPERTY FOR WEAKLY HYPERBOLIC SYSTEMS

ALEXANDER ARBIETO, CARLOS MATHEUS AND MARIA JOSÉ PACIFICO

Abstract. A dynamical system is called partially hyperbolic if it exhibits three invariant directions, one unstable (expanding), one stable (contracting) and one central direction (somewhere in between the other two). We prove that topologically mixing partially hyperbolic diffeomorphisms whose central direction is non-uniformly contracting (negative Lyapunov exponents) almost everywhere have the Bernoulli property: the system is equivalent to an i. i. d. (independently identically distributed) random process. In particular, these systems are mixing: correlations of integrable functions go to zero as time goes to infinity.

We also extend this result in two different ways. Firstly, for 3-dimensional diffeomorphisms, if one requires only non-zero (instead of negative) Lyapunov exponents then one still gets a quasi-Bernoulli property. Secondly, if one assumes accessibility (any two points are joined by some path whose legs are stable segments and unstable segments) then it suffices to requires the mostly contracting property on a positive measure subset, to obtain the same conclusions.

1. Introduction

Chaotic dynamics is associated to loss of memory and creation of information (two aspects of the same phenomenon) as the system evolves time. Indeed, orbits starting at nearby points *forget* this fact rather rapidly; the evolution of each orbit yields *new* information, which can not be deduced from the initial data nor from the evolution of another orbit.

This idea can be formalized in several (non-equivalent) ways. One is encapsulated in the notion of *entropy*, the exponential rate of creation of information by the system. Another, which concerns us more directly here, is through the mixing property: a system is called *mixing* if measurements of any observable quantity at same latter time correlate poorly with the initial measurements of the same, or any other, observable quantity. There are several stronger (and a few weaker) versions of this notion. The strongest is the Bernoulli property: a system is *Bernoulli* if it is ergodically equivalent to an i. i. d. random process. In simple terms, iterations of the system are as chaotic (unpredictable) as successive throws of a coin.

Date: October 13, 2003.

A.A. was partially supported by CNPq-Brazil/Faperj-Brazil, C.M. was partially supported by Faperj-Brazil and M.J.P. was partially supported by CNPq-Brazil/Faperj-Brazil/Pronex Dyn. Systems.

It is now well established that mixing is closely related to hyperbolicity properties of the dynamical system. On one hand, there was the fundamental work of Anosov [3] proving that the geodesic flow on any negatively curved manifold is ergodic. The strategy was to prove that these flows are *uniformly hyperbolic* (meaning that the tangent space transverse to the flow splits into two invariant directions which are expanding and contracted, respectively, at uniform rates, under time evolution) and to deduce ergodicity from it. A powerful machinery developed for hyperbolic systems in the sixties and the seventies shows, in particular, that Anosov flows are Bernoulli.

On the other hand, there was the equally remarkable theory of Kolmogorov, Arnold, Moser showing that most *elliptic* systems are not ergodic, let alone mixing or Bernoulli. For instance, close to an elliptic point most of phase space is occupied by invariant tori restricted to which the dynamics is given by a rigid rotation, up to a smooth change of coordinates.

Roughly speaking, this connection between mixing properties and hyperbolicity goes as follows. Expansion along certain directions of the tangent space means that most nearby points tend to move away from each other, so that their orbits decorrelate rapidly. The same is true for contraction, considering backward iterates. For smooth systems, as we are considering here, this local behavior is reflected at the global level.

Over the last decade, there has been a great deal of attention devoted to investigating the mixing properties of systems lying somewhere in between the two extreme situations, hyperbolic and elliptic, that we discussed before. Most successful attempts dealt with *partially hyperbolic* systems, where one still asks for expanding and contracting invariant directions, but one allows for additional so-called central directions, where the behavior is rather arbitrary. Concrete examples of partially hyperbolic systems arise in several applications, for instance hard ball systems with many balls [16] that model the motion of ideal gases (however hard ball systems correspond to piecewise smooth maps). Our goal in this paper is to prove that, in fact, quite *weak hyperbolicity features suffice for the system to be mixing and even Bernoulli.*

Before we give precise statements of our results, let us mention a few previous related results. On one hand, there are the works of Pugh, Shub [15] and their collaborators, investigating stable ergodicity for conservative (volume-preserving) diffeomorphisms. A dynamical property is *stable* (or *robust*) if it is shared by all systems in a C^1 neighborhood. A key ingredient in this approach is the notion of accessibility: a system is *accessible* if any two points may be joined by a smooth path whose velocity is everywhere contained in the union of the stable and unstable direction. Dolgopyat, Wilkinson [12] recently proved that accessibility holds for generic (residual subset of) C^1 diffeomorphisms, conservative or not. Moreover, Bonatti, Matheus, Viana, Wilkinson [9] proved that generic partially hyperbolic diffeomorphisms with 1-dimensional central direction are stably ergodic.

On the other hand, there is the work of Alves, Bonatti, Viana [2, 10] on the ergodic properties of partially hyperbolic diffeomorphisms, not necessarily conservative. They exploit the combination of partial hyperbolicity and *non-uniform hyperbolicity* (non-zero Lyapunov exponents) to prove existence and finiteness of physical measures for those systems. We prove here that the examples that appear in both papers above have the Bernoulli property. Let us point out that Bochi, Fayad, Pujals [5] prove that generic stably ergodic conservative systems are non-uniformly hyperbolic.

The two approaches have been put together by Burns, Dolgopyat, Pesin [11] in a work which may be considered a predecessor to the present paper. In a few words we push their analysis further to obtain the Bernoulli property rather than just ergodicity.

We point out that ergodicity implies chaotic properties like, for instance, topological transitivity (existence of dense orbits). But the sole assumption of transitiveness does not guaranty that the system is ergodic: Furstenberg exhibited in [13] a minimal but non-ergodic diffeomorphism. So, instead of a topological property of a single system we ask for robustness (it holds in a neighborhood of the system) of such topological behavior in the attempt to derive any statistical/ergodic property. Related to this Bonatti, Diaz and Pujals [8] proved that robustly transitive dissipative diffeomorphisms have some kind of weak hyperbolicity (the tangent bundle splits into two invariant directions, one contracting direction and one central direction). Arbieto and Matheus [4] proved that the same result holds for C^2 robustly transitive conservative diffeomorphisms. Tahzibi [17] proved that robustly transitive partially hyperbolic diffeomorphisms with central direction *mostly contracting* are stably ergodic. Here we extend this last result of Tahzibi replacing the robust transitiveness hypothesis by robust topologically mixing hypothesis, and obtain, instead of ergodicity, the Bernoulli property for these systems. We also prove that the systems in $\mathbb{T}^4$ studied by Tahzibi in [18] have the Bernoulli property.

In this paper we shall prove the Bernoulli property for four different classes of systems. In the first class are the systems that are topologically mixing and partially hyperbolic with *negative* Lyapunov exponent along the central direction for almost every point. Under the additional hypothesis of robustly topological mixing we obtain robustness of the Bernoulli property. In the second class we relax the hypothesis about the Lyapunov exponents asking only *non-zero* Lyapunov exponents along the central direction and obtain that the system is quasi-Bernoulli, meaning that the system is Bernoulli in an ergodic component with measure arbitrarily large. The strategy to obtain that the systems in these two classes are Bernoulli is to use a result by Pesin [14], which in brief terms says that if the system is non-uniformly hyperbolic and every of its iterate is ergodic then the system is Bernoulli. Hence, the goal is to prove that any iterate of a system in these classes are ergodic. For this we use the non-uniformly contracting condition (non-negative Lyapunov exponents along the central direction) to obtain local ergodicity and use the topological mixing property to spread the ergodicity to the whole manifold. Since iterates of a topologically mixing system is still topologically mixing, we repeat this argument and obtain that all iterates are ergodic. To conclude we apply Pesin's result described above.

In the last two classes are the strong partially hyperbolic systems (systems that have two non-zero hyperbolic directions besides a central direction) which are accessible (any two points can be joined by a piecewise smooth path with each piece tangent to one of the hyperbolic directions) and satisfy some additional hypothesis. The main strategy to prove that these systems are Bernoulli is again to use the result of Pesin described before. So we should be able to obtain ergodicity of all the iterates of the system. The additional hypothesis that distinguishes the classes are to guaranty local ergodicity for systems in each class. And accessibility allows to spread the local ergodicity to the whole of M. Since all iterate of an accessible system is still accessible we can repeat the same argument for each iterate and obtain ergodicity for all iterate. Applying Pesin's result we conclude that the system is Bernoulli. We also obtain robustness of the Bernoulli property for systems in these two classes, *without* requiring stable accessibility. For this we analyse the Pinsker partition (the maximal partition with zero entropy) used in the proof of Pesin's result. Actually, we find a uniform bound for the number of atoms of this partition in a neighborhood of the original system. This allow us to use the previous methods to obtain that the system is robustly Bernoulli.

The paper is organized as follows. In the next section we give some definitions and the precise statements of our results. In section 3 we give the proofs of the theorems. In section 4 we study a construction due to Bonatti and Viana [10] and as an application of our methods prove that it yields Bernoulli systems. Finally, in section 5, we point out how obtain some extensions and discuss some open problems.

2. Definitions and Statement of the Results

Throughout we will use the notation $\mathrm{Diff}^{1+}_m(M) := \bigcup_{\alpha>0} \mathrm{Diff}^{1+\alpha}_m$ and the diffeomorphisms considered here will be always in $\mathrm{Diff}^{1+}_m(M)$. We will deal with *robust* properties, but since they arise from different nature we need to specify the topologies involved.

Definition 2.1. *A diffeomorphism f is robustly transitive (resp. robustly topologically mixing) if there exists a neighborhood $\mathcal{U} \subset \mathrm{Diff}^{1+}_m(M)$ in the C^1-topology such that any $g \in \mathcal{U}$ is transitive (resp. topologically mixing).*

Obviously any topological mixing diffeomorphism is transitive.

Definition 2.2. *A diffeomorphism f is robustly ergodic (resp. robustly Bernoulli, robustly mixing) if there exists a neighborhood $\mathcal{U} \subset \mathrm{Diff}^{1+}_m(M)$ in the C^1-topology such that any $g \in \mathcal{U}$ is ergodic (resp. Bernoulli, mixing).*

We recall that by definition, a Bernoulli system is equivalent to a Bernoulli shift. It is easy to see that if f is Bernoulli then it is mixing.

We say that a property is generic robustly if it holds in a neighborhood intersected by a residual set. For example, f is generic transitive if there exists a neighborhood $\mathcal{U} \subset \mathrm{Diff}^{1+}_m(M)$ in the C^1-topology and a residual set $\mathcal{R}$ such that any $g \in \mathcal{U} \cap \mathcal{R}$ is transitive.

Next we state our results, in the different settings of partial hyperbolicity.

Partially Hyperbolic Systems

First let us recall some definitions.

Definition 2.3. *A Df-invariant splitting $TM = E \oplus F$ is a dominated splitting if there is $\lambda < 1$ such that:*

$$\frac{\|Df|_{E_x}\|}{\mathbf{m}(\|Df|_{F_{f(x))}}\|)} \leq \lambda \quad \text{for all} \quad x \in M. \tag{1}$$

We will use also the notion of a $k-$dominated splitting of $E \oplus F$ along the orbit of a point x. We require that for all $n \in \mathbb{Z}$:

$$\frac{\|Df^k_{f^n(x)}|_F\|}{\mathbf{m}(Df^k_{f^n(x)}|_E)} \leq \frac{1}{2},$$

where $\mathbf{m}(A) = \|A^{-1}\|^{-1}$. By a $k-$dominated splitting over an invariant set D we mean a $k-$dominated splitting for all orbits in D

Observe that (1) implies that any possible contraction along F is weaker than any contraction along E. This also implies that successive iterates of a vector in the tangent bundle by the derivative of f eventually lean toward the F direction.

A diffeomorphism f is *partially hyperbolic* if it has a dominated splitting $E \oplus F$ such that at least one of the sub-bundles is hyperbolic (either uniformly contracting or expanding). The complement of the hyperbolic sub-bundle is called the *central bundle* or equivalently *central direction*. We denote by $\mathcal{PH}^r(M)$ (respectively $\mathcal{PH}^r_m(M)$) the set of partially hyperbolic (respectively conservative partially hyperbolic) diffeomorphisms.

We can define the Lyapunov exponents of the system with respect to an invariant measure as the following:

Definition 2.4. *Let $f : M \to M$ be a C^1 diffeomorphism of a compact manifold that preserves a volume m. Oseledets theorem states that, for $m-$almost every point $x \in M$, there exist real numbers $\lambda_1(x) > \cdots > \lambda_{k(x)}(x)$ and*

$$T_xM = E^1_x \oplus \cdots \oplus E^{k(x)}_x$$

such that:

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df^n(x)(v_j)\| = \lambda_j(x) \quad \text{for all} \quad v_j \in E^j_x \backslash \{0\}.$$

For each j, λ_j is the Lyapunov exponent along the sub-bundle E^j and it depends measurably on x.

Definition 2.5. *We say that the central direction of $f \in \mathcal{PH}^r_m(M)$ is non-uniformly contracting if m-almost every point x has negative Lyapunov exponents along the central direction.*

Now we can state our first result.

Theorem A. *Let $f \in \mathrm{Diff}^{1+}_m(M)$ be a topologically mixing partially hyperbolic diffeomorphism with central direction non-uniformly contracting. Then (f, m) is Bernoulli and, in particular mixing.*

From this theorem we obtain immediately an extension of a result by Thazibi [17]:

Corollary 2.6. *If f is robustly topologically mixing and $TM = E^u \oplus E^{cs}$ with central direction E^{cs} non-uniformly contracting then f is robustly Bernoulli.*

We note that in [10] the authors constructed an example in $\mathbb{T}^3$ which is robustly transitive and stably ergodic. In section 4 we show that this example is in fact robustly topologically mixing.

We can relax the hypotheses in Theorem A requiring only non-zero Lyapunov exponents (non-uniformly hyperbolicity). Unfortunately we do not obtain the Bernoulli property in the whole manifold, but it holds for an arbitrarily large region in the sense of Lebesgue measure. To announce our next result let us introduce the following definition.

Definition 2.7. *A diffeomorphism f is ε-Bernoulli if there exist an ergodic component C of the Lebesgue measure m such that $m(C) > 1 - \varepsilon$ and if m_C is the normalization of the Lebesgue measure to C, $(f|_C, m_C)$ is Bernoulli. A diffeomorphism is quasi-robustly Bernoulli if for any $\varepsilon > 0$ there exist an open set $\mathcal{U}_\varepsilon \subset \mathrm{Diff}^{1+}_m(M)$ ε-close to f such that any $g \in \mathcal{U}_\varepsilon$ is ε-Bernoulli.*

With this we obtain an extension of a theorem by Tahzibi:

Theorem B. *If $\dim(M) = 3$ and $\mathcal{U} \subset \mathcal{PH}^{1+}_m(M)$ is an open set such that generically in U any diffeomorphism is non-uniformly hyperbolic and topologically mixing then generically in $\mathcal{U}$ any diffeomorphism is quasi-robustly Bernoulli.*

Strongly Partially Hyperbolic Systems

Now we focus in strongly partially hyperbolic systems, that is, systems that have two genuine hyperbolic directions (contracting and expanding) and a center direction. That is, the tangent bundle admits a dominated splitting $TM = E^u \oplus E^c \oplus E^s$ where E^u (respectively E^s) is uniformly expanding (respectively contracting). This allow us to use accessibility, instead of topological mixing, to spread out the negative Lyapunov exponents as in Theorems 2, 3 and 4 of [11] and obtain the Bernoulli property rather than just ergodicity.

Definition 2.8. *We say that f is accessible if any two points $p, q \in M$ can be joined by piecewise smooth paths such that each piece is a path entirely contained on a stable leaf or a unstable leaf. We call these paths a us-path. We say that f has the essentially accessible if any measurable union of accessible sets (i.e. any two points in each of these sets can be joined by a us-path) must have zero or full measure. Each piece of the us-path is called a leg.*

Next we state the generalizations of the results by Burns, Dolgopyat and Pesin [11].

Theorem C. *Let f be a strongly partially hyperbolic diffeomorphism with negative Lyapunov exponents along the central direction for a positive measure set A and suppose that f is essentially accessible. Then A has full measure. In particular f is Bernoulli and its central direction is non-uniformly contracting.*

Theorem D. *Let f be an accessible strongly partially hyperbolic diffeomorphism satisfying*

$$\int_M \log \|Df|_{E^c_f(x)}\| dm(x) < 0\,.$$

Then f is robustly Bernoulli.

Theorem E. *Let f be an accessible strongly partially hyperbolic diffeomorphism with central direction non-uniformly contracting. Then f is robustly Bernoulli.*

Remark 1. We observe that the hypotheses in the previous theorem as well the hypotheses of corollary 2.6 imply that f is C^1-robustly mostly contracting (see [2]).

We can strength these theorems using a denseness result by Dolgopyat and Wilkinson [12]. This will be done in section 5.

3. Proof of the Theorems

3.1. **Proof of Theorem A.** We will follow Hopf's argument as used in [10],[11] and [17]. For the sake of completeness we give such argument.

By non-uniform hyperbolicity we have a countable number of ergodic components. Now, take an ergodic component C and $R \subset C$ (with full Lebesgue measure in C) the set of regular points in the sense of Birkhoff's, i.e., if $x \in R$ then:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = \lim_{n\to-\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) \text{ for any } \varphi \in C^0(M).$$

Then by Pesin's theory, any $x \in R$ has a local stable manifold $W^{cs}_\varepsilon(x)$ and if m_{cs} is the induced measure in $W^{cs}_\varepsilon(x)$ then m_{cs} is absolutely continuous. This implies that there is a $x \in C$ and $C_x \subset W^{cs}_\varepsilon(x) \cap C \cap R$ such that

$m_{cs}(W^{cs}_\varepsilon(x)\backslash C_x)=0$. By partial hyperbolicity, any point has unstable manifolds with size uniformly away from zero. Now take $U_x=\bigcup_{y\in W^{cs}_\varepsilon(x)}W^u(y)$. Then, by continuity of the unstable foliation, U_x contains an open set. And for every $y\in C_x$ and $z\in W^u(y)$ we have:

$$\begin{aligned}\lim_{n\to-\infty}\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(z)) &= \lim_{n\to-\infty}\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(y)) = \\ =\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(y)) &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(z))\,.\end{aligned}$$

Then, using absolute continuity of W^u, $\bigcup_{y\in C_x}W^u(y)$ has full measure in U_x and hence C contains a total Lebesgue measure subset of the open set U_x. Finally, transitiveness shows that there exists a unique ergodic component with full Lebesgue measure and thus f is ergodic.

Now we use a theorem by Pesin (which is in fact a corollary of Theorem 8.1 in [14]):

Theorem 3.1. *If $f\in\mathrm{Diff}^{1+}_m(M)$ is non-uniformly hyperbolic such that (f^n,m) is ergodic for any $n\geq 0$ then f is Bernoulli.*

Because being topologically mixing and have negative Lyapunov exponents in the E^{cs} direction is an invariant property for all iterates f^k. Then, by the previous argument, all of (f^k,m) are ergodic and non-uniformly hyperbolic then the above theorem shows that (f,m) is Bernoulli. □

Proof of Corollary 2.6. Let U be the open set given by the hypothesis. Then by Theorem A any $g\in U$ is Bernoulli. This implies that f is C^1-robustly Bernoulli. □

The proof above shows the following:

Corollary 3.2. *If $f\in\mathrm{Diff}^{1+}_m(M)$ is generic robustly topologically mixing and $TM=E^u\oplus E^{cs}$ with central direction non-uniformly contracting then f is generic robustly Bernoulli.*

Also [11] uses the same arguments to prove the following:

Theorem 3.3 (Burns, Dolgopyat and Pesin)**.** *If $f\in\mathrm{Diff}^{1+}_m(M)$ has an invariant subset $A\subset M$ with $m(A)>0$, such that $f|_A$ is strongly partially hyperbolic with negative exponents along the central direction then every ergodic component of $f|_A$ (and A) is open (mod 0). If f is topologically transitive then A is dense and $f|_A$ is ergodic.*

This theorem will be used later for the proofs of theorems C, E and D.

3.2. **Proof of Theorem B.** We will follow the arguments in [17]. We can assume that $TM=E^u\oplus E^{cs}$, since the other case is analogous.

By Bochi-Viana's theorem [6], there exists a C^1-residual subset $\mathcal{R}$ of $\mathrm{Diff}^1_m(M)$ such that for every $f\in\mathcal{R}$, the Oseledets splitting is dominated or else trivial, at almost every point. Let $g \in \mathcal{R} \cap \mathcal{U}$ where $\mathcal{U}$ is an open set such that every $g \in \mathcal{U}$ is topologically mixing.

We recall that the residual set given by Bochi-Viana's theorem is characterized as the continuity points of the maps: $\Lambda_i(f) = \lambda_1(f) + \cdots + \lambda_i(f)$, where $\lambda_j(f) = \int_M \lambda_j(x)dm$. Now, let $\mathcal{V}$ be an open set containing g such that for any $f, h \in \mathcal{V}$ we have $|\Lambda_i(f) - \Lambda_i(h)| \leq \delta_0$.

We know that there exists a countable number of ergodic components, and for any ergodic component C we consider the normalized Lebesgue measure m_C on $supp(C)$ and we can use $supp(C)$ instead of C. Recall that the basin of m_C is the set of points z such that $\frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(z)}$ converges to m_C, this set has full measure in C and every orbit in the basin is transitive. Now we use the basin of m_C instead of C and we continue denoting it by C.

An ergodic component C is "good" if $\lambda_2(x) < 0$ (the central exponent) for every $x \in C$ or $\lambda_2(x) > 0$ and $E^2 \oplus E^3$ is dominated. Other components are called "bad" components.

Any "good" component contains open sets (mod 0). Indeed, the case of $\lambda_2 < 0$ is in the proof of Theorem A and the other case follows from the fact that for a conservative system, the dominated splitting $E^{cu} \oplus E^{cs}$ is in fact volume hyperbolic (see [8]). Recall that

Definition 3.4. *A dominated splitting $TM = E^1 \oplus E^2 \cdots \oplus E^k$ is volume hyperbolic if there exist some $K > 0$:*

$$|\det(Df^{-n}|E^k(x)| \leq K\lambda^n\,, \quad \textit{and} \quad |\det(Df^n|E^1(x)| \leq K\lambda^n\,.$$

Because the dimension is 3 and that $\dim E^{cs} = 1$ this sub-bundle is actually uniformly contracting, and so, a stable bundle. Hence, we have in fact a strong stable foliation, and the argument is analogous of the one in the proof of TheoremA (using strong unstable/stable leaves).

We recall that if C is a "good" component for f then it is a "good" component for f^k for any $k \geq 1$ and the same holds for "bad" components. And by topologically mixing there exists only one "good" ergodic component for f^k, $k \geq 1$.

Now we prove that the measure of the union of "bad" components can be made arbitrarily small. Let $\Gamma(f,k)$ be the subset of points such that $E^2 \oplus E^3$ does not admit a $k-$dominated splitting and let $\Gamma(f,\infty) = \cap_{k\in\mathbb{N}}\Gamma(f,k)$. For the "bad" ergodic components C, we have that $\lambda_2 > 0$ and $E^2 \oplus E^3$ does not admit a $k-$dominated splitting over C for any $k \in \mathbb{N}$ and by transitivity every point $x \in C$ doesn't have a k-dominated splitting. This shows that $C \subset \Gamma(f,\infty)$ (mod 0).

Denote $J(f) = \int_{\Gamma(f,\infty)} \frac{\lambda_2-\lambda_3}{2}dm(x)$. Then we can use the following:

Proposition 3.5 (Proposition 4.17 [6]). *Given any $\delta > 0$ and $\varepsilon > 0$, there exists a diffeomorphism f_1, ε near to f such that*

$$\int_M \Lambda_2(f_1,x)dm < \int_M \Lambda_2(f,x)dm(x) - J(f) + \delta.$$

From the above proposition we will deduce that if the measure of bad components is not small enough then after perturbing f a little, the average of $\lambda_1+\lambda_2$ drastically drops. Indeed, as $C\subset\Gamma(f,\infty)$ and on $C, \lambda_2(x)>0$ by the above proposition we get

$$\Lambda_2(f)-\Lambda_2(f_1)\geq\frac{1}{2}\int_C(\lambda_2-\lambda_3)(f)dm-\delta\geq\frac{1}{2}\int_C-\lambda_3(f)dm-\delta$$

$$\geq m(C)\inf_{x\in C}\frac{-\lambda_3(f,x)}{2}-\delta.$$

Now f is volume hyperbolic and partially hyperbolic $TM=E^u\oplus E^{cs}$, so $\det(Df|E^{cs}(x))<\alpha<1$ for all $x\in M$ and we can take α uniform in a C^1 neighborhood of g by continuity on the C^1 topology of $f|_{E^{cs}}(x,f)$. If we take $x\in C$ then:

$$\lambda_2(x)+\lambda_3(x)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log\,\det(Df|E^{cs}(f^i(x)))\leq\log(\alpha)$$

and as $\lambda_2(x)>0$, we have $\lambda_3(x)<\log(\alpha)$ for every $x\in C$. So

$$\inf_{x\in C}\frac{-\lambda_3(f,x)}{2}\geq\frac{-\log(\alpha)}{2}$$

and this estimate is uniform in $\mathcal{V}$. Hence

$$\delta_0\geq\Lambda_2(f)-\Lambda_2(f_1)\geq m(C)\frac{-\log(\alpha)}{2}-\delta.$$

Thus $m(C)\leq\frac{\delta+\delta_0}{-\log(\alpha)}$. Taking δ_0 and δ small, for $f\in\mathcal{V}\cap\mathrm{Diff}^{1+}_m(M)$, $m(C)$ is small enough. We observe that we can do this for the set C_{bad}, the union of all of the "bad" ergodic components. Then the measure of the "good" components is large enough (which are the same for all the iterates of f).

Finally we conclude the proof as follows. Taking $\mathcal{E}_n:=\frac{1}{n}$-ergodic diffeomorphisms in $\mathrm{Diff}^{1+}_m\cap\mathcal{U}$, then $\mathcal{E}_n$ is open and dense in the C^1 induced topology , so $\mathcal{E}=\bigcap\mathcal{E}_n$ is a residual subset and $f\in\mathcal{E}$ is Bernoulli.

3.3. **Proof of theorems C, D, and E.** We follow the proof of [11], so we deal with the notion of ε-accessibility. That is, given $\varepsilon>0$, g is ε-accessible if for any open Ball B of radius ε, the union of points that can be accessible from a point in B is the whole of M. We will also use the following lemmas that can be found in [11]:

Lemma 3.6. *If f is accessible, then for any $\varepsilon>0$ there exists $l>0$ and $R>0$ such that for any $p,q\in M$ there exists a us-path that starts at p, ends within distance $\varepsilon/2$ of q an has at most l legs, each of them with length at most R. And there exist a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^2(M)$ such that any $g\in\mathcal{U}$ is ε-accessible.*

Lemma 3.7. *For any f ε-accessible every orbit is ε-dense (i.e. the set $\{f^n(x)\}_{n\in\mathbb{Z}}$ is an ε-net set). Also, if f is essentially accessible then almost every point has a dense orbit.*

Proof of theorem C. We observe that any stable/unstable leaf of f is also a stable/unstable leaf of any iterate of f, then f is (essentially) accessible if and only if f^k is (essentially) accessible for any $k \geq 0$. Then, the conclusions of Lemma 3.6 and 3.7 hold for any iterate of f (of course the neighborhood $\mathcal{U}$ and the constants can be smaller when the iterate growth).

So for any iterate f^k, $k \geq 0$, almost every point has a dense orbit. Then f^k is ergodic and non-uniformly hyperbolic by Theorem 3.3 for all $k \geq 0$. Thus, Theorem 3.1 implies that f is Bernoulli. □

Proof of theorem D. We will need the following result by Burns-Dolgopyat-Pesin :

Theorem 3.8 (Theorem 4 of [11]). *Let f be a $C^{1+\alpha}$ strongly partially hyperbolic, volume preserving diffeomorphism. Assume that f is accessible and*

$$\int \log \|df|_{E_f^c(x)}\| d\mu < 0.$$

Then f is stably ergodic.

Since any iterate of an acessible diffeomorphism is still acessible we conclude that f^j is stably ergodic for every j. The key point now is to find a neighborhood of f where all g in this neighborhood and all of its iterates g^n are ergodic. For this we need to analyse the key tool used by Pesin in the proof of Theorem 3.1: the *Pinsker partition* (the maximal partition with zero entropy). First of all, Pesin shows that any ergodic component Λ of a non-uniformly hyperbolic conservative $C^{1+\alpha}$ diffeomorphism can be decomposed $\Lambda = \bigcup_{i=1}^{N} \Lambda_i$ into disjoint sets such that $f(\Lambda_i) = \Lambda_{i+1} (i = 1, \dots, N-1)$, $f(\Lambda_N) = \Lambda_1$ and $f^N|\Lambda_1$ is Bernoulli. In Pesin's proof, N is the number of elements of the Pinsker partition of $f|\Lambda$ and this number is bounded from above by $1/\theta$, where θ is the measure of an open set defined by the union of unstable local manifolds along center-stable manifolds. Also, all of the atoms of a Pinsker partition have the same measure. Under the hypotheses that f^j is ergodic for $1 \leq j \leq N$ he obtains that Λ_1 has full measure and so the Pinsker partition is trivial, that is $N = 1$ and hence f is Bernoulli.

Now we shall prove that under our hypotheses the number of atoms of the Pinsker partition is bounded from above by a constant N_0 in a neighborhood of f. Thus, it suffices to prove that there is a small neighborhood $\mathcal{V}$ of f such that for any $g \in \mathcal{V}$, g is ergodic for $1 \leq j \leq N_0$.

For this, recall lemmas 1 and 2 of [11]. The first states that there exists an $\alpha > 0$ such that for any $g \in \mathcal{U}$ there exists a subset A_g with positive measure such that any $x \in A_g$ has *hyperbolic times*, that is,

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j-0}^{n-1} \|Dg|_{E^c_g(g^j(x))}\| \leq -\alpha.$$

This is used to prove:

Lemma 3.9 (Lemma 2 of [11]). *There exists a neighborhood $\mathcal{V}$ of f and $s_0 > 0$ such that if $g \in \mathcal{V}$ is C^2 and $x \in A_g$ then there exists an integer $n \geq 0$ such that the size of $W^{cs}(g^{-n}(x))$ is at least s_0.*

So, if we take an atom of the Pinsker partition of $g \in \mathcal{V}$ which intersects A_g we have, by permutation, a "rectangle" contained in this atom with size at least s_0. So the measure of each atom is at least r_0 (the measure of the rectangle) for some $r_0 > 0$. In particular, the number N of atoms of the Pinsker partition is bounded from above by $1/r_0$ for any $g \in \mathcal{V}$.

Finally take T the intersection of the neighborhoods such that f^j is stably ergodic for $j = 1, ..., N$ and $\mathcal{V}$. For any $g \in T$ we have that g^j is ergodic for any $j = 1, ..., N$ and hence all the iterates of g are ergodic. This shows that g is Bernoulli. □

Proof of theorem E. By theorem C we know that f is Bernoulli and has negative exponents in the central direction almost everywhere. So, by ergodicity, there exists $\beta > 0$ such that

$$\lim_{n\to+\infty} \frac{1}{n} \log \|df^n|_{E^c_f(x)}\| < -\beta \text{ a.e. } x \in M.$$

Hence there exists $n_0 > 0$ such that

$$\int_M \log \|df^{n_0}|_{E^c_f(x)}\| dm \leq -\beta.$$

The same estimate holds for any $n \geq n_0$ and then, by Theorem D, f^n is robustly Bernoulli for $n \geq n_0$. Also, as in the previous proof, if we take $\mathcal{V}$ the neighborhood of f such that if $g \in \mathcal{V}$ then g^{n_0} is Bernoulli and g^{n_0} satisfies the hypothesis of lemma 3.9, we obtain that g satisfies the same conclusion of that lemma. Reasoning as in the previous proof (analyzing the Pinsker partition) we obtain a (uniform) bound for the number of atoms of the Pinsker partition for any $g \in \mathcal{W}$, where $\mathcal{W}$ is a smaller neighborhood of f. Thus there exists $m_0 \geq n_0$ such that if $g, ..., g^{m_0}$ are ergodic then g is Bernoulli. Now we take $\mathcal{V}_i$ the neighborhoods of f such that if $g \in \mathcal{V}_i$ then g^i is Bernoulli and consider $\mathcal{T} = (\bigcap_{i=1}^{m_0} \mathcal{V}_i) \cap \mathcal{W}$. This is a neighborhood of f such that any $g \in \mathcal{T}$ is Bernoulli, completing the proof. □

4. Examples

In this section we prove that an open set of robustly transitive and stably ergodic partially hyperbolic in $\mathbb{T}^3$ studied by Bonatti-Viana [10] are in fact stably Bernoulli.

We recall the construction of Bonatti and Viana [10]. Start with a linear Anosov diffeomorphism in $\mathbb{T}^3$ and fix a fixed point. Then perform a pitchfork bifurcation and obtain 2 hyperbolic fixed points with different indices of stability and make the two contracting eigenvalues of one of these fixed points to be complex. The resulting system is robustly transitive, stably ergodic, partially hyperbolic with central direction mostly contracting. That is, for m-almost every point x the Lyapunov exponents along the central direction are negative. The main property of this construction is that "in a neighborhood of this system every strong-stable leaf is dense in $\mathbb{T}^3$" and this implies robust transitiveness and, since there is uniform expansion in a neighborhood of the starting diffeomorphism, also gives topologically mixing.

We can prove that these systems are Bernoulli by our methods because they are partially hyperbolic with mostly contracting central direction and the denseness of the unstable manifolds holds for every iterate of any system in that neighborhood. So we obtain that every iterate is ergodic. This implies that the system is stably Bernoulli, in particular, robustly topologically mixing, because it preserves Lebesgue measure.

In the same way we can analyze the open set of ergodic diffeomorphisms (non-partially hyperbolic) studied by Tahzibi [18] and get that they are in fact Bernoulli. Indeed, let us define the open set $\mathcal{V}$ considered by Tahzibi. Let f_0 be an Anosov diffeomorphism on $\mathbb{T}^n$ whose foliations lifted to the universal covering are global graphs of C^1 functions. Let $V = \cup V_i$ be a finite union of small balls, such that f_0 has a periodic orbit q outside V. Then $f \in \mathcal{V}$ if:

- TM has small invariant continuous cone fields C^{cu} and C^{cs} containing E^u and E^s (the hyperbolic directions of f_0.
- f is C^1-close to f_0 on V^c. So there exist a $\sigma > 1$ such that
$$\|(DF|_{TxD^{cu}})-1\| < \sigma \text{ and } \|Df|_{TxD^{cs}}\| < \sigma.$$
- There exists some small δ_0 such that for $x \in V$:
$$\|(DF|_{TxD^{cu}})-1\| < 1+\delta_0 \text{ and } \|Df|_{TxD^{cs}}\| < 1+\delta_0.$$

  Where D^{cu} and D^{cs} are disks tangent to C^{cu} and C^{cs}.

Theorem 4.1. *Every $f \in \mathcal{V} \cap \mathrm{Diff}^2_m(\mathbb{T}^n)$ is stably Bernoulli. Also, any $f \in \mathcal{V}$ having volume hyperbolic property for $E^{cu} \oplus E^{cs}$ has an unique SRB measure which is a Bernoulli measure with full Lebesgue measure basin.*

Proof. We proceed as in the proof in [18]. Given any $f \in \mathcal{V}$ we will prove that f is Bernoulli. By the arguments in [18] (using dominated splitting and volume hyperbolicity), there exist a $c_0 > 0$ and a full Lebesgue measure set H such that for $x \in H$ we have
$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|(Df|_{E^{cu}_{f^i(x)}})^{-1}\| \le -c_0,$$

and

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log \|Df|_{E^{cs}_{f^i(x)}}\| \leq -c_0.$$

Then, get a disk tangent to C^{cu} everywhere and intersecting H in a positive Lebesgue measure (of the disk). And construct an invariant measure ν that is an accumulation point of the sequence of averages of forward iterates of Lebesgue measure restricted to the disk. By [2, Proposition 4.1], there exist a cylinder $\mathcal{C}$ (diffeomorphic to $B^u \times B^s$, balls with dimension $\dim(E^{cs})$ and $\dim(E^{cu})$ respectively), and a family $\mathcal{K}_\infty$ of pairwise disjoint disks D_i contained in $\mathcal{C}$ which are graphics over B^u, and such that the union has positive ν measure and ν (restricted to that union) has absolutely continuous conditional measure along the disks in $\mathcal{K}_\infty$. This measure is used to construct a cu-Gibbs state such that an ergodic component has positive measure (with respect to this measure). Hence, can write $M = \cup B(\mu_i)$ where μ_i are cu-Gibbs states (in fact ergodic SRB measures). We observe that Tahzibi constructed stable and unstable manifolds and proved absolute continuity for these systems.

Now we use the fact that if V is small enough then the stable manifold of q intersects any disk tangent to C^{cu} with radius bigger than ε_0 (for some small ε_0), the same holds for the unstable manifold. With this we have

Proposition 4.2 (Proposition 5.1 of [18]). *The stable manifold of q is dense and intersects transversally each D_i.*

Following Tahzibi, with help of this proposition, we can prove that the intersection of the basins $B(\mu_i)$ of each μ_i is non-empty. Because the $\mu_i's$ are ergodic, the $\mu_i's$ are all the same and hence f is ergodic. Now we stress that the construction of the $\mu_i's$ is getting ergodic components of the original measure. So, in the end, all of the $\mu_i's$ are equal to this measure.

Now fix $k > 0$. Since, by construction, ν is invariant for all f^k we can repeat the argument above. For each i we can write $B(\mu_i) = \cup B(\mu_i^k)$ where μ_i^k are the f^k-ergodic components of μ_i, and there exist $D_i^{k,\infty}$ a disk on $\mathcal{K}_\infty$ contained in $B(\mu_i^k)$ with the same property as D_i. The number of ergodic components can grow up, but to prove that their basins intersects depend only from proposition 4.2, which also holds for any iterate f^k. So again, all the μ_i^k are equal to the original measure μ_i, which coincides with ν. Thus, f^k is also ergodic all k.

Now applying Theorem 3.1 we conclude that f is Bernoulli. The dissipative case is analogous. The proof of Theorem 4.1 is complete.

□

Remark 2. We observe that this open set has robustly topologically mixing non partially hyperbolic diffeomorphisms.

5. Final Remarks

We point out as we can obtain "generic" statements of our theorems using some "generic" tools, as for instance, the following theorem:

Theorem 5.1 (Dolgopyat, Wilkinson). *Generically a strongly partially hyperbolic diffeomorphism f is stably accessible.*

So we can drop the accessibility hypothesis over a residual set.

An interesting question is if one can drop the robustly topologically mixing condition in a generic set inside the robustly transitive diffeomorphisms. The results in [1] indicate that there is a generic set of topologically mixing diffeomorphisms inside the set of robustly transitive diffeomorphisms. Using this result together with the result of Bonatti and Crovisier [7] stating that generically a volume preserving diffeomorphism is transitive we can drop the robustly topologically mixing condition over a generic set of partially hyperbolic diffeomorphisms.

Acknowledgments. We are thankful to Marcelo Viana for helpful conversations on the subject.

References

[1] F. Abdenur and A. Avila Robust Transitivity and Topological Mixing for C^1 diffeomorphisms, *in preparation*

[2] J. Alves, C. Bonatti and M. Viana SRB measures for partially hyperbolic systems whose central direction is mostly expanding. *Inven. Math*, 140:351–398, 2000.

[3] D. Anosov Geodesic flows on closed Riemannian manifolds of negative curvature. *Proc. Steklov. Inst. Math.*, 90, 1967.

[4] A. Arbieto and C. Matheus The Pasting Lemma I: The vector field case. *preprint*, 2003.

[5] J. Bochi, B. Fayad and E. Pujals *preprint*, 2003.

[6] J. Bochi and M. Viana. The Lyapunov Exponents of the generic volume preserving and symplectic systems. *preprint*, 2002.

[7] C. Bonatti and S. Crovisier Recurrence and genericite, *preprint.*

[8] C. Bonatti, L. Díaz and E. Pujals A C^1-generic dichotomy for diffeomorphisms: Weak forms of hyperbolicity or infinitely many sinks or sources, *to appear in the Annals of Math.*

[9] C. Bonatti, C. Matheus, M. Viana and A. Wilkinson. Abundance of stable ergodicity. *preprint*, www.impa.br/ viana , 2002.

[10] C. Bonatti and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly contracting. *Israel J. Math.*, 115:157–193, 2000.

[11] K. Burns, D. Dolgopyat and Y. Pesin. Partial Hyperbolicity, Lyapunov Exponents and Stable Ergodicity. *J. Stat. Phys.*, 108, no. 5-6:927–942, 2002.

[12] D. Dolgopyat and A. Wilkinson. Stable accessibility is C^1 dense. *preprint*, .

[13] H. Furstenberg Strictly ergodicity and transformations of the torus. *Amer. J. of Math.*, 83:573–601, 1961.

[14] Y. Pesin. Characteristic Lyapunov exponents and smooth ergodic theory. *Russian Mathematical Surveys*, 32: 55–112, 1977.

[15] C. Pugh and M. Shub Stable Ergodicity and Julienne quasiconformality. *J. Eur. Math. Soc.*, 2, 1-52, 2000.

[16] D. Szasz. Boltzmann's ergodic hypothesis, a conjecture for centuries?. *Hard ball systems and the Lorentz gas*, Encyclopaedia Math. Sci. 101, 421–448, Springer Verlag, Berlin, 2000.
[17] A. Tahzibi. Robust transitivity implies almost robust ergodicity *preprint*, 2003.
[18] A. Tahzibi. Stably ergodic systems which are not partially hyperbolic. *Ph.D. thesis-IMPA, preprint*, www.preprint.impa.br/Shadows/SERIE_C/2002/9.html, 2002.

Alexander Arbieto (alexande@impa.br)
IMPA, Estrada D. Castorina 110, Jardim Botânico, 22460-320 Rio de Janeiro, Brazil

Carlos Matheus (cmateus@impa.br)
IMPA, Estrada D. Castorina 110, Jardim Botânico, 22460-320 Rio de Janeiro, Brazil

Maria José Pacífico (pacifico@im.ufrj.br and pacifico@impa.br)
IM-UFRJ. Cidade Universitária, Ilha do Fundão. Rio de Janeiro, RJ, Brazil. CEP 21945-970. Caixa Postal 68530.