

ON THE CAUCHY PROBLEM FOR A COUPLED SYSTEM OF KDV EQUATIONS

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ABSTRACT. We study some questions related to the well-posedness for the initial value problem associated to the system

$$\begin{cases} u_t + u_{xxx} + a_3 v_{xxx} + uu_x + a_1 vv_x + a_2 (uv)_x = 0, \\ b_1 v_t + v_{xxx} + b_2 a_3 u_{xxx} + vv_x + b_2 a_2 uu_x + b_2 a_1 (uv)_x = 0. \end{cases}$$

Using recent methods, we prove a sharp local result in Sobolev spaces. We also prove global result under some conditions on the coefficients.

1. INTRODUCTION

Let us consider the initial value problem (IVP)

$$\begin{cases} u_t + u_{xxx} + a_3 v_{xxx} + uu_x + a_1 vv_x + a_2 (uv)_x = 0, & x \in \mathbb{R}, \quad t \in \mathbb{R}, \\ b_1 v_t + v_{xxx} + b_2 a_3 u_{xxx} + vv_x + b_2 a_2 uu_x + b_2 a_1 (uv)_x + rv_x = 0, \\ u(x, 0) = \phi(x), \\ v(x, 0) = \psi(x), \end{cases} \quad (1)$$

where $a_1, a_2, a_3, b_1, b_2, r$ are real constants with b_1, b_2 positive, $u = u(x, t)$, $v = v(x, t)$ are real valued functions.

This system was derived by Gear and Grimshaw [11] as a model to describe the strong interaction of two long internal gravity waves in a stratified fluid, where the two waves are assumed to correspond to different modes of the linearized equations of motion. It has the structure of a pair of Korteweg-de Vries (KdV) equations with both linear and nonlinear coupling terms.

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In this work we are interested to study some questions related to the well-posedness of the IVP (1) in low order Sobolev spaces $H^s(\mathbb{R}) \times H^s(\mathbb{R})$. Our notion of well-posedness includes existence, uniqueness, persistence property and continuous dependence of the solution upon the data. If any one of these conditions fails we will say that the IVP is ill-posed.

Bona, Ponce, Saut and Tom [5] showed that the IVP (1) is globally well-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s \geq 1$ whenever $|a_3|\sqrt{b_2} < 1$ neglecting the constant r . We will also neglect r in our analysis (for detail see [5]). Further, Ash, Cohen and Wang [2], using the techniques established by Bourgain [6] and further developed by Kenig, Ponce and Vega ([13], [14]), showed that the IVP (1) is locally well-posed in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ provided $|a_3|\sqrt{b_2} \neq 1$. Also, using the conserved quantity

$$\int_{\mathbb{R}} (b_2 u^2 + b_1 v^2) dx,$$

satisfied by the flow, they proved that the IVP (1) is globally well-posed in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$. Recently, Saut and Tzvetkov [17] considered the IVP (1) without neglecting the constant r and proved global well-posedness in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$.

Using concentration compactness technique, Bona and Chen [4] proved the existence of solitary waves for the system (1) as global minimizers to the constrained variational problem. Later, Albert and Linares [1] proved that the solitary waves are stable in weak sense by considering $a_3^2 b_2 < 1$. Recently, Menzala, Vasconcellos and Zuazua [15] showed that the solutions of the KdV equation in a bounded interval under the effect of a localized damping decay exponentially in time. The method of proof is a combination of multiplier techniques, compactness arguments and the unique continuation property of the KdV equation. A similar result for the IVP (1) was obtained by Bisognin, Bisognin and Menzala [3] whenever the conditions $b_1 = b_2$ and $0 < a_3 < 1$ hold.

Since (1) is a coupled system of KdV equations, it is natural to ask whether it shares similar results like KdV equations, i.e. whether we can lower the Sobolev index in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ as in the case of KdV equation to get well-posedness results? Using the scaling argument we can have an insight to this question. Observe that if (u, v) solves (1) (note that we have neglected r) with initial data (ϕ, ψ) then for $\lambda > 0$ so does (u^λ, v^λ) with initial data $(\phi^\lambda, \psi^\lambda)$; where $u^\lambda(x, t) = \lambda^2 u(\lambda x, \lambda^3 t)$, $v^\lambda(x, t) = \lambda^2 v(\lambda x, \lambda^3 t)$, $\phi^\lambda(x) = \lambda^2 \phi(\lambda x)$ and $\psi^\lambda(x) = \lambda^2 \psi(\lambda x)$. Note that,

$$\|(\phi^\lambda, \psi^\lambda)\|_{\dot{H}^s \times \dot{H}^s} = \lambda^{2s+3} \|(\phi, \psi)\|_{\dot{H}^s \times \dot{H}^s}, \quad (2)$$

where $\dot{H}^s(\mathbb{R})$ denotes the homogeneous Sobolev space of order s . Thus, we see from (2) that the well-posedness result for the IVP (1) could be achieved in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s \geq -3/2$.

Bourgain [7] showed that the well-posedness result for the KdV equation in $H^s(\mathbb{R})$, $s > -3/4$, is essentially optimal if one strengthens the usual notion of well-posedness by requiring the flow-map

$$\phi \mapsto u_\phi(t), \quad |t| < T$$

should act smoothly (for eg. C^3) on the space under consideration (instead of just continuous). This notion of well-posedness seems to be natural because, if one uses the contraction mapping principle to solve the integral equation associated with the Cauchy problem, the flow-map acts smoothly from H^s to itself. In fact, for $s > -3/4$ the flow-map is real analytic (see for eg., [14] [12] [6]).

Motivated by the work of Bourgain in [7], Takaoka [19] showed that the nonlinear Schrödinger equation with derivative in a nonlinear term is ill-posed in $H^s(\mathbb{R})$, $s < 1/2$. Utilizing the same techniques Tzvetkov [20] proved that the KdV equation is locally ill-posed in $H^s(\mathbb{R})$ for $s < -3/4$ if one requires only C^2 regularity of the flow-map in the notion of well-posedness. Following the same scheme, we prove that the IVP (1) is ill-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s < -3/4$. This result is in agreement with the KdV results. So, one expects that the IVP (1) be locally well-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $s > -3/4$. In fact, using bilinear estimates established by Kenig, Ponce and Vega [14], we prove that the IVP (1) is locally well-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > -3/4$. Nakanishi, Takaoka and Tsutsumi [16] produced a counterexample to prove the bilinear estimate established by Kenig, Ponce and Vega [14] fails when $s = -3/4$. Therefore the critical index $s = -3/4$ cannot be achieved using this method. However, Christ, Colliander and Tao [9] showed recently the existence of the solutions to the IVP associated to the KdV equation in $H^{-3/4}(\mathbb{R})$ using a generalized Miura transform to transfer the existing local theory for the modified KdV equation in $H^{1/4}(\mathbb{R})$.

Note that, in the Sobolev spaces of negative index, conservation laws are not available to extend the local solution to the global one. To overcome this difficulty, i.e. lack of conservation laws, quite recently, Colliander, Keel, Staffilani, Takaoka and Tao [10] introduced a variant of Bourgain's method [8] called *I-method* to obtain a global solution to KdV equation in Sobolev spaces of negative index. For this, they introduced a notion of an *almost conserved quantity* by utilizing an appropriate Fourier multiplier operator I . To obtain such almost conserved quantity they exploited some internal cancellation which the KdV equation satisfies. The cancellation plays a main role in this process. In our case, it is not possible to get such cancellation unless the coefficients $a_3 = 0$ and $b_1 = b_2$ (see Lemma 4.2 below). It seems that we cannot get more cancellation in the general case because the IVP under consideration is not completely integrable. Using this method, under above conditions on the coefficients, we prove that the IVP (1) is globally well-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > -3/10$.

Before stating the main results let us introduce some notations which will be used throughout this work.

Notation: We denote by $L_t^p(L_x^q)$, ($1 < p < \infty$) the Banach spaces $L^p(\mathbb{R} : L^q(\mathbb{R}))$ for variables t and x respectively. Let \hat{f} be the Fourier transform of f in both x and t variables

$$\hat{f}(\xi, \tau) = (2\pi)^{-1} \int_{\mathbb{R}^2} e^{-i(x\xi+t\tau)} f(x, t) dx dt.$$

For $-1 < b < 1$, let $X_{s,b}$ denotes the Hilbert space with norm

$$\|f\|_{X_{s,b}} = \left(\int_{\mathbb{R}^2} (1 + |\tau - \xi^3|)^{2b} (1 + |\xi|)^{2s} |\hat{f}(\xi, \tau)|^2 d\xi d\tau \right)^{1/2}.$$

We use C to denote various constants whose exact values are immaterial. Also, we use the notation $A \lesssim B$ if there exists a constant $C > 0$ such that $A < CB$, $A \gtrsim B$ if there exists a constant $C > 0$ such that $A > CB$ and $A \sim B$ if $A \lesssim B$ and $A \gtrsim B$.

The main results of this work are,

THEOREM 1.1. *For any $(\phi, \psi) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > -3/4$ and $b \in (1/2, 1)$, there exist $T = T(\|\phi\|_{H^s}, \|\psi\|_{H^s})$ and a unique solution of (1) in the time interval $[-T, T]$ satisfying*

$$u, v \in C([-T, T]; H^s(\mathbb{R})), \quad (3)$$

$$u, v \in X_{s,b} \subseteq L^p_{x,loc}(\mathbb{R}; L^2_t(\mathbb{R})), \quad \text{for } 1 \leq p \leq \infty, \quad (4)$$

$$(u^2)_x, (v^2)_x \in X_{s,b-1} \quad (5)$$

and

$$u_t, v_t \in X_{s-3,b-1}. \quad (6)$$

Moreover, given $T' \in (0, T)$, the map $(\phi, \psi) \mapsto (u(t), v(t))$ is smooth from $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ to $C([-T', T']; H^s(\mathbb{R})) \times C([-T', T']; H^s(\mathbb{R}))$.

THEOREM 1.2. *The initial value problem (1) is globally well-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > -3/10$ in the case when $a_3 = 0$ and $b_1 = b_2$.*

THEOREM 1.3. *Let $s < -3/4$, then there is no $T > 0$ such that the flow-map*

$$(\phi, \psi) \mapsto (u(t), v(t)), \quad t \in (0, T]$$

be C^2 Frechet-differentiable at the origin from $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ to $H^s(\mathbb{R}) \times H^s(\mathbb{R})$.

Let us recall some properties of the space $X_{s,b}$ regarding the regularity. First, observe that for $f \in X_{s,b}$, one has,

$$\|f\|_{X_{s,b}} = \|(1 + D_t)^b U(t)f\|_{L^2_t(H^s_x)},$$

where $U(t) = e^{-t\partial_x^3}$ is the unitary group associated with the linear problem.

If $b > 1/2$, the previous remark and the Sobolev lemma imply,

$$X_{s,b} \subset C(\mathbb{R}; H^s_x(\mathbb{R})).$$

The remainder of this article is organized as follows. In section 2 we decouple the dispersive terms in (1) with the help of a non-singular linear transformation and recall some linear estimates. In section 3 we prove Theorem 1.1, the local well-posedness result. In section 4 we introduce an almost conserved quantity and prove Theorem 1.2, the global well-posedness result. Finally, we give proof of the local ill-posedness result, Theorem 1.3, in section 5.

2. REDUCTION OF THE PROBLEM AND PRELIMINARY ESTIMATES

As mentioned in the introduction we consider the following IVP

$$\begin{cases} u_t + u_{xxx} + a_3 v_{xxx} + uu_x + a_1 vv_x + a_2 (uv)_x = 0, \\ b_1 v_t + v_{xxx} + b_2 a_3 u_{xxx} + vv_x + b_2 a_2 uu_x + b_2 a_1 (uv)_x = 0, \\ u(x, 0) = \phi(x), \quad v(x, 0) = \psi(x), \end{cases} \quad (7)$$

where a_1, a_2, a_3, b_1, b_2 are real constants with b_1, b_2 positive.

If $a_3 = 0$ there is no coupling in the dispersive terms. So let $a_3 \neq 0$. We are interested to decouple the dispersive terms in the system (7). For this, let $a_3^2 b_2 \neq 1$. Define,

$$\lambda = \left\{ \left(1 - \frac{1}{b_1}\right)^2 + \frac{4b_2 a_3^2}{b_1} \right\}^{1/2} > 0 \quad \text{and} \quad \alpha_{\pm} = \frac{1}{2} \left(1 + \frac{1}{b_1} \pm \lambda\right).$$

Our assumption $a_3^2 b_2 \neq 1$ guarantees that $\alpha_{\pm} \neq 0$. Thus we can write the system (7) in a matrix form and then diagonalize the matrix of coefficients corresponding to the dispersive terms as in [17]. After that we make the change of scale

$$u(x, t) = \tilde{u}(\alpha_+^{-1/3} x, t) \quad \text{and} \quad v(x, t) = \tilde{v}(\alpha_-^{-1/3} x, t).$$

Then we obtain the system of equations

$$\begin{cases} \tilde{u}_t + \tilde{u}_{xxx} + a\tilde{u}\tilde{u}_x + b\tilde{v}\tilde{v}_x + c(\tilde{u}\tilde{v})_x = 0, \\ \tilde{v}_t + \tilde{v}_{xxx} + \tilde{a}\tilde{u}\tilde{u}_x + \tilde{b}\tilde{v}\tilde{v}_x + \tilde{c}(\tilde{u}\tilde{v})_x = 0, \\ \tilde{u}(x, 0) = \tilde{\phi}(x), \quad \tilde{v}(x, 0) = \tilde{\psi}(x), \end{cases} \quad (8)$$

where a, b, c and $\tilde{a}, \tilde{b}, \tilde{c}$, are constants.

REMARK 1. Notice that the nonlinear terms involving the functions \tilde{u} and \tilde{v} are not evaluated at the same point. Therefore those terms are not local anymore. Hence we should be careful in making the estimates. In the existing literature, see for instance [17] and [2], this feature of the nonlinear terms was not pointed out which may lead to wrong conclusions.

REMARK 2. Due to the previous remark, in Proposition 2.1 below, we need to estimate terms of the form $\partial_x(u(Ax, t)v(Bx, t))$ or more generally $\partial_x(u(Ax + C, t)v(Bx + D, t))$. It is not difficult to prove the same inequality since the only changes coming out are from the contributions given by the constants A, B, C and D .

The system (8) has a pair of KdV equations coupled only in the nonlinear terms. It is enough to prove local well-posedness for system (8) because the results for the IVP (7) can be obtained in the obvious way.

Hence, our interest here is to solve system (8) for initial data $(\tilde{\phi}, \tilde{\psi}) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > -3/4$. For this we use the Fourier transform restriction space $X_{s,b}$ discussed above. For the sake of simplicity, from now onwards we will drop $\tilde{}$ and use the notation u, v, ϕ, ψ in the system (8).

Using Duhamel's principle, we study the following system of integral equations equivalent to the system (8),

$$\begin{cases} u(t) = U(t)\phi - \int_0^t U(t-t')F(u, v, u_x, v_x)(t') dt', \\ v(t) = U(t)\psi - \int_0^t U(t-t')G(u, v, u_x, v_x)(t') dt', \end{cases} \quad (9)$$

where $U(t) = e^{(-t\partial_x^3)}$ is the unitary group associated with the linear problem and F and G are respective nonlinearities.

To find a local solution to (8) we can replace (9) with the following system of integral equations,

$$\begin{cases} u(t) = \psi_1(t)U(t)\phi - \psi_1(t) \int_0^t U(t-t')\psi_\delta(t')F(u, v, u_x, v_x)(t') dt', \\ v(t) = \psi_1(t)U(t)\psi - \psi_1(t) \int_0^t U(t-t')\psi_\delta(t')G(u, v, u_x, v_x)(t') dt', \end{cases} \quad (10)$$

where $\psi_1 \in C_0^\infty(\mathbb{R})$, $0 \leq \psi_1 \leq 1$ is a cut-off function given by,

$$\psi_1 = \begin{cases} 1, & |t| < 1 \\ 0, & |t| \geq 2 \end{cases}$$

and $\psi_\delta = \psi_1(t/\delta)$, $0 < \delta \leq 1$.

Now, let us recall some estimates which will be used to prove the local well-posedness result.

LEMMA 2.1. *Let $s \in \mathbb{R}$, $b', b \in (1/2, 1)$ with $b' < b$ and $\delta \in (0, 1)$; then we have,*

$$\|\psi_\delta(t)U(t)\phi\|_{X_{s,b}} \leq C \delta^{(1-2b)/2} \|\phi\|_{H^s}, \quad (11)$$

$$\|\psi_\delta F\|_{X_{s,b-1}} \leq C \delta^{\frac{b-b'}{8(1-b')}} \|F\|_{X_{s,b'-1}}, \quad (12)$$

$$\left\| \psi_\delta(t) \int_0^t U(t-t')F(t') dt' \right\|_{X_{s,b}} \leq C \delta^{(1-2b)/2} \|F\|_{X_{s,b-1}} \quad (13)$$

and

$$\left\| \psi_\delta(t) \int_0^t U(t-t')F(t') dt' \right\|_{H^s} \leq C \delta^{(1-2b)/2} \|F\|_{X_{s,b-1}}. \quad (14)$$

PROPOSITION 2.1. *Let $s > -3/4$, then there exists $b > 1/2$ such that the following bilinear estimate holds,*

$$\|(uv)_x\|_{X_{s,b-1}} \leq C \|u\|_{X_{s,b}} \|v\|_{X_{s,b}}. \quad (15)$$

The proof of Lemma 2.1 can be found in ([13], [14]) and that of Proposition 2.1 in [14], so we skip the details.

3. LOCAL WELL-POSEDNESS RESULT

In this section we give the proof of Theorem 1.1. To do so we first consider the following function space where we seek a solution to the IVP (8). For given $(\phi, \psi) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ and $b > 1/2$, let us define,

$$\mathcal{H}_{MN} := \{(u, v) \in X_{s,b} \times X_{s,b} : \|u\|_{X_{s,b}} \leq M, \|v\|_{X_{s,b}} \leq N\},$$

where $M = 2C_0\|\phi\|_{H^s}$ and $N = 2C_0\|\psi\|_{H^s}$. Then \mathcal{H}_{MN} is a complete metric space with norm,

$$\|(u, v)\|_{\mathcal{H}_{MN}} := \|u\|_{X_{s,b}} + \|v\|_{X_{s,b}}.$$

Without loss of generality, we may assume that $M > 1$ and $N > 1$. For $(u, v) \in \mathcal{H}_{MN}$, let us define the maps,

$$\begin{cases} \Phi_\phi[u, v] = \psi_1(t)U(t)\phi - \psi_1(t) \int_0^t U(t-t')\psi_\delta(t')F(u, v, u_x, v_x)(t') dt' \\ \Psi_\psi[u, v] = \psi_1(t)U(t)\psi - \psi_1(t) \int_0^t U(t-t')\psi_\delta(t')G(u, v, u_x, v_x)(t') dt'. \end{cases} \quad (16)$$

We prove that $\Phi \times \Psi$ maps \mathcal{H}_{MN} into \mathcal{H}_{MN} and is a contraction.

Using (11) and (13) we get from (16),

$$\begin{cases} \|\Phi[u, v]\|_{X_{s,b}} \leq C_0\|\phi\|_{H^s} + C\|\psi_\delta F(u, v, u_x, v_x)\|_{X_{s,b-1}} \\ \|\Psi[u, v]\|_{X_{s,b}} \leq C_0\|\psi\|_{H^s} + C\|\psi_\delta G(u, v, u_x, v_x)\|_{X_{s,b-1}}. \end{cases} \quad (17)$$

Now, using (12) we get from (17) for $b' < b$ and $\theta = \frac{b-b'}{8(1-b')}$,

$$\begin{cases} \|\Phi[u, v]\|_{X_{s,b}} \leq C_0\|\phi\|_{H^s} + C\delta^\theta \|F(u, v, u_x, v_x)\|_{X_{s,b'-1}} \\ \|\Psi[u, v]\|_{X_{s,b}} \leq C_0\|\psi\|_{H^s} + C\delta^\theta \|G(u, v, u_x, v_x)\|_{X_{s,b'-1}}. \end{cases} \quad (18)$$

Using the bilinear estimate (15), the estimate (18) yields,

$$\begin{cases} \|\Phi[u, v]\|_{X_{s,b}} \leq C_0\|\phi\|_{H^s} + C_1\delta^\theta \{\|u\|_{X_{s,b}}^2 + \|v\|_{X_{s,b}}^2 + \|u\|_{X_{s,b}}\|v\|_{X_{s,b}}\} \\ \|\Psi[u, v]\|_{X_{s,b}} \leq C_0\|\psi\|_{H^s} + C_2\delta^\theta \{\|u\|_{X_{s,b}}^2 + \|v\|_{X_{s,b}}^2 + \|u\|_{X_{s,b}}\|v\|_{X_{s,b}}\}. \end{cases} \quad (19)$$

As $(u, v) \in \mathcal{H}_{MN}$, with our choice of M and N we get from (19),

$$\begin{cases} \|\Phi[u, v]\|_{X_{s,b}} \leq \frac{M}{2} + C_1\delta^\theta \{M^2 + N^2 + MN\} \\ \|\Psi[u, v]\|_{X_{s,b}} \leq \frac{N}{2} + C_2\delta^\theta \{M^2 + N^2 + MN\}. \end{cases} \quad (20)$$

If we choose δ such that,

$$\delta^\theta \leq (2 \max\{C_1, C_2\}(M + N)^2)^{-1}$$

then we get from (20),

$$\|\Phi[u, v]\|_{X_{s,b}} \leq M \quad \text{and} \quad \|\Psi[u, v]\|_{X_{s,b}} \leq N.$$

Therefore,

$$(\Phi[u, v], \Psi[u, v]) \in \mathcal{H}_{MN}.$$

Now, we need to show that $\Phi \times \Psi : (u, v) \mapsto (\Phi[u, v], \Psi[u, v])$ is a contraction. For this, let $(u, v), (u_1, v_1) \in \mathcal{H}_{MN}$, then as above using Lemma 2.1 and Proposition 2.1 we get,

$$\begin{cases} \|\Phi[u, v] - \Phi[u_1, v_1]\|_{X_{s,b}} \leq C_1 \delta^\theta (M + N) [\|u - u_1\|_{X_{s,b}} + \|v - v_1\|_{X_{s,b}}] \\ \|\Psi[u, v] - \Psi[u_1, v_1]\|_{X_{s,b}} \leq C_2 \delta^\theta (M + N) [\|u - u_1\|_{X_{s,b}} + \|v - v_1\|_{X_{s,b}}]. \end{cases} \quad (21)$$

If we choose δ such that

$$\delta^\theta \leq (4 \max\{C_1, C_2\} (M + N)^2)^{-1}$$

then we get,

$$\begin{cases} \|\Phi[u, v] - \Phi[u_1, v_1]\|_{X_{s,b}} \leq \frac{1}{4} [\|u - u_1\|_{X_{s,b}} + \|v - v_1\|_{X_{s,b}}] \\ \|\Psi[u, v] - \Psi[u_1, v_1]\|_{X_{s,b}} \leq \frac{1}{4} [\|u - u_1\|_{X_{s,b}} + \|v - v_1\|_{X_{s,b}}]. \end{cases} \quad (22)$$

Therefore the map $\Phi \times \Psi$ is a contraction and we obtain a unique fixed point (u, v) which solves the IVP (8) for $T \leq \delta$. The remainder of the proof follows a standard argument.

4. GLOBAL WELL-POSEDNESS RESULT

This section is devoted to extend the local solution obtained in the previous section to the global one. Using usual conservation laws we have global solution in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s \geq 0$. So, we suppose $s < 0$ throughout this section. Our aim here is to derive an *almost conserved quantity* and use it to prove Theorem 1.2. For this, let us define the Fourier multiplier operator I by,

$$\widehat{Iu}(\xi) = m(\xi) \hat{u}(\xi),$$

where $m(\xi)$ is a smooth and monotone function given by

$$m(\xi) = \begin{cases} 1, & |\xi| < N, \\ N^{-s} |\xi|^s, & |\xi| \geq 2N, \end{cases}$$

with $N \gg 1$ to be fixed later.

Note that, I is the identity operator in low frequencies, $\{\xi : |\xi| < N\}$, and simply an integral operator in high frequencies. In general, it commutes with differential operators and satisfies the following property.

LEMMA 4.1. *Let $-3/4 < s < 0$ and $N \gg 1$. Then the operator I maps $H^s(\mathbb{R})$ to $L^2(\mathbb{R})$ and*

$$\|If\|_{L^2(\mathbb{R})} \lesssim N^{-s} \|f\|_{H^s(\mathbb{R})}. \quad (23)$$

Proof.

$$\begin{aligned} \|If\|_{L^2}^2 &= \|\widehat{If}\|_{L^2}^2 = \|m(\cdot) \hat{f}\|_{L^2}^2 \\ &\leq CN^{-2s} \|f\|_{H^s}^2 + N^{-2s} \int_{|\xi| > 2N} (1 + \xi^2)^s |\hat{f}(\xi)|^2 \left(1 + \frac{1}{\xi^2}\right)^{-s} d\xi \\ &\leq CN^{-2s} \|f\|_{H^s}^2. \end{aligned}$$

□

As discussed in the introduction let us consider the IVP (1) with $a_3 = 0$ and $b_1 = b_2$, that is,

$$\begin{cases} u_t + u_{xxx} + uu_x + a_1vv_x + a_2(uv)_x = 0, \\ b_1v_t + v_{xxx} + vv_x + b_2a_2uu_x + b_2a_1(uv)_x = 0, \\ u(x, 0) = \phi(x), \\ v(x, 0) = \psi(x). \end{cases} \quad (24)$$

After introducing the multiplier operator I , we have the following variant of the local well-posedness for the IVP (24).

THEOREM 4.1. *For any $(\phi, \psi) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > -3/4$, the IVP (24) is locally well-posed in the Banach space*

$$I^{-1}L^2 \times I^{-1}L^2 = \{(\phi, \psi) \in H^s \times H^s, \text{ with norm } \|I\phi\|_{L^2} + \|I\psi\|_{L^2}\}$$

with the existence lifetime satisfying,

$$\delta \gtrsim (\|I\phi\|_{L^2}^2 + \|I\psi\|_{L^2}^2)^{-\theta}, \quad \theta > 0. \quad (25)$$

Moreover,

$$\begin{cases} \|\psi_\delta Iu\|_{X_{0,b}} \leq C\|I\phi\|_{L^2} \\ \|\psi_\delta Iv\|_{X_{0,b}} \leq C\|I\psi\|_{L^2}. \end{cases} \quad (26)$$

The proof of this theorem is not difficult and follows by using the same procedure to prove local well-posedness for the IVP (8) (see the Proof of Theorem 1.1) once we have the bilinear estimate

$$\|\partial_x I(uv)\|_{X_{0,-\frac{1}{2}^+}} \leq C\|Iu\|_{X_{0,\frac{1}{2}^+}}\|Iv\|_{X_{0,\frac{1}{2}^+}}. \quad (27)$$

The proof of the bilinear estimate (27) is easy and follows by using the usual bilinear estimate (15) due to Kenig, Ponce and Vega [14] combined with the following extra smoothing bilinear estimate whose proof is given in Colliander, Keel, Staffilani, Takaoka and Tao [10].

PROPOSITION 4.1. *The bilinear estimate*

$$\|\partial_x \{(IuIv - I(uv))\}\|_{X_{0,-\frac{1}{2}^-}^\delta} \leq CN^{-\frac{3}{4}^+}\|Iu\|_{X_{0,-\frac{1}{2}^+}^\delta}\|Iv\|_{X_{0,-\frac{1}{2}^+}^\delta} \quad (28)$$

holds.

Now we proceed to introduce the almost conserved quantity. Using the Fundamental Theorem of Calculus, the equation and integration by parts we get,

$$\begin{aligned}
\|Iu(\delta)\|_{L^2}^2 &= \|Iu(0)\|_{L^2}^2 + \int_0^\delta \frac{d}{dt}(Iu(t), Iu(t)) dt \\
&= \|Iu(0)\|_{L^2}^2 + 2 \int_0^\delta \left(\frac{d}{dt} Iu(t), Iu(t) \right) dt \\
&= \|Iu(0)\|_{L^2}^2 + 2 \int_0^\delta (I(-u_{xxx} - uu_x - a_1vv_x - a_2(uv)_x), Iu(t)) dt \quad (29) \\
&= \|Iu(0)\|_{L^2}^2 + 2 \int_0^\delta (I(-uu_x - a_1vv_x - a_2(uv)_x), Iu(t)) dt \\
&= \|Iu(0)\|_{L^2}^2 + R_1(\delta),
\end{aligned}$$

where

$$R_1(\delta) = \int_0^\delta \int_{\mathbb{R}} \partial_x(-Iu^2 - a_1Iv^2 - 2a_2I(uv))Iu dxdt. \quad (30)$$

Similarly,

$$\|Iv(\delta)\|_{L^2}^2 = \|Iv(0)\|_{L^2}^2 + R_2(\delta), \quad (31)$$

where

$$R_2(\delta) = \int_0^\delta \int_{\mathbb{R}} \partial_x \left(-\frac{1}{b_1}Iv^2 - \frac{b_2a_2}{b_1}Iu^2 - \frac{2b_2a_1}{b_1}I(uv) \right) Iv dxdt. \quad (32)$$

Let us define $R(\delta) := R_1(\delta) + R_2(\delta)$, so that we have from (29) and (31),

$$\|Iu(\delta)\|_{L^2}^2 + \|Iv(\delta)\|_{L^2}^2 = \|Iu(0)\|_{L^2}^2 + \|Iv(0)\|_{L^2}^2 + R(\delta). \quad (33)$$

We will obtain the so called almost conserved quantity from (33) by treating $R(\delta)$ as an error term. In what follows we prove a cancellation property which plays a vital role in our analysis.

LEMMA 4.2. *The following cancellations hold.*

$$\int_0^\delta \int_{\mathbb{R}} \partial_x(Iu)^2 Iu dxdt = 0 = \int_0^\delta \int_{\mathbb{R}} \partial_x(Iv)^2 Iv dxdt \quad (34)$$

and

$$a_1b_1I_1 + 2a_2b_1I_2 + b_2a_2I_3 + 2b_2a_1I_4 = 0, \quad \text{if } b_1 = b_2, \quad (35)$$

where

$$\begin{aligned}
I_1 &= \int_0^\delta \int_{\mathbb{R}} \partial_x(Iv)^2 Iu dxdt, & I_2 &= \int_0^\delta \int_{\mathbb{R}} \partial_x(IuIv) Iu dxdt, \\
I_3 &= \int_0^\delta \int_{\mathbb{R}} \partial_x(Iu)^2 Iv dxdt & \text{and } I_4 &= \int_0^\delta \int_{\mathbb{R}} \partial_x(IuIv) Iv dxdt.
\end{aligned}$$

Proof. The proof of (34) is trivial and (35) follows by using integration by parts. In fact, for $b_1 = b_2$,

$$\begin{aligned}
& a_1 I_1 + 2a_2 I_2 + a_2 I_3 + a_1 I_4 = \\
& = a_1 \int_0^\delta \int_{\mathbb{R}} \partial_x (Iv)^2 Iv \, dx dt + 2a_2 \int_0^\delta \int_{\mathbb{R}} \partial_x (IuIv) Iv \, dx dt \\
& \quad + a_2 \int_0^\delta \int_{\mathbb{R}} \partial_x (Iu)^2 Iv \, dx dt + 2a_1 \int_0^\delta \int_{\mathbb{R}} \partial_x (IuIv) Iu \, dx dt \\
& = -a_1 \int_0^\delta \int_{\mathbb{R}} (Iv)^2 \partial_x Iu \, dx dt - a_2 \int_0^\delta \int_{\mathbb{R}} Iv \partial_x (Iu)^2 \, dx dt \\
& \quad - a_2 \int_0^\delta \int_{\mathbb{R}} (Iu)^2 \partial_x Iv \, dx dt - a_1 \int_0^\delta \int_{\mathbb{R}} Iu \partial_x (Iv)^2 \, dx dt \\
& = -a_1 \int_0^\delta \int_{\mathbb{R}} \partial_x [Iu(Iv)^2] \, dx dt - a_2 \int_0^\delta \int_{\mathbb{R}} \partial_x [Iv(Iu)^2] \, dx dt \\
& = 0.
\end{aligned}$$

□

REMARK 3. Note that, it is here in Lemma 4.2, where the restriction $b_1 = b_2$ on the coefficients appears. From here onwards we will use this restriction on the coefficients.

Using Lemma 4.2, $R(\delta)$ can be written as,

$$\begin{aligned}
R(\delta) = & \int_0^\delta \int_{\mathbb{R}} \partial_x \{(Iu)^2 - I(u^2)\} Iu \, dx dt + \frac{1}{b_1} \int_0^\delta \int_{\mathbb{R}} \partial_x \{(Iv)^2 - I(v^2)\} Iv \, dx dt \\
& + a_1 \int_0^\delta \int_{\mathbb{R}} \partial_x \{(Iv)^2 - I(v^2)\} Iu \, dx dt + a_2 \int_0^\delta \int_{\mathbb{R}} \partial_x \{(Iu)^2 - I(u^2)\} Iv \, dx dt \\
& + 2a_2 \int_0^\delta \int_{\mathbb{R}} \partial_x \{IuIv - I(uv)\} Iu \, dx dt + 2a_1 \int_0^\delta \int_{\mathbb{R}} \partial_x \{IuIv - I(uv)\} Iv \, dx dt.
\end{aligned} \tag{36}$$

Using Plancherel identity and Cauchy-Schwarz inequality as in [18] we get,

$$\begin{aligned}
& |R(\delta)| \\
& \leq C \left\{ \|\partial_x \{IuIv - I(uv)\}\|_{X_{0,-\frac{1}{2}^-}^\delta} \left(\|Iu\|_{X_{0,-\frac{1}{2}^+}^\delta} + \|Iv\|_{X_{0,-\frac{1}{2}^+}^\delta} \right) \right. \\
& \quad + \left(\|\partial_x \{(Iu)^2 - I(u^2)\}\|_{X_{0,-\frac{1}{2}^-}^\delta} + \|\partial_x \{(Iv)^2 - I(v^2)\}\|_{X_{0,-\frac{1}{2}^-}^\delta} \right) \|Iu\|_{X_{0,-\frac{1}{2}^+}^\delta} \\
& \quad \left. + \left(\|\partial_x \{(Iv)^2 - I(v^2)\}\|_{X_{0,-\frac{1}{2}^-}^\delta} + \|\partial_x \{(Iu)^2 - I(u^2)\}\|_{X_{0,-\frac{1}{2}^-}^\delta} \right) \|Iv\|_{X_{0,-\frac{1}{2}^+}^\delta} \right\}.
\end{aligned} \tag{37}$$

Now, using (37) and Proposition 4.1 the identity (33) yields the following almost conservation law,

$$\begin{aligned} \|Iu(\delta)\|_{L^2}^2 + \|Iv(\delta)\|_{L^2}^2 &\leq \|Iu(0)\|_{L^2}^2 + \|Iv(0)\|_{L^2}^2 \\ &+ CN^{-\frac{3}{4}+} \left\{ \|Iu\|_{X_{0,-\frac{1}{2}+}^\delta}^3 + \|Iu\|_{X_{0,-\frac{1}{2}+}^\delta}^2 \|Iv\|_{X_{0,-\frac{1}{2}+}^\delta} \right. \\ &\quad \left. + \|Iv\|_{X_{0,-\frac{1}{2}+}^\delta}^2 \|Iu\|_{X_{0,-\frac{1}{2}+}^\delta} + \|Iv\|_{X_{0,-\frac{1}{2}+}^\delta}^3 \right\}. \end{aligned} \quad (38)$$

Now we are in position to prove the global well-posedness result.

Proof of Theorem 1.2. To prove the theorem it is enough to show that the local solution to the IVP (24) can be extended to $[0, T]$ for arbitrary $T > 0$. To make the analysis easy we use the scaling introduced in the introduction. That is, if (u, v) solves the IVP (24) with initial data (ϕ, ψ) then for $1 > \lambda > 0$ so does (u^λ, v^λ) ; where $u^\lambda(x, t) = \lambda^2 u(\lambda x, \lambda^3 t)$, $v^\lambda(x, t) = \lambda^2 v(\lambda x, \lambda^3 t)$; with initial data $(\phi^\lambda, \psi^\lambda)$ given by $\phi^\lambda(x) = \lambda^2 \phi(\lambda x)$, $\psi^\lambda(x) = \lambda^2 \psi(\lambda x)$. Observe that, (u, v) exists in $[0, T]$ if and only if (u^λ, v^λ) exists in $[0, \frac{T}{\lambda^3}]$. So we are interested in extending (u^λ, v^λ) to $[0, \frac{T}{\lambda^3}]$.

Using Lemma 4.1 we have,

$$\begin{cases} \|I\phi^\lambda\|_{L^2} \leq C\lambda^{\frac{3}{2}+s}N^{-s} \|\phi\|_{H^s}, \\ \|I\psi^\lambda\|_{L^2} \leq C\lambda^{\frac{3}{2}+s}N^{-s} \|\psi\|_{H^s}. \end{cases} \quad (39)$$

$N = N(T)$ will be selected later, but let us choose $\lambda = \lambda(N)$ right now by requiring that,

$$\begin{cases} C\lambda^{\frac{3}{2}+s}N^{-s} \|\phi\|_{H^s} = \sqrt{\frac{\epsilon_0}{2}} \ll 1, \\ C\lambda^{\frac{3}{2}+s}N^{-s} \|\psi\|_{H^s} = \sqrt{\frac{\epsilon_0}{2}} \ll 1. \end{cases} \quad (40)$$

From (40) we get, $\lambda \sim N^{\frac{2s}{3+2s}}$ and using (40) in (39) we get,

$$\begin{cases} \|I\phi^\lambda\|_{L^2}^2 \leq \frac{\epsilon_0}{2} \ll 1 \\ \|I\psi^\lambda\|_{L^2}^2 \leq \frac{\epsilon_0}{2} \ll 1. \end{cases} \quad (41)$$

Therefore, if we choose ϵ_0 arbitrarily small then from Theorem 4.1 we see that the IVP (24) is well-posed for all $t \in [0, 1]$.

Now, using the almost conserved quantity (38), the identity (41) and Theorem 4.1, we get,

$$\begin{aligned} \|Iu^\lambda(1)\|_{L^2}^2 + \|Iv^\lambda(1)\|_{L^2}^2 &\leq \frac{\epsilon_0}{2} + \frac{\epsilon_0}{2} + CN^{-\frac{3}{4}+} \left[3\frac{\epsilon_0}{2} \left(\frac{\epsilon_0}{2}\right)^{1/2} \right] \\ &\leq \epsilon_0 + CN^{-\frac{3}{4}+} \epsilon_0. \end{aligned} \quad (42)$$

So, we can iterate this process $C^{-1}N^{\frac{3}{4}-}$ times before doubling $\|Iu^\lambda(t)\|_{L^2}^2 + \|Iv^\lambda(t)\|_{L^2}^2$. Using this process we can extend the solution to the time interval $[0, C^{-1}N^{\frac{3}{4}-}]$ by taking

$C^{-1}N^{\frac{3}{4}-}$ time steps of size $O(1)$. As we are interested in extending the solution to the time interval $[0, \frac{T}{\lambda^3}]$, let us select $N = N(T)$ such that, $C^{-1}N^{\frac{3}{4}-} \geq \frac{T}{\lambda^3}$. That is,

$$N^{\frac{3}{4}-} \geq C \frac{T}{\lambda^3} \sim TN^{\frac{-6s}{3+2s}}.$$

Therefore for large N , the existence interval will be arbitrarily large if we choose s such that $s > -3/10$. This completes the proof of the theorem. \square

5. ILL-POSEDNESS RESULT

As in the local well-posedness result, we consider the system (8). Note that we have dropped ‘ \sim ’ and retained the notation u, v, ϕ and ψ . Let $W = (u, v)^T$, then the IVP (8) can be written as,

$$\begin{cases} W_t + W_{xxx} + B(W)W_x = 0, \\ W(x, 0) = W_0(x), \end{cases} \quad (43)$$

where,

$$B(W) = \begin{pmatrix} au + dv & cu + bv \\ \tilde{a}u + \tilde{d}v & \tilde{c}u + \tilde{b}v \end{pmatrix}.$$

For fixed $\Phi \in \dot{H}^s(\mathbb{R}) \times \dot{H}^s(\mathbb{R})$, consider the solution $W = W^\delta$ of the IVP

$$\begin{cases} W_t + W_{xxx} + B(W)W_x = 0, \\ W(x, 0) = \delta\Phi(x), \quad \delta \in \mathbb{R}. \end{cases} \quad (44)$$

We will show that the flow-map $\delta\Phi \mapsto W^\delta(x, t)$ fails to be C^2 at the origin when $s < -3/4$. More precisely, we prove the following theorem which in turn implies Theorem 1.3.

THEOREM 5.1. *Let $s < -3/4$, then there is no $T > 0$ such that the flow-map*

$$\delta\Phi \mapsto W^\delta(t), \quad t \in (0, T]$$

be C^2 Frechet-differentiable at the origin from $\dot{H}^s(\mathbb{R}) \times \dot{H}^s(\mathbb{R})$ to $\dot{H}^s(\mathbb{R}) \times \dot{H}^s(\mathbb{R})$.

Proof. We prove it by contradiction. Suppose that the flow-map be C^2 differentiable at the origin. Using Duhamel’s formula we have,

$$W^\delta(x, t) = \delta U(t)\Phi(x) - \int_0^t U(t-t')B(W^\delta(x, t'))W_x^\delta(x, t') dt', \quad (45)$$

where $U(t)$ is the unitary group associated to the linear problem. Differentiating (45) with respect to δ we get,

$$\left. \frac{\partial W^\delta(x, t)}{\partial \delta} \right|_{\delta=0} = U(t)\Phi(x) := W_1(x, t), \quad (46)$$

$$\left. \frac{\partial^2 W^\delta(x, t)}{\partial \delta^2} \right|_{\delta=0} = -2 \int_0^t U(t-t')B(W_1(x, t'))W_{1x}(x, t') dt' := W_2(x, t). \quad (47)$$

Our assumption of the C^2 regularity of the flow-map at the origin implies that,

$$\|W_2(\cdot, t)\|_{\dot{H}^s \times \dot{H}^s} \leq C \|\Phi\|_{\dot{H}^s \times \dot{H}^s}^2. \quad (48)$$

Now, we look for $\Phi \in \dot{H}^s(\mathbb{R}) \times \dot{H}^s(\mathbb{R})$ so that (48) fails to hold whenever $s < -3/4$. For this, let $I_1 := [-N, -N + \alpha]$, $I_2 := [N + \alpha, N + 2\alpha]$ with $N \gg 1$ and $\alpha \ll 1$, define ϕ by the formula

$$\hat{\phi}(\xi) = \alpha^{-\frac{1}{2}} N^{-s} \{\chi_{I_1}(\xi) + \chi_{I_2}(\xi)\}, \quad (49)$$

and take $\Phi = (\phi, \phi)^T$.

It is easy to see that

$$\|\Phi\|_{\dot{H}^s \times \dot{H}^s} \sim 1. \quad (50)$$

Now we proceed to calculate $\|W_2(\cdot, t)\|_{\dot{H}^s \times \dot{H}^s}$. For this, let us first calculate $W_1(x, t)$ and $W_2(x, t)$.

From (46) we have ,

$$\widehat{W}_1^{(x)}(\xi, t) = e^{it\xi^3} \hat{\Phi}(\xi).$$

Therefore,

$$W_1(x, t) \sim \alpha^{-\frac{1}{2}} N^{-s} \left(\int_{\xi \in I_1 \cup I_2} e^{ix\xi + it\xi^3} d\xi \right).$$

From (47) we get,

$$\begin{aligned} W_2(x, t) &= -2 \int_0^t U(t-t') B(W_1(x, t')) W_{1x}(x, t') dt' \\ &= \int_{\mathbb{R}^2} \xi e^{ix\xi + it\xi^3} \hat{\phi}(\xi - \xi_1) \hat{\phi}(\xi_1) \left(\frac{a' \frac{e^{-3it\xi\xi_1(\xi-\xi_1)} - 1}{3\xi\xi_1(\xi-\xi_1)}}{b' \frac{e^{-3it\xi\xi_1(\xi-\xi_1)} - 1}{3\xi\xi_1(\xi-\xi_1)}} \right) d\xi_1 d\xi, \end{aligned}$$

where $a' = a + b + c + d$ and $b' = \tilde{a} + \tilde{b} + \tilde{c} + \tilde{d}$. Therefore, using (49) we get,

$$W_2(x, t) \sim \alpha^{-1} N^{-2s} \int_{\substack{\xi_1 \in I_1 \cup I_2 \\ \xi - \xi_1 \in I_1 \cup I_2}} \xi e^{ix\xi + it\xi^3} \left(\frac{a' h(\xi, \xi_1, t)}{b' h(\xi, \xi_1, t)} \right) d\xi d\xi_1,$$

where $h(\xi, \xi_1, t) = \frac{e^{-3it\xi\xi_1(\xi-\xi_1)} - 1}{\xi\xi_1(\xi-\xi_1)}$. Hence, formally we have,

$$\begin{aligned} \widehat{W}_2^{(x)}(\xi, t) &\sim \alpha^{-1} N^{-2s} \xi e^{it\xi^3} \left(\frac{a' \sum_{j=1}^3 \int_{A_j(\xi)} h(\xi, \xi_1, t) d\xi_1}{b' \sum_{j=1}^3 \int_{A_j(\xi)} h(\xi, \xi_1, t) d\xi_1} \right) \\ &:= \begin{pmatrix} p_1(\xi, t) + p_2(\xi, t) + p_3(\xi, t) \\ q_1(\xi, t) + q_2(\xi, t) + q_3(\xi, t) \end{pmatrix}, \end{aligned}$$

where,

$$\begin{cases} A_1(\xi) = \{\xi_1 : \xi_1 \in I_1, \xi - \xi_1 \in I_1\} \\ A_2(\xi) = \{\xi_1 : \xi_1 \in I_2, \xi - \xi_1 \in I_2\} \\ A_3(\xi) = \{\xi_1 : \xi_1 \in I_1, \xi - \xi_1 \in I_2 \text{ or } \xi_1 \in I_2, \xi - \xi_1 \in I_1\}. \end{cases}$$

Let $\widehat{f}_j^{(x)} = p_j$ and $\widehat{g}_j^{(x)} = q_j$, $j = 1, 2, 3$, then,

$$\begin{aligned} W_2(x, t) &= \begin{pmatrix} f_1(x, t) \\ g_1(x, t) \end{pmatrix} + \begin{pmatrix} f_2(x, t) \\ g_2(x, t) \end{pmatrix} + \begin{pmatrix} f_3(x, t) \\ g_3(x, t) \end{pmatrix} \\ &:= F_1(x, t) + F_2(x, t) + F_3(x, t). \end{aligned}$$

Let us find an upper bound for $\dot{H}^s \times \dot{H}^s$ norm of F_1 . If $\xi_1 \in I_1$ and $\xi - \xi_1 \in I_1$ then $|\xi_1| \sim |\xi - \xi_1| \sim |\xi| \sim N$ and we get,

$$\begin{aligned} \|F_1\|_{\dot{H}^s \times \dot{H}^s}^2 &= \int_{\mathbb{R}} |\xi|^{2s} (|p_1(\xi, t)|^2 + |q_1(\xi, t)|^2) d\xi \\ &\sim \int_{\mathbb{R}} |\xi|^{2s} \alpha^{-2} N^{-4s} |\xi|^2 (a'^2 + b'^2) \left| \int_{A_1(\xi)} \frac{e^{-3it\xi\xi_1(\xi-\xi_1)} - 1}{\xi\xi_1(\xi-\xi_1)} d\xi_1 \right|^2 d\xi \\ &\leq C\alpha^{-2} N^{-2s-4} (a'^2 + b'^2) \alpha^2 \alpha \\ &= C\alpha N^{-2s-4}. \end{aligned}$$

Therefore,

$$\|F_1\|_{\dot{H}^s \times \dot{H}^s}^2 \leq C\alpha^{1/2} N^{-s-2}.$$

Similarly,

$$\|F_2\|_{\dot{H}^s \times \dot{H}^s}^2 \leq C\alpha^{1/2} N^{-s-2}.$$

Now we find, with proper choice of α and N , the lower bound for the $\dot{H}^s \times \dot{H}^s$ norm of F_3 .

If $\xi_1 \in I_1$ and $\xi - \xi_1 \in I_2$ or $\xi_1 \in I_2$ and $\xi - \xi_1 \in I_1$ then $|\xi_1| \sim N$, $|\xi - \xi_1| \sim N$ and $|\xi| \sim \alpha$. Therefore,

$$|\xi\xi_1(\xi - \xi_1)| \sim N^2\alpha.$$

For $0 < \epsilon \ll 1$, choose N and α such that $N^2\alpha = N^{-\epsilon}$. Hence for $\xi_1 \in A_3(\xi)$ we have, for fixed t and large N ,

$$\left| \frac{e^{-3it\xi\xi_1(\xi-\xi_1)} - 1}{\xi\xi_1(\xi - \xi_1)} \right| \geq C > 0. \quad (51)$$

Now,

$$\|F_3\|_{\dot{H}^s \times \dot{H}^s}^2 \sim \int_{\alpha}^{3\alpha} |\xi|^{2s} \alpha^{-2} N^{-4s} |\xi|^2 (a'^2 + b'^2) \left| \int_{A_3(\xi)} \frac{e^{-3it\xi\xi_1(\xi-\xi_1)} - 1}{\xi\xi_1(\xi - \xi_1)} d\xi_1 \right|^2 d\xi.$$

Using the Mean Value Theorem for integrals and (51) it is easy to see that,

$$\left| \int_{A_3(\xi)} \frac{e^{-3it\xi\xi_1(\xi-\xi_1)} - 1}{\xi\xi_1(\xi - \xi_1)} d\xi_1 \right| \geq C|A_3(\xi)|. \quad (52)$$

Also, it is easy to see that $|A_3(\xi)| \sim \alpha$. In fact, $|A_3(\xi)| \leq |I_1| + |I_2| \leq 2\alpha$. On the other hand, if $\xi \in (\frac{7}{4}\alpha, \frac{9}{4}\alpha) \subset (\alpha, 3\alpha)$ then $[-N + \frac{\alpha}{4}, -N + \frac{3\alpha}{4}] \subset A_3(\xi)$ and we get $|A_3(\xi)| \geq \alpha/2$.

Now using (52), we get,

$$\|F_3\|_{\dot{H}^s \times \dot{H}^s}^2 \geq CN^{-4s}\alpha^{-2}(a'^2 + b'^2) \int_{\frac{7}{4}\alpha}^{\frac{9}{4}\alpha} |\xi|^{2s+2}\alpha^2 d\xi \sim CN^{-4s}\alpha^{2s+3}.$$

Therefore,

$$\|F_3\|_{\dot{H}^s \times \dot{H}^s}^2 \geq CN^{-2s}\alpha^{s+3/2}.$$

Observe that $\text{supp } \widehat{F}_1 \subseteq [-2N, -2N+2\alpha]$, $\text{supp } \widehat{F}_2 \subseteq [2N+2\alpha, 2N+4\alpha]$ and $\text{supp } \widehat{F}_3 \subseteq [\alpha, 3\alpha]$, which are clearly disjoint, therefore using (50) and (48) we obtain,

$$\begin{aligned} 1 &\sim \|\Phi\|_{\dot{H}^s \times \dot{H}^s}^2 \geq \|W_2(\cdot, t)\|_{\dot{H}^s \times \dot{H}^s}^2 \\ &\geq \|F_3(\cdot, t)\|_{\dot{H}^s \times \dot{H}^s}^2 \\ &\geq CN^{-2s}\alpha^{s+3/2} \\ &= CN^{-4s-3}N^{-\epsilon(s+3/2)}. \end{aligned}$$

Hence

$$N^{-4s-3} \leq CN^{\epsilon(s+3/2)}. \quad (53)$$

Case I: If $-3/2 < s < -3/4$ then $s + 3/2 > 0$ and (53) gives

$$N^{(-4s-3)-\epsilon(s+3/2)} \leq C$$

which is a contradiction for $N \gg 1$ if we choose $0 < \epsilon < (-4s - 3)/(s + 3/2)$.

Case II: If $s \leq -3/2$ then $s + 3/2 \leq 0$ and (53) gives $N^{-4s-3} \leq C$, which is again a contradiction for $N \gg 1$.

Hence for $s < -3/4$, (48) fails to hold for our choice of Φ , which completes the proof of the theorem. \square

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