A pasting lemma and some applications for conservative systems

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Abstract

We prove that in a compact manifold of dimension $n \geq 2$, a $C^{1+\alpha}$ volume-preserving diffeomorphisms that are robustly transitive in the C^1-topology have a dominated splitting. Also we prove that for 3-dimensional compact manifolds, an isolated robustly transitive invariant set for a divergence-free vector field can not have a singularity. In particular, we prove that robustly transitive divergence-free vector fields in 3-dimensional manifolds are Anosov. For this, we prove some "pasting" lemma, which allows to make perturbations in conservative systems.

1 Introduction

In this article, we study some properties of conservative dynamical systems over a smooth n-dimensional Riemannian manifold M without boundary, in the discrete and continuous case (here $n \geq 2$). The main technical result in this work is a general tool for perturbing C^2 conservative dynamical systems over (large) compact subsets of phase space, obtaining a globally defined and still conservative perturbations of the original system. In fact, we give several versions of these pasting lemmas, for discrete as well as for continuous time systems (and in this case is enough suppose that the system is C^1). Before hand, let us describe some of its consequences in dynamics. In the rest of the paper, the Riemannian metric is C^∞.

Robustly transitive diffeomorphisms

A dynamical system is *robustly transitive* if any C^1 nearby one has orbits that are dense in the whole ambient space. Recall that a *dominated splitting* is a continuous decomposition $TM = E \oplus F$ of the tangent bundle into continuous subbundles which are invariant under the derivative Df and such that $Df|_E$ is more expanding/less contracting than $Df|_F$ by a definite factor (see precise definitions and more history on section 5).

We prove: "*Let $f : M^n \to M^n$ a $C^{1+\alpha}$ volume preserving diffeomorphism robustly transitive among volume preserving diffeomorphisms, where $n \geq 2$. Then f admits a dominated splitting of the tangent bundle*". This is an extension of a theorem by Bonatti, Diaz and Pujals.

Also this solves a question posed by Tahzibi [T]: every C^1-stably ergodic $C^{1+\varepsilon}$-diffeomorphism has a dominated splitting. We observe that this result was used by Bochi, Fayad and Pujals [BFP] to prove that there exists an open and dense set of C^1-stably ergodic $C^{1+\varepsilon}$-diffeomorphisms that are non-uniformly hyperbolic.

Robustly transitive vector fields

First we prove an extension of a theorem by Doering in the conservative setting: "*Let X be a divergent free vector field robustly transitive among divergence free vector fields in a three dimensional manifold. Then X is Anosov.*

It has been shown by Morales-Pacífico and Pujals [MPP] that robustly transitive sets Λ of a 3-dimensional flows are *singular hyperbolic*: there exists a dominated splitting of the tangent bundle restricted to Λ such that the one-dimensional subbundle is hyperbolic(contracting or expanding and the complementary one is volume hyperbolic). Robust transitivity means that Λ is the maximal invariant set in a neighborhood U and, for any C^1 nearby vector field, the maximal invariant set inside U contains dense orbits.

For generic dissipative flows robust transitivity can not imply hyperbolicity, in view of the phenomena of Lorenz-like attractors containing both equilibria and regular orbits. However we also prove that Lorenz-like sets do not exist for conservative flows: "*If Λ is a robust transitive set within divergence free vector fields then Λ contains no equilibrium points.*

The paper is organized as follows. In section 2 we prove a result of denseness of volume preserve systems with higher differentiability among C^1 volume preserving systems. In section 3 we prove the pasting lemmas for volume preserving diffeomorphisms and vector fields, and for symplectic diffeomorphisms. In section 4, we use the pasting lemma to study robustly transitive volume preserving flows in 3-manifolds. In section 5, we use the pasting lemma to study robustly transitive conservative diffeomorphisms. Finally in the appendix, we investigate why some techniques involved in the proof of the pasting lemma for diffeomorphisms don't work for C^1 diffeomorphisms. Indeed, we given a "non-estimate" argument to get this.

1.1 Outline of the proofs

To prove the statement for diffeomorphisms, we fix f a robuslty transitive volume-preserving diffeomorphism then we use a result of Bonatti-Diaz-Pujals, that says that either any homoclinic class $H(p,f)$ has a dominated splitting or there exists a sequence of volume preserving diffeomorphisms $g_n \to f$ and periodic points x_n of g_n, such that $Dg_n^{p(x_n)}(x_n) = Id$ where $p(x_n)$ is the period of x_n. Now we use the pasting lemma, to obtain another *volume preserving* h close to f such that in a neighborhood U of x_n we have $h = id$. This contradicts the robust transitivity of f. An important observation here is that to use the pasting lemma, we the original diffeomorphism need to be $C^{1+\varepsilon}$.

Now to prove the statement for vector fields, we follow the arguments of Morales-Pacifico-Pujals. The idea is that if there exists a singularity, then this singularity is Lorenz-like, to obtain that we use the pasting lemma to get a contradiction with the robust transitivity if the singularity is not Lorenz-like. Then an abstract lemma says that this set is a proper attractor. So, either the set is the whole manifold or we have a proper attractor. In the first case we prove that the vector field is Anosov, so there are not singularities, in the second we have a contradiction because the vector field is divergence-free. Another point is that for vector fields, we can use the pasting lemma for C^1 vector fields, since we know that C^∞ divergence-free vector fields are dense in C^1 divergence-free vector fields. This regularity is also used to perform a Shilnikov bifurcation in a technical lemma.

Finally, the pasting lemma says that if we have a conservative system (diffeomorphism or flow) in M a smooth manifold, and we have another conservative system defined in some open set U of M, then we can "cut" the first system in some small open set of U and "paste" the second system there, and the result system is again conservative. The idea is to "glue" the two systems using unity partitions, so we have another system that is conservative inside the domains where the unity partitions are equal to 1. But, of course, we have an "annulus" (a domain with boundary in the manifold) where the system is not conservative (divergence not equal to zero in the flow case or determinant of the derivative not equal to one in the diffeomorphism case). Now we will fix this problem, finding another system in this domain which "fix" the non-conservative part, for instance, in the flow case, if $h(x)$ is the divergence of the system in the domain, then we search for a vector field with divergence equal to $-h(x)$, then we sum the two vector fields and we have a conservative vector field in the domain. But of course, we want that at the boundary the vector field is zero and that we can extend it as zero to the rest of the manifold. The problem to find that vector field is then a PDE problem, and we use PDE arguments to solve it and also obtain smallness for the norm of the vector field that solves the equation. The procedure is the same for diffeomorphisms.

2 Denseness Result

Definition 2.1. Let $M = M^n$ a Riemannian manifold without boundary with dimension $n \geq 2$. We say that a vector field X is conservative if div $X = 0$, we denote by $X \in \mathfrak{X}_m(M)$.

Of course if X is conservative, the flow is volume preserving by the Liouville formula.

Now we proving that C^∞ conservative vector fields are C^1-dense in C^1 conservative vector fields:

Theorem 2.2. *$\mathfrak{X}_m^\infty(M)$ is C^1-dense on $\mathfrak{X}_m^1(M)$.*

Proof. Let us fix $X \in \mathfrak{X}_m^1(M)$. Locally we proceed as follows: take a conservative local charts (U, Φ) (Moser's theorem [Mo]) and get η_ε a C^∞ Friedrich's mollifier (i.e. let η a C^∞ function with support in the ball $B(0,1)$ such that $\int \eta = 1$ and take $\eta_\varepsilon(x) = \varepsilon^{-n}\eta(x/\varepsilon)$). If $X = (X_1, \ldots, X_n)$ is the expression of X in local coordinates, we define $X_\varepsilon = (X_1 * \eta_\varepsilon, \ldots X_n * \eta_\varepsilon)$ a C^∞ vector field (where the star operation is the convolution). Now we note that $\frac{d}{dx_i}(X_i * \eta_\varepsilon) = (\frac{d}{dx_i}X_i) * \eta_\varepsilon$, so div $X_\varepsilon = 0$. Furthermore X_ε converges to X in the C^1-topology.

Now we take local charts (U_i, ϕ_i), $i = 1, ..., m$ as above and ξ_i a partition of unity subordinated to U_i, and open sets W_i such that $W_i \subset V_i := supp(\xi_i) \subset U_i$ satisfying $\phi_i(W_i) = B(0, 1/3)$ and $\phi_i(V_i) = B(0, 2/3)$, $\xi_i|_{W_i} = 1$ and $\Omega := M \backslash int(W_i)$ is a manifold with boundary C^∞, here $B(0, r)$ is the ball with center 0 and radii r. And we fix C as the constant given by theorem 2.3 (below).

In each of those charts we get X_i a C^∞ conservative vector fields C^1-close to X by theorem 2.3 such that if we take $Y = \sum_i \xi_i X_i$ a C^∞ vector field and we denote $g = \text{div } Y$ a C^∞ function then:

- g is $\frac{\varepsilon}{2C}$-C^1-close to 0 (where C is the constant in the below theorem). Indeed, we have:

$$g = \sum_i \nabla \xi_i \cdot X_i + \xi_i \cdot \text{div } X_i = \sum_i \nabla \xi_i \cdot X_i.$$

  So, it is suffice to take the X_i's C^1-close enough to X.

- Y is $\frac{\varepsilon}{2}$-C^1-close to X.

By the divergence theorem we get that $\int_\Omega g = 0$, since:

$$\int_\Omega \text{div } Y = \int_{\partial\Omega} Y \cdot N = -\sum_i \int_{\partial W_i} X_i \cdot N = -\sum_i \int_{W_i} \text{div } X_i = 0.$$

Now, we state the following results from PDE's by Dacorogna-Moser [DM, Theorem 2] on the divergence equation which will play also a crucial role in the next section in the proofs of the pasting lemmas:

Theorem 2.3. *Let Ω a manifold with C^∞ boundary. Let $g \in C^{k+\alpha}(\Omega)$ (with $k+\alpha > 0$) such that $\int_\Omega g = 0$. Then there exist v a $C^{k+1+\alpha}$ vector field (with the same regularity at the boundary) such that:*

$$\begin{cases} \text{div } v(x) = g(x), \ x \in \Omega \\ v(x) = 0, \ x \in \partial\Omega \end{cases}$$

Furthermore there exist $C = C(\alpha, k, \Omega) > 0$ such that $\|v\|_{k+1+\alpha} \leq C\|g\|_{k+\alpha}$. Also if g is C^∞ then v is C^∞.

Remark 2.4. In fact Dacorogna-Moser prove this result for open sets in $\mathbb{R}^n$. But the same methods work for manifolds, because it follows from the solvability of $\triangle u = f$ with Neumann's condition. Also, if the manifold has boundary, we obtain solutions that have the required regularity at the boundary (see [DM, p.3] or [H, p.265]; see also [Au, Ch 4,Theorem 4.8], [GT] and [LU]).

So, we get v the vector field given by theorem 2.3 and we define $Z = Y - v$. We observe that Z is a C^∞ vector field because v is C^∞ at the boundary. Also div $Z = 0$. Finally since Y is $\frac{\varepsilon}{2}$-C^1-close to X and v is $\frac{\varepsilon}{2}$-C^1-close to 0 we get Z ε-C^1-close to X. The proof is complete. $\square$

Remark 2.5. As Ali Tahzibi pointed out to the second author after the conclusion of this work, the previous theorem was proved by Zuppa [Zu] (in 1979) with a different proof. In fact, since Dacorogna-Moser's result was not available at that time, Zuppa uses that the Laplacian operator admits a right inverse.

3 The Pasting Lemmas

3.1 Vector Fields

We start proving a weak version of the pasting lemma:

Theorem 3.1 (The $C^{1+\alpha}$-Pasting Lemma for Vector Fields). *Let M^n a compact Riemannian manifold without boundary with dimension $n \geq 2$. Given $\alpha > 0$ and $\varepsilon_0 > 0$ there exists $\delta_0 > 0$ such that if $X_0 \in \mathfrak{X}_m^{1+\alpha}(M)$, K is a compact subset of M and $Y_0 \in \mathfrak{X}_m^{1+\alpha}(M)$ is δ_0-C^1-close to X_0 on a small neighborhood U of K. Then there exist a $Z_0 \in \mathfrak{X}_m^{1+\alpha}(M)$, and V and W such that $K \subset V \subset U \subset W$ satisfying $Z_0|_V = Y_0$, $Z_0|_{int(W^c)} = X_0$ and Z_0 is ε_0-C^1-close to X_0. Furthermore, if X_0 and Y_0 are C^∞ then Z_0 is also C^∞.*

Proof. Let V be a neighborhood of K with C^∞ boundary, compactly contained in U such that U and $int(V^c)$ is a covering of M. Let ξ_1 and ξ_2 be a partition of unity subordinated to this covering such that $\xi_1|_V = 1$ and there exist $W \subset U^c$ such that $\xi_2|_W = 1$. Let $\Omega = M \backslash V \cup W$ be a (non empty) manifold with C^∞ boundary and we fix C as the constant given by theorem 2.3.

Now we choose δ_0 such that if Y_0 is $C^{1+\alpha}$ δ_0-C^1-close to X_0 then $T = \xi_1 Y_0 + \xi_2 X_0$ is $\frac{\varepsilon_0}{2}$-C^1-close to X_0 and $g = \text{div}\, T$ is $\frac{\varepsilon_0}{2C}$-C^1-close to 0. Indeed, we note that $g = \nabla\xi_1 \cdot Y_0 + \nabla\xi_2 \cdot X_0$ is C^1-close to 0 if we get Y_0 sufficiently C^1-close to X_0. Also note that by the divergence theorem (see the proof of the theorem 2.2 above) $\int_\Omega g = 0$. Clearly, g is a $C^{1+\alpha}$ function.

So we take v the vector field given by the theorem 2.3 and extend it as 0 in the rest of the manifold. This extension is a $C^{2+\alpha}$-vector field (because we have regularity of v at the boundary) and $Z_0 = T - v$ is a $C^{1+\alpha}$-vector field ε_0-C^1-close to X_0. Now $Z_0 = Y_0$ in V so $divZ_0 =$ in V, also in W we have $Z_0 = X_0$ so again $divZ_0 = 0$, finally in Ω we have $divZ_0 = div(T - v) = divT - divv =$ since v is a solution of the PDE, so Z_0 satisfies the statement of the theorem. $\square$

We extend this theorem to the C^1 topology using the denseness theorem:

Theorem 3.2 (C^1-Pasting Lemma for vector fields). *Let M^n a compact Riemannian manifold without boundary with dimension $n \geq 2$. Given $\varepsilon > 0$ there exists $\delta > 0$ such that if $X \in \mathfrak{X}^1_m(M)$, K a compact subset of M and $Y \in \mathfrak{X}^{1+\alpha}_m(M)$ is δ-C^1-close to X on a small neighborhood U of K, then there exists a $Z \in \mathfrak{X}^{1+\alpha}_m(M)$ and V such that $K \subset V \subset U$ satisfying $Z|_V = Y$ and Z is ε-C^1-close to X. If $Y \in \mathfrak{X}^\infty_m(M)$ then Z is also in $\mathfrak{X}^\infty_m(M)$.*

Proof. Let ε and set $\varepsilon_0 = \frac{\varepsilon}{2}$ and let δ_0 be given by theorem 3.1. Let $\sigma = \min\{\frac{\varepsilon}{2}, \frac{\delta_0}{2}\}$, and $\delta = \frac{\delta_0}{2}$.

Now get X_0 σ-C^1-close to X by theorem 2.2. So if Y is δ-C^1-close to X then Y is δ_0-C^1-close to X_0. By theorem 3.1 we have Z ε_0-C^1-close to X_0 satisfying the conclusions of the theorem and Z is ε-C^1-close to X. $\square$

Using the same arguments of Dacorogna-Moser, we can produce perturbations in higher topologies, provided that the original systems are smooth enough. We will use the following notation, if r is a real number, we denote by $[r]$ its integer part. In the lemma below $\widetilde{r}$ denotes a real number such that $\widetilde{r} = r$ if r is an integer and $[r] \leq \widetilde{r} < r$ if r is not an integer. So we obtain:

Theorem 3.3 (Pasting Lemma with Higher Differentiability). *Let M^n a compact Riemannian manifold without boundary with dimension $n \geq 2$. Let $X \in \mathfrak{X}^{k+\alpha}_\omega(M)$ (k integer, $0 \leq \alpha < 1$ and $k + \alpha > 1$). Fix $K \subset U$ a neighborhood of a compact set K. Let Y be a vector field defined in $U_1 \subset U$, U_1 an open set containing K. If $Y \in \mathfrak{X}^{k+\alpha}_\omega(M)$ is sufficiently C^r-close to f for $1 \leq r \leq k + \alpha$ (r is real) then there exists some C^r-small $C^{k+\alpha}$-perturbation Z of X and an open set $V \subset U$ containing K such that $Z = X$ outside U and $Z = Y$ on V.*

Furthermore, there exists a constant $C = C(f, \dim(M), K, U, r)$, δ_0 and ϵ_0 such that if Y is $(C \cdot \delta^{\widetilde{r}} \cdot \epsilon)$-$C^r$ close to X, $\delta < \delta_0$, $\epsilon < \epsilon_0$ then Z is ϵ-C^r close to X and support$(Z - X) \subset B_\delta(K)$ (= the δ-neighborhood of K).

3.2 A Weak Pasting Lemma for Conservative Diffeomorphisms

In this subsection we prove a weak version of the pasting lemma for C^2 conservative diffeomorphisms which we will apply to the study of dominated splittings for conservative systems.

First, we note that a similar result as theorem 2.3 holds for diffeomorphisms, as showed by Dacorogna-Moser [DM, Theorem 1, Lemma 4] (see also [RY]):

Theorem 3.4. *Let Ω be a compact manifold with C^∞ boundary. Let $f, g \in C^{k+\alpha}(\Omega)$ $(k+\alpha > 0)$ be such that $f, g > 0$. Then there exists a $C^{k+1+\alpha}$ diffeomorphism φ (with the same regularity at the boundary) such that:*

$$\begin{cases} g(\varphi(x))\det(D\varphi(x)) = \lambda f(x),\ x \in \Omega \\ \varphi(x) = x,\ x \in \partial\Omega \end{cases}$$

where $\lambda = \int g / \int f$. Furthermore there exists $C = C(\alpha, k, \Omega) > 0$ such that $\|\varphi - id\|_{k+1+\alpha} \leq C\|f-g\|_{k+\alpha}$. [1] *Also if f, g are C^∞ then φ is C^∞.*

Remark 3.5. A consequence of the Dacorogna-Moser's theorem is the fact that $\mathrm{Diff}^{1+\alpha}_\omega(M)$ is path connected.

So we obtain a weak version of the pasting lemma that allows us to change the diffeomorphism by its derivative:

Theorem 3.6 (Weak Pasting Lemma). *Let M^n a compact Riemannian manifold without boundary with dimension $n \geq 2$. If f is a C^2-conservative diffeomorphism and x is a point in M, then for any $0 < \alpha < 1, \varepsilon > 0$, there exists a ε-C^1-perturbation g (which is a $C^{1+\alpha}$ diffeomorphism) of f such that, for some small neighborhoods $U \supset V$ of x, $g|_{U^c} = f$ and $g|_V = Df(x)$ (in local charts).*

Proof. First consider a perturbation h of f such that $h(y) = \rho(y)(f(x) + Df(x)(y-x)) + (1-\rho(y))f(y)$ (in local charts), where ρ is a bump function such that $\rho|_{B(x,r/2)} = 1$, $\rho|_{M-B(x,r)} = 0, |\nabla\rho| \leq C/r$ and $|\nabla^2\rho| \leq C/r^2$. Now, we note that $||h-f||_{C^1} \leq C \cdot ||f||_{C^2} \cdot r$ and $||h-f||_{C^2} \leq C \cdot ||f||_{C^2}$, where C is a constant. Then if we denote by $\theta(y) = \det Dh(y)$ the density function of h, by the previous calculation, we obtain that θ is $C \cdot ||f||_{C^2} \cdot r$-$C^0$-close to 1 (the

[1] This constant is invariant by conformal scaling $x \to rx$

density function of f) and $||\theta - 1||_{C^1} \leq C \cdot ||f||_{C^2}$. By the classical property of convexity of the Hölder norms, we obtain that $||\theta - 1||_\alpha \leq C||f||_{C^2} \cdot r^{1-\alpha}$. Hence, increasing the constant C if necessarily, we obtain the same bound for the function $\widehat{\theta}(y) = (\det Dh(h^{-1}(y)))^{-1}$. So, applying the lemma 3.4 for the domain $\Omega = \overline{B(x,r)} - B(x,r/2)$ and the data function $\widehat{\theta}$, we obtain that there exists a $C^{1+\alpha}$ diffeomorphism χ that is a solution to the equation of lemma 3.4 which is $C \cdot ||f||_{C^2} \cdot r^{1-\alpha}$-close to the identity. and is regular at the boundary of Ω where $\chi = id$. Now observe that $g = \chi \circ h$ is conservative and is close to f provided that r is small. Smallness is obvious, and since $g(y) = f(x) + Df(x)(y-x)$ (in local charts) in $B(x,r/2)$ we have that $\det Dg(y) = 1$ there, and in $M - B(x,r)$ we have $g(y) = f(y)$ so also $\det Dg(y) = 1$, finally in Ω we have that $\det Dg(x) = \det D\chi(h(x))Dh(x) = \det D\chi(y)Dh(h^{-1}(y)) = 1$, so g is conservative and this completes the proof. □

Remark 3.7. We observe that the result also holds for $C^{1+\beta}$-diffeomorphisms, instead of C^2-diffeomorphisms. But in this case the perturbation is a $C^{1+\alpha}$-diffeomorphism. Indeed, we can use the following interpolation formula, which can be found in [Z]:

$$\|f\|_{(1-\gamma)a+\gamma b} \leq C\|f\|_a^{1-\gamma} \cdot \|f\|_b^\gamma.$$

Remark 3.8. In section 5 of this paper, we apply this weak pasting lemma to prove that every robust transitive conservative diffeomorphism admits a dominated splitting. We hope that this lemma can be applied to obtain some other interesting consequences. In fact, recently using the pasting lemma, Bochi-Fayad-Pujals were able to show that C^1-stably C^2 ergodic systems are (generically) non-uniformly hyperbolic (see [BFP]).

3.3 A Weak Pasting Lemma for Symplectic Diffeomorphisms

The above perturbation lemma can be done for symplectic diffeomorphisms without loosing the structure outside the neighborhood of a periodic orbit, as follows. We also have the same result as in the conservative vector fields case, if we require more differentiability of the diffeomorphism .

Lemma 3.9. *If f is a C^k-symplectic diffeomorphism ($k \geq 1$), $x \in Per^n(f)$ is a periodic point of f and g is a local diffeomorphism (C^k-close to f) defined in a small neighborhood U of the f-orbit of x, then there exists a C^k-*

symplectic diffeomorphism h (C^k-close to f) and some neighborhood $U \subset V$ of x satisfying $h|_U = g$ and $g|_{V^c} = f$

Proof. Consider the f-orbit $\mathcal{O}(x)$ of x. Since all the perturbations are local, by Darboux's theorem(see page 221 of [KH]), we can use local coordinates near each point in $\mathcal{O}(x)$, say $\varphi_i : U_i \to V_i$, where $f^i(x) \in U_i$ and $0 \in V_i = B(0, \gamma_i) \subset \mathbb{R}^n$. With respect to these coordinates, we have the local maps $f : B(0, \delta_i) \to V_{i+1}$, where $V_n = V_1$, $B(0, \delta_i) \subset V_i$ are small neighborhoods of 0, $f(0) = 0$. In the symplectic case, we fix some bump function $\lambda : \mathbb{R} \to [0, 1]$ such that $\lambda(z) = 1$ for $z \leq 1/2$ and $\lambda(z) = 0$ for $z \geq 1$. Let $S_{1,i}$ be a *generating function* for $g : B(0, \delta_i) \to V_{i+1}$ and $S_{0,i}$ a generating function for $f : B(0, \delta_i) \to V_{i+1}$ (see [X] for definitions and properties of generating functions). Finally, set

$$S(x, y) = \lambda(\frac{2||(x, y)||}{\delta_i}) \cdot S_{1,i}(x, y) + \big[1 - \lambda(\frac{2||(x, y)||}{\delta_i})\big] \cdot S_{0,i}(x, y).$$

In particular, if h is the symplectic map associated to S, then $h = g$ in $B(0, \delta_i/4)$ and $g = f$ outside of $B(0, \delta_i/2)$. To summarize, this construction give us a symplectic ϵ-C^1-perturbation g of f, if the numbers δ_i are small, such that h coincides with g, near to each point $f^i(x)$. □

Remark 3.10. We can not obtain global versions of the pasting lemma for symplectic diffeomorphism using this arguments, as proved for conservative ones, since we used *generating functions* which is a local tool.

In the next sections we study some other consequences of the pasting lemmas for conservative systems.

4 Robustly Transitive Conservative Flows in 3-manifolds are Anosov

Let M^3 be a compact three manifold and ω a smooth volume in M. A vector field $X \in \mathfrak{X}^1_m(M)$ is C^1-*robustly transitive* (in $\mathfrak{X}^1_m(M)$) if there is $\epsilon > 0$ such that every ϵ-C^1-perturbation $Y \in \mathfrak{X}^1_m(M)$ of X is transitive.

In this section we study conservative flows in 3-dimensional manifolds. Recently the following result has been obtained by Morales, Pacífico and Pujals [MPP]: *Any isolated singular C^1 robustly transitive set is a proper* *attractor*. This result is related with a theorem by C. Doering [D]: *A C^1 robustly transitive flow is Anosov*. In the same spirit of the diffeomorphism case, we can use the pasting lemma to prove the conservative version of these results. We now state the theorems:

Theorem A. *Let $X \in \mathfrak{X}^1_m(M)$ be a conservative C^1-robustly transitive vector field in a 3-dimensional compact manifold. Then X is Anosov.*

Theorem B. *If Λ is an isolated robustly transitive set of a conservative vector field $X \in \mathfrak{X}^1_m(M)$ in a 3-dimensional compact manifold then it cannot have a singularity.*

In particular, because the geometrical Lorenz sets are robustly transitive and carrying singularities [V], as a corollary we have:

Corollary 4.1. *There are no geometrical Lorenz sets (in the conservative setting) for C^1 conservative vector fields in 3-dimensional compact manifolds.*

4.1 Some Lemmas

As in the diffeomorphism case we cannot have elliptic periodic orbits (or singularities) in the presence of robust transitivity. In fact, generically we have that any *singularity* is hyperbolic by an argument of Robinson [R], but in the case where we have robust transitivity, we can give another proof using the pasting lemma. More precisely we have the following lemmas:

Lemma 4.2. *If $X \in \mathfrak{X}^1_m(M)$ is C^1-robustly transitive then there are no elliptic singularities (i.e. the spectrum of $DX(p)$ does not intersect S^1).*

Proof. Suppose that there exist an elliptic singularity σ. Then we get K a small compact neighborhood of σ such that X_K is C^1-close to the linear vector field $DX(\sigma)$. Now, we use the theorem 3.2 to obtain a vector field Y C^1-close to X which is $DX(\sigma)$ (in local charts) inside an (possibly smaller) invariant neighborhood. This contradicts transitivity. □

The novelty is that in the robustly transitive case, we also have the same result for periodic orbits:

Lemma 4.3. *If $X \in \mathfrak{X}^1_m(M)$ is C^1-robustly transitive then every periodic orbit is hyperbolic.*

Proof. Suppose that there exist an elliptic periodic orbit p. Then we get K a small compact neighborhood of $O(p)$ (e.g., a tubular neighborhood) such that the linear flow induced by the periodic orbit restricted to K is C^1-close to X. Now, we use the theorem 3.2 to obtain a vector field Y C^1-close to X with a (possibly smaller) invariant neighborhood. This contradicts transitivity. $\square$

Remark 4.4. The above lemmas hold for robustly transitive sets strictly contained in the whole manifold with only minor modifications on the statement.

4.2 Proof of Theorem A

We already know that the singularities are hyperbolic. Also by lemma 4.3 every periodic orbit is hyperbolic. As usual we will denote by $Crit(X)$ the set of periodic orbits and singularities.

Remark 4.5. We observe that the hyperbolicity of hyperbolic periodic orbits needs the robust transitivity and the pasting lemma since Kupka-Smale's theorem is false on dimension three for conservative vector fields.

Now we note that the result below by Doering [D] also holds in the conservative case.

Theorem 4.6. *Let $\mathcal{S}(M^3) = \{X \in \mathfrak{X}^1_m(M);$ every $\sigma \in Crit(X)$ is hyper bolic$\}$. If $X \in \mathcal{S}(M^3)$ then X is Anosov.*

Now if X is robustly transitive then we already know that any critical element is hyperbolic so we have the result.

We only sketch Doering's proof, since is straight forward that it works in the conservative case:

Lemma 4.7. *If $X \in \mathcal{S}(M^3)$ then $M^3 - Sing(X)$ has a dominated splitting. In particular there are no singularities: $Sing(X) = \emptyset$.*

Proof. We start with a result on [D, Proposition 3.5]:

Claim: *There exist $c > 0$ and $0 < \lambda < 1$ and $\mathcal{U}$ neighborhood of X in $\mathcal{S}(M)$ such that there exist (C, λ)-dominated splitting for any $q \in Per(Y)$ and $Y \in \mathcal{U}$* (for the definition of (C, λ) dominated splitting for vector fields see [D]).

Let x be a regular point for any $Y \in \mathcal{U}$, so there are Y_n such that $x \in Per(Y_n)$ (Pugh's Closing Lemma). Now we get the P^{Y_n}-hyperbolic splitting over $O_{Y_n}(x)$ and by compactness of the Grassmannian we have a splitting over $O_X(x)$ (by saturation) then by the Claim we have that this splitting is a (C, λ)-dominated splitting. Now by the abstract lemma of Doering [D, Lemma 3.6]:

Lemma 4.8. *If Λ is an invariant set of regular points of X with a dominated splitting over every orbit and such that every singularity of X in the closure of Λ is a hyperbolic saddle then Λ has a dominated splitting.*

So using the lemma, we obtain a dominated splitting on $M - Sing(x)$.

By hyperbolicity x_0 is an interior point of $(M - Sing(X)) \cup x_0$, but this is a contradiction with the domination by the abstract [D, Theorem 2.1]:

Theorem 4.9. *Let Λ an invariant set of regular points of X and suppose that x_0 is a hyperbolic saddle-type singularity of X. If Λ has a dominated splitting then x_o is not an interior point of $\Lambda \cup \{x_0\}$.*

□

The proof finish with the following result that is also available in the conservative setting (see [Li1], [T1], also see [BDV] and references therein).

Lemma 4.10 (Liao's theorem). *If $X \in \mathcal{S}(M)$ without singularities then X is Anosov.*

4.3 Proof of the Theorem B

We follow the steps of [MPP]. We denote by $\mathcal{U}$ a neighborhood such that every $Y \in \mathcal{U}$ is transitive.

As in the proof of the previous theorem, we can suppose that any singularity or periodic orbit is hyperbolic. The first step is to prove that in fact it is a Lorenz-like singularity. Recall that a singularity σ is *Lorenz-like* if its eigenvalues $\lambda_2(\sigma) \leq \lambda_3(\sigma) \leq \lambda_1(\sigma)$ satisfy either $\lambda_3(\sigma) < 0 \Rightarrow -\lambda_3(\sigma) < \lambda_1(\sigma)$ or $\lambda_3(\sigma) > 0 \Rightarrow -\lambda_3(\sigma) > \lambda_2(\sigma)$.

Lemma 4.11. *Let M a three-dimensional compact manifold boundaryless. If $X \in \mathfrak{X}^1_m(M)$ is C^1-robustly transitive conservative vector field then any singularity σ is Lorenz-like.*

Proof. First of all, the eigenvalues of σ are real. Indeed, if $\omega = a + ib$ is an eigenvalue of σ, then the others are $a - ib$ and $-2a$. Since the singularity is hyperbolic, we can suppose also that $X \in \mathfrak{X}^{\infty}_m(M)$ (using theorem 2.2). By the Connecting Lemma [WX] and [Ha], we can assume that there exist a loop Γ associated to σ which is a Shilnikov Bifurcation. Then by [BS, page 338] there is a vector field C^1-close to X with an elliptic singularity which gives a contradiction with Lemma 4.2(We observe that the regularity for the bifurcation in [BS] is more than or equal to seven).

Let $\lambda_2(\sigma) \leq \lambda_3(\sigma) \leq \lambda_1(\sigma)$ the eigenvalues. Now, $\lambda_2(\sigma) < 0$ and $\lambda_1(\sigma) > 0$, because σ is hyperbolic and there are no sources or sinks (div $X = 0$). Also, using that $\sum \lambda_i = 0$ we have that:

- $\lambda_3(\sigma) < 0 \Rightarrow -\lambda_3(\sigma) < \lambda_1(\sigma)$,
- $\lambda_3(\sigma) > 0 \Rightarrow -\lambda_3(\sigma) > \lambda_2(\sigma)$.

□

Now we give a sufficient condition which guarantees that Λ is the whole manifold.

Theorem 4.12. *If Λ is a transitive isolated set of a conservative vector field X, such that:*

1. *Λ contains robustly the unstable manifold of a critical element $x_0 \in Crit_X(\Lambda)$;*
2. *Every $x \in Crit_X(\Lambda)$ is hyperbolic;*

Then $\Lambda = M$.

Proof. We will use the following abstract lemma, which can be found in [MPP]:

Lemma 4.13 (2.7 of [MPP]). *If Λ is an isolated set of X with U an isolating block and a neighborhood W of Λ such that $X_t(W) \subset U$ for every $t \geq 0$, then Λ is an attracting set of X.*

We will prove that the hypothesis of the previous lemma are satisfied. This will imply that $\Lambda = M$ since there are no attractors for conservative systems. If there are no such W, then there exists $x_n \to x \in \Lambda$ and $t_n > 0$ such that $X_{t_n}(x_n) \in M - U$ and $X_{t_n}(x_n) \to q \in \overline{M - U}$. Now, let $V \subset \overline{V} \subset int(U) \subset U$ a neighborhood of Λ. We have $q \notin \overline{V}$.

There exists a neighborhood $\mathcal{U}_0 \subset \mathcal{U}$ of X such that $\Lambda_Y(U) = \bigcap_{t\geq 0} Y_t(U) \subset V$ for every $Y \in \mathcal{U}_0$. And by hypothesis we have $W^u_Y(x_0(Y)) \subset V$ for a critical element $x_0 \in Crit_Y(\Lambda_Y)$.

If $x \notin Crit_X(\Lambda)$, then let z be such that $\Lambda = \omega_X(z)$ and $p \in W^u_X(x_0) - O_X(x_0)$, so $p \in \Lambda$. Now we have $z_n \in O_X(z)$, $t'_n > 0$ such that $z_n \to p$ and $X_{t'_n}(z_n) \to x$. Now, by the Connecting Lemma [WX] and [Ha] there exist a $Z \in \mathcal{U}_0$ such that $q \in V \supset W^u_Z(x_0(Z))$ and $q \notin \overline{V}$, a contradiction.

If $x \in Crit_X(\Lambda)$ we can use the Hartman-Grobmann theorem to find x'_n in the positive orbit of x_n ,$r \in (W^u_X(x) - O_X(x)) - Crit(X)$ and $t'_n > 0$ such that $x'_n \to r$ and $X_{t'_n}(x'_n) \to q$. This is a reduction to the first case since the following lemma says that $r \in \Lambda$ (the proof of the lemma below is the same as in conservative case, since it only uses the Connecting Lemma and the λ-lemma):

Lemma 4.14 (2.8 of [MPP]). *If Λ satisfies the hypothesis of theorem 4.12, then $W^u_X(x) \subset \Lambda$ for any $x \in Crit_X(\Lambda)$.*

□

Now we prove that there are no singularities. If $\Lambda = M$ then by theorem A X will be Anosov, hence without singularities. So we can assume that Λ is a proper subset with a singularity and find a contradiction. Now we follow the argument on [MPP].

Taking X or $-X$ there exists a singularity σ such that $\dim(W^u_X(\sigma)) = 1$(recall that $\dim M$=3 and σ is Lorenz-like). Let U be an isolating block such that $\Lambda_Y(U)$ is a connected transitive set for any $Y \in \mathcal{U}$. We will prove that $W^u_Y(\sigma(Y)) \subset U$ for any $Y \in \mathcal{U}$, so by theorem 4.12 we have a contradiction.

Suppose that this does not happen for an $Y \in \mathcal{U}$. By the dimensional hypothesis we have two branches w^+ and w^- of $W^u_X(\sigma) - \{\sigma\}$. Let $q^+ \in w^+$ and $q^- \in w^-$ (and let also $q^\pm(Y)$ be the continuation of $q^\pm$ for $Y \in \mathcal{U}$). Because the negative orbit of $q^\pm(Y)$ converges to $\sigma(Y)$ and the unstable manifold is not contained in U there exists $t > 0$ such that either $Y_t(q^+(Y))$ or $Y_t(q^-(Y))$ is not in U. We assume that we are in the first case. We know that there exists a neighborhood $\mathcal{U}' \subset \mathcal{U}$ of Y such that for any $Z \in \mathcal{U}'$ we have $Z_t(q^+(Z)) \notin U$. Let us take $z \in \Lambda_Y(U)$ with dense orbit on $\Lambda_Y(U)$. This implies that $q^-(Y) \in \omega_Y(z)$. So we can find a sequence $z_n \to q^-(Y)$ in $O_Y(z)$ and $t_n > 0$ such that $Y_{t_n}(z_n) \to q$ for some $q \in W^u_Y(\sigma(Y)) - \{\sigma(Y)\}$. We set $p = q^-(Y)$.

By the Connecting Lemma, there exists $Z \in \mathcal{U}'$ such that $q^-(Z) \in W^u_Z(\sigma(Z))$ and using the same arguments of theorem 4.12 we can find $Z' \in \mathcal{U}'$ and $t' > 0$ such that $Z'_{t'}(q^-(Z')) \notin U$. This shows that $\sigma(Z')$ is isolated in $\Lambda_{Z'}(U)$. By connectedness we get that $\Lambda_{Z'}(U)$ is trivial, a contradiction.

5 Robustly Transitive Conservative Diffeomorphisms

Now we study the existence of a dominated splitting for conservative diffeomorphisms. This is a topic of many researches and several results on the dissipative case are well known. To see that transitive systems play an important role, we have the following result of [BDP]: *every C^1 robustly transitive diffeomorphism has a dominated splitting.* This theorem has been preceded by several results in particular cases :

- Mañe [Ma]: *Every C^1-robustly transitive diffeomorphism on a compact surface is an Anosov system, i.e. it has a hyperbolic (hence dominated) splitting.*
- Diaz, Pujals, Ures [DPU]: *There is an open, dense set of 3-dimensional C^1-robustly transitive diffeomorphisms admitting dominated splitting.*

Let f a $C^{1+\alpha}$-diffeomorphism, we say that f is C^1-robustly transitive if for any g a $C^{1+\alpha}$-diffeomorphim that is C^1-close to f we have that g is transitive. In this setting, we use the perturbation lemmas to prove the conservative version of the theorem by Bonatti, Diaz and Pujals [BDP] mentioned above:

Theorem C. *Let f be a $C^{1+\alpha}$-diffeomorphism which is C^1-robustly transitive conservative on a compact manifold M^n with $n \geq 2$. Then f admits a nontrivial dominated splitting defined on the whole M.*

In dimension 2, we can extend Mañé's theorem for C^1 conservative robustly transitive systems.

Theorem 5.1. *A C^1-robustly transitive conservative system $f \in \mathrm{Diff}^1_\omega(M)$ is an Anosov diffeomorphism, where M is a compact surface.*

Also we remark that an immediate corollary of theorem C is :

Corollary 5.2. *Let f be a C^1-stably ergodic diffeomorphism in* $\mathrm{Diff}_\omega^{1+\alpha}(M)$. *Then f admits a dominated splitting.*

This corollary answers positively a question posed by Tahzibi in [T].

5.1 Proofs of Theorem C and Theorem 5.1

First we give some definitions. We denote by $Per^n(f)$ the periodic points of f with period n and $Per^{1\leq n}(f)$ the set of periodic points of f with period at most n. If p is a hyperbolic saddle of f, the homoclinic class of p, $H(p,f)$, is the closure of the transverse intersections of the invariant manifolds of p.

Definition 5.3. A compact invariant set Λ admits a (l,λ)-dominated splitting for f if there exists a decomposition $T_\Lambda M = F \oplus G$ and a number $\lambda < 1$ such that for every $x \in \Lambda$:

$$\|Df^l|_{F(x)}\|\|Df^{-l}|_{G(x)}\| < \lambda.$$

As remarked in [BDP], it is sufficient to show that a conservative robustly transitive diffeomorphism f can not have periodic points $x \in Per^n(f)$ whose derivative $D_x f^n$ is the identity. In fact, this is a consequence of :

Proposition 5.4 (Proposition 7.7 in [BDP]). *Given any $K > 0$ and $\varepsilon > 0$ there is $l(\varepsilon, K) \in \mathbb{N}$ such that for every conservative diffeomorphism f with derivatives Df and Df^{-1} whose norm is bounded by K, and for any saddle p of f having a non-trivial homoclinic class $H(p,f)$, one has that:*

- *Either the homoclinic class $H(p,f)$ admits an $(l,1/2)$-dominated splitting,*
- *or for every neighborhood U of $H(p,f)$ and $k \in \mathbb{N}$ there are a conservative diffeomorphism g $\varepsilon - C^1$-close to f and k periodic points x_i of g arbitrarily close to p, whose orbits are contained in U, such that the derivatives $Dg^{n_i}(x_i)$ are the identity (here n_i is the period of x_i).*

Remark 5.5. We stress that this perturbation g preserves the differentiability class of the original diffeomorphism, but g is only C^1-close to f.

Using the $C^{1+\alpha}$ pasting lemma 3.6 we have immediately the following:

Lemma 5.6. *If f is a $C^{1+\alpha}$-conservative diffeomorphism and $x \in Per^n(f)$ is a periodic point of f such that $Df^n(x)$ is the identity matrix then for any $\epsilon > 0$, there is a ϵ-C^1-perturbation g of f and some neighborhood U of x satisfying $g^n|_U = id$. And g is a volume preserving $C^{1+\alpha}$-diffeomorphism.*

We now prove the theorem C:

Proof. We fix a constant $K > 1$ which bounds the norms of the derivative of f and f^{-1} and l given by proposition 5.4.

Now we use the following lemma whose proof we postpone.

Lemma 5.7. *There exists a small C^1-neighborhood $\mathcal{V}$ of f and $\delta > 0$ such that if $g \in \mathcal{V}$ and $\Lambda \subset M$ is a g-invariant δ-dense set with a (l, λ)-dominated splitting for g then M has an (l, λ_1)-dominated splitting for g (where λ_1 is still less than 1).*

We fix the neighborhood of $\mathcal{V}$ and δ given by the previous lemma. Now we take a δ-dense periodic saddle p using the Closing lemma in the conservative case [PR] (and again we do not loose differentiability in this perturbation) for some g in $\mathcal{V}$ C^1-close to f. So by Lemma 5.6 the second option in Proposition 5.4 cannot be true, because choosing $\varepsilon > 0$ such that all of these perturbations are in the neighborhood of f where we have transitivity, we can find an invariant proper open set for an iterate of f, and this contradicts transitivity. So the homoclinic class $H(p, f)$ has a l-dominated splitting $F \oplus G$. Then by Lemma 5.7 we can extend the splitting for the whole manifold and this defines a global (l, λ_1)-dominated splitting. Hence we find a sequence g_n which converges to f with global (l, λ_1)-dominated splitting, so f also has a global dominated splitting. The proof is complete. $\square$

Now we will prove the lemma, but first we recall the notion of angles between subspaces:

Definition 5.8. Let V a finite dimensional vector space and E a subspace of V. If $F \subset V$ is a subspace with the same dimension of E and such that $F \cap E^{\perp} = \{0\}$ then we define the angle between F and E as $\angle(F, E) = \|L\|$, where $L : E \to E^{\perp}$ is a linear map such that $F = graph(L)$.

Proof of Lemma 5.7. First, we deal with the case that $g = f$. Observe that for every $\varepsilon_0 > 0$ there exists $\gamma_0 > 0$ such that if $d(x, y) < \gamma_0$ then x and y are in a trivializing local chart and $\|Df(x) - Df(y)\| < \varepsilon_0$. Also there exists $\beta > 0$ such that for any (l, λ)-dominated splitting $T_\Lambda M = E \oplus F$ we have $\angle(E, F) > \beta$.

Claim 1. Given $\varepsilon_1 > 0$ there exists $\gamma_1 > 0$ such that for any Λ with $T_\Lambda M = E \oplus F$ an (l, λ)-dominated splitting if $d(x, y) < \gamma_1$ then $dist(F(x), F(y)) < \varepsilon_1$ (here we are using the a distance in the Grassmannian).

Proof. There exists an $m = m(l)$ such that if $v = v_1 + v_2 \in E(x) \oplus F(x)$ then $\|Df^m.v_1\| \geq C\lambda^m \|Df^m(x)|_{F(x)}\|$ (and a similar inequality holds for v_2 and $E(x)$) and such that $C\lambda^m >> \sin\beta$.

Now we take γ_1 such that $d(f^j(x), f^j(y)) < \gamma_0$ for every $j = 0, ..., m$.

Now suppose that $dist(F(y), F(x)) \geq \varepsilon_1$. Take $v \in F(y)$ with $\|v\| = 1$ and write $v = v_1 + v_2 \in E(x) \oplus F(x)$ (because x and y are in the same local chart). So $\|v_2\| \leq C(\beta)$ a small constant, and $\|v_1\| \geq C_1 \sin\alpha$ (where α depends of ε_1, since we are supposing that $F(x)$ is far from $F(y)$). So we have:

$$\begin{array}{rcl} \|Df^m(y).v\| & \geq & \|Df(y)^m.v_1\| - \|Df(y)^m.v_2\| \\ & \simeq & \|Df(x)^m.v_1\| - \|Df(x)^m.v_2\| \\ & \geq & C\lambda^m \|Df(x)^m|_{F(x)}\|.\|v_1\| - \|Df(x)|_{F(x)}\|\|v_2\| \\ & >> & 2\|Df^m(x)|_{F(x)}\|. \end{array}$$

So we have that $m(Df^m|_{F(y)}) >> \|Df^m|_{F(x)}\|$ (here $m(L)$ is the co-norm of the matrix L).

Now, since we are supposing that $dist(F(x), F(y)) > \varepsilon_1$, we can take $w \in F(x)$ with $\|w\| = 1$ and write $w = w_1 + w_2 \in E(y) \oplus F(y)$. So we can perform similar calculations to find that $m(Df^m|_{F(x)}) >> \|Df^m|_{F(y)}\|$. And this gives a contradiction. $\square$

We know that there exists $K > 0$ such that for any Λ with an (l, λ)-dominated splitting then $dist(Df(x)(C(x)), C(f(x))) > K$ for any $x \in \Lambda$ and $Df(x)C(x) \subset C(f(x))$. The next claim says how we can obtain invariance of the cone fields in a neighborhood of Λ.

Claim 2. Let $TM = E \oplus F$ a splitting, such that in an invariant compact set Λ, this is a (l, λ)-dominated splitting. So we have cone fields $C^*(x)$ $(* = s, u)$, such that in Λ, these are invariant cone fields associated to the dominated splitting. Suppose that there exists γ such that, $d(x, y) < \gamma$ implies $dist(C^*(x), C^*(y)) < K/4$ (in the projective metric). Then there exists $\delta > 0$ (which depends only of (l, λ) and the constants of uniform continuity of the diffeomorphism and the derivative) such that in a η-neighborhood of Λ, the cone fields are also invariant.

Proof. We deal with $C^E(x)$, the case $C^F(x)$ is similar. We know that

$$dist(Df(x)(C(x)), C(f(x))) > K$$

for any $x \in \Lambda$ and $Df(x)C(x) \subset C(f(x))$ (here we are using a projective distance on cones). By uniform continuity, there exists δ_1 such that if $d(x,y) < \delta_1$ then

$$dist(C(f(x)), C(f(y)))$$

is small, then (trivializing the tangent bundle) we have that

$$dist(Df(x)(C(x)), C(f(y))) > K/2 \text{ and } Df(x)C(x) \subset C(f(y)).$$

Now, by hypothesis, we have that if $d(x,y) < \gamma$, then we have that

$$dist(Df(x)(C(y)), C(f(y))) > K/4$$

and $Df(x)C(y) \subset C(f(y))$. Finally, by uniform continuity of $Df(x)$, we have that if $d(x,y) < \delta_2$ we have that $dist(Df(y)C(y), C(f(y))) > K/8$ and $Df(y)C(y) \subset C(f(y))$. This proves the claim. $\square$

We will say that a function $H : M \to Y$ is (ε, η)-continuous on $X \subset M$ if for any $a, b \in X$ such that $d_X(a,b) < \eta$ then $d_Y(f(a), f(b)) < \varepsilon$. Now by the first claim, we have that for any Λ with an (l, λ)-dominated splitting, the bundles are (ε, η)-continuous in Λ. Take ε such that if the bundles are $(4\varepsilon, \eta/2)$-continuous then the cone fields are $(K/16, \eta)$-continuous.

So, if we have that Λ is $\eta/2$-dense, then take (U_i) a covering of Λ by balls of radii η, hence is also a covering of M. So each bundle restricted to $U_i \cap \Lambda$ have variation bounded by 2ε. So by Tietze's theorem we can extend the bundles in the whole U_i, obtaining bundles $\widehat{E}$ and $\widehat{F}$ on U_i with variation bounded by 2ε, and so we can glue this bundles on M to obtain $\widehat{E}$ and $\widehat{F}$ bundles which are $(4\varepsilon, \eta/2)$-continuous in M.

By the second claim, using $\gamma = \eta/2$, the cone fields associated to $\widehat{E}$ and $\widehat{F}$ are invariant in a neighborhood of size δ. But the previous analysis holds for any Λ with an (l, λ)-dominated splitting. So if we take Λ also δ-dense. We can extend the splitting $T_\Lambda M = E \oplus F$ to the whole manifold, and obtain invariant cone fields in the whole manifold. This gives another splitting $TM = \widetilde{E} \oplus \widetilde{F}$ which is dominated.

It's clear now, that the proof for f, uses only the constant of uniform continuity of f and Df and also the norm $\|f\|_{C^1}$. But these are uniforms in a C^1-neighborhood of f, then the lemma follows. $\square$

Now we prove the theorem 5.1:

Proof. We will use the following theorem, which holds in the C^1-topology for C^1-diffeomorphisms:

Theorem 5.9 (Theorem 6 of [BDP]). *Let f be a conservative diffeomorphism. Then either:*

1. *for any $k \in \mathbb{N}$ there exists a conservative diffeomorphism g arbitrarily C^1-close to f having k periodic orbits whose derivatives are the identity or;*
2. *M is the union of finitely many invariant compact sets having a dominated splitting.*

If f is transitive and the second possibility occurs then f admits a dominated splitting.

Now to avoid periodic points whose derivative is the identity we use the pasting lemma in the symplectic case since the dimension of the manifold is two, and this can be done for C^1 diffeomorphisms. So we obtain a global dominated splitting. To finish we use:

Proposition 5.10 (Proposition 0.5 of [BDP]). *Let f be a conservative diffeomorphism with a dominated splitting. Then the derivative contracts uniformly the volume of one of the bundles and expands uniformly the volume of the other bundle.*

So because in dimension two, both subbundles are one-dimensional we have hyperbolicity. □

6 Appendix: On a non-estimate theorem for the divergence equation (by David Diica and Yakov Simpson-Weller)

We show that a non-estimate result (similar to the well-known Ornstein's non-estimate theorem) holds for the divergence equation. In particular, the method of construction of solutions of the Jacobian determinant equation by Dacorogna-Moser does not work in the limit case of C^1 diffeomorphisms.

6.1 Introduction and Historical Notes

The main result of this appendix is :

Theorem D. *There exist no operator (even* nonlinear*)*

$$\mathcal{L} : B_r^0 \cap \mathrm{Diff}_\omega^{k+\alpha}(M) \to \mathrm{Diff}_\omega^{k+\alpha}(M)$$

such that div$\mathcal{L}g = g$ *and* $||\mathcal{L}(g_1) - \mathcal{L}(g_2)||_{C^1} \leq ||g_1 - g_2||_{C^0}$ *(i.e.,* $\mathcal{L}$ *is Lipschitz with respect to the* C^0 *norm* $||.||_{C^0}$ *and the* C^1 *norm* $||.||_{C^1}$*).*

The motivation of the previous theorem is the classical problem of denseness of smooth conservative diffeomorphisms of compact manifolds. This problem was successfully solved for $C^{1+\alpha}$ diffeomorphisms, using the regularity estimates for the Laplacian operator. In fact, the problem of denseness of smooth diffeomorphism can be reduced to the problem of solving the Jacobian determinant equation $\det \nabla u = f$, which can be linearized to obtain the equation $\mathrm{div} v = g$. If we look for special solutions $v = \mathrm{grad}\ a$ of this equation, the problem is reduced to solve the equation $\Delta a = g$. The idea is then to "deform" (Moser's "flow idea") the solutions of the equation $\Delta a = g$ to solutions of the Jacobian determinant equation. But, the standard elliptic regularity theory only guarantees solutions for Hölder functions. Using this ideas, Zehnder was able to positively answers the denseness problem for $C^{1+\alpha}$ systems. See, for example, Dacorogna-Moser's work [DM] (and the reference to Zehnder's article therein) for more details.

However, the method used by Dacorogna-Moser and Zenhder to solve the divergence equation (and, in particular, the determinant equation) is *linear*. So, in order to solve the denseness problem for C^1 conservative diffeomorphisms, one can look for *nonlinear* bounded right inverses for the divergence operator. Recently, Bourgain and Brezis [BB] have shown that there are *explicit nonlinear* bounded right inverses for the divergence operator, but there are no bounded linear right inverses. The problem to apply their result is that their theorem is stated for the Sobolev spaces $W^{1,d}$. On the other hand, since Burago-Kleiner and McMullen (see [BK]) showed that for some L^∞ (and even, continuous) functions, the determinant equation admits no solution. The idea of McMullen is that if the equation admits solutions for every L^∞ function, then we have a contradiction with a well-known "non-estimate" result due to Ornstein [O].

In this short note, we consider a lemma due to Burago-Kleiner [BK] in order to show that there are no Lipschitz estimates for the nonlinear right inverses constructed by Bourgain-Brezis. In particular, Dacorogna-Moser's idea can not work for the limit case of C^1 diffeomorphisms, even when we consider the "nonlinear" modification of the idea, which consists in substituting the linear right inverse of the divergence operator in the Hölder case by Lipschitz nonlinear right inverses.

6.2 Main Tools

In this section we state some useful facts which will be used in the proof of the non-estimate theorem. First of all, we recall the following result due to Burago-Kleiner (see [BK, page 275]):

Lemma 6.1 (Burago-Kleiner). *For every $L > 1$, $c > 0$, there exists a smooth function $\rho_{L,c} : I^d \to [1, 1+c]$ (where I^d is the d-dimensional cube, $I = [0,1]$) such that the equation $\det(\nabla\psi) = \rho_{L,c}$ does not admit an L-biLipschitz solution ψ.*

Remark 6.2. As an immediate corollary of this lemma, it follows that the following non-estimate result holds for the Jacobian determinant equation: *There are no bounded right inverses for the Jacobian determinant equation, i.e., there exists no operator $\mathcal{L}$ such that $||\mathcal{L}f||_{C^1} \leq K \cdot ||f||_{C^0}$ for every $f \in C^\infty$ C^0-close to the constant function* 1 *and* $\det \nabla\mathcal{L}(f) = f$.

Now we briefly recall the method (by Dacorogna-Moser [DM]) of construction of solutions of the Jacobian determinant equation

$$\det \nabla u = f$$

using solutions of the divergence equation

$$\mathrm{div} Y = f.$$

Let for any $n \times n$ matrix ζ,

$$Q(\zeta) = \det(I + \zeta) - 1 - \mathrm{tr}(\zeta),$$

where I denotes the identity matrix. Since Q is a sum of monomials of degree t with $2 \leq t \leq n$, we can find a constant $K_2 > 0$ such that if $w_1, w_2 \in C^k(\mathbb{R}^n, \mathbb{R}^n)$ with $||w_1||_0, ||w_2||_0 \leq 1$, then

$$(*)||Q(w_1) - Q(w_2)||_k \leq K_2(||w_1||_0 + ||w_2||_0)||w_1 - w_2||_k.$$

Here k is any real number, but the case of interest for this paper is k integer.

With this notation, Dacorogna-Moser's idea to construct solutions of the Jacobian determinant equation is: denote by $u(x) = x + v(x)$ and *suppose* that there is an operator $\mathcal{L} : \mathrm{Diff}^1_\omega(M) \to \mathrm{Diff}^{1+\alpha}_\omega(M)$ (*not necessarily linear*) such that $\mathrm{div}\mathcal{L}(a) = a$, for every $a \in \mathrm{Diff}^1_\omega(M)$, and $\mathcal{L}$ is Lipschitz continuous. For example, there are bounded linear operators $\mathcal{L}$ as above when $k = r + \alpha$, $r \geq 0$ integer, $\alpha > 0$. Define the operator $N : \mathrm{Diff}^{1+\alpha}_\omega(M) \to \mathrm{Diff}^1_\omega(M)$,

$$N(v) = f - 1 - Q(\nabla v).$$

Clearly any fixed point v of the map $\mathcal{L} \circ N$ (i.e., $v = \mathcal{L}N(v)$) satisfies $\det \nabla u = f$ (recall that $u = \mathrm{id} + v$). So to obtain solutions of the Jacobian determinant equation, it suffices to show that, if $||f - 1||_k$ is small, then $\mathcal{L} \circ N$ is a contraction. But, it is not difficult to see that this follows from the inequality $(*)$ and $||f - 1||_k$ is small (see [DM, page 12] for more details).

Remark 6.3. If $||f - 1||_k$ is not small, Dacorogna-Moser uses a trick which consists of two steps: First, considering the quotient of f by a suitable smooth function g, the quotient has small norm. Using the "flow idea", the case of the determinant equation for a smooth density g is solved (see [DM, page 13]).

After these considerations, we are in position to prove the main theorem of this appendix.

6.3 Proof of theorem D

Our proof of theorem D is by contradiction.

Proof of theorem D. Suppose that there exists some Lipschitz continuous (*not necessarily linear*) operator $\mathcal{L}$ such that $\mathrm{div}\mathcal{L}g = g$ for every $f \in B^0_r \cap \mathrm{Diff}^{k+\alpha}_\omega(M)$ (B^k_r is the r-ball in the space of C^k functions with the C^k-norm). Fix some positive number $L > 0$ such that $L > K_1 \cdot r$, where K_1 is the Lipschitz constant of $\mathcal{L}$. Then, if we consider the smooth function $\rho_{L,c}$ given by Lemma 6.1, where c is sufficiently small, then, by the facts remarked in the previous section, $\mathcal{L} \circ N$ is a contraction ($N(v) = \rho_{L,c} - 1 - Q(\nabla v)$) and, in particular, there exists some fixed point v of $\mathcal{L} \circ N : (B^1_\epsilon \cap \mathrm{Diff}^{k+\alpha}_\omega(M)) \to (B^1_\epsilon \cap \mathrm{Diff}^{k+\alpha}_\omega(M))$ (ε is chosen such that $N : B^1_\epsilon \cap \mathrm{Diff}^{k+\alpha}_\omega(M) \to B^0_r \cap \mathrm{Diff}^{k+\alpha}_\omega(M)$; $\varepsilon > 0$ exists by the inequality $(*)$).

Moreover, $u = \mathrm{id} + v$ solves the equation $\det \nabla u = \rho_{L,c}$, and the C^1-norm of u is at most $K_1 \cdot r$. In particular, since $K_1 \cdot r < L$, u is L-biLipschitz (for r small). But this contradicts the conclusion of lemma 6.1. $\square$

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